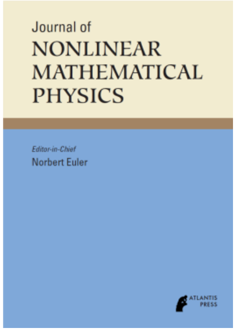

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ON STOCHASTIC DEFORMATIONS OF DYNAMICAL SYSTEMS

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I discuss the connection of the three different questions: The existence of the Gibbs steady state distributions for the stochastic differential equations, the notion and the existence of the conservation laws for such equations, and the convergence of the smooth random perturbations of dynamical systems to stochastic differential equations in the Ito sense. I show that in all cases one needs to include some additional term in the standard form of stochastic equation. I call such approach to describing the influence of the noise on the dynamical systems the "stochastic deformation" to distinguish it from the conventional "stochastic perturbation". I also discuss some consequences of this approach, in particular, a connection between the noise intensity and the temperature. This connection is known in physics (for the case of linear system of differential equations) as "fluctuation-dissipation theorem" (L. D. Landau and E. M. Lifshitz, Course of Theoretical Physics, Vol. 9, Statistical Physics. Part 2). In conclusion, I present an interesting physical example of the dynamics of magnetic dipole in a random magnetic field.

Keywords: Stochastic equations; Gibbs distribution.

Mathematics Subject Classification: 60H10, 65C30

1. Introduction

The stochastic differential equations (SDE) are the standard mathematical models describing the influence of noise on dynamics. To do this, as a rule, physicists simply add to the right side of the initial dynamical system the summand, linearly depending on the (vector) "white noise". Mathematicians, doing the same more rigorously, replace the derivatives by the Ito differentials and add the term linearly depending on the differential of the (vector) Winner process. The theory of the Ito stochastic differential equations is well developed (see, e.g., [4]), and, from the mathematical point of view, these equations do not cause suspicions (but it does not mean, of course, that they always can be solved or even investigated!). Nevertheless, some questions connected with the physical interpretation of such mathematical models remain obscure.

First of all, the Winner process $\xi(t)$ is only an approximative model of a real physical noise (which never has a uniform power spectrum, or, in other words, is never δ-correlated (i.e., never the white noise for which $\langle \xi(t), \xi(t') \rangle = \sigma^2 \delta(t - t')$). So the question is *do the equations (and/or their solutions), containing random, but smooth in time, perturbations and approximating in a sense the Winner process, approximate the Ito equations and/or their solutions?*

The answer (NO!) is known from the mid-1950s, and it was the starting point for criticism of the Ito theory and appearance of the alternative approaches to the stochastic differential equations [6]. In 1965, there appeared a remarkable work [1] in which it was shown how it is possible to resolve the problem: One only needs to add an additional term (Wong–Zakai term) expressed by means of a sensitivity operator, and there appears convergence required! It is remarkable, that this term is proportional to the intensity of the noise and disappears when the noise is "disconnected". The Wong–Zakai term and corresponding theorem was widely used in many subsequent mathematical works by different authors, but seems to be largely unknown to (mathematical) physicists.

The next question concerns the thermodynamic interpretation of the (solutions of the) SDE. This question is discussed in detail in a physical work [9] in the framework of several examples of models of physical devices. Among the three "thermodynamic requirements" discussed there, I select the "requirement ♯2", which asserts that "any system in equilibrium must satisfy the Gibbs distribution". This requirement assumes that, for a dynamical system considered, there exists some function E of phase variables, called "energy", such that the distribution of phase variables in equilibrium of the system with noise is proportional to $e^{-\alpha E}$, where the coefficient α is proportional to the "inverse temperature" of the system. It seems to be very difficult, for an arbitrary dynamical system, to define in pure mathematics terms what this "energy" is and, moreover, such function does not have to exist. This problem is discussed widely in a series of works of P Ao (see [13] and references therein). In these works, we find, in particular, that there exists some "canonical" form of the stochastic differential equation, containing clearly defined "energy", or potential. The proof of this assertion, presented there, is formal, and, as the author pointed out, analytical obstacles may be encountered.

But even if one considers the stochastic equation in the Ao form, it might be shown that the Gibbs distribution with corresponding energy is not, in general, the steady state for the process. The Ao canonical form includes, as particular cases, types of systems widely used in physics, such as Hamiltonian or gradient systems. But, even if one confines oneself to consideration of only such kinds of systems, one finds that, for sufficiently arbitrary noise perturbation, the Gibbs distribution does not satisfy the equation for equilibrium state (the stationary case of the Kolmogorov–Fokker–Planck, a.k.a. KFP, equation).

It is wonderful, however, that if one adds an additional term to the system with noise, the steady state **will be** a Gibbs state. Note that the additional term in this case does not coincide with the Wong–Zakai term, but in both cases one deals with stochastic deformation rather than with stochastic perturbation. In addition to the energy, the Gibbs distribution contains a scalar parameter (called temperature in physics), which must be related with the noise intensity. Such relation is known in physics as an *Einstein relation*, and is one of the assertions of "fluctuation-dissipation theorem" [3]. This physical theorem is proved by physicists only for linear systems with pure additive noise. I obtain this connection as a consequence of the existence of the Gibbs steady state.

The last, but not least, problem concerns the noise influence on the dynamical systems with connections. Mathematically, dynamical systems with connections are considered as systems on manifolds. There exists different mathematical generalizations of the "Winner processes" to the processes on manifolds, see, e.g., [10]. All approaches to the stochastic differential geometry known to me are mathematically very interesting, but rather complicated and not too convenient for describing and investigating physical systems.

On the other hand, the corresponding manifolds often appear as level surfaces of some set of "motion integrals" of the initial dynamical system in $\mathbb{R}^n$. It would be very convenient if (random) trajectories of such systems perturbed with noise would belong to the same surfaces. It is not too difficult to formulate the conditions on the (not random) forces acting on the system with integrals, the forces which ensure conservation of the initial integrals.

But what should one do with the noise? I introduce a very natural and (in my opinion) useful notion of the motion integral for the Ito stochastic differential equation. I was extremely surprised when I found out that, even if the dynamical system in the field of dynamic forces has an integral of motion common for a given class of forces, then the substitution of noise instead of these forces destroys, generally, this integral.

I will show that if one considers both the above-mentioned stochastic deformations (the Wong–Zakai and the "Gibbs" ones), the integrals are conserved! It is very important to note that the deformation terms themselves do not depend on the form of integrals and, moreover, even of the integrals' existence.

Note that, as was mentioned above, apart from the Ito theory, there exists widely used, mainly by physicists, the Stratanovich approach to stochastic differential equations. In this approach, some of the problems listed above are absent, but other troubles appear... I do not discuss here the Stratanovich theory, because it is the Ito construction which is (in my opinion) mainly accepted by mathematicians. One can find a very clear discussion of both these theories in lectures [11]. In particular, in these lectures it is shown that the solution of the Ito equation with Wong–Zakai term is exactly the solution of the corresponding Stratanovich equation. Note also that the conservation of integrals for the Stratanovich equation is the direct consequence of the "chain rule" for the Stratanovich stochastic differential.

It is important that, if one wants to satisfy all requirements, namely, the convergence of the "smoothly perturbed" system to the the stochastic one, the existence of the Gibbs steady state for the stochastic equation, and conservation of the initial integrals, one needs to deform the initial dynamic system by adding the additional term which is the difference between the Wong–Zakai and "Gibbs" terms. It is interesting that in many physical examples, e.g., in the example considered in Sec. 5, this difference is zero...

The structure of this article is as follows. In Sec. 2, I give most of definitions and, for completeness, present the Wong–Zakai approximation theorem in its original form. In Sec. 3, I discuss the existence of Gibbs steady states for different types of dynamical systems with noise and a connection between the noise intensity and the temperature (fluctuation-dissipation theorem). In Sec. 4, I define the notion of a motion integral of stochastic differential equation and show that the same deformation term as in the previous section provides inheritance of integrals of the initial dynamical system. Finally, in Sec. 5, I present the physically interesting example of the dynamical system — the magnetic dipole in the (random) magnetic field. This illustrates our abstract constructions.

2. Dynamic Systems with Noise and the Wong–Zakai Theorem

Let us consider the (system of the) ordinary differential equations in $\mathbb{R}^n$ of the form

$$\frac{dx}{dt} = F(x), \quad \text{where } x \in \mathbb{R}^n. \tag{2.1}$$

In what follows, for the map $F: \mathbb{R}^n \to \mathbb{R}^n$, we assume the uniform Lipschitz condition:

$$\|F(x) - F(y)\| \le C_F \|x - y\| \quad \text{for any } x, y \in \mathbb{R}^n. \tag{2.2}$$

It is well known that the condition (2.2) provides with the global existence of the solution to the Cauchy problem for Eq. (2.1).

Let further $w(t)$ be the standard n-dimensional Winner process and $G: \mathbb{R}^n \to \mathrm{Mat}(n)$ a matrix-valued function called *sensitivity operator* in what follows. We assume that the map G also satisfies the uniform Lipschitz condition, i.e.,

$$\|G(x) - G(y)\| \le C_G \|x - y\| \quad \text{for any } x, y \in \mathbb{R}^n. \tag{2.3}$$

The stochastic Ito differential equation of the form

$$dX = F(X)dt + \sigma G(X)dw \tag{2.4}$$

is usually used as a mathematical model describing how the noise influences the dynamical system (2.1) and is called *the stochastic perturbation* of the Eq. (2.1). The coefficient $\sigma \in \mathbb{R}_+$ in this equation is called *the intensity of noise*. The conditions (2.2), (2.3) yield the existence of solutions of the Eq. (2.4) as realizations of the random process [4]. Note that Eq. (2.4), and its solutions converge, in a sense, as $\sigma \to 0$ to Eq. (2.1), and its solutions.

Together with Eq. (2.4) I will consider the system

$$\frac{dx}{dt} = F(x) + \sigma G(x) f(t), \tag{2.5}$$

where f is, optionally random, vector-function satisfying some continuity conditions. This equation may be considered as a model of action of some "regular" force on the dynamical system (2.1).

For a given real physical system, the structure of Eq. (2.5) is conditioned by physical considerations. It seems that the same considerations predict the form of Eq. (2.4): One only replaces a "regular" force by a "noise". But not all is so simple. Apart from the convergence mentioned above, one might expect that if a random, but "smooth", force f is close to the "derivative" of the Winner process — "white noise" — the Eqs. (2.4), (2.5) are close to each other and so are their solutions. But this expectation is, generally, wrong, as very simple examples show (see, e.g., [1,6]). Indeed, the situation is described by the following Wong–Zakai theorem [1].[a]

Let Ω be a probability space for the Winner process and $\{w^{(j)}: \Omega \times \mathbb{R} \to \mathbb{R}^n \mid j \in \mathbb{N}\}$, be a sequence of the random processes on this space such that

(i) For almost all $\omega \in \Omega$ and all $t \in [a, b] \subset \mathbb{R}$, we have $w^{(j)}(\omega, t) \to w(\omega, t)$ as $j \to \infty$.

[a]In [1], this theorem was proved for the 1-dimensional case. The multi-dimensional and many other generalizations were considered in many subsequent works of different authors.

(ii) There exists the bounded random value k on Ω such that $\|w^{(j)}(\omega, t)\| \le k(\omega)$ for almost all ω and all $t \in [a, b]$.

(iii) The functions $w^{(j)}(\omega, t)$ has, for almost all ω, the piecewise continuous in t derivatives $f^{(j)}(\omega, t) := \frac{dw^{(j)}(\omega,t)}{dt}$ on $[a, b]$.

In this case I say that the sequence $w^{(j)}$ *approximates* the Winner process w. In what follows I use the usual notion $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ for the vector of partial derivatives.

Theorem 2.1. *Let F, G and the vector-functions K_w, $K_{w,i} = G_{jk}\frac{\partial G_{ik}}{\partial x_j}$ satisfy the uniform Lipshitz condition. Let the random vector $x^{(a)}$ be independent of the vectors $w(t) - w(a)$, where $t \in [a, b]$. Let further $x^{(j)}(t)$ be the solution of Eq. (2.5) with f replaced by $f^{(j)}$ and with the initial condition $x^{(j)}(a) = x^{(a)}$.*

Then for almost all ω, there exists a limit

$$X(t) := \lim_{j\to\infty} x^{(j)}(t)$$

and the random process $X(t)$ is the solution of the stochastic Ito equation of the form

$$dX = \left(F(X) + \frac{\sigma^2}{2}K_w\right)dt + \sigma G(X)dw \tag{2.6}$$

with the initial condition $X(a) = x^{(a)}$.

The term $\frac{\sigma^2}{2}K_w$ which distinguishes the Eqs. (2.4) and (2.6) is known as the *Wong–Zakai term*. It is important to note that, same as for Eq. (2.4), the solutions of Eq. (2.6) converge to the solutions of (2.1) as the noise intensity tends to zero. I call the equations of type (2.6) *the stochastic deformation* of the Eq. (2.1) to distinguish between these two stochastic equations, both dealing with noise influence on the dynamical system.

One can see that the stochastic deformation is physically more reasonable than the stochastic perturbation since the "white noise" is just a mathematically convenient approximation of the real physical "noise". In what follows I show that the equations of type (2.6) are more reasonable from other points of view as well. Note that in what follows I will consider the deformations not necessary coinciding with the Wong–Zakai ones.

3. The Gibbs Distribution as the Steady State of Dynamic Systems with Noise

In this section, I consider the existence problem of the Gibbs steady state for the stochastic differential equations. The Gibbs form of the equilibrium distributions of physical systems is their essential property, and hence their mathematical models must preserve it [9]. This property is closely related to the questions of existence of the "energy" for the system considered and the relation between the noise intensity and temperature, which is the essence of the so-called "fluctuation-dissipation theorem" [3].

3.1. *The Gibbs states*

First of all, it is well known [4] that if $X(t)$ is a solution of Eq. (2.4), then the density of the distribution $p(x, t)$ of this random value satisfies the Kolmogorov–Fokker–Planck equation (KFP)

$$\frac{\partial p}{\partial t} = \left(\nabla, \frac{\sigma^2}{2}(\nabla Bp - pF)\right), \quad B := GG^T, \tag{3.1}$$

with the initial condition $p(x, 0) = p_0(x)$ being the distribution of the initial condition $X(0) = X_0$ for the Eq. (2.4). The "*steady state*", or the "*equilibrium state*" of the system (2.4) is, if any exists, the limit

$$p_e(x) = \lim_{t\to\infty} p(x, t).$$

This limit must satisfy the stationary Kolmogorov–Fokker–Planck equation

$$\left(\nabla, \frac{\sigma^2}{2}\nabla Bp_e - p_e F\right) = 0. \tag{3.2}$$

Note that I consider the *family* of the stochastic equations depending on a parameter σ, so the solutions of the Eqs. (2.4), (3.1), as well as the steady state, depend on this parameter. We say that the system has a *Gibbs steady state* if there exists a function $E(x)$ (the *free energy of the system*) independent on the parameter σ and such that

$$p_e(x; \sigma) = C(\sigma)e^{\alpha(\sigma)E(x)}, \tag{3.3}$$

where $C(\sigma)$ is a normalizing factor. In what follows, the function $\alpha(\sigma)$ is supposed to be meromorphic in a vicinity of the point $\sigma = 0$.

A simple analysis of the 1-dimensional case shows that, in general, the steady state is not of the Gibbs form.

Proposition 3.1. *Let F, G satisfy conditions of the Wong–Zakai theorem, $E_0(x) := \int^x \frac{F(\xi)}{G(\xi)^2}d\xi$ and*

$$\int_{-\infty}^{\infty} \frac{1}{G(x)^2}e^{2\sigma^{-2}E_0(x)}dx < \infty.$$

Suppose additionally that

$$\frac{\ln G^2}{E_0} \neq \text{const.} \tag{3.4}$$

Then the steady state of the 1*-dimensional dynamical system with noise is of the Gibbs form if and only if $G(x) = \text{const}$. In this case, $E(x) = E_0(x)$ and $\alpha = 2\sigma^{-2}$.*

Proof. The generic solution of the Eq. (3.2) in the 1-dimensional case is of the form

$$p_e(x) = \frac{1}{G^2}e^{2\sigma^{-2}E_0(x)}\left(c_1(\sigma) + c_2(\sigma)\int^x e^{-2\sigma^{-2}E_0(\eta)}d\eta\right), \tag{3.5}$$

where c_1, c_2 are constants of integration. It is easy to see that, due to conditions on F and G, the function

$$\frac{1}{G^2}e^{2\sigma^{-2}E_0(x)}\int^x e^{-2\sigma^{-2}E_0(\eta)}d\eta$$

is not integrable on $\mathbb{R}$. Hence, one must set $c_2(\sigma) = 0$. Then

$$\frac{1}{\alpha(\sigma)} \ln p_e = c_3(\sigma) - \frac{\ln G^2(x)}{\alpha(\sigma)} + \frac{2}{\sigma^2 \alpha(\sigma)} E_0(x) = \frac{\ln C(\sigma)}{\alpha(\sigma)} + E(x).$$

The last equality is possible if and only if $G(x) = \text{const}$ and $\sigma^2 \alpha(\sigma) = 2$ or $\ln G^2 = \gamma E_0$ with some constant $\gamma \neq 0$. The second option contradicts the condition (3.4) due to the definition of E_0. $\square$

Let us make some remarks concerning the conditions and assertions of this proposition. First of all, if one omits the condition (3.4), the additional possibility for existence of the Gibbs steady state appears: Namely, $\ln G^2 = \gamma E_0$ and $\alpha = 2\sigma^{-2} - \gamma$. As we will see in what follows, such a relation between α and σ is physically meaningless, whereas the relation $\alpha = 2\sigma^{-2}$ is known as the *Einstein relation* and is the assertion of *fluctuation-dissipation theorem*.

On the other hand, the condition $\ln G^2 = \text{const}\ F$ seems to be strange: What is the reason for such a rigid relation between dynamical force and sensitivity operator describing the noise influence on the system?

From this point of view (that of reasonableness and strangeness of conditions), the condition $G = \text{const}$ looks no less strange: Why the noise influence must not depend on system state? As a result, one can say that *in more or less generic case, the* 1*-dimensional equation* (3.2) *does not possess a Gibbs solution*. Unfortunately, I do not know any analog of such assertion in the multi-dimensional case, but I believe it must exist.

To solve the problems in the 1-dimensional case, let us consider the *deformation*, instead of perturbation, of the initial dynamical system of the type

$$dX = \left(F(X) + \frac{\sigma^2}{2} K_G \right) dt + \sigma G(X) dw, \quad \text{with } K_G = \frac{\partial G^2}{\partial X}. \tag{3.6}$$

The corresponding Fokker–Planck equation is of the form (3.2) in which the force F is replaced by the term $F + \frac{1}{2}\sigma^2 K_G$. As a result, we see that the steady state is the solution of the 1-dimensional equation

$$\left(\frac{\sigma^2}{2} G^2 p_e' - p_e F \right)' = 0. \tag{3.7}$$

By repeating the arguments from the proof of Proposition 3.1 it is easy to see that the solution of Eq. (3.7) is of the form

$$p_e(x) = C(\sigma) e^{2\sigma^{-2} E_0(x)},$$

and hence *is of the Gibbs form* for *all* admissible F and G!

The multi-dimensional case is more complicated due to the fact that it is quite difficult to define what is the energy in the generic case. Nevertheless, the following is true:

Theorem 3.1. *Consider the stochastic deformation of the dynamical system* (2.1) *of the form* (3.6) *with*

$$(K_G)_i := \left(\nabla G G^T \right)_i := \frac{\partial}{\partial x_k} G_{kj} G_{ij}. \tag{3.8}$$

Let further there exist a function $E_0(x)$ and an anti-symmetric matrix field $A(x)$ such that

(1) *the function $e^{2\sigma^{-2}E_0(x)}$ is integrable on $\mathbb{R}^n$;*
(2) *we have*

$$F = (GG^T + A)\nabla E_0; \tag{3.9}$$

(3) *the matrix field $A(x)$ satisfies the relation*

$$(\nabla A)_j := \frac{\partial}{\partial x_k} A_{kj}(x) = 0. \tag{3.10}$$

Then there exists the Gibbs steady state of the form (3.3) with $E = E_0$ and $\alpha(\sigma) = 2\sigma^{-2}$.

Proof. The proof is trivial: One must simply substitute the function of the form (3.3) into Eq. (3.2) in which the force F is replaced by $F + \frac{1}{2}\sigma^2 K_G$. Then, using notation $B = GG^T$, one obtains an equation connecting α, σ, and E with F and G:

$$\left(\nabla, \frac{\alpha\sigma^2}{2} B\nabla E - (A+B)\nabla E_0\right) + \alpha\left(\nabla E, \frac{\alpha\sigma^2}{2} B\nabla E - B\nabla E_0\right) = 0.$$

If one puts here $E_0 = E$ and $\frac{\alpha\sigma^2}{2} = 1$, this relation takes the form

$$-\alpha(\nabla E_0, A\nabla E_0) - (\nabla, A\nabla E_0) = 0,$$

which is satisfied identically due to the assumption of the theorem on the field A. □

Comments. First of all, the relation (3.9) is known in physical literature [13]. Moreover, in the series of works of P Ao it is established that, in some sense, this relation *is not* a condition, i.e., for given F and B, there exist a function E and an anti-symmetric matrix field A such that this relation holds. This assertion is not, nevertheless, the theorem, and, even if the relation (3.9) holds, it is *not sufficient* to claim existence of the Gibbs steady state for a stochastic perturbation of the dynamical system. But if one adds the *deformation term* of the form $\frac{1}{2}\sigma^2 K_G$ with vector field K_G defined by relation (3.8), the steady state will be of Gibbs type.

Note also that the deformation (3.6) does not coincide with the Wong–Zakai deformation (2.6), and hence due to the Wong–Zakai theorem, if one approximates the white noise, there is no convergence of the corresponding perturbation of the dynamical system to the Ito equation (3.6). But if one considers, instead of Eq. (2.5), *the deformation* of the dynamical system of the form

$$\frac{dx}{dt} = F + \sigma G(x) f(t) + \frac{\sigma^2}{2} K^G \tag{3.11}$$

with the field $(K^G)_i := G_{ik}\frac{\partial G_{jk}}{\partial x_j}$, then it follows from Theorem 2.1 that the approximations of type (3.11) converge to the Ito equation (3.6). This fact shows that it is useful to consider deformations not only for the Ito equations, but for the dynamical systems also. We will see in what follows that considering such deformations is also reasonable from other points of view.

Finally, let us point once more that all deformation terms depend only on the sensitivity matrix, but not on the dynamic field F. It seems that a given dynamical system admits different stochastic perturbations (and deformations). But the steady state depends on sensitivity operator because the *energy* E depends on it. In the next subsection, I consider the family of *Hamilton systems with dissipation*, which has common steady state after stochastic deformation is performed with it.

3.2. *Hamiltonian systems with dissipation*

Let $n = 2k$ and Ω be the standard symplectic form on $\mathbb{R}^{2k}$. Let $H(x)$ be a sufficiently smooth function on $\mathbb{R}^{2k}$ and $F_H := \Omega\nabla H$. System (2.1) with $F = F_H$ is called the *Hamiltonian system*. It is not difficult to show that the stochastic deformations of the Hamiltonian systems cannot have the Gibbs steady states with $E = H$. Indeed, by substituting the distribution of the form $p_e \sim e^{-\alpha H}$ into Eq. (3.2) for the corresponding stochastic deformation, one obtains

$$\left(\nabla, p_e\left(\frac{\alpha\sigma^2}{2}B\nabla H + \Omega\nabla H\right)\right) = 0, \tag{3.12}$$

where $B = GG^T$. Since B is positive definite, the equality

$$\frac{\alpha\sigma^2}{2}B\nabla H + \Omega\nabla H = 0$$

is impossible, and hence it follows from Eq. (3.12) that B and H are connected by the equation

$$\alpha(B\nabla H, \nabla H) - (\nabla, B\nabla H) = 0$$

which is also impossible because α does not depend on coordinates and must depend on σ.

Let us now introduce "dissipation". The vector field Φ such that $(\nabla H, \Phi) < 0$ is said to be a *dissipation*, and the system (2.1) with $F = \Omega\nabla H + \gamma\Phi$ is said to be a *Hamiltonian system with dissipation*. The parameter $\gamma > 0$ is called the *dissipation coefficient*.

Theorem 3.2. *The stochastic deformation of the form* (3.6) *of the Hamiltonian system with dissipation has a Gibbs steady state of the form*

$$p_e \sim \exp(-\alpha H) \tag{3.13}$$

if and only if the dissipation, energy, and sensitivity are connected by the relation

$$\Phi = -B\nabla H \tag{3.14}$$

whereas the noise intensity, dissipation coefficient, and parameter α *are connected as follows*:

$$\sigma^2 = \frac{2\gamma}{\alpha}. \tag{3.15}$$

Proof. The Kolmogorov–Fokker–Planck equation for the steady state of the stochastic deformation of the Hamiltonian system with dissipation reads

$$\left(\nabla, \frac{\sigma^2}{2}B\nabla p_e - p_e\left(\Omega\nabla H + \gamma\Phi\right)\right) = 0. \tag{3.16}$$

If p_e is of the form (3.13), this equation takes the form

$$\left(-\alpha\nabla H, \frac{\alpha\sigma^2}{2}B\nabla H + \Omega\nabla H + \gamma\Phi\right) + \left(\nabla, \left(\frac{\alpha\sigma^2}{2}B\nabla H + \Omega\nabla H + \gamma\Phi\right)\right) = 0. \tag{3.17}$$

Since Ω is the standard symplectic matrix, Eq. (3.17) implies

$$\left(-\alpha\nabla H, \frac{\alpha\sigma^2}{2}B\nabla H + \gamma\Phi\right) + \left(\nabla, \left(\frac{\alpha\sigma^2}{2}B\nabla H + \gamma\Phi\right)\right) = 0. \tag{3.18}$$

Since $\alpha \to \infty$ as $\sigma \to 0$, the terms in the right-hand side of Eq. (3.18) are of different order in σ, and hence must be equal to zero separately, i.e.,

$$\begin{aligned} &\left(\nabla H, \frac{\alpha\sigma^2}{2}B\nabla H + \gamma\Phi\right) = 0,\\ &\left(\nabla, \left(\frac{\alpha\sigma^2}{2}B\nabla H + \gamma\Phi\right)\right) = 0. \end{aligned} \tag{3.19}$$

Due to the condition on B and Φ (namely, B is positive definite and Φ is a dissipation), the first of these relations may hold only if

$$\Phi = -\frac{\alpha\sigma^2}{2\gamma}B\nabla H. \tag{3.20}$$

Since the dissipation does not depend on σ and γ, we are done. □

Note that the relation (3.14) between the energy and dissipation is also the essence of the fluctuation-dissipation theorem (FDT). Comment: the friction (dissipation) is related to the energy and sensitivity; for quadratic Hamiltonians (= linear systems) and sensitivity G independent of x, this is precisely FDT contained in the textbook by Landau and Lifshitz [3]; in general case, all physicists know this, anyway, as a folklore.

4. Integrals of Motion of the Stochastic Differential Equation

In this section I deal with the systems of type (2.5). A function $S(x) \neq \text{const}$ is called *the common integral* of the system (2.5), if for any "external force" f and any solution $x_f(t)$ of the Eq. (2.5), the function $S(x(t))$ is constant in t. It is evident that S is the common integral if and only if it satisfies the relations

$$(\nabla S, F) = 0, \quad G^T\nabla S = 0. \tag{4.1}$$

If there exist several independent common integrals $\{S_j \,|\, j = 1, \ldots, k\}$ for the system (2.5), then for any f, each trajectory of the system belongs to some common level set of these integrals $\Sigma(c_1, \ldots, c_k) = \{x \in \mathbb{R}^n \,|\, S_j(x) = c_j \text{ for } j = 1, \ldots, k\}$.

What happens with the integrals when the force is replaced by noise? The following definition seems to be very natural, but I did not find it in the literature.

Definition 4.1. A function $S \neq \text{const}$ is said to be an *integral* of the Ito stochastic differential equation (2.4) if, for any σ and any solution $X_\sigma(t)$ of this equation, the Ito differential of the random function $S(X_\sigma(t))$ vanishes, i.e.,

$$dS(X_\sigma(t)) = 0. \tag{4.2}$$