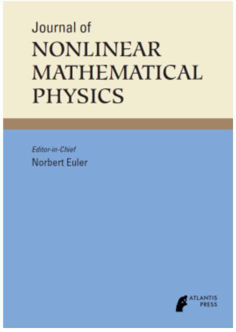


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# ON INVARIANTS OF IMMERSIONS OF AN $n$-DIMENSIONAL MANIFOLD IN AN $n$-DIMENSIONAL PSEUDO-EUCLIDEAN SPACE*

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Let $E_p^n$ be the $n$-dimensional pseudo-Euclidean space of index $p$ and $M(n,p)$ the group of all transformations of $E_p^n$ generated by pseudo-orthogonal transformations and parallel translations. We describe the system of generators of the differential field of all $M(n,p)$-invariant differential rational functions of a map $x : J \to E_p^n$ of an open subset $J \subseteq E_p^n$. Using this result, we prove analogues of the Bonnet theorem for immersions of an $n$-dimensional $C^\infty$-manifold $J$ in $E_p^n$. These analogues are given in terms of the pseudo-Riemannian metric, the volume form, and the connection on $J$ induced by the immersion of $J$ in $E_p^n$.

*Keywords*: Immersion; invariant; pseudo-Riemannian metric; Riemannian curvature tensor.

Mathematics Subject Classification: 53C50 (Primary) 53B30, 53C24, 53C42 (Secondary)

## 1. Introduction

Invariants of immersions of a manifold in a pseudo-Euclidean space play an important role in many areas of mathematics and mathematical physics. Isometric immersions of pseudo-Riemannian space forms have been a subject of wide interest (see [4, 6, 7]).

Let $R$ be the field of real numbers and $p$ an integer such that $0 \le p < n$. Denote by $E_p^n$ the space $R^n$ with the inner product

$$(x,y) = -x_1y_1 - \cdots - x_py_p + x_{p+1}y_{p+1} + \cdots + x_ny_n.$$

Let $O(n,p)$ be the group of all linear transformations $g$ of $E_p^n$ such that $(gx, gy) = (x,y)$. Set

$$M(n,p) = \{F : E_p^n \longrightarrow E_p^n \mid Fx = gx + b, \;\; g \in O(n,p), \;\; b \in E_p^n\}$$

and $SM(n,p) = \{F \in M(n,p) \mid \det g = 1\}$.

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Let $J$ be an $n$-dimensional $C^\infty$-manifold and $x : J \to E_p^n$ be an immersion. This immersion induces the unique pseudo-Riemannian metric, the volume form and the connection on $J$ which we denote by $g\{x\}$, $\Omega(x)$, $C(x)$, respectively.

The fundamental existence and uniqueness theorem of Bonnet for hypersurface immersions (that is for immersions of a connected, simply connected and oriented $n$-dimensional differentiable manifold $J$) in a Euclidean space $E_0^{n+1}$ is known ([14, p. 19]). Two analogues of the Bonnet theorem for immersions of an $n$-dimensional manifold $J$ in a Euclidean space $E_0^n$ were obtained.

The first analogue of the Bonnet theorem is the following (see [8, 16, 17]): *Let $J$ be a connected, simply connected open subset of a Euclidean space $E_0^n$. Then*

(1) (*rigidity property*). *Assume that $x : J \to E_0^n$ and $y : J \to E_0^n$ are immersions such that $g\{x\} = g\{y\}$. Then there exists $F \in M(n, 0)$ such that $y = Fx$;*
(2) (*existence*). *Let $h = \|h_{ij}(u)\|$ be a Riemannian metric in $J$ of class $C^3$ and assume that its Riemannian curvature tensor vanishes. Then there exists an immersion of $J$ in $E_0^n$ such that $g\{x\} = h$.*

Another analogue of the Bonnet theorem is given in the book [1, pp. 69–71]. Note that in this book and papers mentioned below in Introduction, the term "vector field" is used for any map $x : J \to E_0^n$ of an open subset $J \subseteq E_0^n$. This other analogue of the Bonnet theorem is as follows:

*For any $G \subseteq R^3$, let the functions $A_{ik}$ and $B_{ik}$ of Cartesian coordinates be given such that $A_{1k} = -A_{2k}$ for $i = 1, 2$ and $k = 1, 2, 3$. Suppose these functions satisfy the following system*

$$\frac{\partial A_{ik}}{\partial x_l} - \frac{\partial A_{il}}{\partial x_k} + B_{il}B_{jk} - B_{ik}B_{jl} = 0,$$
$$\frac{\partial B_{ik}}{\partial x_l} - \frac{\partial B_{il}}{\partial x_k} + A_{ik}B_{jl} - A_{il}B_{jk} = 0.$$

*Then there exist orthonormal vector fields $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{n} \in G$ such that*

$$\frac{\partial \boldsymbol{a}_i}{\partial x_k} = A_{ik}\boldsymbol{a}_j + B_{ik}\boldsymbol{n}, i \neq j,$$
$$\frac{\partial \boldsymbol{n}}{\partial x_k} = -B_{1k}\boldsymbol{a}_1 - B_{2k}\boldsymbol{a}_2.$$

*The vector fields $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{n}$ are defined uniquely up to their choice at one point.*[a]

We note that the part "*The vector fields $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{n}$ are defined uniquely up to their choice at one point*" of this theorem is not clear. Indeed, the functions

$$A_{1k} = \left(\frac{\partial \mathbf{a}_1}{\partial x_k}, \mathbf{a}_2\right), \quad A_{2k} = \left(\frac{\partial \mathbf{a}_2}{\partial x_k}, \mathbf{a}_1\right), \quad B_{ik} = -\left(\frac{\partial \mathbf{n}}{\partial x_k}, \mathbf{a}_i\right)$$

[a]The meaning of the words "their choice at one point" are hard to decipher, this is a Russian sentence translated word-for-word and the meaning was lost; it should probably be understood as "up to their values at one point".

are $O(3)$-invariant, but they are not invariant with respect to parallel translations in $R^3$. Hence this part of the theorem may be given as follows: "*The vector fields* $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{n}$ *are defined uniquely up to an orthogonal transformation of* $R^3$*.*" This means that the system of functions $A_{ik}, B_{ik}$ is the complete system of joint $O(3)$-invariants of orthonormal vector fields $\mathbf{a}_1, \mathbf{a}_2, \mathbf{n}$.

The aim of the present paper is a generalization of the Bonnet theorem for immersions of an $n$-dimensional manifold $J$ in a pseudo-Euclidean space $E^n_p$. First we describe the system of generators of the differential field of all $H$-invariant differential rational functions of a mapping $x : J \to E^n_p$ of an open subset $J \subseteq E^n_p$ in $E^n_p$ for groups $H = M(n,p)$ and $H = SM(n,p)$. Using these results, we prove analogues of the Bonnet theorem for immersions an $n$-dimensional manifold $J$ in $E^n_p$. These analogues are given in terms of the pseudo-Riemannian metric, the volume form and the connection of an immersion of $J$ in $E^n_p$.

Let $J$ be an open subset of $E^n_0$, $G$ be a group and $\alpha(G)$ be an action of $G$ on the set of all smooth vector fields on $J$. Investigations of the problem of $\alpha(G)$-equivalence of vector fields, $\alpha(G)$-invariant vector fields and $\alpha(G)$-invariants of vector fields have important role in many areas of mathematics.

Let $\rho$ be a linear representation of a group $G$ in $E^n_0$ and $x$ a smooth vector field in $E^n_0$. Let $J = E^n_0$. Consider the action

$$\rho^*(g)(x(a)) := \rho(g)x(\rho(g^{-1})a)$$

of $G$ on the set of all smooth vector fields $x$ in $E^n_0$. The problem of a description of the general form of all $\rho^*(G)$-invariant (that is equivariant) polynomial vector fields for a compact Lie groups $G$ is intensively studied in the bifurcation theory [2, 3, 10]. By using theorems of Schwartz and Poe'naru ([10, Theorems XII4.3 and XII5.2]), this problem reduces to an algebraic problem of invariant theory. The problems of $\rho^*(G)$-equivalence of smooth vector fields and complete systems of $\rho^*(G)$-invariants of polynomial vector fields are investigated in the theory of differential equations [9, 19]. Invariants of vector fields are also studied in differential geometry [1].

This paper is organized as follows. In Sec. 2, we describe the system of generators of the differential field of all $G$-invariant differential rational functions of a map $x : J \to E^n_p$ of an open subset $J \subseteq E^n_p$ for groups $G = M(n,p)$ and $G = SM(n,p)$ (Theorems 2.1 and 2.2).

In Sec. 3, for an $n$-dimensional connected manifold $J$, using results of Sec. 2, we prove

(1) the rigidity theorem for the connection $C(x)$ on $J$ induced by an immersion $x : J \to E^n_p$; (Theorem 3.1);
(2) the rigidity theorem for the pseudo-Riemannian metric $g\{x\}$ on $J$ induced by an immersion $x : J \to E^n_p$; (Theorem 3.2);
(3) the rigidity theorem for the set of all Christoffel symbols of the first and second kinds on $J$ induced by an immersion $x : J \to E^n_p$; (Theorem 3.3);
(4) the rigidity theorem for the pseudo-Riemannian metric $g\{x\}$ and the volume form $\Omega(x)$ on $J$ induced by an immersion $x : J \to E^n_p$; (Corollary 2).

Theorems 3.1–3.3 and Corollary 2 are analogues of the first part of the Bonnet theorem for groups $G = \mathrm{Af}(n)$, $M(n,p)$, $SM(n,p)$.

In Sec. 4, for an $n$-dimensional connected, simply connected manifold $J$, we prove the existence theorem for a pseudo-Riemannian metric on $J$: Let $h = \|h_{ij}(u)\|$ be a pseudo-Riemannian metric in $J$ of class $C^\infty$ and assume that its Riemannian curvature tensor vanishes. Then there exists an immersion $x$ of $J$ in $E_p^n$ such that $g\{x\} = h$ (Theorem 4.1). This theorem is an analogue of the second part of the Bonnet theorem for immersions of the manifold $J$ in $E_p^n$.

We use methods of the invariant theory and the theory of differential equations. A similar approach to the theory of curves was used in the book [12] and papers [13, 18].

In the sequel, $n > 1$. The case $n = 1$ is easily considered.

## 2. Generating Systems of Differential Fields of all $G$-Invariant Differential Rational Functions of an $n$-Parametric Surface

Throughout this section we assume that $J$ is an open subset of $R^n$.

**Definition 1.** A $C^\infty$-mapping $x : J \to E_p^n$ will be called an *$n$-parametric $J$-surface in $E_p^n$* and denoted by $x(u)$. For short, an $n$-parametric $J$-surface $x(u)$ with the domain of the definition $J$ will be called a *$J$-surface*.

Let $x(u)$ be a $J$-surface in $E_p^n$. Denote by $N_0$ the set of all non-negative integers. For $m_i \in N_0$, where $1 \leq i \leq n$, we set

$$x^{(0,0,\dots,0)} = x(u), \quad x^{(m_1,m_2,\dots,m_n)} = \frac{\partial^{m_1+m_2+\cdots+m_n} x(u)}{\partial u_1^{m_1} u_2^{m_2} \cdots \partial u_n^{m_n}}.$$

Below we cite some notation and definitions from the differential algebra (see [11–13]) in a form convenient for our considerations.

**Definition 2.** A polynomial $q(x, x^{(1,0,0,\dots,0)}, x^{(0,1,0,\dots,0)}, \dots, x^{(m_1,m_2,m_3,\dots,m_n)})$ of $x$ and a finite number of partial derivatives $x^{(1,0,0,\dots,0)}, x^{(0,1,0,\dots,0)}, \dots, x^{(m_1,m_2,m_3,\dots,m_n)}$ of $x$ with coefficients from $R$ is called a *differential polynomial of $x$*.

It will be denoted by $q\{x\}$. The set of all differential polynomials of $x$ will be denoted by $R\{x\}$. It is a differential $R$-algebra with respect to the differentiations $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots, \frac{\partial}{\partial u_n}$. This differential $R$-algebra is also an integral domain. Its quotient field will be denoted by $R\langle x\rangle$. It is a differential field with respect to the differentiations $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots, \frac{\partial}{\partial u_n}$. An element $h$ of $R\langle x\rangle$ will be called a *differential rational function of $x$* and denoted by $h\langle x\rangle$.

Let $x(u_1, u_2, \dots, u_n)$, $y(u_1, u_2, \dots, u_n), \dots, z(u_1, u_2, \dots, u_n)$ be a finite number of $J$-surfaces in $R^n$. A differential polynomial and a differential rational function of several $J$-surfaces $x, y, \dots, z$ are similarly defined. They will be denoted by $q\{x, y, \dots, z\}$ and $q\langle x, y, \dots, z\rangle$, respectively. The differential field of all differential rational functions of $x, y, \dots, z$ will be denoted by $R\langle x, y, \dots, z\rangle$.

Let $GL(n)$ be the group of all non-degenerate real $n \times n$-matrices. Set

$$\mathrm{Af}(n) = \{F : R^n \to R^n \mid Fx = gx + b, \;\; g \in GL(n), \;\; b \in R^n\},$$

where $gx$ is the multiplication of a matrix $g$ and a column vector $x \in R^n$. In sequel, we consider the group $O(n,p)$ as the group of all real matrices $g$ such that $g^\top I_p g = I_p$, where $I_p = \|b_{ij}\|$ is the diagonal $n \times n$-matrix such that $b_{ii} = -1$ for all $i = 1, \dots, p$, $b_{jj} = 1$ for all $j = p+1, \dots, n$. Then

$$M(n,p) = \{F : E_p^n \to E_p^n \mid Fx = gx + b, \quad g \in O(n,p), \quad b \in E_p^n\}$$

and $SM(n,p) = \{F \in M(n,p) \mid \det g = 1\}$.

Let $x(u)$ be a $J$-surface in $E_p^n$. Then $Fx(u)$ is also a $J$-surface in $E_p^n$ for all $F \in \mathrm{Af}(n)$. Let $G$ be a subgroup of the group $\mathrm{Af}(n)$.

**Definition 3.** A differential rational function $h\langle x, y, \dots, z\rangle$ is said to be *$G$-invariant* if

$$h\langle gx, gy, \dots, gz\rangle = h\langle x, y, \dots, z\rangle \quad \text{for all } g \in G.$$

The set of all $G$-invariant differential rational functions of $x, y, \dots, z$ will be denoted by $R\langle x, y, \dots, z\rangle^G$. It is a differential subfield of the differential field $R\langle x, y, \dots, z\rangle$.

**Definition 4.** A subset $S$ of $R\langle x, y, \dots z\rangle^G$ is called a *generating system of the differential field* $R\langle x, y, \dots, z\rangle^G$ if the smallest differential subfield of it containing $S$ coincides with $R\langle x, y, \dots, z\rangle^G$.

Let

$$(x, y) = -x_1y_1 - \dots - x_py_p + x_{p+1}y_{p+1} + \dots + x_ny_n$$

be the inner product of vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $E_p^n$. So $(x^{(m_1,m_2,\dots,m_n)}, x^{(p_1,p_2,\dots,p_n)})$ is the inner product of vectors $x^{(m_1,m_2,\dots,m_n)}$, $x^{(p_1,p_2,\dots,p_n)}$ in $E_p^n$.

**Theorem 2.1.** *The system*

$$\left(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x\right), \quad \text{where } 1 \le i \le j \le n, \tag{2.1}$$

*is a generating system of the differential field* $R\langle x\rangle^{M(n,p)}$.

**Proof.** For it, first we prove Lemmas 2.1–2.9. Let $R\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x\rangle$ be the differential field of all differential rational functions of $\frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x$ and $R\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x\rangle^G$ is the differential field of all $G$-invariant differential rational functions of $\frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x$.

**Lemma 2.1.** *We have*

$$R\langle x\rangle^{M(n,p)} = R\left\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x\right\rangle^{M(n,p)} = R\left\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x\right\rangle^{O(n,p)}.$$

**Proof.** Let

$$q\langle x\rangle = q\left(x, \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x, \dots, x^{(m_1,m_2,m_3,\dots,m_n)}\right)$$

be an $M(n,p)$-invariant differential rational function of $x$. Then it is invariant with respect to parallel translations in $E_p^n$. This implies that

$$q\langle x\rangle = q\left(\frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x, \dots, x^{(m_1,m_2,m_3,\dots,m_n)}\right) = q\left\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x\right\rangle.$$

It is also $O(n,p)$-invariant. Hence it is an $O(n,p)$-invariant differential rational function of $\frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x$. Conversely, assume that $q$ is a $O(n,p)$-invariant differential rational function of $\frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x$. Then it is invariant with respect to parallel translations in $E_p^n$. Hence it is $M(n,p)$-invariant. □

**Lemma 2.2.** *Let* $f \in R\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x\rangle^{O(n,p)}$. *Then there exist* $O(n,p)$*-invariant differential polynomials* $f_1, f_2$ *such that* $f = f_1/f_2$.

**Proof.** A proof is similar to the proof in ([12, p. 106]). □

**Lemma 2.3.** *The system of all* $(x^{(m_1,m_2,\dots,m_n)}, x^{(p_1,p_2,\dots,p_n)})$, *where* $m_1+m_2+\dots+m_n \geq 1$, $p_1+p_2+\dots+p_n \geq 1$, *is a generating system of* $R\langle x\rangle^{M(n,p)}$ *as a field.*

**Proof.** Let $R[x^{(m_1,m_2,\dots,m_n)}; m_1+m_2+\dots+m_n \geq 1]^{O(n)}$ be the $R$-algebra of all $O(n,p)$-invariant polynomials of $x^{(m_1,m_2,\dots,m_n)}$, where $m_1+m_2+\dots+m_n \geq 1$. According to the First Main Theorem for $O(n,p)$ (see [21, pp. 53, 65–66]), the system of all $(x^{(m_1,m_2,\dots,m_n)}, x^{(p_1,p_2,\dots,p_n)})$, where $m_1+m_2+\dots+m_n \geq 1$ and $p_1+p_2+\dots+p_n \geq 1$, is a generating system of the $R$-algebra $R[x^{(m_1,m_2,\dots,m_n)}; m_1+m_2+\dots+m_n \geq 1]^{O(n,p)}$. Using Lemmas 2.1 and 2.2, we see that the system of all $(x^{(m_1,m_2,\dots,m_n)}, x^{(p_1,p_2,\dots,p_n)})$, where $m_1+m_2+\dots+m_n \geq 1$ and $p_1+p_2+\dots+p_n \geq 1$, is a generating system of

$$R\left\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x\right\rangle^{O(n,p)} = R\langle x\rangle^{M(n,p)}$$

as a field. □

Denote by $\Delta = \Delta_x$ the determinant $\det \|(a_i, b_j)\|_{i,j=1,2,\dots,n}$, where

$$a_1 = b_1 = \frac{\partial}{\partial u_1}x, \quad a_2 = b_2 = \frac{\partial}{\partial u_2}x, \dots, a_n = b_n = \frac{\partial}{\partial u_n}x.$$

Let $V$ be the system Eq. (2.1). Denote by $R\{V\}$ the differential $R$-subalgebra of $R\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x\rangle^{O(n,p)}$ generated by elements of the system $V$.

**Lemma 2.4.** $\Delta \in R\{V\}$.

**Proof.** By the definition of $V$, we have $(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x) \in V$ for all $1 \leq i,j \leq n$. Hence $\Delta \in R\{V\}$. □

Denote by $R\{V, \Delta^{-1}\}$ the differential $R$-subalgebra of $R\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x\rangle^{O(n,p)}$ generated by elements of the system $V$ and the function $\Delta^{-1}$. According to Lemmas 2.1 and 2.3, to prove our theorem, it suffices to prove that $(x^{(m_1,m_2,\dots,m_n)}, x^{(p_1,p_2,\dots,p_n)}) \in R\{V, \Delta^{-1}\}$ for all $m_i, p_i \in N_0$ such that $m_1+m_2+\dots+m_n \geq 1$ and $p_1+p_2+\dots+p_n \geq 1$.

Let $\det \mathrm{Gr}(y_1, y_2, \dots, y_m; z_1, z_2, \dots, z_m)$ be the determinant of the Gram matrix

$$\mathrm{Gr}(y_1, y_2, \dots, y_m; z_1, z_2, \dots, z_m) := ||(y_i, z_j)||_{i,j=1,2,\dots,m}$$

of vectors $y_1, y_2, \dots, y_m; z_1, z_2, \dots, z_m \in E_p^n$.

A proof of the following is known ([21, p. 75]):

**Lemma 2.5.** *The equality*

$$\det Gr(y_1, y_2, \dots, y_{n+1}; _1, z_2, \dots, z_{n+1}) = \det ||(y_i, z_j)||_{i,j=1,2,\dots,n+1} = 0$$

*holds for all vectors* $y_1, y_2 \dots, y_{n+1}, z_1, z_2 \dots, z_{n+1}$ *in* $R^n$.

**Lemma 2.6.** *Assume that* $b_1, b_2, \dots, b_n; c_1, c_2, \dots, c_n \in N_0$ *are such that*

$$1 \leq b_1 + b_2 + \cdots + b_n, \quad 1 \leq c_1 + c_2 + \cdots + c_n, \quad \left(x^{(b_1,b_2,\dots,b_n)}, \frac{\partial}{\partial u_i} x\right) \in R\{V, \Delta^{-1}\}$$

*and* $(x^{(c_1,c_2,\dots,c_n)}, \frac{\partial}{\partial u_i} x) \in R\{V, \Delta^{-1}\}$ *for all* $i$ *such that* $1 \leq i \leq n$.

*Then* $(x^{(b_1,b_2,\dots,b_n)}, x^{(c_1,c_2,\dots,c_n)}) \in R\{V, \Delta^{-1}\}$.

**Proof.** Applying Lemma 2.5 to vectors

$$\begin{aligned} y_1 = z_1 &= \frac{\partial}{\partial u_1} x, \\ y_2 = z_2 &= \frac{\partial}{\partial u_2} x, \\ &\dots \\ y_n = z_n &= \frac{\partial}{\partial u_n} x, \\ y_{n+1} &= x^{(b_1,b_2,\dots,b_n)}, \\ z_{n+1} &= x^{(c_1,c_2,\dots,c_n)}, \end{aligned}$$

we obtain $\det A = 0$, where

$$A = \|(y_i, z_j)\|_{i,j=1,2,\dots,n+1} = \mathrm{Gr}(y_1, y_2, \dots, y_{n+1}; z_1, z_2, \dots, z_{n+1}).$$

Denote by $D_{n+1j}$ the cofactor of the element $(y_{n+1}, z_j)$ in $A$, where $j = 1, 2, \dots, n+1$. From the equality $\det A = 0$, we obtain the equality

$$\begin{aligned} (y_{n+1}, z_1)D_{n+11} + (y_{n+1}, z_2)D_{n+12} + \cdots + (y_{n+1}, z_n)D_{n+1n} \\ + (y_{n+1}, z_{n+1})D_{n+1n+1} = 0. \end{aligned} \tag{2.2}$$

Since $\Delta = D_{n+1n+1}$, Eq. (2.2) implies that

$$\begin{aligned} (y_{n+1}, z_{n+1}) &= (x^{(b_1,b_2,\dots,b_n)}, x^{(c_1,c_2,\dots,c_n)}) \\ &= -\frac{(y_{n+1}, z_1)D_{n+11} + (y_{n+1}, z_2)D_{n+12} + \cdots + (y_{n+1}, z_n)D_{n+1n}}{\Delta}. \end{aligned} \tag{2.3}$$

In Eq. (2.3), by the assumption of the lemma, $(y_{n+1}, z_j) = (x^{(b_1,b_2,\dots,b_n)}, \frac{\partial}{\partial u_j} x) \in R\{V, \Delta^{-1}\}$ for every $j$ such that $1 \leq j \leq n$. We prove that $D_{n+1s} \in R\{V, \Delta^{-1}\}$ for every $s$ such that $1 \leq s \leq n$. We have

$$D_{n+1s} = (-1)^{n+1+s} \det \mathrm{Gr}(y_1, y_2, \dots, y_n; z_1, z_2, \dots, z_{s-1}, z_{s+1}, \dots, z_n, z_{n+1})).$$

By the definition of $V$, we obtain $(y_i, z_j) \in V \subset R\{V\}$ for all $1 \leq i, j \leq n$. By the assumption of our lemma, we have $(y_i, z_{n+1}) = (\frac{\partial}{\partial u_i} x, x^{(c_1,c_2,\dots,c_n)}) \in R\{V, \Delta^{-1}\}$ for every $1 \leq i \leq n$. Hence $D_{n+1s} \in R\{V, \Delta^{-1}\}$ for every $1 \leq s \leq n$ and Eq. (2.3) implies $(y_{n+1}, z_{n+1}) \in R\{V, \Delta^{-1}\}$. ◻

**Lemma 2.7.** *We have*

$$\left(\frac{\partial^2}{\partial u_i \partial u_j}x, \frac{\partial}{\partial u_l}x\right) = \frac{1}{2}\left(\frac{\partial}{\partial u_j}\left(\frac{\partial}{\partial u_l}x, \frac{\partial}{\partial u_i}x\right) + \frac{\partial}{\partial u_i}\left(\frac{\partial}{\partial u_l}x, \frac{\partial}{\partial u_j}x\right) - \frac{\partial}{\partial u_l}\left(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x\right)\right)$$

*and* $(\frac{\partial^2}{\partial u_i \partial u_j}x, \frac{\partial}{\partial u_l}x) \in R\{V, \Delta^{-1}\}$ *for all* $i, j, l \in \{1, 2, \ldots, n\}$.

**Proof.** We have

$$\begin{aligned}
\frac{\partial}{\partial u_j}\left(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_l}x\right) &= \left(\frac{\partial^2}{\partial u_i \partial u_j}x, \frac{\partial}{\partial u_l}x\right) + \left(\frac{\partial}{\partial u_i}x, \frac{\partial^2}{\partial u_j \partial u_l}x\right),\\
\frac{\partial}{\partial u_i}\left(\frac{\partial}{\partial u_j}x, \frac{\partial}{\partial u_l}x\right) &= \left(\frac{\partial^2}{\partial u_i \partial u_j}x, \frac{\partial}{\partial u_l}x\right) + \left(\frac{\partial}{\partial u_j}x, \frac{\partial^2}{\partial u_i \partial u_l}x\right),\\
\frac{\partial}{\partial u_l}\left(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x\right) &= \left(\frac{\partial^2}{\partial u_i \partial u_l}x, \frac{\partial}{\partial u_j}x\right) + \left(\frac{\partial}{\partial u_i}x, \frac{\partial^2}{\partial u_j \partial u_l}x\right).
\end{aligned} \tag{2.4}$$

Set

$$\begin{aligned}
\frac{\partial}{\partial u_j}\left(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_l}x\right) &= b_1, \quad \left(\frac{\partial^2}{\partial u_i \partial u_j}x, \frac{\partial}{\partial u_l}x\right) = w_1,\\
\frac{\partial}{\partial u_i}\left(\frac{\partial}{\partial u_j}x, \frac{\partial}{\partial u_l}x\right) &= b_2, \quad \left(\frac{\partial^2}{\partial u_j \partial u_l}x, \frac{\partial}{\partial u_i}x\right) = w_2,\\
\frac{\partial}{\partial u_l}\left(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x\right) &= b_3, \quad \left(\frac{\partial^2}{\partial u_i \partial u_l}x, \frac{\partial}{\partial u_j}x\right) = w_3.
\end{aligned}$$

Then system Eq. (2.4) takes the form: $w_1 + w_2 = b_1, w_1 + w_3 = b_2, w_2 + w_3 = b_3$. We consider this system as the system of equations for $w_1, w_2, w_3$. This system has the unique solution $(w_1, w_2, w_3)$, where

$$\begin{aligned}
w_1 &= \frac{1}{2}(b_1 + b_2 - b_3) \in R\{V, \Delta^{-1}\},\\
w_2 &= \frac{1}{2}(b_1 + b_3 - b_2) \in R\{V, \Delta^{-1}\},\\
w_3 &= \frac{1}{2}(b_2 + b_3 - b_1) \in R\{V, \Delta^{-1}\}.
\end{aligned}$$

□

**Lemma 2.8.** *For all* $1 \leq i \leq n$ *and* $b_1, b_2, \ldots, b_n \in N_0$ *such that* $1 \leq b_1 + b_2 + \cdots + b_n$, *we have* $(x^{(b_1, b_2, \ldots, b_n)}, \frac{\partial}{\partial u_i}x) \in R\{V, \Delta^{-1}\}$.

**Proof.** We prove the assertion by induction on $B = b_1 + b_2 + \cdots + b_n$. Let $B = 1$. Then, by the definition of $V$, we have $(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x) \in V \subset R\{V, \Delta^{-1}\}$ for all $1 \leq i, j \leq n$. Hence the assertion holds for $B = 1$. Assume that the assertion holds for $B > 1$.

We prove that $(\frac{\partial}{\partial u_i}x^{(b_1, b_2, \ldots, b_n)}, \frac{\partial}{\partial u_j}x) \in R\{V, \Delta^{-1}\}$ for all $1 \leq j \leq n$ and $b_1, b_2, \ldots, b_n \in N_0$ such that $b_1 + b_2 + \cdots + b_n = B$. For arbitrary $1 \leq i, j \leq n$ and $b_1, b_2, \ldots, b_n \in N_0$ such that $b_1 + b_2 + \cdots + b_n = B$, we have

$$\frac{\partial}{\partial u_i}\left(x^{(b_1, b_2, \ldots, b_n)}, \frac{\partial}{\partial u_j}x\right) = \left(\frac{\partial}{\partial u_i}x^{(b_1, b_2, \ldots, b_n)}, \frac{\partial}{\partial u_j}x\right) + \left(x^{(b_1, b_2, \ldots, b_n)}, \frac{\partial^2}{\partial u_i \partial u_j}x\right). \tag{2.5}$$

By the induction hypothesis on $B$, we have $(x^{(b_1,b_2,\dots,b_n)}, \frac{\partial}{\partial u_l}x)) \in R\{V,\Delta^{-1}\}$ for all $1 \le l \le n$ and $b_s \in N_0$ such that $b_1+b_2+\cdots+b_n = B$. Using Lemmas 2.7 and 2.6, we see that $(x^{(b_1,b_2,\dots,b_n)}, \frac{\partial^2}{\partial u_i \partial u_j}x)) \in R\{V,\Delta^{-1}\}$ for all $1 \le i,j \le n$ and $b_s \in N_0$ such that $b_1+b_2+\cdots+b_n = B$. Hence Eq. (2.5) implies $(\frac{\partial}{\partial u_i}x^{(b_1,b_2,\dots,b_n)}, \frac{\partial}{\partial u_j}x) \in R\{V,\Delta^{-1}\}$. □

**Lemma 2.9.** *For all $b_1,b_2,\dots,b_n,c_1,c_2,\dots,c_n \in N_0$ such that $1 \le b_1+b_2+\cdots+b_n$ and $1 \le c_1+c_2+\cdots+c_n$, we have $(x^{(b_1,b_2,\dots,b_n)}, x^{(c_1,c_2,\dots,c_n)}) \in R\{V,\Delta^{-1}\}$.*

**Proof.** Using Lemmas 2.8 and 2.6, we obtain $(x^{(b_1,b_2,\dots,b_n)}, x^{(c_1,c_2,\dots,c_n)}) \in R\{V,\Delta^{-1}\}$. □

We complete the proof of our theorem. By Lemma 2.4, $\Delta \in R\{V\}$. Since $R\langle V\rangle$ is a field, we obtain $\Delta^{-1} \in R\langle V\rangle$. Hence $R\{V,\Delta^{-1}\} \subset R\langle V\rangle$. By Lemma 2.9,

$$(x^{(b_1,b_2,\dots,b_n)}, x^{(c_1,c_2,\dots,c_n)}) \in R\{V,\Delta^{-1}\} \subset R\langle V\rangle \subset R\langle x\rangle^{M(n,p)}$$

for all $b_1,b_2,\dots,b_n,c_1,c_2,\dots,c_n \in N_0$ such that $1 \le b_1+b_2+\cdots+b_n, 1 \le c_1+c_2+\cdots+c_n$. By Lemma 2.3, the system of all elements $(x^{(m_1,m_2,\dots,m_n)}, x^{(c_1,c_2,\dots,c_n)})$, where $m_1+m_2+\cdots+m_n \ge 1, c_1+c_2+\cdots+c_n \ge 1, m_i \in N_0, c_i \in N_0$, is a generating system of $R\langle x\rangle^{M(n,p)}$ as a field. Hence $R\langle V\rangle = R\langle x\rangle^{M(n,p)}$. The theorem is completed. □

For any $a_m \in E_p^n$, where $m = 1,\dots,n$, the determinant $\det(a_{ij})$ (where the $a_{mi}$ are coordinates of $a_m$) will be denoted by $[a_1 a_2 \cdots a_n]$. So

$$[x^{(m_{11},m_{12},\dots,m_{1n})} x^{(m_{21},m_{22},\dots,m_{2n})} \cdots x^{(m_{n1},m_{n2},\dots,m_{nn})}]$$

is the determinant of the vectors $x^{(m_{i1},m_{i2},\dots,m_{in})}$ for $i = 1,2,\dots,n$.

**Theorem 2.2.** *The system*

$$\left[\frac{\partial}{\partial u_1}x \quad \frac{\partial}{\partial u_2}x \cdots \frac{\partial}{\partial u_n}x\right], \left(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x\right), \quad \text{where } 1 \le i \le j \le n, \ \ i+j < 2n, \tag{2.6}$$

*is a generating system of the differential field $R\langle x\rangle^{SM(n)}$.*

**Proof.** Let $SO(n,p) = \{g \in O(n,p) \mid \det g = 1\}$.

**Lemma 2.10.** $R\langle x\rangle^{SM(n,p)} = R\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x\rangle^{SO(n,p)}$.

**Proof** is similar to the proof of Lemma 2.1. □

**Lemma 2.11.** *Let $f \in R\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \dots, \frac{\partial}{\partial u_n}x\rangle^{SO(n,p)}$. Then $SO(n,p)$-invariant differential polynomials $f_1, f_2$ exist such that $f = f_1/f_2$.*

**Proof** is similar to the proof in ([12, p. 106]). □

**Lemma 2.12.** *The system of all elements*

$$\begin{gathered}[x^{(m_{11},m_{12},\dots,m_{1n})} x^{(m_{21},m_{22},\dots,m_{2n})} \cdots x^{(m_{n1},m_{n2},\dots,m_{nn})}], \\ (x^{(b_1,b_2,\dots,b_n)}, x^{(c_1,c_2,\dots,c_n)}),\end{gathered} \tag{2.7}$$

*where $m_{i1}+m_{i2}+\cdots+m_{in} \ge 1$, $b_1+b_2+\cdots+b_n \ge 1$, $c_1+c_2+\cdots+c_n \ge 1$, is a generating system of $R\langle x\rangle^{SM(n,p)}$ as a field.*

**Proof.** Let $R[x^{(m_1,m_2,\ldots,m_n)}; m_1+m_2+\cdots+m_n \geq 1]^{SO(n,p)}$ be the $R$-algebra of all $SO(n,p)$-invariant polynomials of the system of all $x^{(m_1,m_2,\ldots,m_n)}$, where $m_1+m_2+\cdots+m_n \geq 1$. By the First Main Theorem for $SO(n,p)$ (see [21, pp. 53, 65–66]), the system Eq. (2.7) is a generating system of $R[x^{(m_1,m_2,\ldots,m_n)}; m_1+m_2+\cdots+m_n \geq 1]^{SO(n,p)}$. Lemma 2.11 implies that the system Eq. (2.7) is a generating system of $R\langle x\rangle^{SO(n,p)}$. Then, by Lemma 2.10, the system Eq. (2.7) is a generating system of $R\langle x\rangle^{SM(n,p)}$ as a field. □

Let $Z$ be the system Eq. (2.6) and $R\{Z\}$ be the differential $R$-subalgebra of $R\langle x\rangle^{SM(n,p)}$ generated by elements of the system $Z$. Denote by $\delta = \delta_x$ the determinant of the matrix $\mathrm{Gr}(y_1, y_2, \ldots, y_{n-1}; z_1, z_2, \ldots, z_{n-1})$, where

$$y_1 = z_1 = \frac{\partial}{\partial u_1}x, \quad y_2 = z_2 = \frac{\partial}{\partial u_2}x, \ldots, \; y_{n-1} = z_{n-1} = \frac{\partial}{\partial u_{n-1}}x.$$ □

**Lemma 2.13.** *$\delta \in R\{Z\}$ and $\delta^{-1} \in R\langle Z\rangle$.*

**Proof.** By the definition of $Z$, $(y_i, z_j) = (\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x) \in Z \subset R\{Z\}$ for all $1 \leq i, j \leq n-1$. Hence $\delta \in R\{Z\}$ and $\delta^{-1} \in R\langle Z\rangle$.

In the sequel, we need the following lemma

**Lemma 2.14.** *The equality*

$$(-1)^p [y_1 \cdots y_n][z_1 \cdots z_n] = \det ||(y_i, z_j)||_{i,j=1,2,\ldots,n}$$

*holds for all vectors $y_1, \ldots, y_n, z_1, \ldots, z_n$ in $E_p^n$.*

**Proof.** A proof of the this lemma is similar to the proof in ([12, p. 72]). □

Let $\Delta$ be the function in the proof of Theorem 2.1.

**Lemma 2.15.** *$\Delta \in R\{Z\}$ and $\Delta^{-1} \in R\langle Z\rangle$.*

**Proof.** Using Lemma 2.14 to vectors $y_1 = z_1 = \frac{\partial}{\partial u_1}x, y_2 = z_2 = \frac{\partial}{\partial u_2}x, \ldots, y_n = z_n = \frac{\partial}{\partial u_n}x$, we obtain

$$(-1)^p \left[\frac{\partial}{\partial u_1}x \quad \frac{\partial}{\partial u_2}x \cdots \frac{\partial}{\partial u_n}x\right]^2 = \det \|(y_i, z_j)\|_{i,j=1,2,\ldots n} = \Delta. \tag{2.8}$$

Since $[\frac{\partial}{\partial u_1}x \quad \frac{\partial}{\partial u_2}x \cdots \frac{\partial}{\partial u_n}x] \in Z$, we have $\Delta \in R\{Z\}$ and $\Delta^{-1} \in R\langle Z\rangle$. □

Denote by $R\{Z, \delta^{-1}, \Delta^{-1}\}$ the differential $R$-subalgebra of $R\langle \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x\rangle$ generated by $Z$ and functions $\delta^{-1}, \Delta^{-1}$. By Lemmas 2.11 and 2.12, for a proof of our theorem, it is enough to prove that

$$[x^{(m_{11},m_{12},\ldots,m_{1n})}x^{(m_{21},m_{22},\ldots,m_{2n})} \cdots x^{(m_{n1},m_{n2},\ldots,m_{nn})}] \in R\{Z, \delta^{-1}, \Delta^{-1}\}$$

and

$$(x^{(b_1,b_2,\ldots,b_n)}, x^{(c_1,c_2,\ldots,c_n)}) \in R\{Z, \delta^{-1}, \Delta^{-1}\}$$

for all $m_{ij}, b_i, c_i \in N_0$ such that $m_{i1} + m_{i2} + \cdots + m_{in} \geq 1$, $b_1 + b_2 + \cdots + b_n \geq 1$, $c_1 + c_2 + \cdots + c_n \geq 1$.

Let $V$ be the system in the proof of Theorem 2.1.