

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852

ISSN (Print): 1402-9251

Journal Home Page: <https://www.atlantis-press.com/journals/jnmp>

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To cite this article: Hung-Chu Hsu, Yang-Yih Chen, John R. C. Hsu, Wen-Jer Tseng (2009) Nonlinear Water Waves on Uniform Current in Lagrangian Coordinates, Journal of Nonlinear Mathematical Physics 16:1, 47–61, DOI: <https://doi.org/10.1142/S1402925109000054>

To link to this article: <https://doi.org/10.1142/S1402925109000054>

Published online: 04 January 2021

NONLINEAR WATER WAVES ON UNIFORM CURRENT IN LAGRANGIAN COORDINATES

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Received 28 January 2008

Accepted 2 March 2008

This paper presents a new third-order trajectory solution in Lagrangian form for the water particles in a wave-current interaction flow based on an Euler–Lagrange transformation. The explicit parametric solution highlights the trajectory of a water particle and the wave kinematics above the mean water level and within a vertical water column, which were calculated previously by an approximation method using Eulerian approach. Mass transport associated with a particle displacement can now be obtained directly in Lagrangian form. The angular frequency and Lagrangian mean level of the particle motion in Lagrangian form differ from those of the Eulerian. The variations in the wave profile and the water particle orbits resulting from the interaction with a steady uniform current of different magnitudes are also investigated. Comparison on the wave profiles given by the Eulerian and Lagrangian solution to a third-order reveals that the latter is more accurate than the former in describing the shape of the wave profile. Moreover, the influence of a following current is found to increase the relative horizontal distance traveled by a water particle, while the converse is true in the case of an opposing current.

Keywords: Nonlinear water waves; current; Lagrangian, trajectory; Lagrangian wave frequency; Lagrangian mean level.

1. Introduction

The phenomenon of wave-current interaction has been studied extensively since 1970s. Several theoretical solutions for waves on currents with uniform or sheared profiles have been well documented in the review articles of Peregrine [1], Jonsson [2] and Thomas & Klopman [3]. Reports were also available on experimental studies for combined wave and current covering various aspects of this problem [4–9].

Most previous theories dealing with wave-current interactions have employed the Eulerian description, in which the free surface fluctuations can be expressed in a Taylor series expansion

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relative to a fixed water level (i.e., the still water level). This implicitly assumes that the surface profile of a wave is a differentiable single-valued function. Unlike the Eulerian free surface, which is given as an implicit function, a Lagrangian surface is described through a parametric representation of the position of a particle. The use of Lagrangian coordinates yields the only known nontrivial exact solutions to the governing equations for gravity water waves (i.e., Gerstner's solution for deep-water waves [10] and a recently found edge wave solution along a sloping beach [11]). The main advantage of such description is to allow better flexibility for describing the actual shape of the ocean surface, which will be demonstrated later in this paper. Based on this reason, it has been shown that the Lagrangian description is more appropriate for the motion of the limiting free surface that cannot be captured by the classical Eulerian solutions [12–14]. However, reports on this notable improvement using Lagrangian description were rather limited.

Conventionally, there are two different approaches for the study of the fluid flows in Lagrangian coordinates. One is to directly solve the Lagrangian equations of fluid motion employing a set of partial differential equations that satisfy various boundary conditions. The other approach is to study Lagrangian properties of the waves based on the dynamics of fluid particles upon a transformation of the velocity field initially given by Eulerian description. This second method has been extensively used in studying the mass transport velocity on waves [15]. However, the Euler–Lagrange transformation is a highly nonlinear process, and problem also exists for finding an appropriate transformation from the solution of nonlinear water waves in Eulerian coordinates to a Lagrangian description.

To overcome the difficulty mentioned above, Longuet-Higgins [15] applied a Taylor series expansion to Eulerian velocity and obtained a second order Lagrangian velocity. Subsequently, Longuet-Higgins [16, 17] also presented a simple physical model to obtain the Lagrangian characters including particle motion, mass transport, the Lagrangian wave period and the Lagrangian mean level for the surface waves that cannot directly obtain throughout the entire flow field. Wiegel [18] and Chen *et al.* [14] follow the straightforward expansion of Longuet-Higgins [15] by deriving the horizontal and vertical displacements of a water particle from a still water position to a third-order approximation. These expressions for particle displacements to a third-order contain a resonant term that is non-uniformly valid for large times but physically unrealistic (i.e., Eqs. (2.142) and (2.143) in Wiegel [18], and Chen *et al.* [14]). In order to render an expansion that is uniformly valid, the solutions obtained must be free from secular terms in which a wave property increases with time.

In this paper, we consider that the Lagrangian wave frequency is a function of wave nonlinearity and an arbitrary position for each water particle. The Lindstedt–Poincaré perturbation technique is used to ensure the correctness of the solutions in a Lagrangian system. By equating the instantaneous Eulerian and Lagrangian velocities associated with the same particle; these two different methods for describing the fluid motion may yield similar results, despite the differences in the method of derivation and the final expressions of their solutions. Consequently, an approximate method based upon an expansion of the velocity field in a Taylor series is constructed in this paper. By using a successive expansion in a Taylor series for the water particle path and the period of a particle motion, the new expressions for the Lagrangian particle trajectories, the mass transport velocity and the period of particle motion can be derived from analytical solutions in Eulerian description for a water particle initially located at different vertical level associated with an arbitrary wave height. Moreover, the changes in water surface elevation resulting from the interaction with a following or opposing uniform current is also discussed in this paper.

In Sec. 2 we list the solutions of the wave-current interaction obtained by Chen [19] in Eulerian coordinates. The Euler–Lagrange transformation techniques are then described in Sec. 3 in order to derive a Lagrangian solution in a combined wave-current motion, from which a third-order parametric equation is obtained for the particle trajectory and mass transport. In Sec. 4 the trajectory

of water particles and the variation in water surface elevation over the entire wave-interaction field are discussed. Finally, some concluding remarks are given in Sec. 5.

2. Solution for a Wave-Current Field in Eulerian System

Consider a classical problem of a two-dimensional monochromatic wave with a steady uniform current on an impermeable and horizontal bed (Fig. 1). The fluid is assumed to be inviscid and the flow is irrotational, so that a velocity potential exists in the fluid domain. The Cartesian coordinate system is used, in which the x -axis is in the direction of wave progression, and the y -axis is positive upwards from the still water level. Consider also the condition of weak current, and the effect of current is to change the wavelength of the existing wave properties with a given wave frequency. Based on the analytical solutions of periodic waves propagating over a uniform current in the Eulerian approach obtained by Chen [19], the resulting expressions for the velocity potential and water surface elevation to a third-order approximation are given, respectively as:

$$\begin{aligned}
 \Phi(x, y, t) &= Ux + \frac{a}{k}(\sigma_{w0} - kU) \frac{\cosh k(y+d)}{\sinh kd} \cdot \sin S_1 \\
 &\quad + \frac{3}{8}a^2(\sigma_{w0} - kU) \frac{\cosh 2k(y+d)}{\sinh^4 kd} \cdot \sin 2S_1 - \frac{1}{4}a^2(\sigma_{w0} - kU)^2 \frac{1}{\sinh^2 kd} t \\
 &\quad + \frac{1}{64}ka^3(\sigma_{w0} - kU) \frac{9 - 4 \sinh^2 kd}{\sinh^7 kd} \cosh 3k(y+d) \cdot \sin 3S_1 \\
 &= Ux + \frac{(C_* - U)}{k} [\lambda F_1 \cosh k(y+d) \cdot \sin S_1 + \lambda^2 F_2 \cosh 2k(y+d) \cdot \sin 2S_1 \\
 &\quad + \lambda^3 F_3 \cosh 3k(y+d) \cdot \sin 3S_1] - \frac{1}{4}\lambda^2(C_* - U)^2 \frac{1}{\sinh^2 kd} t
 \end{aligned} \tag{2.1}$$

where $\lambda = ka$, $C_* = \frac{\sigma_{w0}}{k}$, $S_1 = kx - \sigma_w t$, $F_1 = \frac{1}{\sinh kd}$, $F_2 = \frac{3}{8 \sinh^4 kd}$,

$$F_3 = \frac{1}{64 \cosh 3kd} (9 \tanh^{-7} kd + 5 \tanh^{-5} kd - 53 \tanh^{-3} kd + 39 \tanh^{-1} kd)$$

and

$$\begin{aligned}
 \eta(x, t) &= \left(a + \frac{1}{16}k^2a^3 \frac{3 + 14 \sinh^2 kd + 2 \sinh^4 kd}{\sinh^4 kd} \right) \cdot \cos S_1 \\
 &\quad + \frac{1}{4}ka^2 \frac{(3 + 2 \sinh^2 kd) \cosh kd}{\sinh^3 kd} \cdot \cos 2S_1 \\
 &\quad + \frac{1}{64}k^2a^3 \frac{27 + 72 \sinh^2 kd + 72 \sinh^4 kd + 24 \sinh^6 kd}{\sinh^6 kd} \cdot \cos 3S_1
 \end{aligned} \tag{2.2}$$

where

$$\sigma_w = \sigma_{w0} + \frac{1}{16}\lambda^2(\sigma_{w0} - kU) \frac{9 + 8 \sinh^2 kd + 8 \sinh^4 kd}{\sinh^4 kd}. \tag{2.3}$$

$$\sigma_w = \omega + kU, \quad (\sigma_{w0} - kU)^2 = gk \tanh kd. \tag{2.4}$$

In Eqs. (2.1)–(2.4), $\Phi = \Phi(x, y, t)$ and $\eta(x, t)$ are the velocity potential and water surface elevation, respectively; U denotes the speed of a steady uniform current; a is the wave amplitude; C_* is phase speed of the wave; d is the total water depth; g is the acceleration due to gravity and ω is the relative angular frequency. The wave number $k(=2\pi/L)$ and wave frequency $\sigma_w(=2\pi/T_E)$ are a function of wavelength L and wave period T_E , respectively. Equation (2.4) is commonly referred to as the Doppler-shifted solution.

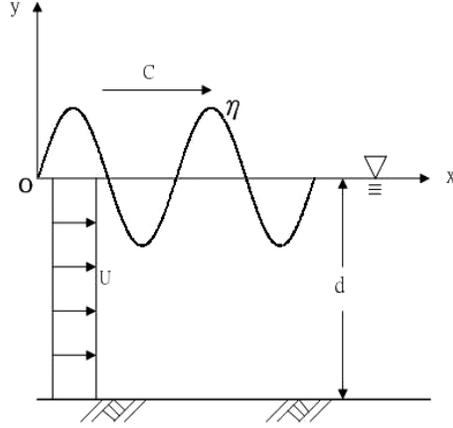


Fig. 1. Definition sketch showing a system of progressive wave train on a uniform current.

3. An Euler–Lagrange Transformation

To trace the motion of a fluid particle, it is necessary to transfer the Eulerian velocity components (u, v) in $\vec{R}(x, y, t) = \vec{i}u(x, y, t) + \vec{j}v(x, y, t)$, at a fix position (x, y) and at a given time t , into a corresponding Lagrangian velocity system, $\vec{W}(\alpha, \beta, t) = \vec{i}U_L(\alpha, \beta, t) + \vec{j}V_L(\alpha, \beta, t)$, with a particle coordinate (α, β) in Lagrangian form in the horizontal and vertical directions, respectively. Upon releasing the water particle from its initial position (α, β) at an initial time $t = t_0$, and equating the Lagrangian velocities to Eulerian velocities at $t = t$,

$$\vec{R}(x, y, t) = \vec{i}u(x, y, t) + \vec{j}v(x, y, t) = \vec{W}(\alpha, \beta, t) = \vec{i}U_L(\alpha, \beta, t) + \vec{j}V_L(\alpha, \beta, t), \quad (3.1)$$

the resultant location of a moving water particle can be obtained. In Eq. (3.1), the $u(x, y, t)$ and $v(x, y, t)$ denote the particle velocity components in the horizontal and vertical direction, respectively, in the Eulerian system, whereas $U_L(\alpha, \beta, t)$ and $V_L(\alpha, \beta, t)$ represent the horizontal and vertical particle velocity components in the Lagrangian system.

For an incompressible fluid the continuity equation provides an invariant condition to conserve the volume of fluid vertically integrated from the horizontal bed to the free surface over one wavelength from mapping the Eulerian to Lagrangian system given by

$$\frac{\partial(x, y)}{\partial(\alpha, \beta)} = J = 1. \quad (3.2)$$

Equation (3.2) indicates that the Jacobian transformation of (x, y) with respect to (α, β) is independent of time and is also the mass conservation or the continuity equation in the Lagrangian system. To solve Eqs. (3.1) and (3.2) together, relevant physical quantities are assumed to be expandable as power series in a small perturbation parameter ε , which also serves as an ordering parameter for the identification of the analytical solution to a specific order of approximation.

The Eulerian velocity components in Eq. (3.1) can be derived from $(u, v) = (\Phi_x, \Phi_y)$. From Eq. (2.1), we have

$$\begin{aligned} u(x, y, t) &= \sum_{n=1}^{\infty} \varepsilon^n u_n(x, y, t) = \Phi_x \\ &= U + (C_* - U)[\lambda F_1 \cosh k(y + d) \cdot \cos S_1 + 2\lambda^2 F_2 \cosh 2k(y + d) \cdot \cos 2S_1 \\ &\quad + 3\lambda^3 F_3 \cosh 3k(y + d) \cdot \sin 3S_1]. \end{aligned} \quad (3.3)$$

$$\begin{aligned}
 v(x, y, t) &= \sum_{n=1}^{\infty} \varepsilon^n v_n(x, y, t) = \Phi_y \\
 &= (C_* - U)[\lambda F_1 \sinh k(y+d) \cdot \sin S_1 + 2\lambda^2 F_2 \sinh 2k(y+d) \cdot \sin 2S_1 \\
 &\quad + 3\lambda^3 F_3 \sinh 3k(y+d) \cdot \sin 3S_1]
 \end{aligned} \tag{3.4}$$

and wave frequency σ_w can be expanded as

$$\sigma_w = \sigma_{w0} + \sum_{n=1}^{\infty} \varepsilon^n \sigma_{wn} = \sum_{n=0}^{\infty} \varepsilon^n \sigma_{wn}. \tag{3.5}$$

In the previous studies [14, 18] for irrotational progressive gravity waves in water of uniform depth, the Eulerian solution for water particle velocity cannot be transformed to the Lagrangian solution to a third-order approximation due to the inclusion of a mixed-secular term, which is a product of algebraic and circular terms (e.g. $t \sin(kx - \sigma_w t)$). This extra term is non-uniformly valid at large time which is physically unrealistic. To improve this situation and render a uniformly valid solution, we impose the assumption that the Lagrangian wave frequency is a function of the nonlinearity and the initial position of each water particle. From this, the particle trajectories $\vec{X} = \{x(\alpha, \beta, t), y(\alpha, \beta, t)\}$, water particle velocities $U_L(\alpha, \beta, t)$ and $V_L(\alpha, \beta, t)$, and angular frequency of the particle motion $\sigma_L(\beta)$ in the Lagrangian system may assume in the form of,

$$x(\alpha, \beta, t) = a + \sum_{n=1}^{\infty} \varepsilon^n [f_n(\alpha, \beta, \sigma_L t) + f'_n(\alpha, \beta, \sigma_{L0} t)] + M(t). \tag{3.6}$$

$$y(\alpha, \beta, t) = b + \sum_{n=1}^{\infty} \varepsilon^n [g_n(\alpha, \beta, \sigma_L t) + g'_n(\alpha, \beta, \sigma_{L0} t)]. \tag{3.7}$$

$$\sigma_L(\beta) = \sum_{n=0}^{\infty} \alpha^n \sigma_{Ln}(\beta). \tag{3.8}$$

$$U_L(\alpha, \beta, t) = x_t = M' + \sum_{n=1}^{\infty} \varepsilon^n \sigma_{L0} (f_{n\sigma_L t} + f'_{n\sigma_{L0} t}) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varepsilon^{n+m} \sigma_{Lm} f_{n\sigma_L t}. \tag{3.9}$$

$$V_L(\alpha, \beta, t) = y_t = \sum_{n=1}^{\infty} \varepsilon^n \sigma_{L0} (g_{n\sigma_L t} + g'_{n\sigma_{L0} t}) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varepsilon^{n+m} \sigma_{Lm} g_{n\sigma_L t}. \tag{3.10}$$

Equations (3.6)–(3.8) give the complete solutions in Lagrangian form with unknowns that can be solved sequentially in each order of ε . Substituting Eqs. (3.6)–(3.8) into the wave phase S_1 in Eq. (2.1) and the function of water depth within the Eulerian frame, we obtain

$$kx - \sigma_w t = \left\{ (k\alpha - \sigma_L t) + \left[k \sum_{n=1}^{\infty} \varepsilon^n (f_n + f'_n) + kM(t) + \sum_{n=0}^{\infty} \varepsilon^n (\sigma_{Ln} - \sigma_{wn}) t \right] \right\}, \tag{3.11}$$

$$k(y+d) = \left[k(\beta+d) + k \sum_{n=1}^{\infty} \varepsilon^n (g_n + g'_n) \right]. \tag{3.12}$$

Further substituting Eqs. (3.6)–(3.8) into Eqs. (3.1) and (3.2), and using Taylor series expansion of functions around a labeled reference position and the period of particle motion, a cascade of ordinary differential equation can be collected for each order in ε that represents different order of approximation. By solving these equations successively all the unknown physical parameters in Lagrangian form can be worked out.

3.1. Zeroth- and first-order approximations

The governing equations in $O(\varepsilon^0)$ and $O(\varepsilon)$ are

$$M'(t) = U, \quad (3.13)$$

$$f_{1\alpha} + f'_{1\alpha} + g_{1\beta} + g'_{1\beta} + [\sigma_{L0\alpha}(f_{1\sigma_L t} + f'_{1\sigma_{L0} t}) + \sigma_{L0\beta}(g_{1\sigma_L t} + g'_{1\sigma_{L0} t})]t = 0, \quad (3.14)$$

$$\begin{aligned} \sigma_{L0}(f_{1\sigma_L t} + f'_{1\sigma_{L0} t}) &= (C_* - U)\lambda F_1 \cosh k(\beta + d) \cdot \cos S \\ &\quad - (C_* - U)\lambda F_1 \cosh k(\beta + d)[kM(t) + (\sigma_{L0} - \sigma_{w0})t] \cdot \sin S. \end{aligned} \quad (3.15)$$

$$\begin{aligned} \sigma_{L0}(g_{1\sigma_L t} + g'_{1\sigma_{L0} t}) &= (C_* - U)\lambda F_1 \sinh k(\beta + d) \cdot \sin S \\ &\quad + (C_* - U)\lambda F_1 \sinh k(\beta + d)[kM(t) + (\sigma_{L0} - \sigma_{w0})t] \cdot \cos S \end{aligned} \quad (3.16)$$

where S is the phase function of particle motion, $S = k\alpha - \sigma_L t$. By nullifying the secular terms in Eqs. (3.14)–(3.16), which otherwise increase linearly with the time, the solutions given by Eqs. (3.13)–(3.16) become

$$M(t) = Ut. \quad (3.17)$$

$$\sigma_{L0} + kU = \sigma_{w0}; \quad \sigma_{L0}^2 = gk \tanh kd. \quad (3.18)$$

$$f_1 = -\frac{C_* - U}{\sigma_{L0}} \lambda F_1 \cosh k(\beta + d) \cdot \sin(k\alpha - \sigma_L t). \quad (3.19)$$

$$g_1 = \frac{C_* - U}{\sigma_{L0}} \lambda F_1 \sinh k(\beta + d) \cdot \cos(k\alpha - \sigma_L t). \quad (3.20)$$

Equation (3.17) is the basic solution that describes the existence of a steady uniform current. In Eq. (3.18), σ_{L0} is the essential Lagrangian wave frequency for water particles relative to the uniform current. From this, it can be demonstrated that the Doppler's effect is not apparent in the Lagrangian dispersion relation. This is because the Lagrangian representation describing fluid particles moving with a uniform current relative to water waves is given in a moving reference frame which does not appear to be transported by the uniform current.

3.2. Second-order approximation

To the second order in $O(\varepsilon^2)$ and by neglecting the secular terms which are non-uniformly valid in longer time, the governing equations are given by

$$\begin{aligned} f_{2\alpha} + f'_{2\alpha} + g_{2\beta} + g'_{2\beta} &= -f_{1\alpha}g_{1\beta} + f_{1\beta}g_{1\alpha} \\ &= \frac{1}{2} \frac{(C_* - U)^2}{\sigma_{L0}^2} k^2 \lambda^2 F_1^2 \cosh 2k(\beta + d) + \frac{1}{2} \frac{(C_* - U)^2}{\sigma_{L0}^2} k^2 \lambda^2 F_1^2 \cos 2S, \end{aligned} \quad (3.21)$$

$$\begin{aligned} \sigma_{L0}(f_{2\sigma_L t} + f'_{2\sigma_{L0} t}) &= (C_* - U)\lambda F_1 \{k(g_1 + g'_1) \sinh k(\beta + d) \cdot \cos S \\ &\quad - k(f_1 + f'_1) \cosh k(\beta + d) \cdot \sin S\} + (C_* - U)2\lambda^2 F_2 \cosh 2k(\beta + d) \cdot \cos 2S, \end{aligned} \quad (3.22)$$

$$\begin{aligned} \sigma_{L0}(g_{2\sigma_L t} + g'_{2\sigma_{L0} t}) &= (C_* - U)\lambda F_1 \{k(g_1 + g'_1) \cosh k(\beta + d) \cdot \sin S \\ &\quad + k(f_1 + f'_1) \sinh k(\beta + d) \cdot \cos S\} + (C_* - U)2\lambda^2 F_2 \sinh 2k(\beta + d) \cdot \sin 2S. \end{aligned} \quad (3.23)$$

The desirable solutions at this order can be obtained from Eqs. (3.21)–(3.23), and the results are

$$\sigma_{L1} = \sigma_{w1} = g'_2 = 0. \quad (3.24)$$

$$f_2 = \frac{1}{4} \frac{(C_* - U)^2}{\sigma_{L0}^2} k \lambda^2 F_1^2 \sin 2S - \frac{(C_* - U)}{\sigma_{L0}} \lambda^2 F_2 \cosh 2k(\beta + d) \cdot \sin 2S. \quad (3.25)$$

$$f'_2 = \frac{1}{2} \frac{(C_* - U)^2}{\sigma_{L0}} k \lambda^2 F_1^2 \cosh 2k(\beta + d) \cdot t. \quad (3.26)$$

$$g_2 = \frac{(C_* - U)}{\sigma_{L0}} \lambda^2 F_2 \sinh 2k(\beta + d) \cdot \cos 2S + \frac{1}{4} \frac{(C_* - U)^2}{\sigma_{L0}^2} k \lambda^2 F_1^2 \sinh 2k(\beta + d). \quad (3.27)$$

The Lagrangian formulation for the particle trajectory at the second-order approximation comprises a periodic component f_2 and non-periodic function f'_2 . The latter increases linearly with time and is independent of the Lagrangian horizontal label a which represents the mass transport, implying that a constant net motion would depend only on the vertical level β where the particle is located and the uniform current. The trajectory is the smallest near the bottom and is not a closed orbit as predicted by the first-order approximation. Moreover, Eq. (3.26), a second-order quantity, renders the same form obtained by Longuet-Higgins [15] for the case of wave alone (i.e., without uniform current). The solution for vertical displacement g_2 includes a second harmonic component and a time-independent term which is a function of wave steepness λ , relative current velocity and the Lagrangian vertical label β . Overall, the expression of g_2 yields vertical shift correction to a second-order which decreases with water depth. Taking the time average of the particle elevation over a given period of a particle motion from Eq. (3.27), it can be shown that the mean level of water particle orbit in Lagrangian approach is higher than that in the Eulerian counterpart as suggested by Longuet-Higgins [16, 17] for two-dimensional progressive water waves.

3.3. Third-order approximation

Upon obtaining the solutions for the pertinent physical parameters to $O(\varepsilon^2)$, those at $O(\varepsilon^3)$ can then be pursued based on the governing equations at $O(\varepsilon^3)$,

$$\begin{aligned} f_{3\alpha} + f'_{3\alpha} + g_{3\beta} + g'_{3\beta} + f_{1\alpha} g_{2\beta} + f_{2\alpha} g_{1\beta} - f_{1\beta} g_{2\alpha} - (f_{2\beta} + f'_{2\beta}) g_{1\alpha} \\ + (\sigma_{L2\alpha} \cdot f_{1\sigma L t} + \sigma_{L2\beta} \cdot g_{1\sigma L t}) t = 0, \end{aligned} \quad (3.28)$$

$$\begin{aligned} & \sigma_{L0}(f_{3\sigma L t} + f'_{3\sigma L 0 t}) + \sigma_{L2} \cdot f_{1\sigma L t} \\ & = (C_* - U) \lambda F_1 \left\{ \frac{1}{2} k^2 (g_1 + g'_1)^2 \cosh k(\beta + d) \cdot \cos S + k(g_2 + g'_2) \sinh k(\beta + d) \cdot \cos S \right. \\ & \quad - [k(f_2 + f'_2) + (\sigma_{L2} - \sigma_{w2})t] \cosh k(\beta + d) \cdot \sin S \\ & \quad \left. - k^2 (g_1 + g'_1)(f_1 + f'_1) \sinh k(\beta + d) \cdot \sin S - \frac{1}{2} [k(f_1 + f'_1)]^2 \cosh k(\beta + d) \cdot \cos S \right\} \\ & \quad + (C_* - U) 2 \lambda^2 F_2 \{ 2k(g_1 + g'_1) \sinh 2k(\beta + d) \cdot \cos 2S \\ & \quad - 2k(f_1 + f'_1) \cosh 2k(\beta + d) \cdot \sin 2S \} + (C_* - U) 3 \lambda^3 F_3 \cosh 3k(\beta + d) \cdot \cos 3S, \end{aligned} \quad (3.29)$$

$$\begin{aligned} & \sigma_{L0}(g_{3\sigma L t} + g'_{3\sigma L 0 t}) + \sigma_{L2} \cdot g_{1\sigma L t} \\ & = (C_* - U) \lambda F_1 \left\{ [k(f_2 + f'_2) + (\sigma_{L2} - \sigma_{w2})t] \sinh k(\beta + d) \cdot \cos S \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}k^2(f_1 + f'_1)^2 \sinh k(\beta + d) \cdot \sin S + k(g_2 + g'_2) \cosh k(\beta + d) \cdot \sin S \\
& + k^2(g_1 + g'_1)(f_1 + f'_1) \cosh k(\beta + d) \cdot \cos S + \frac{1}{2}k^2(g_1 + g'_1)^2 \sinh k(\beta + d) \cdot \sin S \Big\} \\
& + (C_* - U)2\lambda^2 F_2 \{2k(f_1 + f'_1) \sinh 2k(\beta + d) \cdot \cos 2S \\
& + 2k(g_1 + g'_1) \cosh 2k(\beta + d) \cdot \sin 2S\} + (C_* - U)3\lambda^3 F_3 \sinh 3k(\beta + d) \cdot \sin 3S. \quad (3.30)
\end{aligned}$$

Substituting the first- and second-order solutions into Eqs. (3.28)–(3.30), and letting the coefficient of the secular term to zero, the second-order Lagrangian wave frequency correction σ_{L2} can be obtained after a lengthy algebraic manipulation. Finally, the physical parameters to the third-order solutions in Lagrangian form are given as follows,

$$\sigma_{L2} = \sigma_{w2} - \frac{1}{2} \frac{(C_* - U)^2}{\sigma_{L0}} k^2 \lambda^2 F_1^2 \cosh 2k(\beta + d), \quad (3.31)$$

$$\sigma_{w2} = \frac{1}{16} k^2 a^2 (\sigma_{w0} - kU) (9 \tanh^{-4} kd - 10 \tanh^{-2} kd + 9).$$

$$f'_3 = g'_3 = 0. \quad (3.32)$$

$$\begin{aligned}
f_3 = & m_{333} \cosh 3k(\beta + d) \cdot \sin 3S + m_{331} \cosh 3k(\beta + d) \cdot \sin S \\
& + m_{313} \cosh k(\beta + d) \cdot \sin 3S + m_{311} \cosh k(\beta + d) \cdot \sin S. \quad (3.33)
\end{aligned}$$

$$\begin{aligned}
g_3 = & n_{333} \sinh 3k(\beta + d) \cdot \cos 3S + n_{331} \sinh 3k(\beta + d) \cdot \cos S \\
& + n_{313} \sinh k(\beta + d) \cdot \cos 3S + n_{311} \sinh k(\beta + d) \cdot \cos S. \quad (3.34)
\end{aligned}$$

In Eqs. (3.33)–(3.34), the coefficients m_{333} , m_{331} , m_{313} , m_{311} , n_{333} , n_{331} , n_{313} and n_{311} are given respectively by

$$m_{333} = -\frac{C_* - U}{\sigma_{L0}} \lambda^3 F_3, \quad m_{331} = -\left[\frac{5}{2} \frac{(C_* - U)^2}{\sigma_{L0}^2} k \lambda^3 F_1 F_2 + \frac{1}{2} \frac{(C_* - U)^3}{\sigma_{L0}^2} k^3 \lambda^3 F_1^3 \right],$$

$$m_{313} = \frac{5}{6} \frac{(C_* - U)^2}{\sigma_{L0}^2} k \lambda^3 F_1 F_2 - \frac{1}{12} \frac{(C_* - U)^3}{\sigma_{L0}^3} k^2 \lambda^3 F_1^3, \quad m_{311} = \sigma_{w2} \frac{C_* - U}{\sigma_{L0}^2} \lambda F_1,$$

$$n_{333} = \frac{C_* - U}{\sigma_{L0}} \lambda^3 F_3, \quad n_{331} = \frac{3}{2} \frac{(C_* - U)^2}{\sigma_{L0}^2} k \lambda^3 F_1 F_2 + \frac{1}{4} \frac{(C_* - U)^3}{\sigma_{L0}^3} k^2 \lambda^3 F_3,$$

$$n_{313} = -\frac{1}{2} \frac{(C_* - U)^2}{\sigma_{L0}^2} k \lambda^3 F_1 F_2, \quad n_{311} = -\sigma_{w2} \frac{C_* - U}{\sigma_{L0}^2} \lambda F_1.$$

In Eqs. (3.31)~(3.34) above, Eq. (3.31) is a second-order correction to the angular frequency of a particle motion, in which its first term is similar to the second-order Eulerian wave frequency and the second term is a function of its initial elevation and current speed. The result differs from the Eulerian wave frequency σ_{w2} . The third-order solutions for the particle trajectory given by Eqs. (3.33) and (3.34) are periodic functions including both first and third harmonic components.

In summary, the general solutions for the x - and y -components of the water particle trajectory and Lagrangian wave frequency in Lagrangian form up to a third-order can now be completely

formulated as follows,

$$\begin{aligned}
 x(\alpha, \beta, t) = & Ut - \frac{C_* - U}{\sigma_{L0}} \lambda F_1 \cosh k(\beta + d) \cdot \sin S + \frac{1}{4} \frac{(C_* - U)^2}{\sigma_{L0}^2} k \lambda^2 F_1^2 \cdot \sin 2S \\
 & - \frac{C_* - U}{\sigma_{L0}} \lambda^2 F_2 \cosh 2k(\beta + d) \cdot \sin 2S + \frac{1}{2} \frac{(C_* - U)^2}{\sigma_{L0}} k \lambda^2 F_1^2 \cosh 2k(\beta + d) \cdot t \\
 & + m_{333} \cosh 3k(\beta + d) \cdot \sin 3S + m_{331} \cosh 3k(\beta + d) \cdot \sin S \\
 & + m_{313} \cosh k(\beta + d) \cdot \sin 3S + m_{311} \cosh k(\beta + d) \cdot \sin S.
 \end{aligned} \tag{3.35}$$

$$\begin{aligned}
 y(\alpha, \beta, t) = & \frac{C_* - U}{\sigma_{L0}} \lambda F_1 \sinh k(\beta + d) \cdot \cos S + \frac{C_* - U}{\sigma_{L0}} \lambda^2 F_2 \sinh 2k(\beta + d) \cdot \cos 2S \\
 & + \frac{1}{4} \frac{(C_* - U)^2}{\sigma_{L0}^2} k \lambda^2 F_1^2 \sinh 2k(\beta + d) + n_{333} \sinh 3k(\beta + d) \cdot \cos 3S \\
 & + n_{331} \sinh 3k(\beta + d) \cdot \cos S + n_{313} \sinh k(\beta + d) \cdot \cos 3S \\
 & + n_{311} \sinh k(\beta + d) \cdot \cos S.
 \end{aligned} \tag{3.36}$$

$$\sigma_L(\beta) = \sigma_w - kU - \frac{1}{2} \frac{(C_* - U)^2}{\sigma_{L0}} k^2 \lambda^2 F_1^2 \cosh 2k(\beta + d). \tag{3.37}$$

It can be shown that Eq. (3.37) calculates the resultant wave period in Lagrangian form in a combined flow field for all the water particles at different elevations within the fluid domain. Thus, this is a generic expression comparing with that in Longuet-Higgins [16, 17] which was only for the surface particles in two-dimensional gravity waves. This equation also indicates the frequency of particle motion near the surface is smaller than that at the subsurface. The relative ratio of wave period for water particle motion at the free surface between the Lagrangian form $T_L = 2\pi/\sigma_L$ and Eulerian form $T_E = 2\pi/\sigma_w$ for three different current conditions is shown in Fig. 2, in which T_L/T_E is found to increase with a following current (positive U), and to decrease in an opposing current (negative U) for a given wave steepness H/L . This implies that for a coplanar flow the water particles near the surface move forward further over one wave cycle than those against an opposing flow. Moreover, T_L is larger than T_E even with the wave alone (e.g., the case of $U = 0$ in Fig. 2).

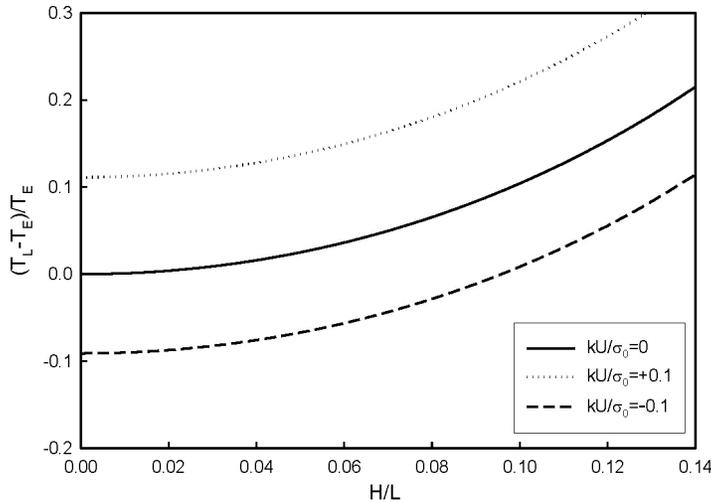
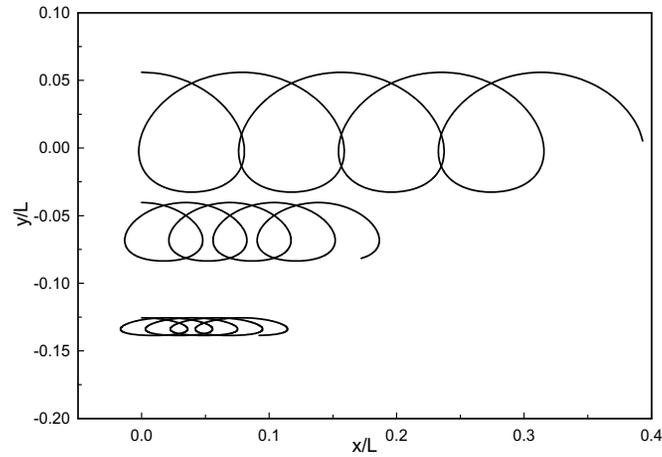


Fig. 2. Relative ratios of wave period for water particle motion at the free surface between Lagrangian and Eulerian approaches for three different current conditions.

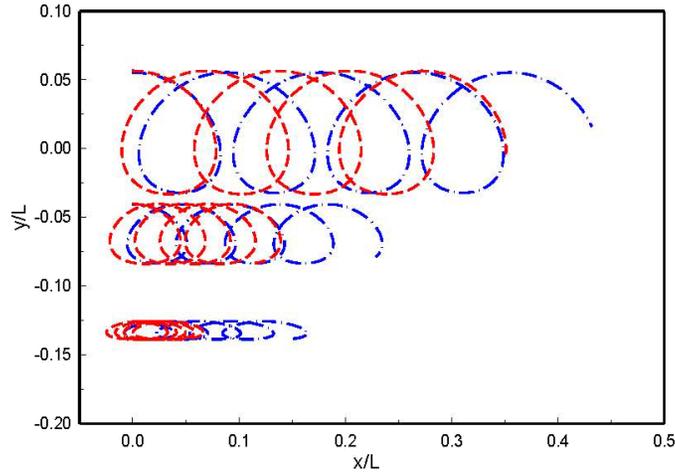
4. Results and Discussions

4.1. Water particle orbits

In Figs. 3(a)–3(d), water particle orbits plotted for three different mean elevations b and different current magnitudes ($F_r = 0, \pm 0.01, \pm 0.03$ and ± 0.05), where Froude number $F_r = U/\sqrt{gd}$, exhibit variations in orbital patterns, both in shapes and sizes, as a function of its original elevation. As can be expected that the orbital displacement based on a third-order solution does not move in a closed orbital motion (Fig. 3(a)). The elongation or shortening of the orbits in the case with following or opposing current is apparent in Figs. 3(b)–3(d). Their orbital dimensions in the cases with positive F_r values reflect the magnitude of a following current, and the converse is true for condition with opposing current. It can be seen in each of the orbits plotted that a water particle advances a distance forward which is commonly referred to as mean horizontal drift or mass transport in the direction of wave propagation. The water particle at the free surface ($\beta = 0$) travels fastest, whilst

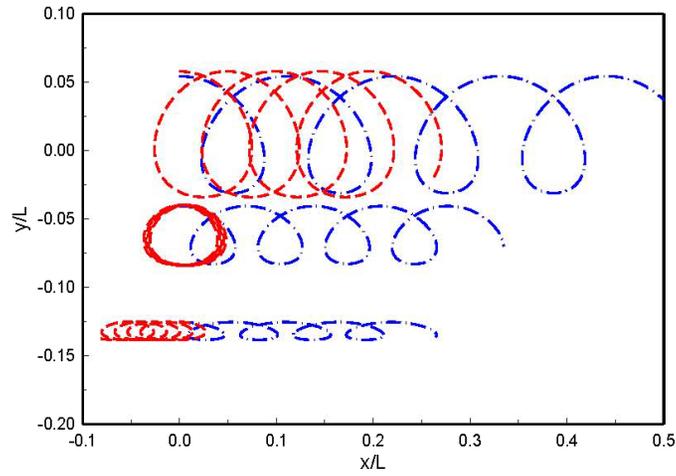


(a)

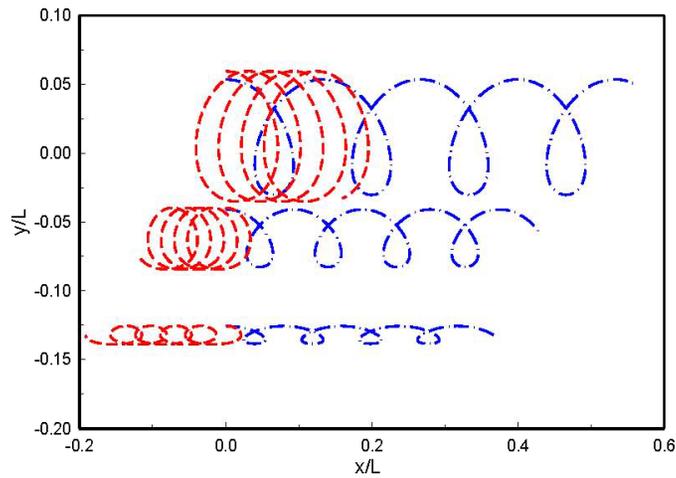


(b)

Fig. 3. Water particle orbits calculated from analytical solutions to a third-order approximation in Lagrangian form at different elevations and current speeds for water depth $d/L = 0.2$ and wave steepness $H/L = 0.04$. (a) Solid line, $F_r = 0$ (b) dash-dotted line, $F_r = 0.01$; dashed line, $F_r = -0.01$ (c) dash-dotted line, $F_r = 0.03$; dashed line, $F_r = -0.03$ (d) dash-dotted line, $F_r = 0.05$; dashed line, $F_r = -0.05$.



(c)



(d)

 Fig. 3. (*Continued*)

that in the interior of the fluid propagates slower. To the third-order approximation, the particle trajectory has non-closed orbit, irrespective of their initial mean locations. The orbital shapes near the bottom are more elliptic since the vertical excursion of the particle is less than its horizontal counterpart. However, for the orbits near or at the still water level, the vertical excursion is greater. Figure 3(a) also shows that the third-order solution has a greater horizontal and vertical excursion than the second-order solution at the free surface. It is noted that the particle orbits plotted in Figs. 3(a)–3(d) resemble in shape the trajectories in an irrotational (Stokes) wave, whose form was recently proved by Constantin [20].

In the case of wave on a following current, the effect of increasing current speed is generally to augment the magnitude and extent of the time-averaged drift velocity, thus resulting in large horizontal distance traveled by a particle compared with the case without current. Again, the converse is true when a wave train encounters an opposing current, which retards the advancement of water particles compared to that without a current or with a following current. As the strength of an opposing current becomes comparable with the wave speed, the water particle at greater depths

beneath the still water level is mainly transport by the opposing current in larger Froude number F_r and the direction of particle movement becomes contrary to wave progression.

4.2. Wave profile

Figure 4 displays the limiting profiles corresponding to a numerical model [21], on the basis of a fifth-order Eulerian solution [22] and a third-order Lagrangian solution presented in this paper at the limiting wave steepness $H/L = 0.1412$ for pure progressive water waves in deep water. It shows that the third-order Lagrangian solution is more accurate than the fifth-order Eulerian approximation of Fenton [22] compared with a numerical result of Williams [21]. Figure 5 provides an additional

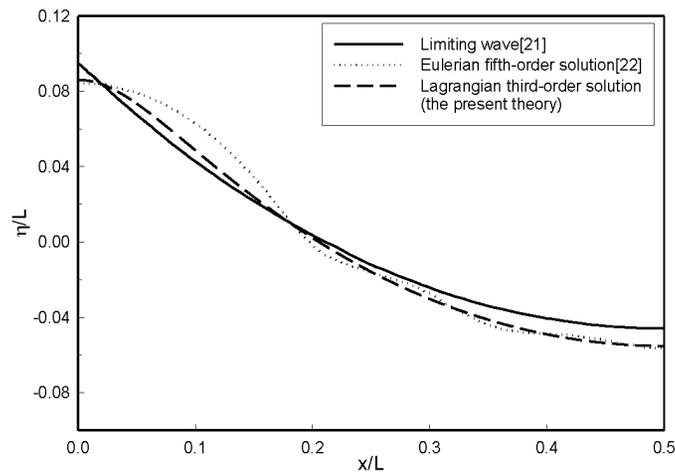


Fig. 4. Comparison on the wave profiles for the fifth-order Eulerian and third-order Lagrangian solutions for limiting wave condition $H/L = 0.1412$ in deep water for progressive water waves. (Solid line: numerical value using Williams [21]; dashed line: third-order Lagrangian solution using the present theory; dotted line: fifth-order Eulerian solution per Fenton [22]).

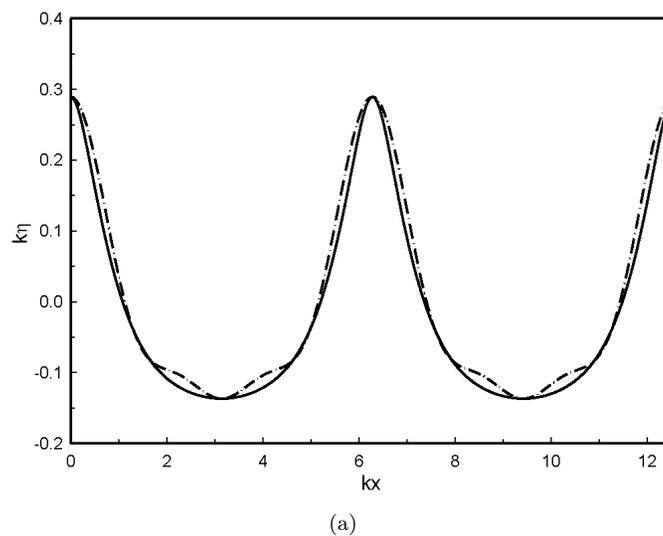
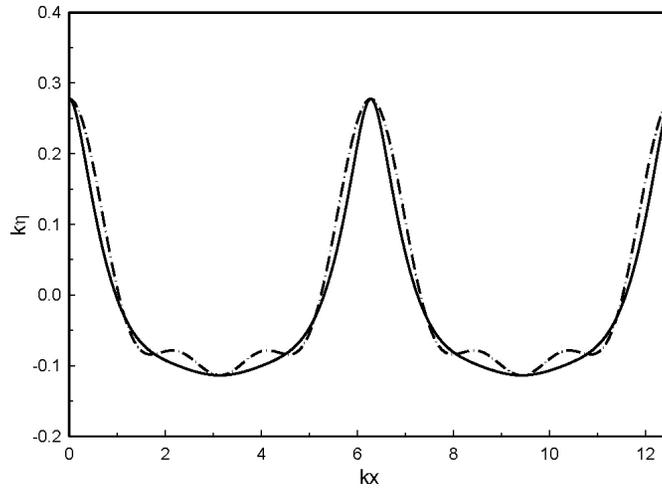
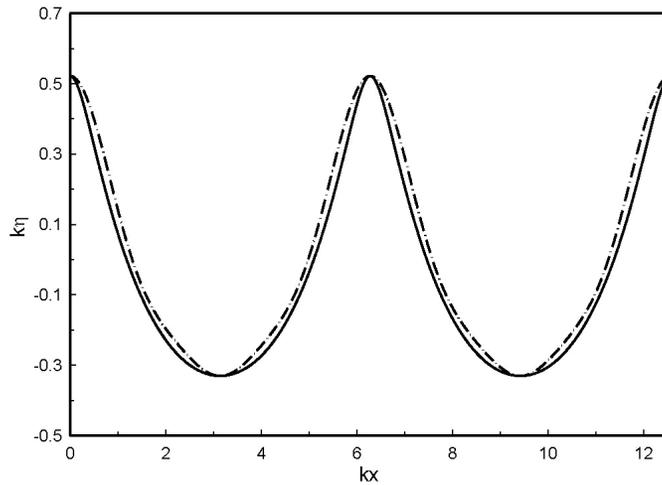


Fig. 5. A comparison on the wave profiles for the Eulerian and Lagrangian solutions both to a third-order under different current conditions (a) $F_r = 0.3$, (b) $F_r = 0.5$, (c) $F_r = -0.2$. Wave conditions $d/L = 0.2$ and $H/L = 0.1$ (Solid line: third-order Lagrangian solution; dash-dotted line: third-order Eulerian solution).



(b)



(c)

Fig. 5. (Continued)

comparison on the wave profiles between Lagrangian and Eulerian solutions, both to a third-order approximation. The results reveal that the height of the wave increases against an opposing current (negative F_r) and decreases on a following current (positive F_r). In Figs. 5(a) and 5(b), the Eulerian wave profiles have anomalous bumps in the trough for the wave conditions tested, which may not be a realistic physical phenomenon for waves of constant form. On the other hand, the Lagrangian wave profiles have sharper crests and broader troughs, as well as exclude any artificial bumps at or near the trough. Clearly the third-order Lagrangian solution is more exact than the Eulerian solution of the same order for describing the shape of the gravity wave. In general, the surface profile is an unknown function in the Eulerian approach, and the boundary conditions at the free surface can only be satisfied in an approximately manner. However, the free surface in the Lagrangian description is represented explicitly by a parametric function for the particles. The advantage of using Lagrangian description is that it allows flexibility for capturing the actual shape and the wave kinematics above mean water level. Thus theoretically, Lagrangian solution can provide better prediction for the wave

profile at large Froude number than the Stokes' expansion to the same order or at higher order (e.g. the fifth order demonstrated in Fig. 4). The wave profiles depicted in Figs. 4 and 5 also show that they are symmetric with respect to the crest line, which were recently proven to hold true for irrotational waves [23, 24].

5. Conclusions

This paper presents the results for the particle trajectories of a nonlinear irrotational water wave train on a uniform current. An explicit expression to a third-order for the Lagrangian parametric solution is obtained by the Euler–Lagrange transformation which was not reported in the literature. This new solution satisfies the original nonlinear equations and the irrotational condition governing the wave motion. In the Lagrangian solution to a second-order, the Lagrangian mean level of a particle orbit over one wave period is found to be higher than that of the Eulerian, and it also has a time-dependent term referred to as the mass transport velocity which are applicable to the entire flow field. The wave frequency associated with water particle motions in Lagrangian form differs from that of the Eulerian, and the former is a function of wave steepness, uniform current speed and the Lagrangian vertical marked label b for each individual particle that can be obtained directly based on the third-order solutions.

Comparison on the wave profiles to a third-order approximation between the Eulerian and Lagrangian descriptions reveals the former has a secondary wave in the trough for large wave in the intermediate to shallow water depth, whereas the latter gives a flatter shape. Therefore, the theoretical solution in Lagrangian form provides not only better but also accurate prediction for the free surface with large Froude number F_r associated with a uniform current than the Eulerian solution to the same order. From the trajectories of water particles resulting from wave-current interaction, it is found that particle displacement near the surface decreases due to its mass transport velocity is resisted by an opposing current. Again in the case with an opposing current, the water particle further beneath the still water level is mainly transported by the opposing current in the direction against the progressive wave, especially with a current in large Froude number F_r . In the cases with a following current, the effect of increasing current speed is generally to increase the magnitude of the time-averaged mass transport velocity since the current is in the same direction as the wave propagation, thus resulting in augmentation to the horizontal distance traveled by a particle. The method we have introduced here can also be implemented to study the problem of particle trajectories within standing or solitary water waves.

Acknowledgments

The authors are indebted to the referee for useful suggestions.

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