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ON INVARIANTS OF IMMERSIONS OF AN n -DIMENSIONAL MANIFOLD IN AN n -DIMENSIONAL PSEUDO-EUCLIDEAN SPACE*

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Let E_p^n be the n -dimensional pseudo-Euclidean space of index p and $M(n, p)$ the group of all transformations of E_p^n generated by pseudo-orthogonal transformations and parallel translations. We describe the system of generators of the differential field of all $M(n, p)$ -invariant differential rational functions of a map $x : J \rightarrow E_p^n$ of an open subset $J \subseteq E_p^n$. Using this result, we prove analogues of the Bonnet theorem for immersions of an n -dimensional C^∞ -manifold J in E_p^n . These analogues are given in terms of the pseudo-Riemannian metric, the volume form, and the connection on J induced by the immersion of J in E_p^n .

Keywords: Immersion; invariant; pseudo-Riemannian metric; Riemannian curvature tensor.

Mathematics Subject Classification: 53C50 (Primary) 53B30, 53C24, 53C42 (Secondary)

1. Introduction

Invariants of immersions of a manifold in a pseudo-Euclidean space play an important role in many areas of mathematics and mathematical physics. Isometric immersions of pseudo-Riemannian space forms have been a subject of wide interest (see [4, 6, 7]).

Let R be the field of real numbers and p an integer such that $0 \leq p < n$. Denote by E_p^n the space R^n with the inner product

$$(x, y) = -x_1y_1 - \cdots - x_py_p + x_{p+1}y_{p+1} + \cdots + x_ny_n.$$

Let $O(n, p)$ be the group of all linear transformations g of E_p^n such that $(gx, gy) = (x, y)$. Set

$$M(n, p) = \{F : E_p^n \longrightarrow E_p^n \mid Fx = gx + b, \quad g \in O(n, p), \quad b \in E_p^n\}$$

and $SM(n, p) = \{F \in M(n, p) \mid \det g = 1\}$.

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Let J be an n -dimensional C^∞ -manifold and $x : J \rightarrow E_p^n$ be an immersion. This immersion induces the unique pseudo-Riemannian metric, the volume form and the connection on J which we denote by $g\{x\}$, $\Omega(x)$, $C(x)$, respectively.

The fundamental existence and uniqueness theorem of Bonnet for hypersurface immersions (that is for immersions of a connected, simply connected and oriented n -dimensional differentiable manifold J) in a Euclidean space E_0^{n+1} is known ([14, p. 19]). Two analogues of the Bonnet theorem for immersions of an n -dimensional manifold J in a Euclidean space E_0^n were obtained.

The first analogue of the Bonnet theorem is the following (see [8, 16, 17]): *Let J be a connected, simply connected open subset of a Euclidean space E_0^n . Then*

- (1) (*rigidity property*). *Assume that $x : J \rightarrow E_0^n$ and $y : J \rightarrow E_0^n$ are immersions such that $g\{x\} = g\{y\}$. Then there exists $F \in M(n, 0)$ such that $y = Fx$;*
- (2) (*existence*). *Let $h = \|h_{ij}(u)\|$ be a Riemannian metric in J of class C^3 and assume that its Riemannian curvature tensor vanishes. Then there exists an immersion of J in E_0^n such that $g\{x\} = h$.*

Another analogue of the Bonnet theorem is given in the book [1, pp. 69–71]. Note that in this book and papers mentioned below in Introduction, the term “vector field” is used for any map $x : J \rightarrow E_0^n$ of an open subset $J \subseteq E_0^n$. This other analogue of the Bonnet theorem is as follows:

For any $G \subseteq \mathbb{R}^3$, let the functions A_{ik} and B_{ik} of Cartesian coordinates be given such that $A_{1k} = -A_{2k}$ for $i = 1, 2$ and $k = 1, 2, 3$. Suppose these functions satisfy the following system

$$\begin{aligned} \frac{\partial A_{ik}}{\partial x_l} - \frac{\partial A_{il}}{\partial x_k} + B_{il}B_{jk} - B_{ik}B_{jl} &= 0, \\ \frac{\partial B_{ik}}{\partial x_l} - \frac{\partial B_{il}}{\partial x_k} + A_{ik}B_{jl} - A_{il}B_{jk} &= 0. \end{aligned}$$

Then there exist orthonormal vector fields $\mathbf{a}_1, \mathbf{a}_2, \mathbf{n} \in G$ such that

$$\begin{aligned} \frac{\partial \mathbf{a}_i}{\partial x_k} &= A_{ik}\mathbf{a}_j + B_{ik}\mathbf{n}, i \neq j, \\ \frac{\partial \mathbf{n}}{\partial x_k} &= -B_{1k}\mathbf{a}_1 - B_{2k}\mathbf{a}_2. \end{aligned}$$

The vector fields $\mathbf{a}_1, \mathbf{a}_2, \mathbf{n}$ are defined uniquely up to their choice at one point.^a

We note that the part “The vector fields $\mathbf{a}_1, \mathbf{a}_2, \mathbf{n}$ are defined uniquely up to their choice at one point” of this theorem is not clear. Indeed, the functions

$$A_{1k} = \left(\frac{\partial \mathbf{a}_1}{\partial x_k}, \mathbf{a}_2 \right), \quad A_{2k} = \left(\frac{\partial \mathbf{a}_2}{\partial x_k}, \mathbf{a}_1 \right), \quad B_{ik} = - \left(\frac{\partial \mathbf{n}}{\partial x_k}, \mathbf{a}_i \right)$$

^aThe meaning of the words “their choice at one point” are hard to decipher, this is a Russian sentence translated word-for-word and the meaning was lost; it should probably be understood as “up to their values at one point”.

are $O(3)$ -invariant, but they are not invariant with respect to parallel translations in R^3 . Hence this part of the theorem may be given as follows: “*The vector fields $\mathbf{a}_1, \mathbf{a}_2, \mathbf{n}$ are defined uniquely up to an orthogonal transformation of R^3 .*” This means that the system of functions A_{ik}, B_{ik} is the complete system of joint $O(3)$ -invariants of orthonormal vector fields $\mathbf{a}_1, \mathbf{a}_2, \mathbf{n}$.

The aim of the present paper is a generalization of the Bonnet theorem for immersions of an n -dimensional manifold J in a pseudo-Euclidean space E_p^n . First we describe the system of generators of the differential field of all H -invariant differential rational functions of a mapping $x : J \rightarrow E_p^n$ of an open subset $J \subseteq E_p^n$ in E_p^n for groups $H = M(n, p)$ and $H = SM(n, p)$. Using these results, we prove analogues of the Bonnet theorem for immersions an n -dimensional manifold J in E_p^n . These analogues are given in terms of the pseudo-Riemannian metric, the volume form and the connection of an immersion of J in E_p^n .

Let J be an open subset of E_0^n , G be a group and $\alpha(G)$ be an action of G on the set of all smooth vector fields on J . Investigations of the problem of $\alpha(G)$ -equivalence of vector fields, $\alpha(G)$ -invariant vector fields and $\alpha(G)$ -invariants of vector fields have important role in many areas of mathematics.

Let ρ be a linear representation of a group G in E_0^n and x a smooth vector field in E_0^n . Let $J = E_0^n$. Consider the action

$$\rho^*(g)(x(a)) := \rho(g)x(\rho(g^{-1})a)$$

of G on the set of all smooth vector fields x in E_0^n . The problem of a description of the general form of all $\rho^*(G)$ -invariant (that is equivariant) polynomial vector fields for a compact Lie groups G is intensively studied in the bifurcation theory [2, 3, 10]. By using theorems of Schwartz and Poe'nanu ([10, Theorems XII4.3 and XII5.2]), this problem reduces to an algebraic problem of invariant theory. The problems of $\rho^*(G)$ -equivalence of smooth vector fields and complete systems of $\rho^*(G)$ -invariants of polynomial vector fields are investigated in the theory of differential equations [9, 19]. Invariants of vector fields are also studied in differential geometry [1].

This paper is organized as follows. In Sec. 2, we describe the system of generators of the differential field of all G -invariant differential rational functions of a map $x : J \rightarrow E_p^n$ of an open subset $J \subseteq E_p^n$ for groups $G = M(n, p)$ and $G = SM(n, p)$ (Theorems 2.1 and 2.2).

In Sec. 3, for an n -dimensional connected manifold J , using results of Sec. 2, we prove

- (1) the rigidity theorem for the connection $C(x)$ on J induced by an immersion $x : J \rightarrow E_p^n$; (Theorem 3.1);
- (2) the rigidity theorem for the pseudo-Riemannian metric $g\{x\}$ on J induced by an immersion $x : J \rightarrow E_p^n$; (Theorem 3.2);
- (3) the rigidity theorem for the set of all Christoffel symbols of the first and second kinds on J induced by an immersion $x : J \rightarrow E_p^n$; (Theorem 3.3);
- (4) the rigidity theorem for the pseudo-Riemannian metric $g\{x\}$ and the volume form $\Omega(x)$ on J induced by an immersion $x : J \rightarrow E_p^n$; (Corollary 2).

Theorems 3.1–3.3 and Corollary 2 are analogues of the first part of the Bonnet theorem for groups $G = \text{Af}(n)$, $M(n, p)$, $SM(n, p)$.

In Sec. 4, for an n -dimensional connected, simply connected manifold J , we prove the existence theorem for a pseudo-Riemannian metric on J : Let $h = \|h_{ij}(u)\|$ be a pseudo-Riemannian metric in J of class C^∞ and assume that its Riemannian curvature tensor vanishes. Then there exists an immersion x of J in E_p^n such that $g\{x\} = h$ (Theorem 4.1). This theorem is an analogue of the second part of the Bonnet theorem for immersions of the manifold J in E_p^n .

We use methods of the invariant theory and the theory of differential equations. A similar approach to the theory of curves was used in the book [12] and papers [13, 18].

In the sequel, $n > 1$. The case $n = 1$ is easily considered.

2. Generating Systems of Differential Fields of all G -Invariant Differential Rational Functions of an n -Parametric Surface

Throughout this section we assume that J is an open subset of R^n .

Definition 1. A C^∞ -mapping $x : J \rightarrow E_p^n$ will be called an n -parametric J -surface in E_p^n and denoted by $x(u)$. For short, an n -parametric J -surface $x(u)$ with the domain of the definition J will be called a J -surface.

Let $x(u)$ be a J -surface in E_p^n . Denote by N_0 the set of all non-negative integers. For $m_i \in N_0$, where $1 \leq i \leq n$, we set

$$x^{(0,0,\dots,0)} = x(u), \quad x^{(m_1,m_2,\dots,m_n)} = \frac{\partial^{m_1+m_2+\dots+m_n} x(u)}{\partial u_1^{m_1} \partial u_2^{m_2} \dots \partial u_n^{m_n}}.$$

Below we cite some notation and definitions from the differential algebra (see [11–13]) in a form convenient for our considerations.

Definition 2. A polynomial $q(x, x^{(1,0,0,\dots,0)}, x^{(0,1,0,\dots,0)}, \dots, x^{(m_1,m_2,m_3,\dots,m_n)})$ of x and a finite number of partial derivatives $x^{(1,0,0,\dots,0)}, x^{(0,1,0,\dots,0)}, \dots, x^{(m_1,m_2,m_3,\dots,m_n)}$ of x with coefficients from R is called a *differential polynomial* of x .

It will be denoted by $q\{x\}$. The set of all differential polynomials of x will be denoted by $R\{x\}$. It is a differential R -algebra with respect to the differentiations $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots, \frac{\partial}{\partial u_n}$. This differential R -algebra is also an integral domain. Its quotient field will be denoted by $R\langle x \rangle$. It is a differential field with respect to the differentiations $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots, \frac{\partial}{\partial u_n}$. An element h of $R\langle x \rangle$ will be called a *differential rational function* of x and denoted by $h\langle x \rangle$.

Let $x(u_1, u_2, \dots, u_n), y(u_1, u_2, \dots, u_n), \dots, z(u_1, u_2, \dots, u_n)$ be a finite number of J -surfaces in R^n . A differential polynomial and a differential rational function of several J -surfaces $x, y, \dots, z$ are similarly defined. They will be denoted by $q\{x, y, \dots, z\}$ and $q\langle x, y, \dots, z \rangle$, respectively. The differential field of all differential rational functions of $x, y, \dots, z$ will be denoted by $R\langle x, y, \dots, z \rangle$.

Let $GL(n)$ be the group of all non-degenerate real $n \times n$ -matrices. Set

$$Af(n) = \{F : R^n \rightarrow R^n \mid Fx = gx + b, \quad g \in GL(n), \quad b \in R^n\},$$

where gx is the multiplication of a matrix g and a column vector $x \in R^n$. In sequel, we consider the group $O(n, p)$ as the group of all real matrices g such that $g^\top I_p g = I_p$, where $I_p = \|b_{ij}\|$ is the diagonal $n \times n$ -matrix such that $b_{ii} = -1$ for all $i = 1, \dots, p$, $b_{jj} = 1$ for

all $j = p + 1, \dots, n$. Then

$$M(n, p) = \{F : E_p^n \rightarrow E_p^n \mid Fx = gx + b, \quad g \in O(n, p), \quad b \in E_p^n\}$$

and $SM(n, p) = \{F \in M(n, p) \mid \det g = 1\}$.

Let $x(u)$ be a J -surface in E_p^n . Then $Fx(u)$ is also a J -surface in E_p^n for all $F \in \text{Af}(n)$. Let G be a subgroup of the group $\text{Af}(n)$.

Definition 3. A differential rational function $h\langle x, y, \dots, z \rangle$ is said to be G -invariant if

$$h\langle gx, gy, \dots, gz \rangle = h\langle x, y, \dots, z \rangle \quad \text{for all } g \in G.$$

The set of all G -invariant differential rational functions of $x, y, \dots, z$ will be denoted by $R\langle x, y, \dots, z \rangle^G$. It is a differential subfield of the differential field $R\langle x, y, \dots, z \rangle$.

Definition 4. A subset S of $R\langle x, y, \dots, z \rangle^G$ is called a *generating system of the differential field* $R\langle x, y, \dots, z \rangle^G$ if the smallest differential subfield of it containing S coincides with $R\langle x, y, \dots, z \rangle^G$.

Let

$$(x, y) = -x_1y_1 - \dots - x_py_p + x_{p+1}y_{p+1} + \dots + x_ny_n$$

be the inner product of vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in E_p^n . So $(x^{(m_1, m_2, \dots, m_n)}, x^{(p_1, p_2, \dots, p_n)})$ is the inner product of vectors $x^{(m_1, m_2, \dots, m_n)}, x^{(p_1, p_2, \dots, p_n)}$ in E_p^n .

Theorem 2.1. *The system*

$$\left(\frac{\partial}{\partial u_i} x, \frac{\partial}{\partial u_j} x \right), \quad \text{where } 1 \leq i \leq j \leq n, \quad (2.1)$$

is a generating system of the differential field $R\langle x \rangle^{M(n, p)}$.

Proof. For it, first we prove Lemmas 2.1–2.9. Let $R\langle \frac{\partial}{\partial u_1} x, \frac{\partial}{\partial u_2} x, \dots, \frac{\partial}{\partial u_n} x \rangle$ be the differential field of all differential rational functions of $\frac{\partial}{\partial u_1} x, \frac{\partial}{\partial u_2} x, \dots, \frac{\partial}{\partial u_n} x$ and $R\langle \frac{\partial}{\partial u_1} x, \frac{\partial}{\partial u_2} x, \dots, \frac{\partial}{\partial u_n} x \rangle^G$ is the differential field of all G -invariant differential rational functions of $\frac{\partial}{\partial u_1} x, \frac{\partial}{\partial u_2} x, \dots, \frac{\partial}{\partial u_n} x$.

Lemma 2.1. *We have*

$$R\langle x \rangle^{M(n, p)} = R\left\langle \frac{\partial}{\partial u_1} x, \frac{\partial}{\partial u_2} x, \dots, \frac{\partial}{\partial u_n} x \right\rangle^{M(n, p)} = R\left\langle \frac{\partial}{\partial u_1} x, \frac{\partial}{\partial u_2} x, \dots, \frac{\partial}{\partial u_n} x \right\rangle^{O(n, p)}.$$

Proof. Let

$$q\langle x \rangle = q\left(x, \frac{\partial}{\partial u_1} x, \frac{\partial}{\partial u_2} x, \dots, \frac{\partial}{\partial u_n} x, \dots, x^{(m_1, m_2, m_3, \dots, m_n)}\right)$$

be an $M(n, p)$ -invariant differential rational function of x . Then it is invariant with respect to parallel translations in E_p^n . This implies that

$$q\langle x \rangle = q\left(\frac{\partial}{\partial u_1} x, \frac{\partial}{\partial u_2} x, \dots, \frac{\partial}{\partial u_n} x, \dots, x^{(m_1, m_2, m_3, \dots, m_n)}\right) = q\left\langle \frac{\partial}{\partial u_1} x, \frac{\partial}{\partial u_2} x, \dots, \frac{\partial}{\partial u_n} x \right\rangle.$$

It is also $O(n, p)$ -invariant. Hence it is an $O(n, p)$ -invariant differential rational function of $\frac{\partial}{\partial u_1} x, \frac{\partial}{\partial u_2} x, \dots, \frac{\partial}{\partial u_n} x$. Conversely, assume that q is a $O(n, p)$ -invariant differential rational

function of $\frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x$. Then it is invariant with respect to parallel translations in E_p^n . Hence it is $M(n, p)$ -invariant. $\square$

Lemma 2.2. *Let $f \in R\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x \rangle^{O(n,p)}$. Then there exist $O(n, p)$ -invariant differential polynomials f_1, f_2 such that $f = f_1/f_2$.*

Proof. A proof is similar to the proof in ([12, p. 106]). $\square$

Lemma 2.3. *The system of all $(x^{(m_1, m_2, \dots, m_n)}, x^{(p_1, p_2, \dots, p_n)})$, where $m_1 + m_2 + \dots + m_n \geq 1$, $p_1 + p_2 + \dots + p_n \geq 1$, is a generating system of $R\langle x \rangle^{M(n,p)}$ as a field.*

Proof. Let $R[x^{(m_1, m_2, \dots, m_n)}; m_1 + m_2 + \dots + m_n \geq 1]^{O(n)}$ be the R -algebra of all $O(n, p)$ -invariant polynomials of $x^{(m_1, m_2, \dots, m_n)}$, where $m_1 + m_2 + \dots + m_n \geq 1$. According to the First Main Theorem for $O(n, p)$ (see [21, pp. 53, 65–66]), the system of all $(x^{(m_1, m_2, \dots, m_n)}, x^{(p_1, p_2, \dots, p_n)})$, where $m_1 + m_2 + \dots + m_n \geq 1$ and $p_1 + p_2 + \dots + p_n \geq 1$, is a generating system of the R -algebra $R[x^{(m_1, m_2, \dots, m_n)}; m_1 + m_2 + \dots + m_n \geq 1]^{O(n,p)}$. Using Lemmas 2.1 and 2.2, we see that the system of all $(x^{(m_1, m_2, \dots, m_n)}, x^{(p_1, p_2, \dots, p_n)})$, where $m_1 + m_2 + \dots + m_n \geq 1$ and $p_1 + p_2 + \dots + p_n \geq 1$, is a generating system of

$$R\left\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x \right\rangle^{O(n,p)} = R\langle x \rangle^{M(n,p)}$$

as a field. $\square$

Denote by $\Delta = \Delta_x$ the determinant $\det \|(a_i, b_j)\|_{i,j=1,2,\dots,n}$, where

$$a_1 = b_1 = \frac{\partial}{\partial u_1}x, \quad a_2 = b_2 = \frac{\partial}{\partial u_2}x, \dots, a_n = b_n = \frac{\partial}{\partial u_n}x.$$

Let V be the system Eq. (2.1). Denote by $R\{V\}$ the differential R -subalgebra of $R\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x \rangle^{O(n,p)}$ generated by elements of the system V .

Lemma 2.4. $\Delta \in R\{V\}$.

Proof. By the definition of V , we have $(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x) \in V$ for all $1 \leq i, j \leq n$. Hence $\Delta \in R\{V\}$. $\square$

Denote by $R\{V, \Delta^{-1}\}$ the differential R -subalgebra of $R\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x \rangle^{O(n,p)}$ generated by elements of the system V and the function Δ^{-1} . According to Lemmas 2.1 and 2.3, to prove our theorem, it suffices to prove that $(x^{(m_1, m_2, \dots, m_n)}, x^{(p_1, p_2, \dots, p_n)}) \in R\{V, \Delta^{-1}\}$ for all $m_i, p_i \in N_0$ such that $m_1 + m_2 + \dots + m_n \geq 1$ and $p_1 + p_2 + \dots + p_n \geq 1$.

Let $\det \text{Gr}(y_1, y_2, \dots, y_m; z_1, z_2, \dots, z_m)$ be the determinant of the Gram matrix

$$\text{Gr}(y_1, y_2, \dots, y_m; z_1, z_2, \dots, z_m) := \|(y_i, z_j)\|_{i,j=1,2,\dots,m}$$

of vectors $y_1, y_2, \dots, y_m; z_1, z_2, \dots, z_m \in E_p^n$.

A proof of the following is known ([21, p. 75]):

Lemma 2.5. *The equality*

$$\det \text{Gr}(y_1, y_2, \dots, y_{n+1}; z_1, z_2, \dots, z_{n+1}) = \det \|(y_i, z_j)\|_{i,j=1,2,\dots,n+1} = 0$$

holds for all vectors $y_1, y_2, \dots, y_{n+1}, z_1, z_2, \dots, z_{n+1}$ in R^n .

Lemma 2.6. Assume that $b_1, b_2, \dots, b_n; c_1, c_2, \dots, c_n \in N_0$ are such that

$$1 \leq b_1 + b_2 + \dots + b_n, \quad 1 \leq c_1 + c_2 + \dots + c_n, \quad \left(x^{(b_1, b_2, \dots, b_n)}, \frac{\partial}{\partial u_i} x \right) \in R\{V, \Delta^{-1}\}$$

and $(x^{(c_1, c_2, \dots, c_n)}, \frac{\partial}{\partial u_i} x) \in R\{V, \Delta^{-1}\}$ for all i such that $1 \leq i \leq n$.

Then $(x^{(b_1, b_2, \dots, b_n)}, x^{(c_1, c_2, \dots, c_n)}) \in R\{V, \Delta^{-1}\}$.

Proof. Applying Lemma 2.5 to vectors

$$\begin{aligned} y_1 &= z_1 = \frac{\partial}{\partial u_1} x, \\ y_2 &= z_2 = \frac{\partial}{\partial u_2} x, \\ &\dots \\ y_n &= z_n = \frac{\partial}{\partial u_n} x, \\ y_{n+1} &= x^{(b_1, b_2, \dots, b_n)}, \\ z_{n+1} &= x^{(c_1, c_2, \dots, c_n)}, \end{aligned}$$

we obtain $\det A = 0$, where

$$A = \|(y_i, z_j)\|_{i,j=1,2,\dots,n+1} = \text{Gr}(y_1, y_2, \dots, y_{n+1}; z_1, z_2, \dots, z_{n+1}).$$

Denote by D_{n+1j} the cofactor of the element (y_{n+1}, z_j) in A , where $j = 1, 2, \dots, n+1$. From the equality $\det A = 0$, we obtain the equality

$$\begin{aligned} (y_{n+1}, z_1)D_{n+11} + (y_{n+1}, z_2)D_{n+12} + \dots + (y_{n+1}, z_n)D_{n+1n} \\ + (y_{n+1}, z_{n+1})D_{n+1n+1} = 0. \end{aligned} \quad (2.2)$$

Since $\Delta = D_{n+1n+1}$, Eq. (2.2) implies that

$$\begin{aligned} (y_{n+1}, z_{n+1}) &= (x^{(b_1, b_2, \dots, b_n)}, x^{(c_1, c_2, \dots, c_n)}) \\ &= -\frac{(y_{n+1}, z_1)D_{n+11} + (y_{n+1}, z_2)D_{n+12} + \dots + (y_{n+1}, z_n)D_{n+1n}}{\Delta}. \end{aligned} \quad (2.3)$$

In Eq. (2.3), by the assumption of the lemma, $(y_{n+1}, z_j) = (x^{(b_1, b_2, \dots, b_n)}, \frac{\partial}{\partial u_j} x) \in R\{V, \Delta^{-1}\}$ for every j such that $1 \leq j \leq n$. We prove that $D_{n+1s} \in R\{V, \Delta^{-1}\}$ for every s such that $1 \leq s \leq n$. We have

$$D_{n+1s} = (-1)^{n+1+s} \det \text{Gr}(y_1, y_2, \dots, y_n; z_1, z_2, \dots, z_{s-1}, z_{s+1}, \dots, z_n, z_{n+1}).$$

By the definition of V , we obtain $(y_i, z_j) \in V \subset R\{V\}$ for all $1 \leq i, j \leq n$. By the assumption of our lemma, we have $(y_i, z_{n+1}) = (\frac{\partial}{\partial u_i} x, x^{(c_1, c_2, \dots, c_n)}) \in R\{V, \Delta^{-1}\}$ for every $1 \leq i \leq n$. Hence $D_{n+1s} \in R\{V, \Delta^{-1}\}$ for every $1 \leq s \leq n$ and Eq. (2.3) implies $(y_{n+1}, z_{n+1}) \in R\{V, \Delta^{-1}\}$. $\square$

Lemma 2.7. *We have*

$$\left(\frac{\partial^2}{\partial u_i \partial u_j} x, \frac{\partial}{\partial u_l} x \right) = \frac{1}{2} \left(\frac{\partial}{\partial u_j} \left(\frac{\partial}{\partial u_l} x, \frac{\partial}{\partial u_i} x \right) + \frac{\partial}{\partial u_i} \left(\frac{\partial}{\partial u_l} x, \frac{\partial}{\partial u_j} x \right) - \frac{\partial}{\partial u_l} \left(\frac{\partial}{\partial u_i} x, \frac{\partial}{\partial u_j} x \right) \right)$$

and $(\frac{\partial^2}{\partial u_i \partial u_j} x, \frac{\partial}{\partial u_l} x) \in R\{V, \Delta^{-1}\}$ for all $i, j, l \in \{1, 2, \dots, n\}$.

Proof. We have

$$\begin{aligned} \frac{\partial}{\partial u_j} \left(\frac{\partial}{\partial u_i} x, \frac{\partial}{\partial u_l} x \right) &= \left(\frac{\partial^2}{\partial u_i \partial u_j} x, \frac{\partial}{\partial u_l} x \right) + \left(\frac{\partial}{\partial u_i} x, \frac{\partial^2}{\partial u_j \partial u_l} x \right), \\ \frac{\partial}{\partial u_i} \left(\frac{\partial}{\partial u_j} x, \frac{\partial}{\partial u_l} x \right) &= \left(\frac{\partial^2}{\partial u_i \partial u_j} x, \frac{\partial}{\partial u_l} x \right) + \left(\frac{\partial}{\partial u_j} x, \frac{\partial^2}{\partial u_i \partial u_l} x \right), \\ \frac{\partial}{\partial u_l} \left(\frac{\partial}{\partial u_i} x, \frac{\partial}{\partial u_j} x \right) &= \left(\frac{\partial^2}{\partial u_i \partial u_l} x, \frac{\partial}{\partial u_j} x \right) + \left(\frac{\partial}{\partial u_i} x, \frac{\partial^2}{\partial u_j \partial u_l} x \right). \end{aligned} \quad (2.4)$$

Set

$$\begin{aligned} \frac{\partial}{\partial u_j} \left(\frac{\partial}{\partial u_i} x, \frac{\partial}{\partial u_l} x \right) &= b_1, \quad \left(\frac{\partial^2}{\partial u_i \partial u_j} x, \frac{\partial}{\partial u_l} x \right) = w_1, \\ \frac{\partial}{\partial u_i} \left(\frac{\partial}{\partial u_j} x, \frac{\partial}{\partial u_l} x \right) &= b_2, \quad \left(\frac{\partial^2}{\partial u_j \partial u_l} x, \frac{\partial}{\partial u_i} x \right) = w_2, \\ \frac{\partial}{\partial u_l} \left(\frac{\partial}{\partial u_i} x, \frac{\partial}{\partial u_j} x \right) &= b_3, \quad \left(\frac{\partial^2}{\partial u_i \partial u_l} x, \frac{\partial}{\partial u_j} x \right) = w_3. \end{aligned}$$

Then system Eq. (2.4) takes the form: $w_1 + w_2 = b_1$, $w_1 + w_3 = b_2$, $w_2 + w_3 = b_3$. We consider this system as the system of equations for w_1, w_2, w_3 . This system has the unique solution (w_1, w_2, w_3) , where

$$\begin{aligned} w_1 &= \frac{1}{2}(b_1 + b_2 - b_3) \in R\{V, \Delta^{-1}\}, \\ w_2 &= \frac{1}{2}(b_1 + b_3 - b_2) \in R\{V, \Delta^{-1}\}, \\ w_3 &= \frac{1}{2}(b_2 + b_3 - b_1) \in R\{V, \Delta^{-1}\}. \end{aligned} \quad \square$$

Lemma 2.8. *For all $1 \leq i \leq n$ and $b_1, b_2, \dots, b_n \in N_0$ such that $1 \leq b_1 + b_2 + \dots + b_n$, we have $(x^{(b_1, b_2, \dots, b_n)}, \frac{\partial}{\partial u_i} x) \in R\{V, \Delta^{-1}\}$.*

Proof. We prove the assertion by induction on $B = b_1 + b_2 + \dots + b_n$. Let $B = 1$. Then, by the definition of V , we have $(\frac{\partial}{\partial u_i} x, \frac{\partial}{\partial u_j} x) \in V \subset R\{V, \Delta^{-1}\}$ for all $1 \leq i, j \leq n$. Hence the assertion holds for $B = 1$. Assume that the assertion holds for $B > 1$.

We prove that $(\frac{\partial}{\partial u_i} x^{(b_1, b_2, \dots, b_n)}, \frac{\partial}{\partial u_j} x) \in R\{V, \Delta^{-1}\}$ for all $1 \leq j \leq n$ and $b_1, b_2, \dots, b_n \in N_0$ such that $b_1 + b_2 + \dots + b_n = B$. For arbitrary $1 \leq i, j \leq n$ and $b_1, b_2, \dots, b_n \in N_0$ such that $b_1 + b_2 + \dots + b_n = B$, we have

$$\frac{\partial}{\partial u_i} \left(x^{(b_1, b_2, \dots, b_n)}, \frac{\partial}{\partial u_j} x \right) = \left(\frac{\partial}{\partial u_i} x^{(b_1, b_2, \dots, b_n)}, \frac{\partial}{\partial u_j} x \right) + \left(x^{(b_1, b_2, \dots, b_n)}, \frac{\partial^2}{\partial u_i \partial u_j} x \right). \quad (2.5)$$

By the induction hypothesis on B , we have $(x^{(b_1, b_2, \dots, b_n)}, \frac{\partial}{\partial u_l} x) \in R\{V, \Delta^{-1}\}$ for all $1 \leq l \leq n$ and $b_s \in N_0$ such that $b_1 + b_2 + \dots + b_n = B$. Using Lemmas 2.7 and 2.6, we see that $(x^{(b_1, b_2, \dots, b_n)}, \frac{\partial^2}{\partial u_i \partial u_j} x) \in R\{V, \Delta^{-1}\}$ for all $1 \leq i, j \leq n$ and $b_s \in N_0$ such that $b_1 + b_2 + \dots + b_n = B$. Hence Eq. (2.5) implies $(\frac{\partial}{\partial u_i} x^{(b_1, b_2, \dots, b_n)}, \frac{\partial}{\partial u_j} x) \in R\{V, \Delta^{-1}\}$. $\square$

Lemma 2.9. *For all $b_1, b_2, \dots, b_n, c_1, c_2, \dots, c_n \in N_0$ such that $1 \leq b_1 + b_2 + \dots + b_n$ and $1 \leq c_1 + c_2 + \dots + c_n$, we have $(x^{(b_1, b_2, \dots, b_n)}, x^{(c_1, c_2, \dots, c_n)}) \in R\{V, \Delta^{-1}\}$.*

Proof. Using Lemmas 2.8 and 2.6, we obtain $(x^{(b_1, b_2, \dots, b_n)}, x^{(c_1, c_2, \dots, c_n)}) \in R\{V, \Delta^{-1}\}$. $\square$

We complete the proof of our theorem. By Lemma 2.4, $\Delta \in R\langle V \rangle$. Since $R\langle V \rangle$ is a field, we obtain $\Delta^{-1} \in R\langle V \rangle$. Hence $R\{V, \Delta^{-1}\} \subset R\langle V \rangle$. By Lemma 2.9,

$$(x^{(b_1, b_2, \dots, b_n)}, x^{(c_1, c_2, \dots, c_n)}) \in R\{V, \Delta^{-1}\} \subset R\langle V \rangle \subset R\langle x \rangle^{M(n, p)}$$

for all $b_1, b_2, \dots, b_n, c_1, c_2, \dots, c_n \in N_0$ such that $1 \leq b_1 + b_2 + \dots + b_n, 1 \leq c_1 + c_2 + \dots + c_n$. By Lemma 2.3, the system of all elements $(x^{(m_1, m_2, \dots, m_n)}, x^{(c_1, c_2, \dots, c_n)})$, where $m_1 + m_2 + \dots + m_n \geq 1, c_1 + c_2 + \dots + c_n \geq 1, m_i \in N_0, c_i \in N_0$, is a generating system of $R\langle x \rangle^{M(n, p)}$ as a field. Hence $R\langle V \rangle = R\langle x \rangle^{M(n, p)}$. The theorem is completed. $\square$

For any $a_m \in E_p^n$, where $m = 1, \dots, n$, the determinant $\det(a_{ij})$ (where the a_{mi} are coordinates of a_m) will be denoted by $[a_1 a_2 \dots a_n]$. So

$$[x^{(m_{11}, m_{12}, \dots, m_{1n})} x^{(m_{21}, m_{22}, \dots, m_{2n})} \dots x^{(m_{n1}, m_{n2}, \dots, m_{nn})}]$$

is the determinant of the vectors $x^{(m_{i1}, m_{i2}, \dots, m_{in})}$ for $i = 1, 2, \dots, n$.

Theorem 2.2. *The system*

$$\left[\frac{\partial}{\partial u_1} x \quad \frac{\partial}{\partial u_2} x \cdots \frac{\partial}{\partial u_n} x \right], \left(\frac{\partial}{\partial u_i} x, \frac{\partial}{\partial u_j} x \right), \quad \text{where } 1 \leq i \leq j \leq n, \quad i + j < 2n, \quad (2.6)$$

is a generating system of the differential field $R\langle x \rangle^{SM(n)}$.

Proof. Let $SO(n, p) = \{g \in O(n, p) \mid \det g = 1\}$.

Lemma 2.10. $R\langle x \rangle^{SM(n, p)} = R\langle \frac{\partial}{\partial u_1} x, \frac{\partial}{\partial u_2} x, \dots, \frac{\partial}{\partial u_n} x \rangle^{SO(n, p)}$.

Proof is similar to the proof of Lemma 2.1. $\square$

Lemma 2.11. *Let $f \in R\langle \frac{\partial}{\partial u_1} x, \frac{\partial}{\partial u_2} x, \dots, \frac{\partial}{\partial u_n} x \rangle^{SO(n, p)}$. Then $SO(n, p)$ -invariant differential polynomials f_1, f_2 exist such that $f = f_1/f_2$.*

Proof is similar to the proof in ([12, p. 106]). $\square$

Lemma 2.12. *The system of all elements*

$$[x^{(m_{11}, m_{12}, \dots, m_{1n})} x^{(m_{21}, m_{22}, \dots, m_{2n})} \dots x^{(m_{n1}, m_{n2}, \dots, m_{nn})}], \\ (x^{(b_1, b_2, \dots, b_n)}, x^{(c_1, c_2, \dots, c_n)}), \quad (2.7)$$

where $m_{i1} + m_{i2} + \dots + m_{in} \geq 1, b_1 + b_2 + \dots + b_n \geq 1, c_1 + c_2 + \dots + c_n \geq 1$, is a generating system of $R\langle x \rangle^{SM(n, p)}$ as a field.

Proof. Let $R[x^{(m_1, m_2, \dots, m_n)}; m_1 + m_2 + \dots + m_n \geq 1]^{SO(n, p)}$ be the R -algebra of all $SO(n, p)$ -invariant polynomials of the system of all $x^{(m_1, m_2, \dots, m_n)}$, where $m_1 + m_2 + \dots + m_n \geq 1$. By the First Main Theorem for $SO(n, p)$ (see [21, pp. 53, 65–66]), the system Eq. (2.7) is a generating system of $R[x^{(m_1, m_2, \dots, m_n)}; m_1 + m_2 + \dots + m_n \geq 1]^{SO(n, p)}$. Lemma 2.11 implies that the system Eq. (2.7) is a generating system of $R\langle x \rangle^{SO(n, p)}$. Then, by Lemma 2.10, the system Eq. (2.7) is a generating system of $R\langle x \rangle^{SM(n, p)}$ as a field. $\square$

Let Z be the system Eq. (2.6) and $R\{Z\}$ be the differential R -subalgebra of $R\langle x \rangle^{SM(n, p)}$ generated by elements of the system Z . Denote by $\delta = \delta_x$ the determinant of the matrix $\text{Gr}(y_1, y_2, \dots, y_{n-1}; z_1, z_2, \dots, z_{n-1})$, where

$$y_1 = z_1 = \frac{\partial}{\partial u_1} x, \quad y_2 = z_2 = \frac{\partial}{\partial u_2} x, \dots, \quad y_{n-1} = z_{n-1} = \frac{\partial}{\partial u_{n-1}} x. \quad \square$$

Lemma 2.13. $\delta \in R\{Z\}$ and $\delta^{-1} \in R\langle Z \rangle$.

Proof. By the definition of Z , $(y_i, z_j) = (\frac{\partial}{\partial u_i} x, \frac{\partial}{\partial u_j} x) \in Z \subset R\{Z\}$ for all $1 \leq i, j \leq n-1$. Hence $\delta \in R\{Z\}$ and $\delta^{-1} \in R\langle Z \rangle$.

In the sequel, we need the following lemma

Lemma 2.14. *The equality*

$$(-1)^p [y_1 \cdots y_n] [z_1 \cdots z_n] = \det \|(y_i, z_j)\|_{i,j=1,2,\dots,n}$$

holds for all vectors $y_1, \dots, y_n, z_1, \dots, z_n$ in E_p^n .

Proof. A proof of the this lemma is similar to the proof in ([12, p. 72]). $\square$

Let Δ be the function in the proof of Theorem 2.1.

Lemma 2.15. $\Delta \in R\{Z\}$ and $\Delta^{-1} \in R\langle Z \rangle$.

Proof. Using Lemma 2.14 to vectors $y_1 = z_1 = \frac{\partial}{\partial u_1} x, y_2 = z_2 = \frac{\partial}{\partial u_2} x, \dots, y_n = z_n = \frac{\partial}{\partial u_n} x$, we obtain

$$(-1)^p \left[\frac{\partial}{\partial u_1} x \quad \frac{\partial}{\partial u_2} x \cdots \frac{\partial}{\partial u_n} x \right]^2 = \det \|(y_i, z_j)\|_{i,j=1,2,\dots,n} = \Delta. \quad (2.8)$$

Since $[\frac{\partial}{\partial u_1} x \quad \frac{\partial}{\partial u_2} x \cdots \frac{\partial}{\partial u_n} x] \in Z$, we have $\Delta \in R\{Z\}$ and $\Delta^{-1} \in R\langle Z \rangle$. $\square$

Denote by $R\{Z, \delta^{-1}, \Delta^{-1}\}$ the differential R -subalgebra of $R\langle \frac{\partial}{\partial u_1} x, \frac{\partial}{\partial u_2} x, \dots, \frac{\partial}{\partial u_n} x \rangle$ generated by Z and functions δ^{-1}, Δ^{-1} . By Lemmas 2.11 and 2.12, for a proof of our theorem, it is enough to prove that

$$[x^{(m_{11}, m_{12}, \dots, m_{1n})} x^{(m_{21}, m_{22}, \dots, m_{2n})} \dots x^{(m_{n1}, m_{n2}, \dots, m_{nn})}] \in R\{Z, \delta^{-1}, \Delta^{-1}\}$$

and

$$(x^{(b_1, b_2, \dots, b_n)}, x^{(c_1, c_2, \dots, c_n)}) \in R\{Z, \delta^{-1}, \Delta^{-1}\}$$

for all $m_{ij}, b_i, c_i \in N_0$ such that $m_{i1} + m_{i2} + \dots + m_{in} \geq 1$, $b_1 + b_2 + \dots + b_n \geq 1$, $c_1 + c_2 + \dots + c_n \geq 1$.

Let V be the system in the proof of Theorem 2.1.

Lemma 2.16. $(\frac{\partial}{\partial u_n}x, \frac{\partial}{\partial u_n}x) \in R\{Z, \delta^{-1}\}$ and $R\{V, \Delta^{-1}\} \subset R\{Z, \delta^{-1}, \Delta^{-1}\}$.

Proof. For $i = 1, 2, \dots, n$, denote by D_{ni} the cofactor of the element (y_n, z_j) of the matrix $A = \|(y_i, z_j)\|_{i,j=1,2,\dots,n}$ in Eq. (2.8). Then we obtain the equality

$$\Delta = (y_n, z_1)D_{n1} + (y_n, z_2)D_{n2} + \dots + (y_n, z_{n-1})D_{nn-1} + (y_n, z_n)D_{nn}. \quad (2.9)$$

Since $\delta = D_{nn} \neq 0$, equalities Eqs. (2.8) and (2.9) imply

$$\begin{aligned} (y_n, z_n) &= \left(\frac{\partial}{\partial u_n}x, \frac{\partial}{\partial u_n}x \right) \\ &= \Delta \delta^{-1} - (y_n, z_1)D_{n1}\delta^{-1} - (y_n, z_2)D_{n2}\delta^{-1} - \dots - (y_n, z_{n-1})D_{nn-1}\delta^{-1}. \end{aligned} \quad (2.10)$$

In Eq. (2.10), $(y_n, z_j) = (\frac{\partial}{\partial u_n}x, \frac{\partial}{\partial u_j}x) \in Z \subset R\{Z\}$ for every $1 \leq j \leq n-1$ by the definition of Z . We prove that $D_{ns} \in R\{Z\}$ for every $1 \leq s \leq n-1$. We have

$$D_{ns} = (-1)^{n+s} \det \text{Gr}(y_1, y_2, \dots, y_{n-1}; z_1, z_2, \dots, z_{s-1}, z_{s+1}, \dots, z_n).$$

The elements of D_{ns} have the following forms (y_i, z_j) and (y_i, z_n) , where $i < n, j < n$. By the definition of Z , we have $(y_i, z_n) = (y_n, z_i) \in Z \subset R\{Z\}$. Hence Eq. (2.10) implies $(y_n, z_n) \in R\{Z, \delta^{-1}\}$. Using $V \subset Z \cup \{(y_n, z_n)\}$, we obtain $R\{V, \Delta^{-1}\} \subset R\{Z, \delta^{-1}, \Delta^{-1}\}$. $\square$

Lemma 2.17. We have $(x^{(b_1, b_2, \dots, b_n)}, x^{(c_1, c_2, \dots, c_n)}) \in R\{Z, \delta^{-1}, \Delta^{-1}\}$ for all $b_i, c_i \in N_0$ such that $0 < b_1 + b_2 + \dots + b_n$ and $0 < c_1 + c_2 + \dots + c_n$.

Proof. By Lemma 2.16, we have $R\{V, \Delta^{-1}\} \subset R\{Z, \delta^{-1}, \Delta^{-1}\}$. Hence, by Lemma 2.9, we obtain $(x^{(b_1, b_2, \dots, b_n)}, x^{(c_1, c_2, \dots, c_n)}) \in R\{Z, \delta^{-1}, \Delta^{-1}\}$ for all $b_i, c_i \in N_0$ such that $0 < b_1 + b_2 + \dots + b_n$ and $0 < c_1 + c_2 + \dots + c_n$. $\square$

Lemma 2.18. We have

$$[x^{(m_{11}, m_{12}, \dots, m_{1n})} x^{(m_{21}, m_{22}, \dots, m_{2n})} \dots x^{(m_{n1}, m_{n2}, \dots, m_{nn})}] \in R\{Z, \delta^{-1}, \Delta^{-1}\}$$

for all $m_{ij} \in N_0$ such that $m_{i1} + m_{i2} + \dots + m_{in} \geq 1$ and $i = 1, 2, \dots, n$.

Proof. Applying Lemma 2.14 to vectors $z_1 = x^{(m_{11}, m_{12}, \dots, m_{1n})}$, $z_2 = x^{(m_{21}, m_{22}, \dots, m_{2n})}$, $\dots$, $z_n = x^{(m_{n1}, m_{n2}, \dots, m_{nn})}$ and $y_1 = \frac{\partial}{\partial u_1}x$, $y_2 = \frac{\partial}{\partial u_2}x$, $\dots$, $y_n = \frac{\partial}{\partial u_n}x$ we see that

$$(-1)^p [y_1 \dots y_n] [z_1 \dots z_n] = \det \|(y_i, z_j)\|_{i,j=1,2,\dots,n}. \quad (2.11)$$

Since $\Delta = (-1)^p [y_1 \dots y_n]^2$, Eq. (2.11) implies

$$[z_1 \dots z_n] = \Delta^{-1} [y_1 \dots y_n] \det \|(y_i, z_j)\|_{i,j=1,2,\dots,n}.$$

By Lemma 2.17, $(y_i, z_j) = (\frac{\partial}{\partial u_i}x, x^{(m_{j1}, m_{j2}, \dots, m_{jn})}) \in R\{Z, \delta^{-1}, \Delta^{-1}\}$ for all $i, j = 1, 2, \dots, n$. Since $[y_1 \dots y_n] \in Z \subset R\{Z, \delta^{-1}, \Delta^{-1}\}$, we obtain $[z_1 \dots z_n] \in R\{Z, \delta^{-1}, \Delta^{-1}\}$. $\square$

We complete the proof of our theorem. By Lemmas 2.13 and 2.15, $\delta^{-1}, \Delta^{-1} \in R\langle Z \rangle$. Hence $R\{Z, \delta^{-1}, \Delta^{-1}\} \subset R\langle Z \rangle$. By Lemma 2.17,

$$(x^{(b_1, b_2, \dots, b_n)}, x^{(c_1, c_2, \dots, c_n)}) \in R\{Z, \delta^{-1}, \Delta^{-1}\} \subset R\langle Z \rangle$$

for all $b_1, b_2, \dots, b_n, c_1, c_2, \dots, c_n \in N_0$ such that $1 \leq b_1 + b_2 + \dots + b_n, 1 \leq c_1 + c_2 + \dots + c_n$. By Lemma 2.18,

$$[x^{(m_{11}, m_{12}, \dots, m_{1n})} x^{(m_{21}, m_{22}, \dots, m_{2n})} \dots x^{(m_{n1}, m_{n2}, \dots, m_{nn})}] \in R\{Z, \delta^{-1}, \Delta^{-1}\} \subset R\langle Z \rangle$$

for all $m_{ij} \in N_0$ such that $m_{i1} + m_{i2} + \dots + m_{in} \geq 1, i = 1, 2, \dots, n$. Hence Lemmas 2.10–2.12 imply that $R\langle Z \rangle = R\langle x \rangle^{SM(n,p)}$. The proof of the theorem is completed. $\square$

3. Rigidity Theorems for Immersions of a Manifold

Throughout this section we assume that J is a connected C^∞ -manifold of dimension n . Let $\{(V_\lambda, \phi_\lambda), \lambda \in \Lambda\}$ be the collection of all coordinate neighborhoods of J .

Definition 5 ([5, p. 70]). A C^∞ -mapping $x : J \rightarrow E_p^n$ will be called a C^∞ -immersion of J in E_p^n (J -immersion, for short) if the rank of x at each point of J is equal to n .

Let x be a J -immersion, $(V_\lambda, \phi_\lambda)$ be a coordinate neighborhood of J and $x^{(u)}$ be the expressing of x in the local coordinates in V_λ . Using the determinant $[\frac{\partial}{\partial u_1} x^{(u)}, \frac{\partial}{\partial u_2} x^{(u)}, \dots, \frac{\partial}{\partial u_n} x^{(u)}]$, we define the alternating covariant tensor field of order n on J (an n -form on J) by the following expression in local coordinates

$$\left[\frac{\partial}{\partial u_1} x^{(u)} \frac{\partial}{\partial u_2} x^{(u)} \dots \frac{\partial}{\partial u_n} x^{(u)} \right] du_1 \wedge \dots \wedge du_n.$$

Definition 6. This n -form on J will be called the volume form induced by a J -immersion x . We denote it by $\Omega(x)$.

Proposition 3.1. Let x be a J -immersion in E_p^n . Then the volume form $\Omega(x)$ is nonzero at any point of J .

Proof. Since x is a J -immersion in E_p^n , we have $[\frac{\partial}{\partial u_1} x^{(u)}, \frac{\partial}{\partial u_2} x^{(u)}, \dots, \frac{\partial}{\partial u_n} x^{(u)}] \neq 0$ at every point of $\phi_\lambda(V_\lambda)$. $\square$

Corollary 3.1. Let a C^∞ -manifold J of dimension n such that it has a J -immersion in E_p^n . Then it is orientable.

Proof. Follows from Definition V.7.5, Theorem V.7.6 in ([5, pp. 214–215]) and Proposition 3.1. $\square$

Thus every non-orientable C^∞ -manifold of dimension n has not an immersion in E_p^n . But there exists an orientable C^∞ -manifold J of dimension n without J -immersions in E_p^n . For example, a compact orientable C^∞ -manifold of dimension n has not an immersion in E_p^n (see [5, p. 80]).

Let x be a J -immersion in E_p^n , $(V_\lambda, \phi_\lambda)$ a coordinate neighborhood of J and $x^{(u)}$ the expressing of x in the local coordinates in V_λ . Set $D_i x^{(u)} = \frac{\partial}{\partial u_i} x^{(u)}$. Let $\{D^1 x^{(u)}, \dots, D^n x^{(u)}\}$ be the system which is biorthonormal to $\{D_1 x^{(u)}, \dots, D_n x^{(u)}\}$ with respect to the inner product in E_p^n that is $(D_i x^{(u)}(a), D^j x^{(u)}(a)) = \delta_i^j$ for all $a \in J$ and $i, j = 1, \dots, n$. Consider

the following derivation formulas (see [20, p. 116]) for the restriction of the J -immersion x to the coordinate neighborhood V_λ :

$$\frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} x^{(u)} = \sum_{s=1}^n \Gamma_{ij}^s \{x, V_\lambda\} D_s x^{(u)} \quad \text{for } i, j = 1, 2, \dots, n. \quad (3.1)$$

$$\frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} x^{(u)} = \sum_{s=1}^n \Gamma_{ij,s} \{x, V_\lambda\} D^s x^{(u)} \quad \text{for } i, j = 1, 2, \dots, n. \quad (3.2)$$

The functions $\Gamma_{ij,s} \{x, V_\lambda\}$ and $\Gamma_{ij}^s \{x, V_\lambda\}$ will be called, respectively, the *Christoffel symbols of the first and second kind* of the J -immersion x on V_λ .

Proposition 3.2. *The family $\{\Gamma_{ij}^s \{x, V_\lambda\} \mid \lambda \in \Lambda, \ i, j, s = 1, 2, \dots, n\}$ is a connection on J .*

Proof. Follows from Eq. (1.20b) in ([20, p. 118]). $\square$

The family $\{\Gamma_{ij}^s \{x, V_\lambda\} \mid \lambda \in \Lambda, \ i, j, s = 1, 2, \dots, n\}$ will be called the *connection induced by the J -immersion x* . We denote it by $C(x) = \{\Gamma_{ij}^s \{x\}\}$.

Proposition 3.3. *Let x be a J -immersion in E_p^n . Then the torsion tensor and the Riemannian curvature tensor of the connection $C(x)$ are equal to zero.*

Proof. Let x be a J -immersion in E_p^n and $x^{(u)}$ be the expressing of x in the local coordinates in V_λ . By Eq. (3.1), we see that $\Gamma_{ij}^s \{x\} = \Gamma_{ji}^s \{x\}$ for all $i, j, s = 1, 2, \dots, n$. Hence the torsion tensor is equal to zero.

Equation (3.1) imply the following equalities (here $i, j, k = 1, 2, \dots, n$):

$$\begin{aligned} \frac{\partial}{\partial u_k} \left(\frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} x^{(u)} \right) &= \frac{\partial}{\partial u_k} \left(\sum_{s=1}^n \Gamma_{ij}^s \{x\} D_s x^{(u)} \right) \\ &= \sum_{s=1}^n \left(\frac{\partial}{\partial u_k} (\Gamma_{ij}^s \{x\}) D_s x^{(u)} + \Gamma_{ij}^s \{x\} \frac{\partial}{\partial u_k} D_s x^{(u)} \right), \\ \frac{\partial}{\partial u_j} \left(\frac{\partial}{\partial u_i} \frac{\partial}{\partial u_k} x^{(u)} \right) &= \frac{\partial}{\partial u_j} \left(\sum_{s=1}^n \Gamma_{ik}^s \{x\} D_s x^{(u)} \right) \\ &= \sum_{s=1}^n \left(\frac{\partial}{\partial u_j} (\Gamma_{ik}^s \{x\}) D_s x^{(u)} + \Gamma_{ik}^s \{x\} \frac{\partial}{\partial u_j} D_s x^{(u)} \right). \end{aligned}$$

Since

$$\frac{\partial}{\partial u_k} \left(\frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} x^{(u)} \right) = \frac{\partial}{\partial u_j} \left(\frac{\partial}{\partial u_i} \frac{\partial}{\partial u_k} x^{(u)} \right)$$

for all $i, j, k = 1, 2, \dots, n$, we obtain

$$\begin{aligned} & \sum_{s=1}^n \frac{\partial}{\partial u_k} (\Gamma_{ij}^s \{x\}) D_s x^{(u)} + \sum_{s=1}^n \Gamma_{ij}^s \{x\} \frac{\partial}{\partial u_k} D_s x^{(u)} \\ &= \sum_{s=1}^n \frac{\partial}{\partial u_j} (\Gamma_{ik}^s \{x\}) D_s x^{(u)} + \sum_{s=1}^n \Gamma_{ik}^s \{x\} \frac{\partial}{\partial u_j} D_s x^{(u)}. \end{aligned}$$

Using the equality $\frac{\partial}{\partial u_k} D_s x^{(u)} = \frac{\partial}{\partial u_k} \frac{\partial}{\partial u_s} x^{(u)}$ and Eq. (3.1), we have

$$\begin{aligned} & \sum_{s=1}^n \frac{\partial}{\partial u_k} (\Gamma_{ij}^s \{x\}) D_s x^{(u)} + \sum_{s=1}^n \Gamma_{ij}^s \{x\} \left(\sum_{l=1}^n \Gamma_{ks}^l \{x\} D_l x^{(u)} \right) \\ &= \sum_{s=1}^n \frac{\partial}{\partial u_j} (\Gamma_{ik}^s \{x\}) D_s x^{(u)} + \sum_{s=1}^n \Gamma_{ik}^s \{x\} \left(\sum_{l=1}^n \Gamma_{js}^l \{x\} D_l x^{(u)} \right). \end{aligned}$$

This equality implies

$$\begin{aligned} & \sum_{s=1}^n \frac{\partial}{\partial u_k} (\Gamma_{ij}^s \{x\}) D_s x^{(u)} + \sum_{s=1}^n \left(\sum_{r=1}^n \Gamma_{ij}^r \{x\} \Gamma_{kr}^s \{x\} \right) D_s x^{(u)} \\ &= \sum_{s=1}^n \frac{\partial}{\partial u_j} (\Gamma_{ik}^s \{x\}) D_s x^{(u)} + \sum_{s=1}^n \left(\sum_{r=1}^n \Gamma_{ik}^r \{x\} \Gamma_{jr}^s \{x\} \right) D_s x^{(u)}. \end{aligned}$$

Then

$$\sum_{s=1}^n \left(\left(\frac{\partial}{\partial u_k} \Gamma_{ij}^s \{x\} - \frac{\partial}{\partial u_j} \Gamma_{ik}^s \{x\} \right) + \sum_{r=1}^n (\Gamma_{ij}^r \{x\} \Gamma_{kr}^s \{x\} - \Gamma_{ik}^r \{x\} \Gamma_{jr}^s \{x\}) \right) D_s x^{(u)} = 0.$$

Since x is a J -immersion in E_p^n , vectors $D_1 x^{(u)}, \dots, D_n x^{(u)}$ are linearly independent for all $u \in \phi_\lambda(V_\lambda)$. Hence

$$\frac{\partial}{\partial u_k} \Gamma_{ij}^s \{x\} - \frac{\partial}{\partial u_j} \Gamma_{ik}^s \{x\} + \sum_{r=1}^n (\Gamma_{ij}^r \{x\} \Gamma_{kr}^s \{x\} - \Gamma_{ik}^r \{x\} \Gamma_{jr}^s \{x\}) = 0. \quad \square$$

Let x be a J -immersion in E_p^n . Then Fx is also a J -immersion in E_p^n for all $F \in \text{Af}(n)$. Let G be a subgroup of the group $\text{Af}(n)$.

Definition 7. J -immersions x and y in E_p^n will be called G -equivalent if there exists $F \in G$ such that $y(a) = Fx(a)$ for all $a \in J$. This being the case, we write $x \stackrel{G}{\sim} y$.

Theorem 3.1. Let x and y be J -immersions in E_p^n . Then the following two conditions are equivalent:

- (1) $\Gamma_{ij}^s \{x\} = \Gamma_{ij}^s \{y\}$ for all $\lambda \in \Lambda$ and $i, j, s = 1, 2, \dots, n$;
- (2) $x \stackrel{\text{Af}(n)}{\sim} y$.

Proof. (2) $\Rightarrow$ (1). Let $x \stackrel{\text{Af}(n)}{\sim} y$. Assume that λ, i, j be fixed. We consider

$$\frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} x^{(u)} = \sum_{s=1}^n \Gamma_{ij}^s \{x, V_\lambda\} D_s x^{(u)}$$

as a system of linear equations for $\Gamma_{ij}^s \{x, V_\lambda\}$, where $s = 1, 2, \dots, n$. Since

$$\left[D_1 x^{(u)}, D_2 x^{(u)}, \dots, D_n x^{(u)} \right] = \left[\frac{\partial}{\partial u_1} x^{(u)} \frac{\partial}{\partial u_2} x^{(u)} \dots \frac{\partial}{\partial u_n} x^{(u)} \right]$$

is the determinant of this system and x is a J -immersion, $[D_1 x^{(u)}, D_2 x^{(u)}, \dots, D_n x^{(u)}] \neq 0$ for all $u \in \phi_\lambda(V_\lambda)$. Hence the system has the following solutions:

$$\Gamma_{ij}^s \{x, V_\lambda\} = \frac{\left[D_1 x \dots D_{s-1} x \frac{\partial}{\partial u_i} \left(\frac{\partial}{\partial u_j} x \right) D_{s+1} x \dots D_n x \right]}{\left[D_1 x D_2 x \dots D_n x \right]}, \quad (3.3)$$

where $s = 1, 2, \dots, n$. By Eq. (3.3), $\Gamma_{ij}^s \{x, V_\lambda\}$ is $\text{Af}(n)$ -invariant. Hence

$$\Gamma_{ij}^s \{x(u), V_\lambda\} = \Gamma_{ij}^s \{y(u), V_\lambda\} \quad \text{for all } i, j, s = 1, 2, \dots, n \quad \text{and} \quad u \in \phi_\lambda(V_\lambda).$$

(1) $\Rightarrow$ (2). Assume that $\Gamma_{ij}^s \{x(u), V_\lambda\} = \Gamma_{ij}^s \{y(u), V_\lambda\}$ holds for all $i, j, s = 1, 2, \dots, n$ and $u \in \phi_\lambda(V_\lambda)$. Set $A(x) = \|D_1 x D_2 x \dots D_n x\|$ and $\frac{\partial}{\partial u_i} A(x) = \left\| \frac{\partial}{\partial u_i} D_1 x \frac{\partial}{\partial u_i} D_2 x \dots \frac{\partial}{\partial u_i} D_n x \right\|$, where we consider $D_i x$ as a column-vector. Equation (3.1) implies $A(x)^{-1} \frac{\partial}{\partial u_i} A(x) = \|\Gamma_{ij}^s \{x, V_\lambda\}\|_{s,j=1,\dots,n}$. Since $\Gamma_{ij}^s \{x(u), V_\lambda\} = \Gamma_{ij}^s \{y(u), V_\lambda\}$ for all $i, j, s = 1, 2, \dots, n$, we get

$$A(x(u))^{-1} \frac{\partial}{\partial u_i} A(x(u)) = A(y(u))^{-1} \frac{\partial}{\partial u_i} A(y(u))$$

for all $i = 1, 2, \dots, n$ and $u \in \phi_\lambda(V_\lambda)$.

The equality $A(x)^{-1} \frac{\partial}{\partial u_i} A(x) = A(y)^{-1} \frac{\partial}{\partial u_i} A(y)$ implies

$$\begin{aligned} \frac{\partial}{\partial u_i} (A(y) A(x)^{-1}) &= \left(\frac{\partial}{\partial u_i} A(y) \right) A(x)^{-1} + A(y) \frac{\partial}{\partial u_i} (A(x)^{-1}) \\ &= \left(\frac{\partial}{\partial u_i} A(y) \right) A(x)^{-1} - A(y) A(x)^{-1} \left(\frac{\partial}{\partial u_i} A(x) \right) A(x)^{-1} \\ &= A(y) \left(A(y)^{-1} \frac{\partial}{\partial u_i} A(y) - A(x)^{-1} \frac{\partial}{\partial u_i} A(x) \right) A(x)^{-1} = 0 \end{aligned}$$

for all $u \in \phi_\lambda(V_\lambda)$. Using this equality for all $i = 1, 2, \dots, n$, we see that $A(y(u)) A(x(u))^{-1}$ is not depend on u for every connected component of $\phi_\lambda(V_\lambda)$. Since J is connected, it is easy to see that $A(y(a)) A(x(a))^{-1}$ does not depend on $a \in J$.

Let $F = A(y) A(x)^{-1}$. Since $\det A(x(u)) \neq 0$ and $\det A(y(u)) \neq 0$ for all $u \in J$, we have $\det F \neq 0$ and $A(y(a)) = F A(x(a))$ for all $a \in J$. The equality $A(y(u)) = F A(x(u))$ implies $\frac{\partial}{\partial u_i} y(u) = F \frac{\partial}{\partial u_i} x(u)$ for all $i = 1, 2, \dots, n$ and $u \in \phi_\lambda(V_\lambda)$. Then there exists a constant vector $b \in E_p^n$ such that $y(u) = F x(u) + b$ for all $u \in \phi_\lambda(V_\lambda)$. Since J is connected, we see that b does not depend on $\lambda \in \Lambda$. $\square$

Let x be a J -immersion in E_p^n and $g_{ij}\{x(u), V_\lambda\} = (\frac{\partial}{\partial u_i}x(u), \frac{\partial}{\partial u_j}x(u))$. The form

$$I_\lambda(x) = \sum_{i,j=1}^n g_{ij}\{x(u), V_\lambda\} du_i du_j$$

is said to be the *first fundamental form of a J -immersion x in the coordinate neighborhood V_λ* . By Theorem V.2.3 and Corollary V.2.5 in ([5, p. 183]), $g\{x\} = \{I_\lambda(x), \lambda \in \Lambda\}$ is a pseudo-Riemannian metric on J . This pseudo-Riemannian metric on J will be called *induced by the J -immersion x* .

Theorem 3.2. *Let x and y be J -immersions in E_p^n . Then the following two conditions are equivalent:*

(1) *Pseudo-Riemannian metrics $g\{x\}$ and $g\{y\}$ on J coincide, that is*

$$I_\lambda(x) = I_\lambda(y) \quad \text{for all } \lambda \in \Lambda; \quad (3.4)$$

(2) $x \stackrel{M(n,p)}{\sim} y$.

Proof. (2) $\Rightarrow$ (1). Since functions $I_\lambda(x)$ are $M(n,p)$ -invariant, this implication is obvious.

(1) $\Rightarrow$ (2). Let λ be fixed and $\Delta_x(u) = \det \|g_{ij}(x(u), V_\lambda)\|$. Equalities Eq. (3.4) imply $\Delta_x(u) = \Delta_y(u)$ for all $u \in J$. Since $\Delta_x(u) \neq 0$ and $\Delta_y(u) \neq 0$ for J -immersions x and y , we get

$$\Delta_x(u)^{-1} = \Delta_y(u)^{-1} \quad (3.5)$$

for all $u \in J$. Let V be the system $\{g_{ij}(x(u), V_\lambda) \mid 1 \leq i \leq j \leq n\}$ and $f\{x\} \in R\{V, \Delta^{-1}\}$. Then Eqs. (3.4) and (3.5) imply

$$f\{x(u)\} = f\{y(u)\} \quad \text{for all } u \in \phi_\lambda(V_\lambda). \quad (3.6)$$

Lemma 3.1. $\Gamma_{ij}^s\{x, V_\lambda\} \in R\{V, \Delta^{-1}\}$ for all $1 \leq i, j, s \leq n$.

Proof. Equation (3.3) implies

$$\Gamma_{ij}^s\{x, V_\lambda\} = \frac{(-1)^p [D_1 x \cdots D_{s-1} x \frac{\partial}{\partial u_i} (\frac{\partial}{\partial u_j} x) D_{s+1} x \cdots D_n x] [D_1 x D_2 x \cdots D_n x]}{(-1)^p [D_1 x D_2 x \cdots D_n x]^2}. \quad (3.7)$$

Using Lemmas 2.14 and 2.9, we see that

$$(-1)^p \left[D_1 x \cdots D_{s-1} x \frac{\partial}{\partial u_i} \left(\frac{\partial}{\partial u_j} x \right) D_{s+1} x \cdots D_n x \right] [D_1 x D_2 x \cdots D_n x] \in R\{V, \Delta^{-1}\}.$$

By Lemma 2.14 and Eq. (2.8), $\Delta_x(u) = (-1)^p [D_1 x D_2 x \cdots D_n x]^2$. Hence $\Gamma_{ij}^s\{x, V_\lambda\} \in R\{V, \Delta^{-1}\}$ for all $1 \leq i, j, s \leq n$. $\square$

By Lemma 3.1, $\Gamma_{ij}^s\{x, V_\lambda\} \in R\{V, \Delta^{-1}\}$ for all $i, j, s = 1, 2, \dots, n$. Then Eq. (3.6) implies $\Gamma_{ij}^s\{x, V_\lambda\} = \Gamma_{ij}^s\{y, V_\lambda\}$ for all $\lambda \in \Lambda$ and $i, j, s = 1, 2, \dots, n$. By Theorem 3.1, there exists $F \in GL(n)$ and $b \in E_p^n$ such that $y(a) = Fx(a) + b$ for all $a \in J$. We prove that $F \in O(n, p)$.

Let $A(x)^\top$ be the transposed of the matrix $A(x)$ in Theorem 3.1. Using the equality

$$A(x)^\top I_p A(x) = \|g_{ij}\{x, V_\lambda\}\|_{i,j=1,2,\dots,n}$$

and Eq. (3.4), we obtain $A(x)^\top I_p A(x) = A(y)^\top I_p A(y)$. Since x is a J -immersion, we have $\det A(x(u)) \neq 0$ for all $u \in \phi_\lambda(V_\lambda)$. Then equalities $A(x)^\top I_p A(x) = A(y)^\top I_p A(y)$ and $A(y) = FA(x)$ imply $F^\top I_p F = I_p$. Hence $F \in O(n, p)$. $\square$

Let x and y be J -immersions in E_p^n such that $x \stackrel{M(n,p)}{\sim} y$. Since $M(n, p) \subset \text{Af}(n)$, by Theorem 3.1, $\Gamma_{ij}^s\{x, V_\lambda\} = \Gamma_{ij}^s\{y, V_\lambda\}$ for all $\lambda \in \Lambda$ and $i, j, s = 1, 2, \dots, n$. The equality Eq. (3.2) implies

$$\Gamma_{ij,s}\{x, V_\lambda\} = \left(\frac{\partial^2}{\partial u_i \partial u_j} x, D_s x(u) \right) = \left(\frac{\partial^2}{\partial u_i \partial u_j} x, \frac{\partial}{\partial u_s} x \right). \quad (3.8)$$

Using Lemma 2.7, we obtain

$$\Gamma_{ij,s}\{x, V_\lambda\} = \frac{1}{2} \left(\frac{\partial}{\partial u_j} g_{si}\{x, V_\lambda\} + \frac{\partial}{\partial u_i} g_{sj}\{x, V_\lambda\} - \frac{\partial}{\partial u_s} g_{ij}\{x, V_\lambda\} \right)$$

for all $i, j, s = 1, 2, \dots, n$, where $g_{ij}\{x, V_\lambda\} = (\frac{\partial}{\partial u_i} x(u), \frac{\partial}{\partial u_j} x(u))$. Since, by Eq. (3.8), $\Gamma_{ij,s}\{x, V_\lambda\}$ is $M(n, p)$ -invariant, we have $\Gamma_{ij,s}\{x, V_\lambda\} = \Gamma_{ij,s}\{y, V_\lambda\}$ for all $\lambda \in \Lambda$ and $i, j, s = 1, 2, \dots, n$. The converse statement is true in the following form:

Theorem 3.3. *Let x and y be J -immersions in E_p^n . Assume that the following conditions hold:*

- (1) $\Gamma_{ij}^s\{x, V_\lambda\} = \Gamma_{ij}^s\{y, V_\lambda\}$ for all $\lambda \in \Lambda$ and $i, j, s = 1, 2, \dots, n$;
- (2) $\Gamma_{ij,s}\{x, V_\lambda\} = \Gamma_{ij,s}\{y, V_\lambda\}$ $i \in \{1, 2, \dots, n\}$ for all $\lambda \in \Lambda$ and $i, j, s = 1, 2, \dots, n$;
- (3) *there exist $\lambda \in \Lambda, u_0 \in \phi_\lambda(V_\lambda)$ such that $\det(\frac{\partial}{\partial u_i} A(x(u_0))) \neq 0$ for some $i \in \{1, 2, \dots, n\}$, where $A(x)$ is the matrix in Theorem 3.1.*

Then $x \stackrel{M(n,p)}{\sim} y$.

Proof. Let λ and u_0 be as in condition 3. As in Theorem 3.1, condition 1 imply existence $F \in GL(n)$ and $b \in R^n$ such that $y(a) = Fx(a) + b$ for all $a \in J$. We prove that $F \in O(n, p)$. Using Eq. (3.8), we have

$$\|\Gamma_{ij,s}\{x(u), V_\lambda\}\|_{s,j=1,2,\dots,n} = A(x(u))^\top I_p \frac{\partial}{\partial u_i} A(x(u)).$$

By condition 2, we have

$$A(x(u))^\top I_p \frac{\partial}{\partial u_i} A(x(u)) = A(y(u))^\top I_p \frac{\partial}{\partial u_i} A(y(u))$$

for all $u \in \phi_\lambda(V_\lambda)$ and $i \in \{1, 2, \dots, n\}$. Since $A(y(u)) = FA(x(u))$ and $\det A^\top(x(u)) = \det A(x(u)) \neq 0$ for all $u \in \phi_\lambda(V_\lambda)$, we get $I_p \frac{\partial}{\partial u_i} A(x(u_0)) = F^\top I_p F \frac{\partial}{\partial u_i} A(x(u_0))$. Since $\det(\frac{\partial}{\partial u_i} A(x(u_0))) \neq 0$, this equality implies $F^\top I_p F = I_p$. Hence $F \in O(n, p)$. $\square$

Corollary 3.2. *Let x and y be J -immersions in E_p^n . Then the following two conditions are equivalent:*

- (1) $g\{x\} = g\{y\}$ and $\Omega(x) = \Omega(y)$;
- (2) $x \stackrel{SM(n,p)}{\sim} y$.

Proof. (2) $\Rightarrow$ (1). Since functions $I_\lambda(x)$ and $\Omega(x)$ are $SM(n,p)$ -invariant, this implication is obvious.

(1) $\Rightarrow$ (2). Assume that $g\{x\} = g\{y\}$ and $\Omega(x) = \Omega(y)$. By $g\{x\} = g\{y\}$, Theorem 3.2 implies $x \stackrel{M(n,p)}{\sim} y$. Hence there exist $F \in O(n,p)$ and $b \in E_p^n$ such that $y(a) = Fx(a) + b$ for all $a \in J$. Using this equality, $\Omega(x) \neq 0$, expressions $\Omega(x) = \Omega(y)$ and $\Omega(y) = (\det F)\Omega(x)$ in local coordinates, we obtain $\det F = 1$. This means that $x \stackrel{SM(n,p)}{\sim} y$. $\square$

4. Pseudo-Riemannian Metrics with the Zero Riemannian Curvature Tensor on a Connected, Simply Connected Manifold

Let J be a connected C^∞ -manifold of dimension n , $\{(V_\lambda, \phi_\lambda), \lambda \in \Lambda\}$ the collection of all coordinate neighborhoods of J , x a J -immersion in E_p^n , and $x^{(u)}(a)$ the expressing of x in the local coordinates $u = (u_1, u_2, \dots, u_n)$ in V_λ . Consider the function $g_{ij}\{x\} = (\frac{\partial}{\partial u_i}x^{(u)}(a), \frac{\partial}{\partial u_j}x^{(u)}(a))$ and the matrix $g\{x\} = \|g_{ij}\{x\}\|_{i,j=1,2,\dots,n}$ on the coordinate neighborhood V_λ . Set $g_l\{x\} := (g_{1l}\{x\}, g_{2l}\{x\}, \dots, g_{nl}\{x\})^\top$ and $\Gamma_{sj}\{g\{x\}\} := (\Gamma_{sj,1}\{x\}, \Gamma_{sj,2}\{x\}, \dots, \Gamma_{sj,n}\{x\})^\top$.

Proposition 4.1. *Let $\Gamma_{js}^i\{x\}$ be the function Eq. (3.3). Then*

$$\Gamma_{js}^i\{x\} = \frac{[g_1\{x\}g_2\{x\} \cdots g_{i-1}\{x\}\Gamma_{js}\{g\{x\}\}g_{i+1}\{x\} \cdots g_n\{x\}]}{[g_1\{x\}g_2\{x\} \cdots g_n\{x\}]}.$$

Proof. A proof follows from Eq. (3.7) and Lemma 2.14. $\square$

Let h be a pseudo-Riemannian metric on J and $h^{(u)}(a) = \|h_{ij}^{(u)}(a)\|_{i,j=1,2,\dots,n}$ be the expressing of h in local coordinates in V_λ . Then $h_{ij}^{(u)}(a)$ is a C^∞ -function on $\phi_\lambda(V_\lambda)$, $\det \|h_{ir}^{(u)}(a)\|_{i,r=1,2,\dots,n} \neq 0$, $h_{ir}^{(u)}(a) = h_{ri}^{(u)}(a)$ for all $u \in \phi_\lambda(V_\lambda)$ and $i, r = 1, 2, \dots, n$. The Christoffel symbol $\Gamma_{sj,l}$ for h is defined as follows:

$$\Gamma_{sj,l}\{h\} = \frac{1}{2} \left(\frac{\partial}{\partial u_s} h_{lj} + \frac{\partial}{\partial u_j} h_{ls} - \frac{\partial}{\partial u_l} h_{js} \right).$$

Set

$$\Gamma_{js}\{h\} = (\Gamma_{js,1}\{h\}, \Gamma_{js,2}\{h\}, \dots, \Gamma_{js,n}\{h\})^\top,$$

set also $h_i := (h_{1i}, h_{2i}, \dots, h_{ni})^\top$ and $[h_1 h_2 \cdots h_n] =: \det \|h_{ir}(u)\|_{i,r=1,2,\dots,n}$. The Christoffel symbol Γ_{js}^i for h is defined as follows:

$$\Gamma_{js}^i\{h\} = \frac{[h_1 h_2 \cdots h_{i-1} \Gamma_{js}\{h\} h_{i+1} \cdots h_n]}{[h_1 h_2 \cdots h_n]}.$$

Proposition 4.2. $\Gamma_{js}^i\{h\} = \Gamma_{sj}^i\{h\}$ for all $i, j, s = 1, 2, \dots, n$.

Proof. By definition of the functions $h_{ir}^{(u)}(a)$, we have $h_{ir}^{(u)}(a) = h_{ri}^{(u)}(a)$ for all $i, r = 1, 2, \dots, n$. This equality implies $\Gamma_{sj,l}\{h\} = \Gamma_{js,l}\{h\}$ and $\Gamma_{js}\{h\} = \Gamma_{sj}\{h\}$ for all $j, s = 1, 2, \dots, n$. Then, by definition of $\Gamma_{js}^i\{h\}$, we obtain $\Gamma_{js}^i\{h\} = \Gamma_{sj}^i\{h\}$. $\square$

Proposition 4.3. *The system*

$$\Gamma_{sj}\{h\} = \sum_{i=1}^n \Gamma_{ij}^s\{h\} h_i \quad \text{where } j, s = 1, 2, \dots, n, \quad (4.1)$$

of equalities holds for any matrix $h(u) = \|h_{i,j}(u)\|_{i,j=1,2,\dots,n}$ such that $h_{i,j}(u)$ is a C^∞ -function on $\phi_\lambda(V_\lambda)$, $\det \|h_{ir}(u)\|_{i,r=1,2,\dots,n} \neq 0$ and $h_{i,r}(u) = h_{r,i}(u)$ for all $i, r = 1, 2, \dots, n$ and $u \in \phi_\lambda(V_\lambda)$.

Proof. The definition of the functions $\{\Gamma_{ij}^s\{h\} \mid i = 1, 2, \dots, n\}$ implies system Eq. (4.1). $\square$

Proposition 4.4. *Let $\|c_{ij}\|_{i,j=1,2,\dots,n}$ be a real symmetric matrix such that $\det \|c_{ir}\|_{i,r=1,2,\dots,n} \neq 0$ and $\|c_{ir}(u)\|_{i,r=1,2,\dots,n}$ is congruent to the matrix I_p . Then there exists a system of vectors $d_1, d_2, \dots, d_n$ in E_p^n such that $(d_i, d_j) = c_{ij}$ for all $i, j = 1, 2, \dots, n$.*

Proof. For vectors $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, $\dots$, $e_n = (0, 0, 0, \dots, 1)$ in E_p^n , we have $(e_i, e_i) = -1$ for $i = 1, 2, \dots, p$, $(e_i, e_i) = 1$ for $i = p+1, \dots, n$, and $(e_i, e_j) = 0$ for $i \neq j$. Set $C = \|c_{ij}\|_{i,j=1,2,\dots,n}$. Since $\det C \neq 0$ and the matrix C is congruent to the matrix I_p , there exists an $n \times n$ -matrix B such that $\det B \neq 0$ and $B^\top C B = I_p$. Then $C = (B^{-1})^\top I_p B^{-1}$. Set $B^{-1} = \|\gamma_{ik}\|_{i,k=1,2,\dots,n}$. Hence, for vectors $d_k = \sum_{i=1}^n \gamma_{ik} e_i$, where $k = 1, 2, \dots, n$, we have $(d_i, d_j) = c_{ij}$ for all $i, j = 1, 2, \dots, n$. $\square$

We assume that in each chart of J with local coordinates $(u_1, u_2, \dots, u_n) = u$ an $(n \times n)$ -matrix C^∞ -function $\xi^{(u)}(a) = \|\xi_1^{(u)}(a) \xi_2^{(u)}(a) \dots \xi_n^{(u)}(a)\|$ is given, where $\xi_i^{(u)}(a)$ is a column of $\xi^{(u)}(a)$. The following definition is taken from ([20, p. 56]):

Definition 8. A collection of an $(n \times n)$ -matrix C^∞ -function $\xi^{(u)}(a) = \|\xi_1^{(u)}(a) \xi_2^{(u)}(a) \dots \xi_n^{(u)}(a)\|$ on J will be called a *covariant tensor field of rank 1* if it is transformed according to law

$$\xi_i^{(v)} = \sum_{s=1}^n \frac{\partial u_s}{\partial v_i} \xi_s^{(u)}$$

when passing from one chart to another; here $v_1, v_2, \dots, v_n$ and $u_1, u_2, \dots, u_n$ are, respectively, “old” and “new” coordinates in the intersection of the charts.

Theorem 4.1. *Let J be a connected, simply connected C^∞ -manifold. Assume that h be a pseudo-Riemannian metric of the index p on J such that its Riemannian curvature tensor vanishes. Then there exists a J -immersion x in E_p^n such that the pseudo-Riemannian metric $g\{x\}$ on J induced by x is equal to h .*

Proof. Let $\mu \in \Lambda$ be fixed and $a_0 \in V_\mu$. Consider the matrix $h(a_0) = \|h_{ij}(a_0)\|_{i,j=1,2,\dots,n}$. According to Proposition 4.4, there exists a system of vectors $d_1, d_2, \dots, d_n$ in tangent space

of the manifold J at the point a_0 such that $(d_i, d_j) = h_{ij}(a_0)$ for all $i, j = 1, 2, \dots, n$. Set $Q_s\{h\} = \|\Gamma_{lj}^s(h)\|_{l,j=1,2,\dots,n}$ for $s = 1, 2, \dots, n$. Consider the following system of equations

$$\frac{\partial}{\partial u_s} \xi^{(u)} = \xi^{(u)} Q_s\{h(u)\}, \quad \text{where } s = 1, 2, \dots, n, \quad (4.2)$$

for an $(n \times n)$ -matrix C^∞ -function $\xi^{(u)}(a) = \|\xi_1^{(u)}(a)\xi_2^{(u)}(a)\dots\xi_n^{(u)}(a)\|$ on J , where $\xi^{(u)}$ is a covariant tensor field of the 1st-rank on J . It is easy to see that the form of the system Eq. (4.2) of equations is the same in “old” and “new” coordinates in the intersection of the charts. Since the Riemannian curvature tensor of h is equal to zero, the following system of equalities

$$\frac{\partial}{\partial u_s} Q_l\{h(u)\} - \frac{\partial}{\partial u_l} Q_s\{h(u)\} = [Q_l\{h(u)\}, Q_s\{h(u)\}], \quad \text{where } l, s = 1, 2, \dots, n, \quad (4.3)$$

holds, where $[Q_l\{h\}, Q_s\{h\}]$ denotes $Q_l\{h\}Q_s\{h\} - Q_l\{h\}Q_s\{h\}$. By Eq. (4.3) and according to the theory of linear differential equations, there exist a neighborhood $W \subseteq V_\mu$ of the point a_0 and an $(n \times n)$ -matrix C^∞ -function $\xi^{(u)} = \|\xi_1^{(u)}(a)\xi_2^{(u)}(a)\dots\xi_n^{(u)}(a)\|$ on W such that $\xi_i^u(a_0) = d_i$ for all $i = 1, 2, \dots, n$, that $\det(\xi^{(u)}(a)) \neq 0$ for all $a \in W$, and $\xi^{(u)}(a)$ is a solution of the system Eq. (4.2) on W . Using connectedness and simply connectedness of the manifold J , according to the theory of linear differential equations on manifolds (see [15]), we see that the unique an $(n \times n)$ -matrix C^∞ -function $\xi(a) = \|\xi_1(a)\xi_2(a)\dots\xi_n(a)\|$ on J exists such that $\xi(a)$ is a covariant tensor field of the 1st-rank on J and $\xi(a)$ is a solution of Eq. (4.2) on V_λ for every $\lambda \in \Lambda$, $\det(\xi(a)) \neq 0$ for all $a \in J$ and $\xi(a) = \xi^{(u)}(a)$ for all $a \in W$. Below we consider this solution $\xi(a)$.

By $Q_s\{h(u)\} = \|\Gamma_{lj}^s(h(u))\|_{l,j=1,2,\dots,n}$ and Eq. (4.2), we obtain

$$\frac{\partial}{\partial u_i} \xi_j^{(u)}(a) = \sum_{i=1}^n \Gamma_{ij}^s\{h(u)\} \xi_s^{(u)}(a), \quad \frac{\partial}{\partial u_j} \xi_i^{(u)}(a) = \sum_{i=1}^n \Gamma_{ji}^s\{h(u)\} \xi_s^{(u)}(a)$$

in each chart of J with local coordinates $u = (u_1, u_2, \dots, u_n)$. These equalities and the equality $\Gamma_{ij}^s\{h(u)\} = \Gamma_{ji}^s\{h(u)\}$ in Proposition 4.2 imply

$$\frac{\partial}{\partial u_i} \xi_j^{(u)}(a) = \frac{\partial}{\partial u_j} \xi_i^{(u)}(a) \quad \text{for all } i, j = 1, 2, \dots, n \quad (4.4)$$

in each chart of J with local coordinates u . Since the $(n \times n)$ -matrix-valued C^∞ -function $\xi(a) = \|\xi_1(a)\xi_2(a)\dots\xi_n(a)\|$ on J is a covariant tensor field of rank 1 on J and $\det(\xi(a)) \neq 0$ for all $a \in J$, Eq. (4.4) implies an existence of a J -immersion $x(a)$ such that $\frac{\partial}{\partial u_j} x^{(u)}(a) = \xi_j^{(u)}(a)$ for all $j = 1, 2, \dots, n$ in each chart of J in local coordinates u , where $x^{(u)}(a)$ is the expression of $x(a)$ in coordinates u . Since $(\xi_i^{(u)}(a_0), \xi_j^{(u)}(a_0)) = h_{ij}(a_0)$, we obtain $(\frac{\partial}{\partial u_i} x^{(u)}(a_0), \frac{\partial}{\partial u_j} x^{(u)}(a_0)) = h_{ij}(a_0)$ for all $i, j = 1, 2, \dots, n$ in the chart V_μ of J with local coordinates u . Below we consider this J -immersion $x(a)$.

Let V_τ be an arbitrary chart of J with local coordinates $u = (u_1, u_2, \dots, u_n)$ and

$$A^{(u)}(x) = \left\| \frac{\partial}{\partial u_1} x^{(u)}(a) \frac{\partial}{\partial u_2} x^{(u)}(a) \dots \frac{\partial}{\partial u_n} x^{(u)}(a) \right\|,$$

where $\frac{\partial}{\partial u_j} x^{(u)}(a)$ as a column-vector of $A^{(u)}(x)$.

Consider

$$\frac{\partial}{\partial u_s} A^{(u)}(x) = \left\| \frac{\partial}{\partial u_s} \left(\frac{\partial}{\partial u_1} x^{(u)}(a) \right) \frac{\partial}{\partial u_s} \left(\frac{\partial}{\partial u_2} x^{(u)}(a) \right) \cdots \frac{\partial}{\partial u_s} \left(\frac{\partial}{\partial u_n} x^{(u)}(a) \right) \right\|.$$

For $\xi_i^{(u)}(a) = \frac{\partial}{\partial u_i} x^{(u)}(a)$, the system Eq. (4.2) has the following form:

$$\frac{\partial}{\partial u_s} A^{(u)}(x) = A^{(u)}(x) Q_s \{h(u)\} \quad \text{for } s = 1, 2, \dots, n. \quad (4.5)$$

Let $A^{(u)}(x)^\top$ be the transpose matrix of $A^{(u)}(x)$. Then Eq. (4.5) implies

$$A^{(u)}(x)^\top \frac{\partial}{\partial u_s} A^{(u)}(x) = A^{(u)}(x)^\top A^{(u)}(x) Q_s \{h(u)\} \quad \text{for } s = 1, 2, \dots, n.$$

In terms of $g\{x\}$, $g_{ij}\{x\}$ and $\Gamma_{sj,l}\{x\}$, this system has the following form:

$$\Gamma_{sj,l}\{g\} = \sum_{i=1}^n \Gamma_{ij}^s\{h\} g_{li} \quad \text{for } j, l, s = 1, 2, \dots, n. \quad (4.6)$$

According to Proposition 4.3, the following system of equations

$$\Gamma_{sj,l}\{h\} = \sum_{i=1}^n \Gamma_{ij}^s\{h\} h_{li} \quad \text{for } j, l, s = 1, 2, \dots, n. \quad (4.7)$$

holds for the pseudo-Riemannian metric h of the index p on J .

By Lemma 2.7 and the definition of $\Gamma_{sj,l}\{g\{x\}\}$, we obtain

$$\begin{aligned} \Gamma_{sj,l}\{g\{x\}\} &= \left(\frac{\partial}{\partial u_s} \frac{\partial}{\partial u_j} x, \frac{\partial}{\partial u_l} x \right) = \frac{1}{2} \left(\frac{\partial}{\partial u_s} g_{lj}\{x\} + \frac{\partial}{\partial u_j} g_{ls}\{x\} - \frac{\partial}{\partial u_l} g_{js}\{x\} \right), \\ \Gamma_{jl,s}\{g\{x\}\} &= \left(\frac{\partial}{\partial u_j} \frac{\partial}{\partial u_l} x, \frac{\partial}{\partial u_s} x \right) = \frac{1}{2} \left(\frac{\partial}{\partial u_j} g_{sl}\{x\} + \frac{\partial}{\partial u_l} g_{sj}\{x\} - \frac{\partial}{\partial u_s} g_{lj}\{x\} \right). \end{aligned}$$

These equalities and Eq. (4.6) imply

$$\frac{\partial}{\partial u_j} g_{ls}\{x\} = \Gamma_{sj,l}\{g\{x\}\} + \Gamma_{jl,s}\{g\{x\}\} = \sum_{i=1}^n \Gamma_{ij}^s\{h\} g_{li}\{x\} + \sum_{i=1}^n \Gamma_{il}^j\{h\} g_{si}\{x\}.$$

Hence we have the following system of differential equations for $g_{ls}\{x\}$:

$$\frac{\partial}{\partial u_j} g_{ls}\{x\} = \sum_{i=1}^n \Gamma_{ij}^s\{h\} g_{li}\{x\} + \sum_{i=1}^n \Gamma_{il}^j\{h\} g_{si}\{x\} \quad \text{for } j, l, s = 1, 2, \dots, n.$$

Similarly, using the definition of $\Gamma_{lj}^s\{h\}$ and Eq. (4.7), we obtain the following system of differential equations for h_{ls} :

$$\frac{\partial}{\partial u_j} h_{ls} = \sum_{i=1}^n \Gamma_{ij}^s\{h\} h_{li} + \sum_{i=1}^n \Gamma_{il}^j\{h\} h_{si}, \quad \text{where } j, l, s = 1, 2, \dots, n. \quad (4.8)$$

Hence the system $\{g_{ls}\{x\}$, where $l, s = 1, 2, \dots, n\}$ is a solution of Eq. (4.8). Since

$$g_{ls}\{x(a_0)\} = \left(\frac{\partial}{\partial u_l} x^{(u)}(a_0), \frac{\partial}{\partial u_s} x^{(u)}(a_0) \right) = h_{ls}(a_0),$$

using the uniqueness theorem for linear differential equations on a connected, simply connected manifold, we obtain $g_{ls}\{x(a)\} = h_{ls}(a)$ for all $a \in J$ and $l, s = 1, 2, \dots, n$. $\square$

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