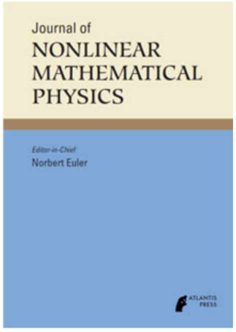

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THE BOGOMOLNY DECOMPOSITION FOR SYSTEMS OF TWO GENERALIZED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER*

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Using a concept of strong necessary conditions we derive the Bogomolny decomposition for systems of two generalized elliptic and parabolic nonlinear partial differential equations (NPDE) of the second order. The generalization means that the equation coefficients depend on the field variables. According to the Cinquini-Cibrario criteria [18–20] the first type is characterized to be an elliptic, whereas the second one is a parabolic system. As a result we derive conditions for existence of the Bogomolny relationships.

Keywords: Nonlinear partial differential equations; variational methods; Bogomolny decomposition.

1. Introduction

An important class of nonlinear partial differential equations (NPDE) can be solved by using relatively simple methods from the calculus of variations. The method basis on the assumption that the considered NPDE

$$\hat{O}[u] = 0 \tag{1.1}$$

is the necessary condition for unknown u to be critical point of a generating functional $\Phi[u]$, where $\hat{O}[\cdot]$ denotes nonlinear partial differential operator. Expressing (1.1) by $\Phi[u]$ we get:

$$\frac{\delta\Phi[u]}{\delta u} = 0. \tag{1.2}$$

Usually it is difficult to solve (1.1) directly, however it may be easier somehow to derive critical points of the functional $\Phi[\cdot]$.

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The two first achievements along this line basing on the calculus of variations has been reported by Rund [1] and Bogomolny [2]. Rund considered a pair of NPDEs:

$$\hat{O}[u] = 0, \quad \hat{O}[v] = 0 \tag{1.3}$$

where u and v are unknown. The generating functional for both equations is of the following form:

$$\Phi[u, v] = \int_X (\mathcal{L}(u, \{u'\}) - \mathcal{L}(v, \{v'\}))\, dx, \tag{1.4}$$

where X is bounded, simply-connected region in the domain $\mathbb{R}^m$ of the independent variables x_α, $(\alpha = 1, \dots, m)$, dx represents m-dimensional volume element of X, u and v are smooth enough functions satisfying the requisite boundary conditions on ∂X, $\{u'\}$ is the set of partial derivatives of u of the orders required for derivation of (1.3). Main point of this approach consists in deriving such relations between $(u, \{u'\})$ and $(v, \{v'\})$ that the following difference $(\mathcal{L}(u, \{u'\}) - \mathcal{L}(v, \{v'\}))$ becomes a divergence, where $\mathcal{L}(\cdot, \{\cdot\}) \in \mathcal{C}^2$. The found relations occur to be the variational Bäcklund transformations.

The Bogomolny approach consists in splitting the generating functional of the considered equations

$$\hat{O}_1[u, v] = 0, \quad \hat{O}_2[u, v] = 0 \tag{1.5}$$

into the positive determined functional $\Phi_+[u, v]$ and the topological invariant $I[u, v]$:

$$\Phi[u, v] = \Phi_+[u, v] + I[u, v], \tag{1.6}$$

where

$$\Phi_+[u, v] = \int_X F(u, \{u'\}, v, \{v'\})^2\, dx \geq 0, \quad I[u, v] = \int_X \mathcal{L}_0(u, \{u'\}, v, \{v'\})\, dx. \tag{1.7}$$

$I[u, v]$ only depends upon the boundary conditions on ∂X and therefore

$$\frac{\delta I[u, v]}{\delta u} \equiv 0, \quad \frac{\delta I[u, v]}{\delta v} \equiv 0, \tag{1.8}$$

i.e. it is constant with respect to a local variations of u and v. Since $\Phi_+[u, v]$ is positive determined, one can write down the following bound for $\Phi[u, v]$:

$$\Phi[u, v] \geq I[u, v] \tag{1.9}$$

and the necessary condition for its minimum:

$$F(u, \{u'\}, v, \{v'\}) = 0. \tag{1.10}$$

The order of (1.10) is less then the order of the considered equations (1.5). Moreover, in certain cases this equation splits into two simpler selves consistent equations. During the three last decades more advanced methods for obtaining the Bogomolny relationships (1.9) and (1.10) have been developed [3–10]. In 1979 the first attempt to combine both the Rund and the Bogomolny methods has been done [11]. During the two last decades these results have been improved and generalized [12–17]. The developed formalism named the strong necessary conditions method (SNCM), uniquely treats the Rund and the Bogomolny approaches. In the light of this formalism the Bäcklund transformations and the Bogomolny relationships are particular examples of the dual equations resulting from SNCM. Moreover it has been shown that set of the Bogomolny equations (1.10) becomes the Bäcklund transformation when the coupled system (1.5) splits into a set of independent equations [15, 17].

However, the most of cited papers concern semi-linear partial differential equations. In this paper we extend SNCM into the quasilinear partial differential equations.

We consider so called generalized systems of nonlinear partial differential equations with coefficients dependent on unknown functions [16].

The paper is organized as follows: in Sec. 2 we derive the conditions of the existence of the Bogomolny decomposition for the two types of NPDE's characterized by the property of the highest degree operator: elliptic and parabolic. In Sec. 3 we illustrate the application of obtained results on the two-dimensional static Heisenberg model.

2. The Bogomolny Decomposition for Generalized Systems of NPDE's of the Second Order

We assume that the considered system of NPDE's forms the necessary conditions for an extremum of a functional to exist. Since we relate the developed method to the classical solitons theory we assume the structure of the Euclidean space for a domain of the independent variables. Both variables x and t are equivalent. Therefore we determine the following asymptotic conditions for the dependent variables:

$$\lim_{x\to\pm\infty} u(x,t) = \lim_{t\to\pm\infty} u(x,t) = \alpha, \tag{2.1}$$

$$\lim_{x\to\pm\infty} v(x,t) = \lim_{t\to\pm\infty} v(x,t) = \beta, \tag{2.2}$$

where α and β arbitrary constants.

For the classification of NPDE's system we apply characteristics according to Cinquini-Cibrario [18–20].

2.1. *Generalized elliptic systems of NPDE of the second order*

Let us take into account the following generating functional:

$$\Phi[u,v] = \int_{E^2} \mathcal{L}\, dx\, dt \tag{2.3}$$

where

$$\mathcal{L} = \frac{1}{2}[A_1(u,v)u_{,x}u_{,t} - A_2(u,v)v_{,x}v_{,t}] - V(u,v), \tag{2.4}$$

and where

$$u(x,t) \in \mathcal{C}^2(\mathbb{R}\times\mathbb{R}), \quad v(x,t) \in \mathcal{C}^2(\mathbb{R}\times\mathbb{R}), \tag{2.5}$$

$$V(u,v) \in L^1(\mathbb{R}\times\mathbb{R}) \bigwedge V(u,v) \in \mathcal{C}^1(\mathbb{R}\times\mathbb{R}), \tag{2.6}$$

$$A_i(u,v) \in \mathcal{C}^1(\mathbb{R}\times\mathbb{R}), \quad i = 1,2. \tag{2.7}$$

A system of NPDE's being the necessary conditions for an extremum of (2.3) to exist (the Euler–Lagrange equations) is of the following form:

$$\frac{1}{2}(A_{1,u}u_{,x}u_{,t} + A_{1,v}(u_{,x}v_{,t} + u_{,t}v_{,x}) + A_{2,u}v_{,x}v_{,t}) + A_1 u_{,xt} = -V_{,u} \tag{2.8}$$

$$\frac{1}{2}(A_{1,v}u_{,x}u_{,t} + A_{2,u}(u_{,x}v_{,t} + u_{,t}v_{,x}) + A_{2,v}v_{,x}v_{,t}) + A_2 v_{,xt} = V_{,v}. \tag{2.9}$$

We are interested in the system of (2.8) and (2.9). We will derive conditions for the coefficients $A_{1,2}(u,v)$ and the potential $V(u,v)$ for which (2.8), (2.9) can be solved by the Bogomolny decomposition. In further considerations we follow the procedure described in [12–14, 17]. In the concept of the strong necessary conditions the topological invariants I_i, defined on the same domain as $\Phi[u,v]$ play a crucial role. The following set of topological invariants:

$$
\begin{aligned}
I_1 &= \int_{E^2} G_1(u,v)(u_{,x}v_{,t} - u_{,t}v_{,x})dx\,dt \\
I_2 &= \int_{E^2} D_x G_2(u,v)dx\,dt \\
I_3 &= \int_{E^2} D_t G_3(u,v)dx\,dt
\end{aligned}
\tag{2.10}
$$

is complete for the set (2.8), (2.9). For details we refer to [17]. Using (2.10) we perform gauge transformation on the functional (2.3):

$$\widetilde{\mathcal{L}} = \mathcal{L} + I_1 + I_2 + I_2 \tag{2.11}$$

and taking into account (2.4) we obtain:

$$
\begin{aligned}
\widetilde{\Phi}[u,v] = \int_{E^2}\bigg(&\frac{1}{2}(A_1(u,v)u_{,x}u_{,t} - A_2(u,v)v_{,x}v_{,t}) - V(u,v) \\
&+ G_1(u,v)(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_x G_2(u,v) + D_t G_3(u,v)\bigg)\,dx\,dt.
\end{aligned}
\tag{2.12}
$$

The strong necessary conditions for (2.11) are:

$$\widetilde{\mathcal{L}}_{,u} = 0, \quad \widetilde{\mathcal{L}}_{,v} = 0, \tag{2.13}$$

$$\widetilde{\mathcal{L}}_{,u_{,t}} = 0, \quad \widetilde{\mathcal{L}}_{,u_{,x}} = 0, \tag{2.14}$$

$$\widetilde{\mathcal{L}}_{,v_{,t}} = 0, \quad \widetilde{\mathcal{L}}_{,v_{,x}} = 0. \tag{2.15}$$

Taking into account the form of (2.12) we get:

$$\frac{1}{2}(A_{1,u}u_{,x}u_{,t} - A_{2,u}v_{,x}v_{,t}) - V_{,u} + G_{1,u}(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_x G_{2,u} + D_t G_{3,u} = 0, \tag{2.16}$$

$$\frac{1}{2}(A_{1,v}u_{,x}u_{,t} - A_{2,v}v_{,x}v_{,t}) - V_{,v} + G_{1,v}(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_x G_{2,v} + D_t G_{3,v} = 0, \tag{2.17}$$

$$\frac{1}{2}A_1 u_{,t} + G_1 v_{,t} + G_{2,u} = 0, \tag{2.18}$$

$$\frac{1}{2}A_1 u_{,x} - G_1 v_{,x} + G_{3,u} = 0, \tag{2.19}$$

$$-\frac{1}{2}A_2 v_{,t} - G_1 u_{,t} + G_{2,v} = 0, \tag{2.20}$$

$$-\frac{1}{2}A_2 v_{,x} + G_1 u_{,x} + G_{3,v} = 0. \tag{2.21}$$

The Eqs. (2.16)–(2.21) have to be self-consistent. Formally, we have six simultaneous equations for the five unknown functions: u, v, G_1, G_2, G_3. The set (2.16)–(2.21) becomes the Bogomolny transformations if there exists such an Ansatz for the G_1, G_2, G_3 for which the above equations reduce themselves to two equations for the u and v [12–14, 17]. In this procedure the reduction of the number of independent equations which can be achieved by an appropriate choice of G_1, G_2, G_3 plays an essential role. Only for very special relation between $A_i(u,v)$ and $V(u,v)$ such an Ansatz exists [17]. In the most cases of $A_i(u,v)$ and $V(u,v)$ the system (2.16)–(2.21) cannot be reduced to the Bogomolny equations. The (2.16), (2.17) and (2.18)–(2.21) require different ways of treatment. In order to make all (2.16), (2.17) and (2.18)–(2.21) self-consistent we make (2.18)–(2.21) linearly dependent and then we reduce (2.16), (2.17) to a tautology. The linear dependence of (2.18)–(2.21) can be of different ranks. At this stage of development of the strong necessary conditions's we cannot determine a criterion for the rank reduction. Therefore we scan the following set of possibilities: rank $= 1, 2, 3, 4$. The rank $= 1$ creates too strong constrains ($A_1 = A_2 = 0$) whereas rank $= 3$ generates to many constrains which are in contradiction. The rank $= 4$ is trivial and be unable to satisfy (2.16), (2.17) by reducing them to a tautology. Only the system (2.18)–(2.21) of the rank $= 2$ leads to the solution of the considered problem:

$$\mathrm{rank}\begin{bmatrix} 0 & 1/2A_1 & 0 & G_1 & -G_{2u} \\ 1/2A_1 & 0 & -G_1 & 0 & -G_{3u} \\ 0 & -G_1 & 0 & -1/2A_2 & -G_{2v} \\ G_1 & 0 & -1/2A_2 & 0 & -G_{3v} \end{bmatrix} = 2. \tag{2.22}$$

From (2.22) we derive the following conditions:

$$-4{G_1}^2 + A_1A_2 = 0, \tag{2.23}$$

$$-2G_{3u}G_1 + A_1G_{3v} = 0, \tag{2.24}$$

$$2G_{2u}G_1 + A_1G_{2v} = 0, \tag{2.25}$$

where $G_1 \neq 0, A_1A_2 > 0$. The system of conditions (2.23)–(2.25) leads immediately to the equations for $G_2(u, v)$ and $G_3(u, v)$:

$$\begin{aligned} -G_{3u}\sqrt{A_1A_2} + G_{3v}A_1 &= 0, \\ G_{2u}\sqrt{A_1A_2} + G_{2v}A_1 &= 0. \end{aligned} \tag{2.26}$$

Satisfying the condition (2.22) we reduce the subsystem (2.18)–(2.21) to the two independent equations which will play the role of the sought Bogomolny decomposition. The reduced subsystem (2.18)–(2.19) becomes the Bogomolny equations under the condition which makes (2.16), (2.17) a tautology. Therefore, the strong necessary conditions are reduced to (2.18)–(2.19). In order to satisfy (2.16), (2.17) to be a tautology we apply the following procedure:

(1) Solve (2.18)–(2.19) with respect to $u_{,x}$ and $u_{,t}$,

$$\begin{aligned} u_{,x} &= 2\frac{G_1v_{,x} - G_{3,u}}{A_1} \\ u_{,t} &= -2\frac{G_1v_{,t} + G_{2,u}}{A_1}. \end{aligned} \tag{2.27}$$

(2) Substitute the solutions for $u_{,x}$ and $u_{,t}$ into (2.16), (2.17).
(3) Collect coefficients of all powers of $v_{,x}$ and $v_{,t}$ (including mixed terms).
(4) Put zero for all coefficient derived in item (3) and by this way obtain additional constrains for G_1, G_2, G_3, A_1, A_2 and V.
(5) Eliminate as much as possible variables by applying (2.23)–(2.25).
(6) Some of relations derived in item (4) occur to be a tautology, whereas the remaining ones constitute supplementary constrains:

$$2\frac{A_{1,u}G_{3,u}G_{2,u}}{A_1^2} - V_{,u} - 2\frac{G_{2,uu}G_{3,u}}{A_1} - 2\frac{G_{3,uu}G_{2,u}}{A_1} = 0 \tag{2.28}$$

$$2\frac{A_{1,v}G_{3,u}G_{2,u}}{A_1^2} - V_{,v} - 2\frac{G_{2,uv}G_{3,u}}{A_1} - 2\frac{G_{3,uv}G_{2,u}}{A_1} = 0. \tag{2.29}$$

(7) Satisfying both (2.23)–(2.25) and the supplementary constrains (2.28), (2.29) one can accept (2.18)–(2.19) to be the Bogomolny equations.

In the considered case it is possible to integrate (2.28), (2.29) into the following compact form:

$$V(u,v) = -2\frac{G_{3,u}(u,v)G_{2,u}(u,v)}{A_1(u,v)} + C, \tag{2.30}$$

where C is an arbitrary constant.

In order to present a working example we assume the following forms for the equation coefficients:

$$\sqrt{A_1 A_2} = f(u+v) \tag{2.31}$$

$$A_1 = g(u+v). \tag{2.32}$$

Substituting (2.31) and (2.32) to (2.26) we derive the following solutions for G_2 and G_3:

$$G_2(u,v) = F_2\left(u - \int^{u+v} \frac{f(a)}{f(a)+g(a)}da\right) \tag{2.33}$$

$$G_3(u,v) = F_3\left(u - \int^{u+v} \frac{f(a)}{f(a)-g(a)}da\right) \tag{2.34}$$

where F_2 and F_3 are arbitrary functions of $\mathcal{C}^2$. The (2.30), (2.33) and (2.34) constitute the explicit relation between $V(u,v), A_1(u,v)$ and $A_2(u,v)$ which is a sufficient condition for the Bogomolny decomposition to exist. Note, that due to the assumptions (2.31) and (2.32) the obtained relation is particular. In a general case the corresponding conditions for G_2 and G_3 are established implicitly by (2.26) and (2.30).

2.2. *Generalized parabolic systems of NPDE of the second order*

We search for the conditions for an existence of the Bogomolny equations for the following systems:

$$\frac{1}{2}(A_{1,u}u_{,x}^2 + A_{2,u}u_{,t}^2 - A_{3,u}v_{,x}^2 - A_{4,u}v_{,t}^2) + A_{1,v}u_{,x}v_{,x} + A_{2,v}u_{,t}v_{,t} + A_1 u_{,xx} + A_2 u_{,tt} = -V_{,u} \tag{2.35}$$

$$-\frac{1}{2}(A_{1,v}u_{,x}^2 + A_{2,v}u_{,t}^2 - A_{3,v}v_{,x}^2 - A_{4,v}v_{,t}^2) + A_{3,u}u_{,x}v_{,x} + A_{4,u}u_{,t}v_{,t} + A_3 v_{,xx} + A_4 v_{,tt} = -V_{,v}. \tag{2.36}$$

The set (2.35), (2.36) is generated by the following functional:

$$\Phi[u,v] = \int_{E^2}\left(\frac{1}{2}(A_1(u,v)u_{,x}^2 + A_2(u,v)u_{,t}^2 + A_3(u,v)v_{,x}^2 + A_4(u,v)v_{,t}^2) - V(u,v)\right) dx\,dt, \tag{2.37}$$

where (2.1), (2.2), (2.5) and (2.6) still hold as well as

$$A_i(u,v) \in \mathcal{C}^1(\mathbb{R}\times\mathbb{R}), \quad i = 1,2,3,4. \tag{2.38}$$

After the transformation: $\widetilde{\Phi} = \Phi + \int_{E^2}(G_1(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_xG_2 + D_tG_3)dx\,dt$ we apply the concept of strong necessary conditions and we derive:

$$\frac{1}{2}(A_{1,u}u_{,x}^2 + A_{2,u}u_{,t}^2 + A_{3,u}v_{,x}^2 + A_{4,u}v_{,t}^2) - V_{,u} + G_{1,u}(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_xG_{2,u} + D_tG_{3,u} = 0 \tag{2.39}$$

$$\frac{1}{2}(A_{1,v}u_{,x}^2 + A_{2,v}u_{,t}^2 + A_{3,v}v_{,x}^2 + A_{4,v}v_{,t}^2) - V_{,v} + G_{1,v}(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_xG_{2,v} + D_tG_{3,v} = 0 \tag{2.40}$$

$$A_1u_{,x} + G_1v_{,t} + G_{2,u} = 0 \tag{2.41}$$

$$A_2u_{,t} - G_1v_{,x} + G_{3,u} = 0 \tag{2.42}$$

$$A_3v_{,x} - G_1u_{,t} + G_{2,v} = 0 \tag{2.43}$$

$$A_4v_{,t} + G_1u_{,x} + G_{3,v} = 0. \tag{2.44}$$

In order to derive the Bogomolny decomposition from (2.39)–(2.44) we repeat the procedure described in Subsec. 2.1. A condition for reduction of (2.41)–(2.44) from 4 to 2 independent equations reads:

$$\operatorname{rank}\begin{bmatrix} A_1 & 0 & 0 & G_1 & G_{2,u} \\ 0 & A_2 & -G_1 & 0 & G_{3,u} \\ 0 & -G_1 & A_3 & 0 & G_{2,v} \\ G_1 & 0 & 0 & A_4 & G_{3,v} \end{bmatrix} = 2. \tag{2.45}$$

The following constrains for $A_i(u,v)$ and $G_i(u,v)$ result from (2.45):

$$A_1 \neq 0, \quad A_2 \neq 0, \qquad -A_3A_2 + {G_1}^2 = 0, \quad A_4A_1 - {G_1}^2 = 0 \tag{2.46}$$

$$G_{2,v}A_2 + G_1G_{3,u} = 0, \quad -G_{3,v}A_1 + G_1G_{2,u} = 0. \tag{2.47}$$

It results from (2.46) that the following constraint for $A_i(u,v)$ is required:

$$A_3A_2 = A_4A_1 > 0. \tag{2.48}$$

The next steps of the applied procedure assume that (2.39) and (2.40) reduce to a tautology. Taking into account (2.41) and (2.42) as well as (2.46) and (2.47) we derive a final condition for V and A_i:

$$\frac{{G_{3,u}}^2}{A_2} + \frac{{G_{2,u}}^2}{A_1} + 2V = \text{const.} \tag{2.49}$$

However, the presence of G_2 and G_3 in (2.49) does not relate V and A_i directly. Therefore, the condition can be obtained by solving (2.46) and (2.47) with respect to G_2 and G_3.

3. The Heisenberg Ferromagnet

Applying the results of Subsec. 2.2 we consider a σ^2 model with an arbitrary potential energy $V(u,v)$. The model is equivalent to the classical Heisenberg model in 2-dimensions with an extra interaction term $V(u,v)$, where u and v are stereographic coordinates. This model for $V(u,v) = 0$ has been solved by applying the Bogomolny decomposition [21,22]. Here we derive the general form of $V(u,v)$

which conserves integrability of the 2-d Heisenberg model in the sense of the Bogomolny decomposition. The Hamiltonian of the considered σ^2 model is defined in the two-dimensional Euclidean space and takes the following form:

$$H = \int_{E^2} \left(\frac{1}{4} S^\alpha_{,i} S^\alpha_{,i} + \tilde{V}(S^{(1)}, S^{(2)}, S^{(3)}) \right) d^2x, \tag{3.1}$$

where $\alpha = 1, 2, 3$ and $i = 1, 2$. $S^\alpha_{,i}$ means the derivative of the α component with respect to the i Euclidean coordinate. The classical spin field is normalized to the unity $(S^{(1)})^2+(S^{(2)})^2+(S^{(3)})^2 = 1$. Using a stereographic projection from the unit sphere in spin field space to the complex plane one can project the classical spin field down to a single complex field given by [21]:

$$u + i\,v = \frac{S^{(1)} + iS^{(2)}}{1 + S^{(3)}}. \tag{3.2}$$

Expressing (3.1) by (3.2) we get the following form for H:

$$H = \int_{E^2} \left(\frac{u^2_{,x} + u^2_{,y} + v^2_{,x} + v^2_{,y}}{(1 + u^2 + v^2)^2} + V(u, v) \right) d^2x, \tag{3.3}$$

where $V(u, v) = \tilde{V}(S^{(1)}, S^{(2)}, S^{(3)})$. We study equilibrium configurations in a model represented by the static energy functional (3.3). To obtain the field equations we apply the necessary condition for an extremum of (3.3) to exists. It leads to the Euler–Lagrange equations:

$$\frac{4u[(\nabla u)^2 - (\nabla v)^2] + 8(v\nabla u \cdot \nabla v)}{(1 + u^2 + v^2)^3} - \frac{2\triangle u}{(1 + u^2 + v^2)^2} + V(u, v)_{,u} = 0 \tag{3.4}$$

$$\frac{4v[(\nabla v)^2 - (\nabla u)^2] + 8(u\nabla v \cdot \nabla u)}{(1 + u^2 + v^2)^3} - \frac{2\triangle v}{(1 + u^2 + v^2)^2} + V(u, v)_{,v} = 0. \tag{3.5}$$

The Hamiltonian (3.3) is a particular case of (2.37), where $A_1 = A_2 = A_3 = A_4 = 2(1+u^2+v^2)^{-2}$. Therefore, in order to derive a sufficient condition for the integrability of (3.4) and (3.5) in the meaning of the Bogomolny decomposition we directly apply the results from Subsec. 2.2. The (2.48) condition is satisfied automatically. From (2.46) we find that $G_1 = \pm A_1$. This result enables a solution of (2.47) in the following forms $G_2 = \Re G(u + iv)$ and $G_3 = \Im G(u + iv)$ where $G(u + iv)$ is an arbitrary analytic function. Thus, the sufficient condition takes the following form:

$$V(u, v) = \frac{1}{4} G_{,u} G^*_{,u} (1 + u^2 + v^2)^2. \tag{3.6}$$

Therefore, there exists a denumerable number of integrable models defined by (3.3). Let us consider the simplest nontrivial case: $G(u + iv) = V_0^{1/2}(u + iv)$, where V_0 is positive determined constants. Then G_2, G_3 and (3.6) reduce to the following forms: $G_2 = V_0^{1/2} u, G_3 = V_0^{1/2} v$ and

$$V(u, v) = \frac{1}{4} V_0 (1 + u^2 + v^2)^2. \tag{3.7}$$

Finally we obtain the Bogomolny equations:

$$u_{,x} + v_{,y} + \frac{1}{2} V_0^{\frac{1}{2}} (1 + u^2 + v^2)^2 = 0, \quad u_{,y} - v_{,x} = 0. \tag{3.8}$$

We find (3.8) very convenient for numerical calculations. However, here we give only proof of the following theorem:

Theorem 1. *Each solution of the Bogomolny equations* (3.8) *satisfies the Euler–Lagrange equations* (3.4), (3.5).

Proof. Expressing the second derivatives and some combinations of the first derivatives in (3.4), (3.5) by the function W appearing in (3.8):

$$W = \frac{1}{2} V_0^{\frac{1}{2}} (1 + u^2 + v^2)^2 \tag{3.9}$$

and by the potential (3.7) we will reduce (3.4) and (3.5) to tautologies. From (3.8) we derive the following relations:

$$\begin{aligned} u[(\nabla u)^2 - (\nabla v)^2] &= uW^2 - 2uv_{,y}W \\ v[(\nabla v)^2 - (\nabla u)^2] &= vW^2 - 2vu_{,x}W, \end{aligned} \tag{3.10}$$

$$\triangle u = W_{,x}, \quad \triangle v = W_{,y}. \tag{3.11}$$

Inserting (3.7), (3.10), (3.11) and (3.9) to (3.4), (3.5) we get the tautologies. □

4. Conclusions

We have shown that due to the strong necessary condition concept the derivation of the Bogomolny relationships for the quasi-linear systems of the second order can be reduced to an algorithm formulated by the following steps:

(1) Construct the Φ functional (2.3) that the Lagrange–Euler equations are equivalent to the considered set of NPDE's (2.8), (2.9).
(2) Derive the strong necessary conditions: (a) so called the first conditions (2.16), (2.17), (b) so called the second conditions (2.18)–(2.21).
(3) Reduce the number of the second conditions: (a) determine an appropriate rank of the reduced conditions (2.22), (b) transforming (2.45) to the Gauss–Jordan matrix form derive the conditions for G_1, G_2, G_3 which reduce the number of the independent second conditions to $n =$ rank.
(4) Reduce the first strong conditions to a tautology by the procedure described in Subsec. 2.1 formulated by the $1 - 7$ items. The algorithm described here can be extended into other classes of sets of NPDE's by an extension of the strong necessary conditions into the semi-strong necessary conditions [12–14, 17].

Derived here algorithm is easy to implement in symbolic calculations (Maple, Mathematica).

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