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ISOCHRONOUS DYNAMICAL SYSTEM AND DIOPHANTINE RELATIONS. I.

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We identify a *solvable* dynamical system — interpretable to some extent as a many-body problem — and point out that — for an appropriate assignment of its parameters — it is *entirely isochronous*, namely *all* its nonsingular solutions are *completely periodic* (i.e., *periodic* in *all* degrees of freedom) with the same *fixed* period (independent of the initial data). We then identify its *equilibrium* configurations and investigate its behavior in their neighborhood. We thereby identify certain matrices — of arbitrary order — whose eigenvalues are all *rational* numbers: a *Diophantine* finding.

Keywords: Dynamical systems; integrable; isochronous; *Diophantine*; matrices; eigenvalues; conjectures; nonlinear harmonic oscillators.

1. Introduction

The strategy underlying the *Diophantine* findings (results and conjectures) obtained in this paper takes as starting point a dynamical system whose time evolution is demonstrably *entirely isochronous*, namely *all* its solutions are *completely periodic* (i.e., periodic in *all* its degrees of freedom) with the same *fixed* period (generally an *integer* multiple of a basic period) in its *entire* phase space, except possibly for a sector with vanishing dimensionality where the solutions become *singular*. Such systems — especially when they are characterized by equations of motion of Newtonian type — can justifiably be considered as describing *nonlinear harmonic oscillators* [1]. Suppose moreover that such a system possesses (at least) one, *explicitly known*, equilibrium configuration. It is then possible to investigate its behavior near equilibrium via the standard approach, namely by assuming that the coordinates are infinitesimally close to their values at equilibrium and thereby linearizing the equations of motion. In this manner it is generally found that the behavior of the system near (a stable) equilibrium is *multiply periodic*, the frequencies of the basic oscillations coinciding with the *eigenvalues* of a matrix constructed from the equations of motion and evaluated at the equilibrium configuration (hence, under the above assumptions, an *explicitly known* matrix). But if the system is *entirely isochronous*, its behavior near equilibrium must be as well *isochronous*: hence *all* its basic frequencies must be *integer* multiples of a basic frequency. The outcome of this approach is therefore to identify a matrix (of arbitrary order) *all* eigenvalues of which are *rational* numbers: a *Diophantine* finding!

An approach to implement this strategy begins by manufacturing a dynamical system which is *solvable.* Here and throughout the term *solvable* is employed to identify dynamical systems whose initial-value problem can be solved by purely *algebraic* techniques, typically by finding the eigenvalues of matrices explicitly known for all time in terms of the initial data, or equivalently the zeros of explicitly time-dependent polynomials. Next, one "ω-modifies" this *solvable* system so that the (still *solvable*) ω-modified dynamical system thereby obtained is *entirely isochronous* — as can be demonstrated by exploiting the *solvable* character of the system. A technique to perform this second step is by now well known, and its applicability is sufficiently wide to justify the assertion that *isochronous systems are not rare.* Indeed several *solvable* and *entirely isochronous* dynamical systems have been recently identified in this manner, which are naturally interpretable as many-body problems, their time evolution being characterized by Newtonian equations of motion with one-body and two-body forces (for reviews of these recent results see Refs. [2–4]). These findings are of interest in themselves, but they also yielded — via the strategy outlined above — several *Diophantine* findings and conjectures (for a review of these results see Appendix C of Ref. [4]; and for consequential developments involving orthogonal polynomials see [5–7]).

The main technique to manufacture these *solvable* systems starts from a *solvable* matrix evolution equation and then focusses on the evolution of its *eigenvalues.* Indeed — as it is by now well known: see for instance Ref. [4] — this often leads to Newtonian equations of motion characterizing a many-body problem in which the coordinates $z_n(t)$ of the moving particles are identified with the *eigenvalues* of the evolving matrix. It is generally convenient — and we shall follow this practice hereafter — to consider these coordinates as *complex* numbers, hence to imagine that the many-body problem thereby manufactured describes point-like particles moving in the *complex* z-plane. (It is also generally possible — and significant — to identify this *complex* plane with a *physical* plane on which move particles whose positions are identified by *real* two-vectors; we will not elaborate on this aspect in this paper). Generally these many-body problems also feature *auxiliary variables*, which might be interpreted as "time-dependent coupling constants" or equivalently as "internal parameters", whose time evolution is determined by additional equations of motion, nonlinearly coupled to the equations of motion satisfied by the particle coordinates $z_n(t)$. Only in special cases, via some appropriate *ansatz* which turns out — as it were, "miraculously" — to be compatible with these equations of motion, one can express in terms of the particle coordinates $z_n(t)$ (and possibly their time derivatives $\dot{z}_n(t)$) these additional variables, getting thereby rid of them, hence obtaining a *solvable* many-body problem whose equations of motion *only* involves the coordinates $z_n(t)$ (and possibly the velocities $\dot{z}_n(t)$) of the moving particles — and possibly in addition some *arbitrary constants.*

This last step requires that such an *ansatz* exist, and that someone discover it. Several successful examples have been recently reported [8–14] (for a review see Ref. [4]), and they also led, as already mentioned above, to *Diophantine* findings and conjectures. However, even when this last step does not seem to be feasible, the interpretation of these types of *solvable* models as many-body problems with additional time-dependent *auxiliary* variables besides the particle coordinates $z_n(t)$ and the corresponding velocities $\dot{z}_n(t)$ is also valid [15–18]. Moreover, the elimination of the auxiliary variables — while important to yield a many-body problem characterized by neater equations of motion — is by no means essential in order to apply the strategy outlined above and thereby to arrive at *Diophantine* findings — indeed, even the fact that the equations of motion of the dynamical system under consideration have a Newtonian look has no relevance from this point of view, although it may promote the interest *per se* of these dynamical systems. What is essential is to manufacture an *entirely isochronous* system and then to identify *explicitly* a (nontrivial) *equilibrium* configuration of it. The first of these two steps can be realized in many ways, since no "miracle" is now required in order to proceed; the feasibility of the second step must be investigated on a case-by-case basis. Of course whether the *entirely isochronous* dynamical systems thereby obtained, and the *Diophantine* findings arrived at in this manner, are deemed "interesting", is a value judgement that can just as well be issued only *a posteriori* and on a case-by-case basis — and is anyway in the eye of the beholder.

The results of this paper are reported in the following Sec. 2, and they are proved in Sec. 3. A terse Sec. 4 ("Outlook") completes this paper.

2. Results

In this section we introduce our *solvable* and *entirely isochronous* dynamical system, we identify its *equilibria*, we discuss its behavior in their neighborhood, and we report the *Diophantine* findings arrived at in this manner. The proofs of these results requiring any elaboration are postponed to the following Sec. 3.

2.1. *A solvable, entirely isochronous, dynamical system*

We take as starting point the *second-order* matrix ODE

$$\nu \ddot{U} = 2i\lambda\omega\dot{U} + (\lambda^2 - \nu^2)\omega^2 U + (\nu - 1)\dot{U}U^{-1}\dot{U}. \tag{1}$$

Here and hereafter $U \equiv U(t)$ is a time-dependent $N \times N$ matrix, with N an *arbitrary positive integer*; superimposed dots denote time derivatives; i denotes the *imaginary* unit, $i^2 = -1$; and λ, ν, ω are three scalar *constants*: the third, ω, is hereafter assumed to be *positive*, and we associated with it the basic period

$$T = \frac{2\pi}{\omega}; \tag{2}$$

the other two, λ and ν, are *a priori* arbitrary, but we generally assume that they are *rational* numbers, this being *sufficient* — as we shall immediately see — to guarantee that *all* the *nonsingular* solutions of this matrix ODE, (1), evolve *periodically* with a period $\tilde{T}$ which is an *integer multiple* of the basic period T.

The solution of the initial-value problem for this matrix ODE, (1), is indeed provided by the following (explicit!) formula (see Eq. (5.1–2) of Ref. [19]; or verify by explicit computation):

$$U(t) = \exp(i\lambda\omega t)\left[\cos(\omega t) + \frac{\sin(\omega t)}{\nu\omega}\{\dot{U}(0)[U(0)]^{-1} - i\lambda\omega\}\right]^{\nu} U(0). \tag{3}$$

It is clear from it that, for *generic* initial data (of course such that $U(0)$ is invertible), the matrix $U(t)$ is periodic, $U(t + \tilde{T}) = U(t)$ with $\tilde{T} = pT$, the *positive integer* p being simply related to the denominators of the two *rational* numbers λ and ν; although there are some *exceptional* initial data for which the time evolution of $U(t)$ runs into singularities (except when ν is an *integer*). It is also plain that, while the condition that both λ and ν be *rational* is sufficient to guarantee that the matrix ODE (1) be *entirely isochronous*, the condition that (only) λ be *rational* is sufficient in order that this matrix ODE be *isochronous*: indeed, even if the requirement that the parameter ν be *rational* were dropped, there clearly would still exist an *open*, *fully dimensional*, sector in the phase space of the initial data $U(0)$, $\dot{U}(0)$ such that the corresponding solution (3) is *periodic* (with period $\tilde{T} = pT$, the *positive* integer p being then characterized by the requirement that $p\lambda$ be *integer*).

Next, we introduce the N eigenvalues $z_n(t)$ of the $N \times N$ matrix $U(t)$ by setting

$$U(t) = R(t)Z(t)[R(t)]^{-1}, \tag{4a}$$

with

$$Z(t) = \mathrm{diag}[z_n(t)]. \tag{4b}$$

Here $R(t)$ is of course the $N \times N$ matrix that diagonalizes the matrix $U(t)$. Here and hereafter it is understood that the index n (and likewise indices such as m, ℓ, see below) range from 1 to N.

Given the matrix $U(t)$, the evaluation of its N eigenvalues $z_n(t)$, namely of the diagonal matrix $Z(t)$, is of course a *purely algebraic* task; and of course, since the matrix $U(t)$ is *periodic* with a period $\tilde{T}$ that is an *integer* multiple of T, *all* its eigenvalues $z_n(t)$ shall also be *periodic* functions of time with periods that are *integer* multiples of T (these periods might themselves be *integer* multiples of the period $\tilde{T}$ due to an exchange of the eigenvalues through the time evolution; for special initial data it might happen that the time evolution of these eigenvalues runs into singularities, due to a collision of one or more of them, but this shall not happen for *generic* initial data). Likewise, it is a *purely algebraic* task to evaluate the diagonalizing $N \times N$ matrix $R(t)$, and this matrix shall also be *periodic* in time with a period that is an *integer* multiple of the basic period T. However this assertion is only valid modulo the property of this matrix to be defined up to multiplication from the right by an *arbitrary diagonal* matrix $D(t)$,

$$D(t) = \text{diag}[d_n(t)], \tag{5}$$

since, thanks to (4b), it makes no difference if in the right-hand side of (4a) $R(t)$ is replaced by $R(t)D(t)$ and simultaneously $[R(t)]^{-1}$ is of course replaced by $[D(t)]^{-1}[R(t)]^{-1}$.

Next, we introduce the $N \times N$ matrix

$$M(t) = [R(t)]^{-1}\dot{R}(t). \tag{6}$$

The indeterminacy in the definition of the $N \times N$ matrix $R(t)$ due to the possibility of multiplying it from the right by an *arbitrary diagonal* matrix $D(t)$ entails that this matrix $M(t)$ is defined up to the "gauge" replacement

$$M(t) \Rightarrow \tilde{M}(t) \equiv [D(t)]^{-1}M(t)D(t) + [D(t)]^{-1}\dot{D}(t), \tag{7a}$$

namely (componentwise)

$$\mu_n(t) \Rightarrow \tilde{\mu}_n(t) \equiv \mu_n(t) + \frac{\dot{d}_n(t)}{d_n(t)}, \tag{7b}$$

$$M_{nm}(t) \Rightarrow \tilde{M}_{nm}(t) \equiv [d_n(t)]^{-1}M_{nm}(t)d_m(t), \quad n \neq m. \tag{7c}$$

Here and hereafter, for notational convenience, we denote as $\mu_n(t)$ (respectively $\tilde{\mu}_n(t)$) the N diagonal elements of the $N \times N$ matrix $M(t)$ (respectively $\tilde{M}(t)$),

$$M_{nn}(t) \equiv \mu_n(t), \quad \tilde{M}_{nn}(t) \equiv \tilde{\mu}_n(t). \tag{8}$$

It is clear from (7b) that these quantities $\tilde{\mu}_n(t)$ remain undetermined (due to our freedom to assign the diagonal matrix $D(t)$), namely that we retain the privilege to assign them at our convenience — provided we simultaneously take account of the corresponding modification of the off-diagonal elements of the matrix $\tilde{M}(t)$, see (7c). And it is as well plain that, up to this indeterminacy (but hereafter we assume to restrict the selection of the diagonal matrix $D(t)$ so that its diagonal elements are periodic, $d_n(t+T) = d_n(t)$), the matrix $M(t)$ shall also be *periodic* with a period that is an *integer* multiple of the basic period T.

It can then be shown (see the following Sec. 3) that the matrix evolution equation (1) implies that the N quantities $z_n(t)$, see (4b), and the $N(N-1)$ quantities

$$Y_{nm}(t) = [z_n(t) - z_m(t)]M_{nm}(t), \quad m \neq n \tag{9}$$

evolve according to the following $N + N(N-1) = N^2$ equations of motion:

$$\nu \ddot{z}_n = 2i\lambda\omega\dot{z}_n + (\lambda^2 - \nu^2)\omega^2 z_n + (\nu - 1)\frac{\dot{z}_n^2}{z_n} + \sum_{\ell=1,\ell\neq n}^{N}\left[\frac{Y_{n\ell}Y_{\ell n}}{z_n - z_\ell}\left(\nu + 1 + (\nu - 1)\frac{z_n}{z_\ell}\right)\right], \tag{10a}$$

$$\dot{Y}_{nm} = \left[-\frac{\dot{z}_n - \dot{z}_m}{z_n - z_m} + \frac{2i\lambda\omega}{\nu} + \frac{\nu - 1}{\nu} \left(\frac{\dot{z}_n}{z_n} + \frac{\dot{z}_m}{z_m} \right) - \mu_n + \mu_m \right] Y_{nm} - \sum_{\ell=1, \ell \neq n,m}^{N} \left[Y_{n\ell} Y_{\ell m} \left(\frac{1}{z_n - z_\ell} + \frac{1}{z_m - z_\ell} + \frac{\nu - 1}{\nu z_\ell} \right) \right], \quad n \neq m. \tag{10b}$$

Note that the first set of these ODEs, (10a), look like N Newtonian equations of motion for N "particle coordinates" z_n, except that they also feature the $N(N-1)$ "auxiliary variables" Y_{nm} — playing the role of "time-dependent coupling constants" — whose time evolution is then specified by the $N(N-1)$ (first-order) ODEs (10b). Of course the previous discussion entails that — remarkably — the *generic* solution of this system of N^2 *autonomous* ODEs is *completely periodic* with a period that is an *integer* multiple of the basic period T.

2.2. *Equilibria*

Clearly the equilibrium configurations of the system (10) are provided by the formulae

$$z_n(t) = \frac{x_n}{i\omega}, \quad Y_{nm}(t) = y_{nm}, \quad \mu_n(t) = i\omega\gamma_n, \tag{11}$$

with the $N + N(N-1) = N^2$ (time-independent) quantities x_n, y_{nm} solutions of the system of N^2 algebraic equations

$$(\lambda^2 - \nu^2)x_n = \sum_{\ell=1, \ell \neq n}^{N} \left[\frac{y_{n\ell} y_{\ell n}}{x_n - x_\ell} \left(\nu + 1 + (\nu - 1)\frac{x_n}{x_\ell} \right) \right], \tag{12a}$$

$$\frac{2\lambda}{\nu} = \gamma_n - \gamma_m + \sum_{\ell=1, \ell \neq n,m}^{N} \left\{ \frac{y_{n\ell} y_{\ell m}}{y_{nm}} \left[\frac{1}{x_n - x_\ell} + \frac{1}{x_m - x_\ell} + \frac{\nu - 1}{\nu x_\ell} \right] \right\}, \quad n \neq m, \tag{12b}$$

where we retain the privilege to assign at our convenience the N constants γ_n.

As shown in the following section, a solution of these equations (hereafter identified as **Case 1**) reads

$$x_n = \exp\left(\frac{2\pi i n}{N} \right), \quad y_{nm} = \pm\frac{2\nu}{N} \exp\left[\frac{\pi i (n + m)}{N} \right], \quad \gamma_n = 0, \tag{13a}$$

provided

$$\lambda = \pm\frac{\nu}{N}(2 - N). \tag{13b}$$

There is no restriction on the parameter ν, except the requirement that it be *rational*, implying of course that λ is as well *rational*.

In the following section it is also shown that a second class of solutions (hereafter identified as **Case 2**) exists provided

$$\lambda = 0, \quad \nu = 1, \tag{14a}$$

as detailed by the following formulae:

$$y_{nm} = \frac{1}{x_n - x_m}, \quad \gamma_n = \frac{x_n^2}{3} + c, \tag{14b}$$

where c is an irrelevant arbitrary parameter and the numbers x_n are now the N zeros of the Hermite polynomial of order N:

$$H_N(x_n) = 0. \tag{14c}$$

2.3. *Behavior near equilibria*

To investigate the behavior of the system (10) in the neighborhood of its equilibria one sets (see (11))

$$z_n(t) = \frac{x_n + \epsilon w_n(t)}{i\omega}, \quad Y_{nm}(t) = y_{nm} + \epsilon w_{nm}(t), \quad \mu_n(t) = i\omega\gamma_n, \tag{15}$$

treating ϵ as an (infinitesimally) small parameter. Note that, for simplicity, we keep fixed the variables $\mu_n(t)$ at their equilibrium values.

Insertion of this *ansatz* in the equations of motion (10) yields the following (constant-coefficient, linearized) system of $N + N(N-1) = N^2$ ODEs for the $N + N(N-1) = N^2$ dependent variables $w_n(t)$ and $w_{nm}(t)$ (with $n \neq m$):

$$\nu\ddot{w}_n - 2i\lambda\omega\dot{w}_n - \omega^2\left[\sum_{\ell=1}^{N} A_{n,\ell}w_\ell + \sum_{\ell,j=1,\ell\neq j}^{N} B_{n,\ell j}w_{\ell j}\right] = 0, \tag{16a}$$

$$\dot{w}_{nm} + \sum_{\ell=1}^{N} C_{nm,\ell}\dot{w}_\ell + i\omega\left[\sum_{\ell=1}^{N} D_{nm,\ell}w_\ell + \sum_{\ell,j=1;\ell\neq j}^{N} E_{nm,\ell j}w_{\ell j}\right] = 0, \quad n \neq m, \tag{16b}$$

where the $N \times N$ matrix A, the $N \times [N(N-1)]$ matrix B, the two $[N(N-1)] \times N$ matrices C and D, and the $[N(N-1)] \times [N(N-1)]$ matrix E are defined componentwise in terms of the equilibrium data as follows:

$$A_{n,\ell} = \delta_{n\ell}\left[\lambda^2 - \nu^2 + 2\nu\sum_{j=1,j\neq n}^{N}\frac{y_{nj}y_{jn}}{(x_n - x_j)^2}\right] - (1-\delta_{n\ell})\frac{y_{n\ell}y_{\ell n}}{(x_n - x_\ell)^2}\left[\nu + 1 + 2(\nu-1)\frac{x_n}{x_\ell} - (\nu-1)\left(\frac{x_n}{x_\ell}\right)^2\right], \tag{17a}$$

$$B_{n,\ell j} = -\frac{y_{n\ell}\delta_{nj}}{x_n - x_\ell}\left(\nu + 1 + (\nu-1)\frac{x_n}{x_\ell}\right) - \frac{y_{jn}\delta_{n\ell}}{x_n - x_j}\left(\nu + 1 + (\nu-1)\frac{x_n}{x_j}\right), \quad \ell \neq j, \tag{17b}$$

$$C_{nm,\ell} = \left[\frac{\delta_{n\ell} - \delta_{m\ell}}{x_n - x_m} - \frac{\nu-1}{\nu}\left(\frac{\delta_{n\ell}}{x_n} + \frac{\delta_{m\ell}}{x_m}\right)\right]y_{nm}, \quad n \neq m, \tag{17c}$$

$$D_{nm,\ell} = -\sum_{j=1,j\neq n,m}^{N}\left\{y_{nj}y_{jm}\left[\frac{\delta_{n\ell}}{(x_n - x_j)^2} + \frac{\delta_{m\ell}}{(x_m - x_j)^2}\right]\right\} + (1-\delta_{n\ell})(1-\delta_{m\ell})y_{n\ell}y_{\ell m}\left[\frac{1}{(x_n - x_\ell)^2} + \frac{1}{(x_m - x_\ell)^2} - \frac{\nu-1}{\nu x_\ell^2}\right], \quad n \neq m, \tag{17d}$$

$$E_{nm,\ell j} = \delta_{n\ell}\delta_{mj}\left(-\frac{2\lambda}{\nu} + \gamma_n - \gamma_m\right) + \delta_{n\ell}(1-\delta_{mj})y_{jm}\left(\frac{1}{x_n - x_j} + \frac{1}{x_m - x_j} + \frac{\nu-1}{\nu x_j}\right) + \delta_{mj}(1-\delta_{n\ell})y_{n\ell}\left(\frac{1}{x_n - x_\ell} + \frac{1}{x_m - x_\ell} + \frac{\nu-1}{\nu x_\ell}\right), \quad n \neq m, \quad \ell \neq j. \tag{17e}$$

Here and hereafter δ_{nm} denotes the Kronecker symbol, $\delta_{nm} = 1$ if $n = m$, $\delta_{nm} = 0$ if $n \neq m$.

It is therefore clear that around its equilibria our system oscillates with the $N^2 + N = S$ basic eigenfrequencies $\eta_s\omega$,

$$w_n(t) = \sum_{s=1}^{S} a_{ns}\exp(i\eta_s\omega t), \quad w_{nm}(t) = \sum_{s=1}^{S} b_{nms}\exp(i\eta_s\omega t), \tag{18a}$$

where the S numbers η_s are the S roots of the following polynomial equation, of degree S in the variable η:

$$\det \begin{bmatrix} (\nu\eta^2 - 2\lambda\eta)\mathbf{1} + A & B \\ \eta C + D & \eta \mathbf{I} + E \end{bmatrix} = 0. \tag{18b}$$

The (block) structure of the square matrix of order N^2 whose determinant constitutes the left-hand side of this equation is implied by the definitions of the matrices A, B, C, D and E, as specified above, while of course $\mathbf{1}$ respectively $\mathbf{I}$ denote the unit matrices of order N respectively $N(N-1)$.

2.4. *Diophantine findings*

It is plain from the above treatment — in particular, from the *isochronous* character of the system under consideration, implying of course that also its behavior around its equilibria must be *completely periodic* — that the $S = N^2 + N$ roots of the Eq. (18b) must *all* be *rational* numbers: a *Diophantine* finding.

Let us now report the explicit form — whose derivation is outlined in the following Sec. 3 — of the matrices $A,\ B,\ C,\ D,\ E$ characterizing this *Diophantine* finding, corresponding to the two equilibria reported above.

In **Case 1:**

$$\begin{aligned} A_{n,\ell} = &\left(-\frac{2}{3}\right)\alpha^2(N-1)(\alpha N + \alpha N^2 + 6)\delta_{n\ell} \\ &+ (1-\delta_{n\ell})\alpha^2\left\{\sin\left[\frac{(n-\ell)\pi}{N}\right]\right\}^{-2}\left\{\alpha N + 1 + 2(\alpha N - 1)\exp\left[\frac{2\pi i(n-\ell)}{N}\right]\right. \\ &\left. - (\alpha N - 1)\exp\left[\frac{4\pi i(n-\ell)}{N}\right]\right\}, \end{aligned} \tag{19a}$$

$$\begin{aligned} B_{n,\ell j} = &\frac{i\alpha\delta_{nj}[\alpha N + 1 + (\alpha N - 1)\exp\{2\pi i(n-\ell)/N\}]}{\sin[\pi(n-\ell)/N]} \\ &+ \frac{i\alpha\delta_{n\ell}[\alpha N + 1 + (\alpha N - 1)\exp\{2\pi i(n-j)/N\}]}{\sin[\pi(n-j)/N]}, \quad \ell \neq j, \end{aligned} \tag{19b}$$

$$\begin{aligned} C_{nm,\ell} = &-\frac{i\alpha(\delta_{n\ell} - \delta_{m\ell})}{\sin[\pi(n-m)/N]} - \frac{2(\alpha N - 1)}{N}\left(\delta_{n\ell}\exp\left[\frac{\pi i(m-n)}{N}\right]\right. \\ &\left. + \delta_{m\ell}\exp\left[\frac{\pi i(n-m)}{N}\right]\right), \quad n \neq m, \end{aligned} \tag{19c}$$

$$\begin{aligned} D_{nm,\ell} = &\alpha^2\left(\frac{N^2-1}{3} - \left\{\sin\left[\frac{\pi(n-m)}{N}\right]\right\}^{-2}\right)\cdot\left(\delta_{n\ell}\exp\left[\frac{\pi i(m-n)}{N}\right] + \delta_{m\ell}\exp\left[\frac{\pi i(n-m)}{N}\right]\right) \\ &- (1-\delta_{n\ell})(1-\delta_{m\ell})\alpha^2\left\{\frac{\exp[\pi i(m-n)/N]}{\sin^2[\pi(n-\ell)/N]} + \frac{\exp[\pi i(n-m)/N]}{\sin^2[\pi(m-\ell)/N]}\right. \\ &\left. + \frac{4(\alpha N - 1)}{\alpha N}\exp[\pi i(n+m-2\ell)/N]\right\}, \quad n \neq m, \end{aligned} \tag{19d}$$

$$\begin{aligned} E_{nm,\ell j} = &\delta_{n\ell}\delta_{mj}\frac{2(N-2)}{N} - i\delta_{n\ell}(1-\delta_{mj})\alpha\left\{\frac{\exp[\pi i(m-n)/N]}{\sin[\pi(n-j)/N]} + \frac{1}{\sin[\pi(m-j)/N]}\right. \\ &\left. + 2i\frac{\alpha N - 1}{\alpha N}\exp[\pi i(m-j)/N]\right\} - i\delta_{mj}(1-\delta_{n\ell})\alpha\left\{\frac{\exp[\pi i(n-m)/N]}{\sin[\pi(m-\ell)/N]}\right. \\ &\left. + \frac{1}{\sin[\pi(n-\ell)/N]}, +2i\frac{\alpha N - 1}{\alpha N}\exp[\pi i(n-\ell)/N]\right\}, \quad n \neq m, \quad \ell \neq j, \end{aligned} \tag{19e}$$

where for notational convenience we set

$$\alpha = \frac{\nu}{N}. \tag{19f}$$

For $N = 2$, $N = 3$ respectively $N = 4$ the explicit evaluation of the determinants (see (18b) with (19)) yields

$$\det\begin{bmatrix}(\nu\eta^2 - 2\lambda\eta)\mathbf{1} + A & B \\ \eta C + D & \eta\mathbf{I} + E\end{bmatrix} = \det\begin{pmatrix} -\nu(\nu + \frac{\nu^2}{2} - \eta^2) & \nu^2(1-\frac{\nu}{2}) & -i\nu & -i\nu \\ \nu^2(1-\frac{\nu}{2}) & -\nu(\nu+\frac{\nu^2}{2}-\eta^2) & i\nu & i\nu \\ i\eta(1-\frac{\nu}{2}) & -i\eta(1-\frac{\nu}{2}) & \eta & 0 \\ i\eta(1-\frac{\nu}{2}) & -i\eta(1-\frac{\nu}{2}) & 0 & \eta \end{pmatrix}$$
$$= (\nu\eta)^2(\eta-2)(\eta+2)(\eta-\nu)(\eta+\nu), \tag{20a}$$

$$\det\begin{bmatrix}(\nu\eta^2 - 2\lambda\eta)\mathbf{1} + A & B \\ \eta C + D & \eta\mathbf{I} + E\end{bmatrix} = (\nu\eta)^3(\eta-2)(\eta+2)^2\left(\eta - \frac{2\nu}{3}\right)\left(\eta + \frac{2\nu}{3}\right)$$
$$\cdot\left(\eta - \frac{4\nu}{3}\right)\left(\eta + \frac{4\nu}{3}\right)\left(\eta - \frac{2\nu}{3} + 2\right)\left(\eta + \frac{2\nu}{3} + 2\right), \tag{20b}$$

respectively

$$\det\begin{bmatrix}(\nu\eta^2 - 2\lambda\eta)\mathbf{1} + A & B \\ \eta C + D & \eta\mathbf{I} + E\end{bmatrix} = (\nu\eta)^4(\eta-2)(\eta+2)^3\left(\eta - \frac{\nu}{2}\right)\left(\eta + \frac{\nu}{2}\right)(\eta-\nu)(\eta+\nu)\left(\eta - \frac{3\nu}{2}\right)$$
$$\cdot\left(\eta + \frac{3\nu}{2}\right)\left(\eta - \frac{\nu}{2} + 2\right)^2\left(\eta + \frac{\nu}{2} + 2\right)^2(\eta - \nu + 2)(\eta + \nu + 2). \tag{20c}$$

The 9×9 respectively 16×16 matrices in the left-hand side of the last two, (20b) respectively (20c), of these 3 formulae are too large to be displayed.

These findings suggest the following:

Conjecture 2.4.1. For arbitrary N

$$\det\begin{bmatrix}(\nu\eta^2 - 2\lambda\eta)\mathbf{1} + A & B \\ \eta C + D & \eta\mathbf{I} + E\end{bmatrix} = (\nu\eta)^N(\eta-2)(\eta+2)^{N-1}\prod_{k=1}^{N-1}\left[\left(\eta - \frac{2k\nu}{N}\right)\left(\eta + \frac{2k\nu}{N}\right)\right] \tag{21}$$
$$\cdot\prod_{k=1}^{N-2}\left[\left(\eta - \frac{2k\nu}{N} + 2\right)^{N-1-k}\left(\eta + \frac{2k\nu}{N} + 2\right)^{N-1-k}\right]. \quad \square$$

Let us recall that the left-hand side of this formula is the determinant of a specific $N^2 \times N^2$ block matrix, see (19), while the right-hand side is a factorized polynomial of order $N^2 + N$ in the variable η. To arrive at this result the assumption that ν be a *rational* number played a role, but obviously this conjecture is applicable even if this requirement does not hold.

In **Case 2:**

$$A_{n,\ell} = -\delta_{n\ell}\left[1 + \frac{2[2(N+2) - x_n^2][2(N-1) - x_n^2]}{45}\right] + \frac{2(1-\delta_{n\ell})}{(x_n - x_\ell)^4}, \tag{22a}$$

$$B_{n,\ell j} = -2\delta_{nj}(x_n - x_\ell)^{-2} + 2\delta_{n\ell}(x_n - x_j)^{-2}, \quad \ell \neq j, \tag{22b}$$

$$C_{nm,\ell} = (\delta_{n\ell} - \delta_{m\ell})(x_n - x_m)^{-2}, \quad n \neq m, \tag{22c}$$

$$D_{nm,\ell} = (\delta_{n\ell} + \delta_{m\ell}) \left[\frac{4}{(x_n - x_m)^4} - \frac{2N+1}{3(x_n - x_m)^2} \right] + \frac{\delta_{n\ell} x_n^2 + \delta_{m\ell} x_m^2}{3(x_n - x_m)^2} - \frac{\delta_{n\ell} x_n - \delta_{m\ell} x_m}{2(x_n - x_m)}$$
$$- \frac{(1-\delta_{n\ell})(1-\delta_{m\ell})}{(x_n - x_\ell)(x_m - x_\ell)} \left[\frac{1}{(x_n - x_\ell)^2} + \frac{1}{(x_m - x_\ell)^2} \right], \quad n \neq m, \tag{22d}$$

$$E_{nm,\ell j} = \delta_{n\ell}\delta_{mj} \frac{x_n^2 - x_m^2}{3} - \frac{\delta_{n\ell}(1-\delta_{mj})}{x_m - x_j} \left(\frac{1}{x_n - x_j} + \frac{1}{x_m - x_j} \right)$$
$$+ \frac{\delta_{mj}(1-\delta_{n\ell})}{x_n - x_\ell} \left(\frac{1}{x_n - x_\ell} + \frac{1}{x_m - x_\ell} \right), \quad n \neq m, \quad \ell \neq j. \tag{22e}$$

For $N = 2$, $N = 3$ respectively $N = 4$ the explicit evaluation of the determinants (see (18b) with (22)) yields

$$\det \begin{bmatrix} (\nu\eta^2 - 2\lambda\eta)\mathbf{1} + A & B \\ (\eta C + D) & (\eta \mathbf{I} + E) \end{bmatrix} = \det \begin{pmatrix} \eta^2 - \frac{3}{2} & \frac{1}{2} & 1 & -1 \\ \frac{1}{2} & \eta^2 - \frac{3}{2} & -1 & 1 \\ \frac{1}{2}\eta & -\frac{1}{2}\eta & \eta & 0 \\ -\frac{1}{2}\eta & \frac{1}{2}\eta & 0 & \eta \end{pmatrix}$$
$$= \eta^2(\eta-1)(\eta+1)(\eta-2)(\eta+2), \tag{23a}$$

$$\det \begin{bmatrix} (\nu\eta^2 - 2\lambda\eta)\mathbf{1} + A & B \\ (\eta C + D) & (\eta \mathbf{I} + E) \end{bmatrix}$$
$$= \det \begin{pmatrix} \eta^2 - \frac{25}{9} & \frac{8}{9} & \frac{8}{9} & \frac{4}{3} & \frac{4}{3} & -\frac{4}{3} & 0 & -\frac{4}{3} & 0 \\ \frac{8}{9} & \eta^2 - \frac{35}{18} & \frac{1}{18} & -\frac{4}{3} & 0 & \frac{4}{3} & \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{8}{9} & \frac{1}{18} & \eta^2 - \frac{35}{18} & 0 & -\frac{4}{3} & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{2}{3}\eta + \frac{2}{9} & -\frac{2}{3}\eta + \frac{1}{18} & -\frac{5}{18} & \eta - \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 1 \\ \frac{2}{3}\eta + \frac{2}{9} & -\frac{5}{18} & -\frac{2}{3}\eta + \frac{1}{18} & -\frac{1}{2} & \eta - \frac{1}{2} & 0 & 1 & 0 & 0 \\ -\frac{2}{3}\eta + \frac{2}{9} & \frac{2}{3}\eta + \frac{1}{18} & -\frac{5}{18} & 0 & 0 & \eta + \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{8}{9} & \frac{1}{6}\eta - \frac{4}{9} & -\frac{1}{6}\eta - \frac{4}{9} & 0 & 0 & 0 & \eta & 0 & 0 \\ -\frac{2}{3}\eta + \frac{2}{9} & -\frac{5}{18} & \frac{2}{3}\eta + \frac{1}{18} & 0 & 0 & \frac{1}{2} & 0 & \eta + \frac{1}{2} & -1 \\ \frac{8}{9} & -\frac{1}{6}\eta - \frac{4}{9} & \frac{1}{6}\eta - \frac{4}{9} & 0 & 0 & 0 & 0 & 0 & \eta \end{pmatrix}$$
$$= \eta^4(\eta-1)^2(\eta+1)^2(\eta-2)(\eta+2)(\eta-3)(\eta+3), \tag{23b}$$

respectively

$$\det \begin{bmatrix} (\nu\eta^2 - 2\lambda\eta)\mathbf{1} + A & B \\ (\eta C + D) & (\eta \mathbf{I} + E) \end{bmatrix}$$
$$= \eta^6(\eta-1)^3(\eta+1)^3(\eta-2)^2(\eta+2)^2(\eta-3)(\eta+3)(\eta-4)(\eta+4). \tag{23c}$$

The 16×16 matrix in the left-hand side of the last, (23c), of these 3 formulae is too large to be displayed.

These findings suggest the following:

Conjecture 2.4.2. For arbitrary N

$$\det \begin{bmatrix} (\nu\eta^2 - 2\lambda\eta)\mathbf{1} + A & B \\ \eta C + D & \eta \mathbf{I} + E \end{bmatrix} = \eta^{2(N-1)}(\eta - N)(\eta + N) \prod_{k=1}^{N-1} [(\eta - k)(\eta + k)]^{N-k}. \tag{24}$$

□

Let us recall that the left-hand side of this formula is the determinant of a specific $N^2 \times N^2$ block matrix, see (22) where the numbers x_n are the N zeros of the Hermite polynomial of order N (see (14c)), while the right-hand side is a factorized polynomial of order $N^2 + N$ in the variable η.

3. Proofs

The starting point to obtain the equations of motion (10) is the observation that, via the definition (6), time-differentiation of (4a) yields

$$\dot{U} = R\{\dot{Z} + [M, Z]\}R^{-1}, \tag{25a}$$

$$\ddot{U} = R\{\ddot{Z} + [\dot{M}, Z] + 2[M, \dot{Z}] + [M, [M, Z]]\}R^{-1}. \tag{25b}$$

Here and hereafter the notation $[A, B]$ denotes the commutator of the two matrices A, B:

$$[A, B] \equiv AB - BA. \tag{26}$$

Via these formulae the $N \times N$ matrix ODE (1) yields

$$\nu\{\ddot{Z} + [\dot{M}, Z] + 2[M, \dot{Z}] + [M, [M, Z]]\} = 2i\lambda\omega\{\dot{Z} + [M, Z]\} + (\lambda^2 - \nu^2)\omega^2 Z + (\nu - 1)\{\dot{Z} + [M, Z]\}Z^{-1}\{\dot{Z} + [M, Z]\}, \tag{27}$$

and it is then easily seen, via (4b), (8) and (9), that the diagonal and off-diagonal parts of this matrix ODE yield the N^2 equations of motion (10).

The verification that the formulae (13) of **Case 1** provide a solution of the system of N^2 algebraic equations (11) is reduced to a trivial computation via the identities [20]

$$\sum_{\ell=1, \ell\neq n}^{N} \frac{\exp(\frac{2\pi i\ell}{N})}{\exp(\frac{2\pi in}{N}) - \exp(\frac{2\pi i\ell}{N})} = -\frac{N-1}{2}, \tag{28a}$$

$$\sum_{\ell=1, \ell\neq n}^{N} \frac{\exp(\frac{2\pi in}{N})}{\exp(\frac{2\pi in}{N}) - \exp(\frac{2\pi i\ell}{N})} = \frac{N-1}{2}. \tag{28b}$$

And the corresponding matrices (17), see (19), are immediately yielded by (13) via the trigonometric identity [21, 22]

$$\sum_{\ell=1}^{N-1} \left[\sin\left(\frac{\pi\ell}{N}\right)\right]^{-2} = \frac{N^2 - 1}{3}. \tag{28c}$$

Likewise, the verification that the formulae (14) of **Case 2** provide another solution of the system of N^2 algebraic equation (11) is a matter of trivial algebra using the identity

$$\frac{x_n - x_m}{(x_n - x_\ell)(x_m - x_\ell)} = -\left(\frac{1}{x_n - x_\ell} - \frac{1}{x_m - x_\ell}\right) \tag{29}$$

(to be conveniently inserted in (12b) with (14)) and the second and third of the following 4 formulae (see Appendix C of Ref. [19])

$$\sum_{\ell=1, \ell\neq n}^{N} \frac{1}{x_n - x_\ell} = x_n, \tag{30a}$$

$$\sum_{\ell=1, \ell\neq n}^{N} \frac{1}{(x_n - x_\ell)^2} = \frac{2(N-1) - x_n^2}{3}, \tag{30b}$$

$$\sum_{\ell=1, \ell\neq n}^{N} \frac{1}{(x_n - x_\ell)^3} = \frac{x_n}{2}, \tag{30c}$$

$$\sum_{\ell=1,\ell\neq n}^{N} \frac{1}{(x_n - x_\ell)^4} = \frac{[2(N+2) - x_n^2][2(N-1) - x_n^2]}{45}, \tag{30d}$$

satisfied by the N zeros of the Hermite polynomial of degree N. And also the derivation (via the 4 formulae (30)) of the expressions (22) from (17) with (14) is a matter of trivial if tedious algebra, using repeatedly (in particular to get (22d)) the identity (29).

4. Outlook

As explained in the introductory Sec. 1, the technique described and utilized in this paper can be used relatively widely (although its application is not quite trivial), yielding *Diophantine* findings (results and conjectures). This has provided the motivation to append the Roman numeral I to this paper: indeed a second paper following the same pattern and arriving thereby at additional *Diophantine* findings is in the pipeline [20]. As already mentioned in the introductory Sec. 1, the interest of these findings is in the eye of the beholder: we feel this kind of findings deserve to be ascertained and to be eventually recorded in standard compilations of mathematical formulae, as done for instance in [22].

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