Invariants of Lie Algebras Extended Over Commutative Algebras Without Unit

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We establish results about the second cohomology with coefficients in the trivial module, symmetric invariant bilinear forms, and derivations of a Lie algebra extended over a commutative associative algebra without unit. These results provide a simple unified approach to a number of questions treated earlier in completely separated ways: periodization of semisimple Lie algebras (Anna Larsson), derivation algebras, with prescribed semisimple part, of nilpotent Lie algebras (Benoist), and presentations of affine Kac–Moody algebras.

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0. Introduction

In this paper we consider current Lie algebras, i.e., Lie algebras of the form $L \otimes A$, where L is a Lie algebra, A is a commutative associative algebra, and the multiplication in $L \otimes A$ is defined by the formula

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab$$

for any $x, y \in L$, $a, b \in A$. Note that A can be considered as the algebra of functions on the spectrum of A, and $L \otimes A$ can therefore be interpreted as the algebra of "currents", as physicists say, on this spectrum.

Berezin and Karpelevich [3] were among the first to study certain class of representations of current algebras over local finite-dimensional commutative algebras A. In that, somewhat obscure, paper they showed that cohomology of such algebras can be reduced to cohomology of L. In what follows we will consider other types of commutative algebras A and the description of cohomology in our case is more involved and interesting. We are primarily interested in the second cohomology of $L \otimes A$ with trivial coefficients, the space of symmetric invariant bilinear forms on $L \otimes A$, and the algebras of derivations of $L \otimes A$.

These invariants were determined for numerous particular cases of current Lie algebras (see, for example, [28]), the general formulae for the second homology with trivial coefficients in terms of invariants of L and A were obtained in [12], [29] and [24], and similar formulae for the space of symmetric invariant bilinear forms and derivation algebras were obtained in [29] and [30], respectively.

So why return to these settled questions? In all considerations until now, the algebra A was supposed to have a unit. However, there are many interesting examples of current algebras where A is not unital. For example, in [17], the so-called periodization of semisimple Lie algebras $\mathfrak{g}$ was considered, which is nothing but $\mathfrak{g} \otimes t\mathbb{C}[t]$. It is known that the second homology of any nilpotent Lie algebra with trivial coefficients has interpretation in terms of presentation of the algebra, so allowing A to be nilpotent allows us to obtain presentation of $L \otimes A$ irrespective of the properties of L.

It turns out that elementary arguments similar to those in [30] allow us to extend the above mentioned results to the case of non-unital A. In particular, concerning the second cohomology and symmetric invariant bilinear forms, we provide another proof, considerably shorter than all the previous ones even in the case of unital A.

The contents of this paper are as follows. In Secs. 1–3 we establish the general formulae for 2-cocycles, symmetric invariant bilinear forms, and get partial results about derivations of the current Lie algebras, respectively. This is followed by applications: in Sec. 4 we reprove the result from [17] about presentations of periodizations of the semisimple Lie algebras. In passing, we also mention how to derive from our results the theorem from [2] about semisimple components of the derivation algebras of certain current Lie algebras, and the Serre defining relations between Chevalley generators of the non-twisted affine Kac–Moody algebra.

In all these cases, the absence of unit in A is essential. All these proofs are significantly shorter than the original ones, and reveal various almost trivial, but so far unnoticed or unpublished, links between different concepts and results. These links are, perhaps, the main virtue of this paper.

It seems that most if not everything considered here can be extended in a straightforward way to twisted, Leibniz and super settings, but we will not venture into this, at least for now.

Notation and Conventions

All algebras and vector spaces are defined over an arbitrary field K of characteristic different from 2 and 3, unless stated otherwise (some of the results are valid in characteristic 3, but we will not go into this).

In what follows, L denotes a Lie algebra, A is an associative commutative algebra.

Given an L-module M, let $B^n(L, M)$, $Z^n(L, M)$, $C^n(L, M)$ and $H^n(L, M)$ denote the space of nth degree coboundaries, cocycles, cochains, and cohomology of L with coefficients in M, respectively (we will be mainly interested in the particular cases of degree 2 and the trivial module K, or degree 1 and the adjoint module or its dual). Note that $C^2(L, K)$ is the space of all skew-symmetric bilinear forms on L. The space of all symmetric bilinear forms on L will be denoted as $S^2(L, K)$.

Let $\mathcal{Z}(L)$, $[L, L]$ and $\mathrm{Der}(L)$ denote the center, the commutant (the derived algebra), and the Lie algebra of derivations of L, respectively. Similarly, let $\mathrm{Ann}(A) = \{a \in A \mid Aa = 0\}$

and AA denote the annulator and the square of A, respectively, and let $HC^*(A)$ denotes its cyclic cohomology.

A bilinear form $\varphi : L \times L \to K$ is said to be *cyclic* if

$$\varphi([x, y], z) = \varphi([z, x], y)$$

for any $x, y, z \in L$. Note that if φ is symmetric, this condition is equivalent to the *invariance* of the form φ:

$$\varphi([x, y], z) + \varphi(y, [x, z]) = 0,$$

while if φ is skew-symmetric, the notions of cyclic and invariant forms differ. Let $\mathcal{B}(L)$ denote the space of all symmetric bilinear invariant (=cyclic) forms on L.

Similarly, a bilinear form $\alpha : A \times A \to K$ is said to be *cyclic* if

$$\alpha(ab, c) = \alpha(ca, b),$$

and *invariant* if

$$\alpha(ab, c) = \alpha(a, bc)$$

for any $a, b, c \in A$. If the form α is symmetric, it is cyclic if and only if it is invariant.

1. The Second Cohomology

Theorem 1.1. *Let L be a Lie algebra, A an associative commutative algebra, and at least one of L and A be finite-dimensional. Then each cocycle in $Z^2(L \otimes A, K)$ can be represented as the sum of decomposable cocycles $\varphi \otimes \alpha$, where $\varphi : L \times L \to K$ and $\alpha : A \times A \to K$ are of one of the following 8 types:*

(i) $\varphi([x, y], z) + \varphi([z, x], y) + \varphi([y, z], x) = 0$ *and α is cyclic,*
(ii) φ *is cyclic and $\alpha(ab, c) + \alpha(ca, b) + \alpha(bc, a) = 0$,*
(iii) $\varphi([L, L], L) = 0$,
(iv) $\alpha(AA, A) = 0$,

where each of these 4 types splits into two subtypes: with φ skew-symmetric and α symmetric, and with φ symmetric and α skew-symmetric.

Proof. Each cocycle $\Phi \in Z^2(L \otimes A, K)$, being an element of

$$\mathrm{End}(L \otimes A \otimes L \otimes A, K) \simeq \mathrm{End}(L \otimes L, K) \otimes \mathrm{End}(A \otimes A, K),$$

can be written in the form $\Phi = \sum_{i \in I} \varphi_i \otimes \alpha_i$, where $\varphi_i : L \times L \to K$ and $\alpha_i : A \times A \to K$ are bilinear maps, and I is a finite set of indices (this is the place where the assumption of finite-dimensionality is needed). Using this representation, write the cocycle equation for an arbitrary triple $x \otimes a$, $y \otimes b$, $z \otimes c$, where $x, y, z \in L$, $a, b, c \in A$:

$$\sum_{i \in I} \varphi_i([x, y], z) \otimes \alpha_i(ab, c) + \varphi_i([z, x], y) \otimes \alpha_i(ca, b) + \varphi_i([y, z], x) \otimes \alpha_i(bc, a) = 0. \quad (1.1)$$

Symmetrizing this equality with respect to x, y, we get:

$$\sum_{i \in I} (\varphi_i([x, z], y) + \varphi_i([y, z], x)) \otimes (\alpha_i(bc, a) - \alpha_i(ca, b)) = 0.$$

On the other hand, cyclically permuting x, y, z in (1.1) and summing up the 3 equalities obtained, we get:

$$\sum_{i \in I} (\varphi_i([x,y],z) + \varphi_i([z,x],y) + \varphi_i([y,z],x)) \otimes (\alpha_i(ab,c) + \alpha_i(bc,a) + \alpha_i(ca,b)) = 0.$$

Applying Lemma 1.1 from [30] to the last two equalities, we get a partition of the index set $I = I_1 \cup I_2 \cup I_3 \cup I_4$ such that

$$
\begin{array}{lll}
\varphi_i([x,z],y) + \varphi_i([y,z],x) = 0, & \varphi_i([x,y],z) + \varphi_i([z,x],y) + \varphi_i([y,z],x) = 0 & \text{for } i \in I_1, \\
\varphi_i([x,z],y) + \varphi_i([y,z],x) = 0, & \alpha_i(ab,c) + \alpha_i(bc,a) + \alpha_i(ca,b) = 0 & \text{for } i \in I_2, \\
\varphi_i([x,y],z) + \varphi_i([z,x],y) + \varphi_i([y,z],x) = 0, & \alpha_i(bc,a) - \alpha_i(ca,b) = 0 & \text{for } i \in I_3, \\
\alpha_i(bc,a) - \alpha_i(ca,b) = 0, & \alpha_i(ab,c) + \alpha_i(bc,a) + \alpha_i(ca,b) = 0 & \text{for } i \in I_4.
\end{array}
$$

It is obvious that if the characteristic of K is different from 3, then $\varphi_i([L,L],L) = 0$ for $i \in I_1$, and $\alpha_i(AA, A) = 0$ for $i \in I_4$. It is obvious also that $\varphi_i \otimes \alpha_i$ satisfies the cocycle equation (1.1) for each $i \in I_1, I_2, I_3, I_4$.

Now write the condition of skew-symmetry of Φ:

$$\sum_{i \in I} \varphi_i(x,y) \otimes \alpha_i(a,b) + \varphi_i(y,x) \otimes \alpha_i(b,a) = 0 \tag{1.2}$$

and symmetrize it with respect to x, y:

$$\sum_{i \in I} (\varphi_i(x,y) - \varphi_i(y,x)) \otimes (\alpha_i(a,b) - \alpha_i(b,a)) = 0$$

$$\sum_{i \in I} (\varphi_i(x,y) + \varphi_i(y,x)) \otimes (\alpha_i(a,b) + \alpha_i(b,a)) = 0.$$

From the last two equalities, using again Lemma 1.1 from [30], we see that each set I_1, I_2, I_3, I_4 can be split further into two subsets, one having skew-symmetric φ_i and symmetric α_i, and the other one having symmetric φ_i and skew-symmetric α_i. $\qquad\square$

Remark 1.1. As all our bilinear maps are K-valued, the cocycles of the form $\varphi \otimes \alpha$ are, of course, just products of bilinear maps $\varphi\alpha$. However, we have retained the symbol $\otimes$, to make it easier to track dependence on the more general situation of [30].

Remark 1.2. The second subdivision in the statement of Theorem 1.1, which follows from equality (1.2), is merely a manifestation of the vector space isomorphism

$$C^2(L \otimes A, K) \simeq (S^2(L,K) \otimes C^2(A,K)) \oplus (C^2(L,K) \otimes S^2(A,K)). \tag{1.3}$$

Let $d\Omega$ be the 2-coboundary defined by a given linear map $\Omega : L \otimes A \to K$. The latter can be written in the form $\Omega = \sum_{i \in I} \omega_i \otimes \beta_i$ for some linear maps $\omega_i : L \to K$ and $\beta_i : A \to K$. Then

$$d\Omega(x \otimes a, y \otimes b) = \sum_{i \in I} \omega_i([x,y]) \otimes \beta_i(ab),$$

i.e., coboundaries always lie in the direct summand $C^2(L,K) \otimes S^2(A,K)$. Consequently, nonzero cocycles from different direct summands in (1.3) can never be cohomologically

dependent, and the cocycles from the direct summand $S^2(L, K) \otimes C^2(A, K)$ are cohomologically independent if and only if they are linearly independent.

One may try to formulate Theorem 1.1 as a statement about $H^2(L \otimes A, K)$, but in full generality this will lead only to cumbersome complications. In each case of interest, one can easily obtain an information about the cohomology. For example, assuming A contains a unit, one immediately sees that cocycles of type (i) necessarily have φ skew-symmetric and α symmetric, cocycles of type (ii) necessarily have φ symmetric and α skew-symmetric, and cocycles of type (iv) vanish. This leads to a known formula for $H^2(L \otimes A, K)$, where the cocycles of type (i) contribute to the term $H^2(L, K) \otimes A^*$, the cocycles of type (ii) contribute to the term $\mathcal{B}(L) \otimes HC^1(A)$, and the cocycles of type (iii) are non-essential (in terminology of [29]).

Another, more concrete, application is given in Sec. 4.

2. Symmetric Invariant Bilinear Forms

Theorem 2.1. *Let L be a Lie algebra, A an associative commutative algebra, and at least one of L and A be finite-dimensional. Then each symmetric invariant bilinear form on $L \otimes A$ can be represented as a sum of decomposable forms $\varphi \otimes \alpha$, $\varphi : L \times L \to K$, $\alpha : A \times A \to K$ of one of the following 6 types:*

(i) *both φ and α are cyclic,*
(ii) *$\varphi([L, L], L) = 0$,*
(iii) *$\alpha(AA, A) = 0$,*

where each of these 3 types splits into two subtypes: with both φ and α symmetric, and with both φ and α skew-symmetric.

Proof. The proof is absolutely similar to that of Theorem 1.1. As in the proof of Theorem 1.1, we may write a symmetric invariant bilinear form Φ on $L \otimes A$ as $\sum_{i \in I} \varphi_i \otimes \alpha_i$ for suitable bilinear maps $\varphi_i : L \times L \to K$ and $\alpha_i : A \times A \to K$. The invariance condition, written for a given triple $x \otimes a$, $y \otimes b$, $z \otimes c$, reads:

$$\sum_{i \in I} \varphi_i([x, y], z) \otimes \alpha_i(ab, c) + \varphi_i([x, z], y) \otimes \alpha_i(ca, b) = 0. \tag{2.1}$$

Symmetrizing this with respect to x, y, we get:

$$\sum_{i \in I} (\varphi_i([x, z], y) + \varphi_i([y, z], x)) \otimes \alpha_i(ca, b) = 0.$$

Hence the index set can be partitioned $I = I_1 \cup I_2$ in such a way that

$$\varphi_i([x, z], y) + \varphi_i([y, z], x) = 0$$

for any $i \in I_1$, and $\alpha_i(AA, A) = 0$ for any $i \in I_2$. Then (2.1) can be rewritten as

$$\sum_{i \in I_1} \varphi_i([x, y], z) \otimes (\alpha_i(ab, c) - \alpha_i(ca, b)) = 0.$$

Hence there is a partition $I_1 = I_{11} \cup I_{12}$ such that $\varphi_i([L, L], L) = 0$ for any $i \in I_{11}$, and $\alpha_i(ab, c) = \alpha_i(ac, b)$ for any $i \in I_{12}$.

The condition of symmetry of Φ:

$$\sum_{i \in I} \varphi_i(x,y) \otimes \alpha_i(a,b) - \varphi_i(y,x) \otimes \alpha_i(b,a) = 0,$$

being symmetrized with respect to x, y, allows us to partition further each of the sets I_{11}, I_{12}, I_2 into two subsets, one having both φ_i and α_i symmetric, and the other one having both φ_i and α_i skew-symmetric. $\qquad\qquad\qquad\square$

This generalizes [29, Theorem 4.1], where a similar statement is proved for A unital.

3. Derivations

Naturally, one may try to apply the same approach to description of the derivations of a given current algebra $L \otimes A$ (for unital A, see [30, Corollary 2.2]). Indeed, each derivation D of $L \otimes A$, being an element of

$$\mathrm{End}(L \otimes A, L \otimes A) \simeq \mathrm{End}(L, L) \otimes \mathrm{End}(A, A),$$

can be expressed in the form $D = \sum_{i \in I} \varphi_i \otimes \alpha_i$, where $\varphi_i : L \to L$ and $\alpha_i : A \to A$ are linear maps. The condition that D is a derivation, written for an arbitrary pair $x \otimes a$ and $y \otimes b$, where $x, y \in L$ and $a, b \in A$, reads:

$$\sum_{i \in I} \varphi_i([x,y]) \otimes \alpha_i(ab) - [\varphi_i(x), y] \otimes \alpha_i(a)b - [x, \varphi_i(y)] \otimes a\alpha_i(b) = 0.$$

Symmetrizing this equality with respect to a, b (this is equivalent to symmetrization with respect to x, y):

$$\sum_{i \in I} ([\varphi_i(x), y] - [x, \varphi_i(y)]) \otimes (a\alpha_i(b) - b\alpha_i(a)) = 0,$$

we get a partition of the index set I into two subsets with conditions $[\varphi_i(x), y] = [x, \varphi_i(y)]$ and $a\alpha_i(b) = b\alpha_i(a)$, respectively. But as there are only two variables in each of L and A, no other symmetrization is possible, so the last equality is all what we can get in this way.

The failure of this method can be also explained by looking at the simple example of the Lie algebra $sl(2) \otimes tK[t]$. In $sl(2)$, fix a basis $\{e_-, h, e_+\}$ with multiplication

$$[h, e_-] = -e_-, \quad [h, e_+] = e_+, \quad [e_-, e_+] = h. \tag{3.1}$$

It is easy to check that the map defined, for any $f(t) \in tK[t]$, by the formulae

$$e_- \otimes f(t) \mapsto e_- \otimes t\left(\frac{df(t)}{dt} - f(t)\right)$$

$$e_+ \otimes f(t) \mapsto e_+ \otimes t\left(\frac{df(t)}{dt} + f(t)\right)$$

$$h \otimes f(t) \mapsto h \otimes t\frac{df(t)}{dt}$$

is a derivation of $sl(2) \otimes tK[t]$. It is obvious that this map is not a decomposable one, i.e., of the form $\varphi \otimes \alpha$ for some $\varphi : sl(2) \to sl(2)$ and $\alpha : tK[t] \to tK[t]$. But for this approach to succeed, all the maps in question should be representable in such a way in the end.

However, under additional assumption on $L \otimes A$, we can derive information about $\mathrm{Der}(L \otimes A)$ from the results of the preceding sections, using the relationship between $H^1(L, L^*)$, $H^2(L, K)$, and $\mathcal{B}(L)$. In the literature, this relationship was noted many times in a slightly different form, and goes back to the classical works of Koszul and Hochschild–Serre [14]. Namely, there is an exact sequence

$$0 \to H^2(L, K) \xrightarrow{u} H^1(L, L^*) \xrightarrow{v} \mathcal{B}(L) \xrightarrow{w} H^3(L, K) \qquad (3.2)$$

where for the representative $\varphi \in Z^2(L, K)$ of a given cohomology class, we have to take the class of $u(\varphi)$, the latter being given by

$$(u(\varphi)(x))(y) = \varphi(x, y)$$

for any $x, y \in L$, v is sending the class of a given cocycle $D \in Z^1(L, L^*)$ to the bilinear form $v(D) : L \times L \to K$ defined by the formula

$$v(D)(x, y) = D(x)(y) + D(y)(x),$$

and w is sending a given symmetric bilinear invariant form $\psi : L \times L \to K$ to the class of the cocycle $\omega \in Z^3(L, K)$ defined by

$$\omega(x, y, z) = \psi([x, y], z)$$

(see, for example, [5], where a certain long exact sequence is obtained, of which this one is the beginning, and references therein for many earlier particular versions; this exact sequence was also established in [24, Proposition 7.2] with two additional terms on the right).

In the case where $L \simeq L^*$ as L-modules, the sequence (3.2) provides a way to evaluate $H^1(L, L)$ given $H^2(L, K)$ and $\mathcal{B}(L)$. The L-module isomorphism $L \simeq L^*$ implies the existence of a symmetric invariant non-degenerate form $\langle \cdot, \cdot \rangle$ on L. In terms of this form, u is sending the class of a given cocycle $\varphi \in Z^2(L, K)$ to the class of the cocycle $u(\varphi) \in Z^1(L, L)$ defined by

$$\langle (u(\varphi))(x), y \rangle = \varphi(x, y),$$

and v is sending the class of a given cocycle $D \in Z^1(L, L)$ to the bilinear form $v(D) : L \times L \to K$ defined by the formula

$$v(D)(x, y) = \langle D(x), y \rangle + \langle x, D(y) \rangle.$$

Turning to current Lie algebras, we will make even stronger assumption: that $L \simeq L^*$ and $A \simeq A^*$. Then, utilizing the results of preceding sections about $H^2(L \otimes A, K)$ and $\mathcal{B}(L \otimes A)$, we will derive results about $\mathrm{Der}(L \otimes A)$.

In the literature, given $H^1(L, L^*)$, the space $H^2(L, K)$ was computed for various Lie algebras L (see, for example, [28], [5] and references therein). Here we utilize this connection in the other direction.

Theorem 3.1. *Let L be a non-Abelian Lie algebra, A an associative commutative algebra, both L and A finite-dimensional and with symmetric invariant non-degenerate bilinear form.*

Then each derivation of $L \otimes A$ can be represented as the sum of decomposable linear maps $d \otimes \beta$, where $d : L \to L$ and $\beta : A \to A$ are of one of the following types:

(i) $d([x,y]) = \lambda([d(x),y] + [x,d(y)])$, $\beta(ab) = \mu\beta(a)b$ for certain $\lambda, \mu \in K$ such that $\lambda\mu = 1$,

(ii) $d([x,y]) = \lambda[d(x),y]$, $\beta(ab) = \mu(\beta(a)b + a\beta(b))$ for certain $\lambda, \mu \in K$ such that $\lambda\mu = 1$,

(iii) $[d(x),y] + [x,d(y)] = 0$, $\beta(AA) = 0$, $\beta(a)b = a\beta(b)$,

(iv) $d([L,L]) = 0$, $[d(x),y] + [x,d(y)] = 0$, $\beta(a)b = a\beta(b)$,

(v) $d([L,L]) = 0$, $[d(x),x] = 0$, $\beta(a)b + a\beta(b) = 0$,

(vi) $[d(x),x] = 0$, $\beta(AA) = 0$, $\beta(a)b + a\beta(b) = 0$,

(vii) $d([L,L]) = 0$, $d(L) \subseteq \mathcal{Z}(L)$,

(viii) $d([L,L]) = 0$, $\beta(A) \subseteq \mathrm{Ann}(A)$,

(ix) $d(L) \subseteq \mathcal{Z}(L)$, $\beta(AA) = 0$,

(x) $\beta(AA) = 0$, $\beta(A) \subseteq \mathrm{Ann}(A)$.

Proof. By abuse of notation, let $\langle \cdot , \cdot \rangle$ denote a symmetric invariant non-degenerate bilinear form both on L and A. Obviously, the tensor product of these forms defines a symmetric invariant non-degenerate bilinear form on $L \otimes A$, for which by even bigger abuse of notation we will use the same symbol:

$$\langle x \otimes a, y \otimes b \rangle = \langle x, y \rangle \langle a, b \rangle.$$

We have $L^* \simeq L$ as L-modules, $A^* \simeq A$ as A-modules, and $(L \otimes A)^* \simeq L \otimes A$ as $L \otimes A$-modules.

As a vector space, $H^1(L \otimes A, L \otimes A)$ can be represented as the direct sum of $\mathrm{Ker}\, v$ and $\mathrm{Im}\, v$, and the exact sequence (3.2) tells that $\mathrm{Ker}\, v = \mathrm{Im}\, u$ and $\mathrm{Im}\, v = \mathrm{Ker}\, w$.

By Theorem 1.1, $H^2(L \otimes A, K)$ is spanned by cohomology classes which can be represented by decomposable cocycles $\varphi \otimes \alpha$ for appropriate $\varphi : L \times L \to L$ and $\alpha : A \times A \to K$. For each such pair φ and α, there are unique linear maps $d : L \to L$ and $\beta : A \to A$ such that

$$\langle d(x), y \rangle = \varphi(x,y) \tag{3.3}$$

for any $x, y \in L$, and

$$\langle \beta(a), b \rangle = \alpha(a,b) \tag{3.4}$$

for any $a, b \in A$. Hence the decomposable linear map $d \otimes \beta : L \otimes A \to L \otimes A$ satisfies

$$\langle (d \otimes \beta)(x \otimes a), y \otimes b \rangle = (\varphi \otimes \alpha)(x \otimes a, y \otimes b),$$

i.e., coincides with $u(\varphi \otimes \alpha)$. Thus, $\mathrm{Im}\, u$ is spanned by cohomology classes whose representatives are decomposable derivations.

Similarly, by the proof of Theorem 2.1, $\mathcal{B}(L \otimes A)$ is spanned by decomposable elements $\varphi \otimes \alpha$, and $\mathrm{Ker}\, w$ is spanned by such elements of types (ii) and (iii), i.e., either $\varphi([L,L],L) = 0$ or $\alpha(AA,A) = 0$. Again, for each such element we can find $d : L \to L$ and $\beta : A \to A$ satisfying (3.3) and (3.4) respectively. Furthermore, we may assume that for each such $\varphi \otimes \alpha$, the maps φ and α are either both symmetric, or both skew-symmetric, and hence both d and β are either self-adjoint or skew-self-adjoint, respectively, with respect

to $\langle \cdot, \cdot \rangle$. In both cases we have:

$$\langle (d \otimes \beta)(x \otimes a), y \otimes b \rangle + \langle x \otimes a, (d \otimes \beta)(y \otimes b) \rangle$$
$$= \langle d(x), y \rangle \langle \beta(a), b \rangle + \langle x, d(y) \rangle \langle a, \beta(b) \rangle = 2 \langle d(x), y \rangle \langle \beta(a), b \rangle$$
$$= 2\varphi(x, y) \otimes \alpha(a, b) \tag{3.5}$$

for any $x, y \in L$ and $a, b \in A$.

The condition $\varphi([L, L], L) = 0$ ensures that $d([L, L]) = 0$, and an equivalent condition $\varphi(L, [L, L]) = 0$ ensures that

$$\langle [d(x), z], y \rangle = -\langle d(x), [y, z] \rangle = 0,$$

implying $d(L) \subseteq \mathcal{Z}(L)$, and hence $d \otimes \beta$ is a derivation of $L \otimes A$. Quite analogously, the condition $\alpha(AA, A) = \alpha(A, AA) = 0$ implies also that $d \otimes \beta$ is a derivation of $L \otimes A$. The equality (3.5) ensures that v maps the cohomology class of this derivation to $2\varphi \otimes \alpha$. Thus $\operatorname{Im} v$ is spanned by images of cohomology classes whose representatives are decomposable derivations.

Putting all this together, we see that $H^1(L \otimes A, L \otimes A)$ is spanned by the cohomology classes whose representatives are decomposable derivations. As inner derivations of $L \otimes A$ are, obviously, also spanned by decomposable inner derivations $ad\,(x \otimes a) = ad\,x \otimes R_a$, where R_a is the multiplication on $a \in A$, any derivation of $L \otimes A$ is representable as the sum of decomposable derivations.

The rest is easy. The condition that $d \otimes \beta$ is a derivation, reads:

$$d([x, y]) \otimes \beta(ab) - [d(x), y] \otimes \beta(a)b - [x, d(y)] \otimes a\beta(b) = 0 \tag{3.6}$$

for any $x, y \in L$, $a, b \in A$. Symmetrizing (3.6) as in the beginning of this section, we see that either $[d(x), y] = [x, d(y)]$ for any $x, y \in L$, or $\beta(a)b = a\beta(b)$ for any $a, b \in A$. The equation (3.6) is equivalent to

$$d([x, y]) \otimes \beta(ab) - [d(x), y] \otimes (\beta(a)b + a\beta(b)) = 0$$

in the first case, and to

$$d([x, y]) \otimes \beta(ab) - ([d(x), y] + [x, d(y)]) \otimes \beta(a)b = 0$$

in the second case. Now trivial case-by-case considerations involving vanishing and linear dependence of the linear operators occurring as tensor product factors in these two equalities, produce the final list of derivations. $\qquad\square$

Theorem 3.1 can be applied, for example, to the Lie algebra $\mathfrak{g} \otimes tK[t]/(t^n)$, $n > 2$, where $\mathfrak{g}$ is a semisimple finite-dimensional Lie algebra over any field of characteristic 0, to obtain a very short proof of the result of Benoist [2] about realization of any semisimple Lie algebra as semisimple part of the Lie algebra of derivations of a nilpotent Lie algebra (another short proof with direct calculation of $\operatorname{Der}(\mathfrak{g} \otimes tK[t]/(t^3))$ follows from [19, Proposition 3.5]). Indeed, as noted, for example, in [1, Lemma 2.2], $tK[t]/(t^n)$ possesses a symmetric nondegenerate invariant bilinear form B, hence $\mathfrak{g} \otimes tK[t]/(t^n)$ possesses such a form (being the product of the Killing form on $\mathfrak{g}$ and B), so Theorem 3.1 is applicable. As $\mathfrak{g}$ is perfect and centerless, the derivations of types (iv), (v), (vii), (viii), (ix) vanish. The remaining types

can be handled, for example, by appealing to results of [15], [9] or [19], which imply that in the case $\mathfrak{g} \not\simeq sl(2)$, the corresponding mappings d vanish also for types (ii) and (iii), and for the rest of the types are either inner derivations of $\mathfrak{g}$, or multiplications by scalar. Then, performing elementary calculations with conditions imposed on β's in the remaining types, and rearranging the obtained spaces of derivations, we get the following isomorphism of vector spaces:

$$\mathrm{Der}(\mathfrak{g} \otimes tK[t]/(t^n)) \simeq (\mathfrak{g} \otimes K[t]/(t^n)) \oplus (\mathrm{End}(\mathfrak{g})/\mathfrak{g}) \oplus K. \tag{3.7}$$

Elements of the first summand are assembled from types (i) and (x), and act on $\mathfrak{g} \otimes tK[t]/(t^n)$ as the Lie multiplication by an element of $\mathfrak{g}$ and the associative commutative multiplication by an element of $K[t]/(t^n)$. Elements of the second summand are assembled from types (i), (vi) and (x), and act by the rule

$$x \otimes t \;\mapsto F(x) \otimes t^{n-1}$$
$$x \otimes t^k \mapsto 0 \quad \text{if } k \geq 2,$$

where $x \in \mathfrak{g}$, $F \in \mathrm{End}(\mathfrak{g})$, and elements of $\mathfrak{g}$ are assumed to be embedded into $\mathrm{End}(\mathfrak{g})$ as inner derivations. Elements of the third, one-dimensional, summand are assembled from types (i) and (vi), and are proportional to the following derivation:

$$x \otimes t \;\mapsto\; x \otimes t^{n-2}$$
$$x \otimes t^2 \mapsto 2x \otimes t^{n-1}$$
$$x \otimes t^k \mapsto 0 \quad \text{if } k \geq 3,$$

where $x \in \mathfrak{g}$.

All this implies that, as a Lie algebra, $\mathrm{Der}(\mathfrak{g} \otimes tK[t]/(t^n))$ splits into the semidirect sum of the semisimple part isomorphic to $\mathfrak{g}$ (identified with the part $\mathfrak{g} \otimes 1$ of the first summand in (3.7)), and the nilpotent radical consisting of $\mathfrak{g} \otimes tK[t]/(t^n)$ from the first summand, and the whole second and third summands.

The case $\mathfrak{g} = sl(2)$ can be treated separately and easily.

4. Periodization of Semisimple Lie Algebras

For a given Lie algebra L, its *periodization* is an $\mathbb{N}$-graded Lie algebra, with component in each degree isomorphic to L. In other words, the periodization of L is $L \otimes tK[t]$.

In [17], Anna Larsson studied periodization of semisimple finite-dimensional Lie algebras $\mathfrak{g}$ over any field K of characteristic 0. She proved that, unless $\mathfrak{g}$ contains direct summands isomorphic to $sl(2)$, its periodization possesses a presentation with only quadratic relations. Since generators and relations of (generalized graded) nilpotent Lie algebras can be interpreted as the first and second homology, Larsson's statement can be formulated in homological terms.

Note that in [18] this result was generalized to some classes of Lie superalgebras. We do not touch upon superalgebras here, and it seems to be an interesting task to tackle the results and questions from [18] from this paper's viewpoint. Interest in periodizations of Lie superalgebras, and whether they admit generators subject to quadratic relations, arose from

an earlier work of Löfwall and Roos [23] about some amazing Hopf algebras (for further details, see [17] and [18]).

As noted in [17], the whole space $H^*(\mathfrak{g} \otimes t\mathbb{C}[t], \mathbb{C})$ was studied by much more sophisticated methods in the celebrated paper by Garland and Lepowsky [11] (actually, a particular case interesting for us here was already sketched in [10]; the case of $\mathfrak{g} = sl(2)$ was also treated in [8, p. 233]). Garland and Lepowsky determined the eigenvectors of the Laplacian on the corresponding Chevalley–Eilenberg (co)chain complex. However, to extract from [11] exact results about (co)homology of interest requires nontrivial combinatorics with the Weyl group, as demonstrated in [13], and case-by-case analysis for each series of the simple Lie algebras. Here we derive results for the second cohomology in a uniform way from the results of Sec. 1 using elementary methods providing an alternative short proof of Larsson's result. Also, our approach clearly shows why the case of $sl(2)$ is exceptional.

Theorem 4.1 (Larsson). *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field K of characteristic 0. Then*

$$H^2(\mathfrak{g} \otimes tK[t], K) \simeq C^2(\mathfrak{g}, K)/B^2(\mathfrak{g}, K) \oplus (\oplus S_\alpha),$$

where the second direct sum is taken over all simple direct summands of $\mathfrak{g}$ isomorphic to $sl(2)$, and each S_α is a certain 5-dimensional space of symmetric bilinear forms on the corresponding direct summand. The basic cocycles can be chosen among cocycles of the form

$$\Phi(x \otimes t^i, y \otimes t^j) = \begin{cases} \varphi(x, y) & \text{if } i = j = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where $x, y \in \mathfrak{g}$ and φ is a skew-symmetric bilinear form on $\mathfrak{g}$, and the cocycles Ψ whose only non-vanishing values on all pairs of the simple direct summands of $\mathfrak{g}$ are determined by the formula

$$\Psi(x \otimes t^i, y \otimes t^j) = \begin{cases} \psi(x, y) & \text{if } i = 1, \ j = 2, \\ -\psi(x, y) & \text{if } i = 2, \ j = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

where x, y belong to the corresponding $sl(2)$-direct summand, and ψ is a symmetric bilinear form on this direct summand satisfying

$$\psi(e_-, e_+) = \frac{1}{2}\psi(h, h) \tag{4.3}$$

in the standard $sl(2)$-basis (3.1).

Proof. Consider the cocycles appearing in Theorem 1.1 for $L = \mathfrak{g}$ and $A = tK[t]$ case-by-case.

Cocycles of type (i). Writing the cyclicity condition of α for the triple t^{i-1}, t^j, t, we get

$$\alpha(t^{i+j-1}, t) = \alpha(t^i, t^j) \tag{4.4}$$

for any $i \geq 2, j \geq 1$.

(ia). φ is skew-symmetric, thus $\varphi \in Z^2(\mathfrak{g}, K)$, and α is symmetric. Since $H^2(\mathfrak{g}, K) = 0$, it follows that $\varphi = d\omega$ for some linear map $\omega : \mathfrak{g} \to K$. Define a linear map $\Omega : \mathfrak{g} \otimes tK[t] \to K$ by setting, for any $x \in \mathfrak{g}$,

$$\Omega(x \otimes t^i) = \begin{cases} 0 & \text{if } i = 1, \\ \omega(x)\alpha(t^{i-1}, t) & \text{if } i \geq 2. \end{cases}$$

Then, taking (4.4) into account, we get $\varphi \otimes \alpha = d\Omega$, i.e., cocycles of this type are trivial.

(ib). φ is symmetric, α is skew-symmetric.

If $\mathfrak{g} = sl(2)$, direct calculation shows that the space of corresponding φ's coincides with the space of all symmetric bilinear forms on $sl(2)$ satisfying the condition (4.3), and hence is 5-dimensional (an equivalent calculation is contained in [6, Theorem 6.5]). This case is exceptional, as shows the following

Lemma 4.1 (Dzhumadil'daev–Bakirova). *Let* $\mathfrak{g} \not\simeq sl(2)$ *be simple. Then any symmetric bilinear form* φ *on* $\mathfrak{g}$ *satisfying*

$$\varphi([x,y],z) + \varphi([z,x],y) + \varphi([y,z],x) = 0 \tag{4.5}$$

for any $x, y, z \in \mathfrak{g}$ *vanishes.*

This Lemma is proved in [7] by considering the Chevalley basis of $\mathfrak{g}$ and performing computations with the corresponding root system. In [6] and [7], symmetric bilinear forms satisfying the condition (4.5) are called *commutative* 2-*cocycles* and arise naturally in connection with classification of algebras satisfying skew-symmetric identities. We will give a different proof which stresses the connection with yet other notions and results.

Proof. Consider a map from the space of bilinear forms on a Lie algebra L to the space of linear maps from L to L^*, by sending a bilinear form φ to the linear map $D : L \to L^*$ such that $D(x)(y) = \varphi(x, y)$. It is easy to see that a symmetric bilinear form φ satisfying (4.5) maps to a linear map D satisfying

$$D([x,y]) = -y \bullet D(x) + x \bullet D(y),$$

where $\bullet$ denotes the standard L-action on the dual module L^* (this is completely analogous to the embedding of $H^2(L, K)$ into $H^1(L, L^*)$ mentioned in Sec. 3).

For any finite-dimensional simple Lie algebra, there is an isomorphism of $\mathfrak{g}$-modules $\mathfrak{g} \simeq \mathfrak{g}^*$, and we have an embedding of the space of bilinear forms in question into the space of linear maps $D : \mathfrak{g} \to \mathfrak{g}$ satisfying the condition

$$D([x,y]) = -[D(x), y] - [x, D(y)].$$

Such maps are called *antiderivations* and were studied in several papers. In particular, in [15, Theorem 5.1] it is proved that central simple classical Lie algebras of dimension > 3 do not have nonzero antiderivations (generalizations of this result in different directions obtained further in the series of papers by Filippov, of which [9] is the latest, and in [19]). Hence, these Lie algebras do not have nonzero symmetric bilinear forms satisfying (4.5) either. □

Coupled with the well-known fact that $H^2(\mathfrak{g}, K) = 0$, Lemma 4.1 implies that *any* (symmetric, skew-symmetric or mixed) bilinear form φ on $\mathfrak{g} \not\simeq sl(2)$ satisfying the cocycle equation (4.5), is a 1-coboundary. This can be also compared with the fact that Leibniz cohomology (and, in particular, the second Leibniz cohomology with trivial coefficients) of $\mathfrak{g}$ vanishes (for homological version, see [26, Proposition 2.1] or [25]). The condition for a bilinear form φ to be a Leibniz 2-cocycle can be expressed as

$$\varphi([x,y],z) + \varphi([z,x],y) - \varphi(x,[y,z]) = 0.$$

In the general case, where $\mathfrak{g}$ is a direct sum of simple ideals $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, it is easy to see that $\varphi(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ for $i \neq j$, and hence the space of symmetric bilinear forms on $\mathfrak{g}$ satisfying (4.5) decomposes into the direct sum of appropriate spaces on each of $\mathfrak{g}_i$. The latter are determined by (4.3) if $\mathfrak{g}_i \simeq sl(2)$ and vanishes otherwise.

Now, turning to α's, and permuting i and j in (4.4), we get $\alpha(t^i, t^j) = 0$ for all $i, j \geq 2$ and $\alpha(t^k, t) = 0$ for all $k \geq 3$. Conversely, it is easy to see that a skew-symmetric map α satisfying these conditions is cyclic. Hence, the space of skew-symmetric cyclic maps on $tK[t]$ is 1-dimensional and each cocycle of this type can be written in the form (4.2).

Cocycles of type (ii).
(iia). φ is skew-symmetric, α is symmetric.

Lemma 4.2. *Any skew-symmetric cyclic bilinear form on $\mathfrak{g}$ vanishes.*

Proof. The proof is almost identical to the proof of the well-known fact that any skew-symmetric invariant bilinear form on $\mathfrak{g}$ vanishes (see, for example, [4, Chapter 1, §6, Exercises 7(b) and 18(a, b)]). Namely, let φ be a skew-symmetric cyclic form on $\mathfrak{g}$. First, let $\mathfrak{g}$ be simple. There is a linear map $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\varphi(x,y) = \langle \sigma(x), y \rangle$, where $\langle \cdot, \cdot \rangle$ is the Killing form on $\mathfrak{g}$. Then, for any $x, y, z \in \mathfrak{g}$:

$$\langle \sigma([x,y]), z \rangle = \varphi([x,y], z) = -\varphi(y, [z,x]) = -\langle \sigma(y), [z,x] \rangle = -\langle [x, \sigma(y)], z \rangle.$$

Hence σ anticommutes with each $ad\,x$ and, in particular, $[\sigma(x), x] = 0$ for any $x \in \mathfrak{g}$. But then by [2, Lemme 2] (or by more general results from [19]), σ belongs to the centroid of $\mathfrak{g}$. Hence σ is a scalar and necessarily vanishes.

In the general case where $\mathfrak{g}$ is the direct sum of simple ideals $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, it is easy to see that $\varphi(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ for any $i \neq j$. But by just proved φ vanishes also on each $\mathfrak{g}_i$, and hence vanishes on the whole of $\mathfrak{g}$. $\square$

(iib). φ is symmetric and α is skew-symmetric, so $\alpha \in HC^1(tK[t])$. There is an isomorphism of graded algebras

$$HC^*(K[t]) \simeq HC^*(K) \oplus HC^*(tK[t])$$

(for a general relationship between cyclic homology of augmented algebra and its augmentation ideal, see [22, §4]; the cohomological version can be obtained in exactly the same way). On the other hand,

$$HC^*(K[t]) \simeq HC^*(K) \oplus \text{(terms concentrated in degree 0)}$$

(see, for example, [21, §3.1.7]). Hence $HC^1(tK[t]) = 0$. Of course, the vanishing of $HC^1(tK[t])$ can be established also by direct easy calculations.

Cocycles of type (iii) vanish.

Cocycles of type (iv). Obviously, $\alpha(t^i, t^j) = 0$ for $(i, j) \neq (1, 1)$. Hence α is symmetric, and each cocycle of this type has the form (4.1).

To summarize: cocycles of type (ia), (ii) and (iii) either are trivial or vanish, and cocycles of type (ib) are given by formula (4.2) and vanish if $\mathfrak{g}$ does not contain direct summands isomorphic to $sl(2)$. So in the latter case, all nontrivial cocycles are of type (iv) and given by formula (4.1). Considering the natural grading in $\mathfrak{g} \otimes tK[t]$ by degrees of t, observing that the second cohomology is finite-dimensional and hence is dual to the second homology, and turning to interpretation of the 2-cycles as relations between generators, we get the assertion proved in [17]: that $\mathfrak{g} \otimes tK[t]$ admits a presentation with quadratic relations, provided $\mathfrak{g}$ does not contain direct summands isomorphic to $sl(2)$.

Let us decide now when cocycles given by (4.1) and (4.2) are cohomologically independent. According to Remark 1.2 in Sec. 1, any cohomological dependency beyond linear dependency can occur for cocycles of type (4.1) only. Writing, as in Remark 1.2, $\Phi = d\Omega$ for $\Omega = \sum_{i \in I} \omega_i \otimes \beta_i$, where $\omega_i : \mathfrak{g} \to \mathfrak{g}$ and $\beta_i : tK[t] \to tK[t]$ are some linear maps, we get:

$$\sum_{i \in I} \omega_i([x, y])\beta_i(t^2) = \varphi(x, y)$$

and

$$\sum_{i \in I} \omega_i([x, y])\beta_i(t^k) = 0$$

for $k \geq 3$. Hence it is clear that cocycles of type (4.1) are cohomologically independent if and only if the corresponding skew-symmetric bilinear forms are independent modulo 2-coboundaries. $\qquad\square$

In [17], the author speculates about the possibility to derive Theorem 4.1 from the standard presentation of the affine Kac–Moody algebra. Let us indicate briefly how one can do the opposite: namely, how Theorem 4.1 allows us to recover the Serre relations between Chevalley generators of non-twisted affine Kac–Moody algebras.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over an algebraically closed field K of characteristic 0, and $\widehat{L} = \mathfrak{g} \otimes K[t, t^{-1}]$ be a loop algebra, aka "centerless, derivation-free" part of an affine non-twisted Kac–Moody algebra. It is well-known that it admits a triangular decomposition which, with slight rearrangements of terms, can be written in the form

$$\widehat{L} = ((\mathfrak{g} \otimes t^{-1}K[t^{-1}]) \oplus (\mathfrak{n}_- \otimes 1)) \oplus (\mathfrak{h} \otimes 1) \oplus ((\mathfrak{g} \otimes tK[t]) \oplus (\mathfrak{n}_+ \otimes 1)),$$

where $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is the triangular decomposition of the simple Lie algebra $\mathfrak{g}$ ([16, §7.6]; all direct sums are direct sums of vector spaces).

Consider the Hochschild–Serre spectral sequence abutting to $H^*((\mathfrak{g} \otimes tK[t]) \oplus (\mathfrak{n}_+ \otimes 1), K)$ with respect to the ideal $\mathfrak{g} \otimes tK[t]$. The general result implicitly contained in [14] and explicitly, for example, in [27, Lemma 1], tells that this spectral sequence degenerates at the E_2 term. For the sake of simplicity, let us exclude the case where $\mathfrak{g} = sl(2)$. Then the

E_2 terms affecting the second cohomology in question are:

$$E_\infty^{02} = E_2^{02} = H^2(\mathfrak{n}_+, H^0(\mathfrak{g} \otimes tK[t], K)) \simeq H^2(\mathfrak{n}_+, K),$$
$$E_\infty^{11} = E_2^{11} = H^1(\mathfrak{n}_+, H^1(\mathfrak{g} \otimes tK[t], K)) \simeq H^1(\mathfrak{n}_+, \mathfrak{g}),$$
$$E_\infty^{20} = E_2^{20} = H^0(\mathfrak{n}_+, H^2(\mathfrak{g} \otimes tK[t], K)) \simeq (C^2(\mathfrak{g}, K)/B^2(\mathfrak{g}, K))^{\mathfrak{n}_+}.$$

The first and second isomorphisms here are obvious, the third one follows from Theorem 4.1. Here the first term corresponds to relations between elements of $\mathfrak{n}_+ \otimes 1$, which are (classical) Serre relations for the finite-dimensional Lie algebra $\mathfrak{g}$, the second term corresponds to relations between elements of $\mathfrak{g} \otimes tK[t]$ and $\mathfrak{n}_+ \otimes 1$, and the third one corresponds to relations between elements of $\mathfrak{g} \otimes tK[t]$. Expressing these elements by means of Chevalley generators in terms of the corresponding Cartan matrix, we get the corresponding part of the Serre relations.

Repeating similar reasonings for the "minus" part, and completing relations in an obvious manner between the "plus" and "minus" parts and the "Cartan subalgebra" $\mathfrak{h} \otimes 1$, we get the complete set of defining relations for $\widehat{L}$. The whole affine non-twisted Kac–Moody algebra is obtained from $\widehat{L}$ by the well-known construction which adds central extension and derivation, and its presentation readily follows from the presentation of $\widehat{L}$.

This approach is by no means new. For example, we find in [20, Remark in §2]: "Similar calculations by induction on the rank for simple finite-dimensional and loop algebras give the shortest known to us proof of completeness of the Serre defining relations". "Induction on the rank" means induction with repetitive application of the Hochschild–Serre spectral sequence relative to a Kac–Moody algebra build upon the simple finite-dimensional subalgebra of $\mathfrak{g}$. Here we use the Hochschild–Serre spectral sequence in a different way, and only once, thus getting even shorter proof.

It is mentioned in [16, §9.16] that "a simple cohomological proof" of the completeness of the Serre defining relations "was found by O. Mathieu (unpublished)". We presume that the approach outlined here is similar to that unpublished proof.

Acknowledgments

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