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THE ADJOINT REPRESENTATION OF QUANTUM ALGEBRA $U_q(\mathfrak{sl}(2))$

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Starting from any representation of the Lie algebra \mathfrak{g} on the finite dimensional vector space V we can construct the representation on the space $\text{Aut}(V)$. These representations are of the type of ad . That is one of the reasons, why it is important to study the adjoint representation of the Lie algebra \mathfrak{g} on the universal enveloping algebra $U(\mathfrak{g})$. A similar situation is for the quantum groups $U_q(\mathfrak{g})$. In this paper, we study the adjoint representation for the simplest quantum algebra $U_q(\mathfrak{sl}(2))$ in the case that q is not a root of unity.

Keywords: Quantum algebra; enveloping algebra; adjoint representation.

1. Introduction

It is a well-known fact that the adjoint representation of the algebra $\mathfrak{sl}(2)$ on its enveloping algebra $U(\mathfrak{sl}(2))$ is fully reducible and can be decomposed into a direct sum of finite dimensional representations, [1]. Specifically, let the generators e , f and h fulfil the commutation relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Then the center $U(\mathfrak{sl}(2))$ is generated by an element

$$C = ef + fe + \frac{1}{2}h^2,$$

and for any $n, k \in \mathbb{N}_0$, $\mathbb{N}_0 = 0, 1, 2, \dots$, the vector spaces

$$\mathcal{V}_{(n,k)} = \text{ad}_{U(\mathfrak{sl}(2))}(e^n C^k)$$

are invariant with respect to the adjoint representation and $(2n+1)$ -dimensional. Moreover, we can write

$$U(\mathfrak{sl}(2)) = \bigoplus_{n,k=0}^{\infty} \mathcal{V}_{(n,k)}. \tag{1.1}$$

A similar situation is for complex semisimple Lie algebras, see e.g. [1–4].

On the other hand, in the case of the quantum group $U_q(\mathfrak{sl}(2))$ there exists an element a for which the space $\text{ad}_{U_q(\mathfrak{sl}(2))}a$ is infinite dimensional. The aim of this paper is to find the decomposition of the quantum group $U_q(\mathfrak{sl}(2))$, where q is not a root of unity, which is similar to the decomposition (1.1).

2. The Quantum Group $U_q(\mathfrak{sl}(2))$

Let us have the quantum group $U_q(\mathfrak{sl}(2))$, [5], which is generated by the elements E, F, K and K^{-1} fulfilling the relations

$$\begin{aligned} KE &= q^2EK, & KF &= q^{-2}FK, \\ EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}, & KK^{-1} &= K^{-1}K = 1. \end{aligned} \quad (2.1)$$

The coproduct Δ , the antipod S and the counit ϵ are defined by

$$\begin{aligned} \Delta E &= E \otimes K + 1 \otimes E, & S(E) &= -EK^{-1}, & \epsilon(E) &= 0, \\ \Delta F &= F \otimes 1 + K^{-1} \otimes F, & S(F) &= -KF, & \epsilon(F) &= 0, \\ \Delta K &= K \otimes K, & S(K) &= K^{-1}, & \epsilon(K) &= 1, \\ \Delta K^{-1} &= K^{-1} \otimes K^{-1}, & S(K^{-1}) &= K, & \epsilon(K^{-1}) &= 1. \end{aligned}$$

It is a well-known fact that the basis in $U_q(\mathfrak{sl}(2))$ consists of the elements $E^{n_1}F^{n_2}K^{n_3}$, where $n_1, n_2 \in \mathbb{N}_0$ and $n_3 \in \mathbb{Z}$.

The adjoint action of the Drinfeld–Jimbo algebras is given by the formula (see e.g. [6])

$$\text{ad}_a b = \sum a_{(1)} b S(a_{(2)}).$$

In particular, in the case of the algebra $U_q(\mathfrak{sl}(2))$ we have

$$\begin{aligned} \text{ad}_E a &= EaK^{-1} - aEK^{-1}, & \text{ad}_F a &= Fa - K^{-1}aKF, \\ \text{ad}_K a &= KaK^{-1}, & \text{ad}_{K^{-1}} a &= K^{-1}aK. \end{aligned} \quad (2.2)$$

This shows that $\text{ad}_E E = \text{ad}_F(KF) = 0$ holds. We denote

$$\begin{aligned} X &= E, & Y &= KF, \\ Z &= \text{ad}_E Y = q^{-2}EF - FE = \frac{K - K^{-1}}{q - q^{-1}} - q^{-1}(q - q^{-1})EF, \end{aligned} \quad (2.3)$$

and take the element

$$C = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2}, \quad (2.4)$$

which generates the center of $U_q(\mathfrak{sl}(2))$.

From Eqs. (2.3) and (2.4) we obtain

$$K = \frac{q - q^{-1}}{q + q^{-1}} (qZ + (q - q^{-1})C). \quad (2.5)$$

Notice that the element K^{-1} cannot be expressed by these equations.

3. The New Basis in $U_q(\mathfrak{sl}(2))$

The next step is to express $U_q(\mathfrak{sl}(2))$ using the elements X, Y, Z, C and $W = K^{-1}$. From relations (2.3), (2.4), (2.5) and the commutation relations (2.1) we obtain

$$\begin{aligned}
 CX &= CX, & CY &= YC, & CZ &= ZC, & CW &= WC, \\
 WX &= q^{-2}XW, & WY &= q^2YW, & WZ &= ZW, \\
 ZX &= q^2XZ + (q - q^{-1})^2XC, & YZ &= q^2ZY + (q - q^{-1})^2YC, \\
 YX - XY &+ \frac{q - q^{-1}}{q + q^{-1}}(qZ + (q - q^{-1})C)Z = 0, \\
 (qZ + (q - q^{-1})C)W - \frac{q + q^{-1}}{q - q^{-1}} &= 0, \\
 (qZ + (q - q^{-1})C)(q^{-1}Z - (q - q^{-1})C) &+ q(q + q^{-1})^2XY + \left(\frac{q + q^{-1}}{q - q^{-1}}\right)^2 = 0.
 \end{aligned} \tag{3.1}$$

The original elements E, F, K and K^{-1} in $U_q(\mathfrak{sl}(2))$ can be written in the following way:

$$E = X, \quad F = WY, \quad K = \frac{q - q^{-1}}{q + q^{-1}}(qZ + (q - q^{-1})C), \quad K^{-1} = W.$$

If we denote by \mathcal{T} a free algebra generated by X, Y, Z, C and W , and take the two-sided ideal \mathcal{J} in \mathcal{T} generated by relations (3.1), the algebra $U_q(\mathfrak{sl}(2))$ is equal to \mathcal{T}/\mathcal{J} .

It is easy to show the following analogy of the Poincaré–Birkhoff–Witt theorem:

Lemma. *The elements $X^{n_1}Y^{n_2}Z^{n_3}C^{n_4}W^{n_5}$, where $n_k \geq 0$, $n_1n_2 = 0$ and $n_3n_5 = 0$, form the basis in $U_q(\mathfrak{sl}(2))$.*

4. The Adjoint Representation in the New Basis

The adjoint representation (2.2) can be rewritten in the form

$$\begin{aligned}
 \text{ad}_E a &= XaW - aXW, & \text{ad}_F a &= WYa - WaY, \\
 \text{ad}_K a &= \frac{q - q^{-1}}{q + q^{-1}}(qZ + (q - q^{-1})C)aW, \\
 \text{ad}_{K^{-1}} a &= \frac{q - q^{-1}}{q + q^{-1}}Wa(qZ + (q - q^{-1})C).
 \end{aligned} \tag{4.1}$$

4.1. The finite dimensional representations

From relations (3.1) and (4.1) it is possible by direct calculations, to show that the vector space \mathcal{V}_N with the basis $X^{n_1}Y^{n_2}Z^{n_3}C^{n_4}$, where $n_1n_2 = 0$ and $n_1 + n_2 + n_3 + n_4 \leq N$, is invariant with respect to the adjoint representation for any N . The dimension of this space is

$$\dim \mathcal{V}_N = \frac{1}{6}(N + 1)(N + 2)(2N + 3).$$

In the space \mathcal{V}_N there are the highest weight elements $X^{n_1}C^{n_4}$, $n_1 + n_4 \leq N$. It is well known, see e.g. [5, p. 61], that the invariant subspace

$$\mathcal{V}_{(n_1, n_4)} = \text{ad}_{U_q(\mathfrak{sl}(2))}(X^{n_1}C^{n_4})$$

has the dimension $2n_1 + 1$ and is irreducible. Since

$$\sum_{n_1 + n_4 \leq N} (2n_1 + 1) = \frac{1}{6}(N + 1)(N + 2)(2N + 3) = \dim \mathcal{V}_N,$$

the subspace \mathcal{V}_{fin} with the basis $X^{n_1}Y^{n_2}Z^{n_3}C^{n_4}$, $n_1n_2 = 0$, can be decomposed into a direct sum of invariant subspaces

$$\mathcal{V}_{\text{fin}} = \bigoplus_{n_1, n_4} \mathcal{V}_{(n_1, n_4)}.$$

The eigenvalue of ad_C on the space $\mathcal{V}_{(n_1, n_4)}$ is determined by the relation

$$\text{ad}_C(X^{n_1}C^{n_4}) = \frac{q^{2n_1+1} + q^{-2n_1-1}}{(q - q^{-1})^2} X^{n_1}C^{n_4}.$$

4.2. The infinite dimensional representations

By direct calculations we obtain the action of the adjoint representation on the elements of the basis $X^{n_1}C^{n_4}W^{n_5}$ and $Y^{n_2}C^{n_4}W^{n_5}$ for $n_5 \geq 1$ of the following form (in the formulas where X^{n_1-1} resp. Y^{n_2-1} occurs we suppose that $n_1 > 0$ resp. $n_2 > 0$):

$$\begin{aligned} \text{ad}_E(X^{n_1}C^{n_4}W^{n_5}) &= (1 - q^{-2n_5})X^{n_1+1}C^{n_4}W^{n_5+1}, \\ \text{ad}_F(X^{n_1}C^{n_4}W^{n_5}) &= \frac{q^{-2n_1+n_5+1}[n_5]}{q - q^{-1}} X^{n_1-1}C^{n_4}W^{n_5+1} \\ &\quad + (1 - q^{-2n_1+2n_5})X^{n_1-1}C^{n_4+1}W^{n_5} \\ &\quad - \frac{q^{n_5-1}[2n_1 - n_5]}{q - q^{-1}} X^{n_1-1}C^{n_4}W^{n_5-1}, \\ \text{ad}_K(X^{n_1}C^{n_4}W^{n_5}) &= q^{2n_1}X^{n_1}C^{n_4}W^{n_5}, \\ \text{ad}_{K^{-1}}(X^{n_1}C^{n_4}W^{n_5}) &= q^{-2n_1}X^{n_1}C^{n_4}W^{n_5}, \\ \text{ad}_E(Y^{n_2}C^{n_4}W^{n_5}) &= -\frac{q^{-n_5-1}[n_5]}{q - q^{-1}} Y^{n_2-1}C^{n_4}W^{n_5+1} \\ &\quad + (q^{-2n_2} - q^{-2n_5})Y^{n_2-1}C^{n_4+1}W^{n_5} \\ &\quad + \frac{q^{-2n_2-n_5+1}[2n_2 - n_5]}{q - q^{-1}} Y^{n_2-1}C^{n_4}W^{n_5-1}, \\ \text{ad}_F(Y^{n_2}C^{n_4}W^{n_5}) &= q^{2n_2+2}(1 - q^{2n_5})Y^{n_2+1}C^{n_4}W^{n_5+1}, \\ \text{ad}_K(Y^{n_2}C^{n_4}W^{n_5}) &= q^{-2n_2}Y^{n_2}C^{n_4}W^{n_5}, \\ \text{ad}_{K^{-1}}(Y^{n_2}C^{n_4}W^{n_5}) &= q^{2n_2}Y^{n_2}C^{n_4}W^{n_5}. \end{aligned}$$

By using these relations we obtain

$$\begin{aligned} \text{ad}_C(X^{n_1}C^{n_4}W^{n_5}) &= \frac{q^{2n_1-2n_5+1} + q^{-2n_1+2n_5-1}}{(q - q^{-1})^2} X^{n_1}C^{n_4}W^{n_5} \\ &\quad + q^{-2n_1}[n_5][n_5 + 1]X^{n_1}C^{n_4}W^{n_5+2} \\ &\quad + q^{-n_1}[n_5][n_1 - n_5](q - q^{-1})^2 X^{n_1}C^{n_4+1}W^{n_5+1}, \\ \text{ad}_C(Y^{n_2}C^{n_4}W^{n_5}) &= \frac{q^{2n_2-2n_5+1} + q^{-2n_2+2n_5-1}}{(q - q^{-1})^2} Y^{n_2}C^{n_4}W^{n_5} \\ &\quad + q^{2n_2}[n_5][n_5 + 1]Y^{n_2}C^{n_4}W^{n_5+2} \\ &\quad + q^{n_2}[n_5][n_2 - n_5](q - q^{-1})^2 Y^{n_2}C^{n_4+1}W^{n_5+1}. \end{aligned}$$

4.2.1. *The eigenvalues and the eigenfunctions of ad_C*

Since \mathcal{V}_{fin} is invariant subspace, we can define factor-representation of ad on the space

$$\mathcal{W} = U_q(\mathfrak{sl}(2))/\mathcal{V}_{\text{fin}}.$$

We denote this representation by the same symbol ad , no confusion arises from this abuse of notation.

In the space \mathcal{W} we introduce the notation $T_n = [n-1]!W^n$ for $n \geq 1$ and the basis consisting of elements

$$X^{n_1}T_{n_5}C^{m_4}, \quad T_{n_5}Y^{n_2}C^{m_4} = [n_5-1]!W^{n_5}Y^{n_2}C^{m_4} = [n_5-1]!q^{2n_2n_5}Y^{n_2}C^{m_4}W^{n_5}.$$

In this basis the representation has the form

$$\begin{aligned} \text{ad}_E(X^{n_1}T_{n_5}C^{m_4}) &= q^{-n_5}(q-q^{-1})X^{n_1+1}T_{n_5+1}C^{m_4}, \\ \text{ad}_F(X^{n_1}T_{n_5}C^{m_4}) &= \frac{q^{-2n_1+n_5+1}}{q-q^{-1}}X^{n_1-1}T_{n_5+1}C^{m_4} \\ &\quad + q^{-n_1+n_5}[n_1-n_5](q-q^{-1})X^{n_1-1}T_{n_5}C^{m_4+1} \\ &\quad - \frac{q^{n_5-1}[2n_1-n_5][n_5-1]}{q-q^{-1}}X^{n_1-1}T_{n_5-1}C^{m_4}, \\ \text{ad}_K(X^{n_1}T_{n_5}C^{m_4}) &= q^{2n_1}X^{n_1}T_{n_5}C^{m_4}, \\ \text{ad}_{K^{-1}}(X^{n_1}T_{n_5}C^{m_4}) &= q^{-2n_1}X^{n_1}T_{n_5}C^{m_4}, \\ \text{ad}_E(T_{n_5}Y^{n_2}C^{m_4}) &= -\frac{q^{-2n_2+n_5+1}}{q-q^{-1}}T_{n_5+1}Y^{n_2-1}C^{m_4} \\ &\quad - q^{-n_2+n_5}[n_2-n_5](q-q^{-1})T_{n_5}Y^{n_2-1}C^{m_4+1} \\ &\quad + \frac{q^{n_5-1}[2n_2-n_5][n_5-1]}{q-q^{-1}}T_{n_5-1}Y^{n_2-1}C^{m_4}, \\ \text{ad}_F(T_{n_5}Y^{n_2}C^{m_4}) &= -q^{-n_5}(q-q^{-1})T_{n_5+1}Y^{n_2+1}C^{m_4}, \\ \text{ad}_K(T_{n_5}Y^{n_2}C^{m_4}) &= q^{-2n_2}T_{n_5}Y^{n_2}C^{m_4}, \\ \text{ad}_{K^{-1}}(T_{n_5}Y^{n_2}C^{m_4}) &= q^{2n_2}T_{n_5}Y^{n_2}C^{m_4}, \\ \text{ad}_C(X^{n_1}T_{n_5}C^{m_4}) &= \frac{q^{2n_1-2n_5+1} + q^{-2n_1+2n_5-1}}{(q-q^{-1})^2}X^{n_1}T_{n_5}C^{m_4} + q^{-2n_1}X^{n_1}T_{n_5+2}C^{m_4} \\ &\quad + q^{-n_1}[n_1-n_5](q-q^{-1})^2X^{n_1}T_{n_5+1}C^{m_4+1}, \\ \text{ad}_C(T_{n_5}Y^{n_2}C^{m_4}) &= \frac{q^{2n_2-2n_5+1} + q^{-2n_2+2n_5-1}}{(q-q^{-1})^2}T_{n_5}Y^{n_2}C^{m_4} + q^{-2n_2}T_{n_5+2}Y^{n_2}C^{m_4} \\ &\quad + q^{-n_2}[n_2-n_5](q-q^{-1})^2T_{n_5+1}Y^{n_2}C^{m_4+1}. \end{aligned}$$

To find the eigenfunctions of ad_C , we add certain vectors of the form

$$\sum_{n_1=0}^{N_1} \sum_{n_5=1}^{\infty} \sum_{n_4=0}^{\infty} a_{n_1, n_5, n_4} X^{n_1} T_{n_5} C^{m_4} \quad \text{and} \quad \sum_{n_2=0}^{N_2} \sum_{n_5=1}^{\infty} \sum_{n_4=0}^{\infty} b_{n_2, n_5, n_4} T_{n_5} C^{m_4} Y^{n_2}$$

to the space \mathcal{W} and denote this new space by $\overline{\mathcal{W}}$. In the space $\overline{\mathcal{W}}$ we find the elements

$$\begin{aligned} |N_1, 0, N_4, S\rangle &= \sum_{k=0}^{\infty} \sum_{r=0}^k X^{N_1} (a_{k,r}^{(N_1, N_4, S)} T_{N_1+S+2k} C^{N_4+2r} \\ &\quad + b_{k,r}^{(N_1, N_4, S)} T_{N_1+S+2k+1} C^{N_4+2r+1}), \\ |0, N_2, N_4, S\rangle &= \sum_{k=0}^{\infty} \sum_{r=0}^k (c_{k,r}^{(N_2, N_4, S)} T_{N_2+S+2k} C^{N_4+2r} \\ &\quad + d_{k,r}^{(N_2, N_4, S)} T_{N_2+S+2k+1} C^{N_4+2r+1}) Y^{N_2}, \end{aligned}$$

where $N_1 + S \geq 1$, $N_2 + S \geq 1$ and $a_{0,0}^{(N_1, N_4, S)} = 1$, for which the relations

$$\begin{aligned} \text{ad}_C |N_1, 0, N_4, S\rangle &= \frac{q^{2S-1} + q^{-2S+1}}{(q - q^{-1})^2} |N_1, 0, N_4, S\rangle, \\ \text{ad}_C |0, N_2, N_4, S\rangle &= \frac{q^{2S-1} + q^{-2S+1}}{(q - q^{-1})^2} |0, N_2, N_4, S\rangle \end{aligned}$$

are valid.

For the constants $a_{k,r}^{(N_1, N_4, S)}$ and $b_{k,r}^{(N_1, N_4, S)}$ we obtain the system of equations

$$\begin{aligned} a_{k+1,0}^{(N_1, N_4, S)} [2k+2][2S+2k+1] + q^{-2N_1} a_{k,0}^{(N_1, N_4, S)} &= 0, \\ a_{k+1,k+1}^{(N_1, N_4, S)} [2k+2][2S+2k+1] &= q^{-N_1} b_{k,k}^{(N_1, N_4, S)} [S+2k+1](q - q^{-1})^2, \\ a_{k+1,r}^{(N_1, N_4, S)} [2k+2][2S+2k+1] + q^{-2N_1} a_{k,r}^{(N_1, N_4, S)} \\ &= q^{-N_1} b_{k,r-1}^{(N_1, N_4, S)} [S+2k+1](q - q^{-1})^2 \quad \text{for } 1 \leq r \leq k, \\ b_{k,k}^{(N_1, N_4, S)} [2k+1][2S+2k] &= q^{-N_1} a_{k,k}^{(N_1, N_4, S)} [S+2k](q - q^{-1})^2, \\ b_{k+1,r}^{(N_1, N_4, S)} [2k+3][2S+2k+2] + q^{-2N_1} b_{k,r}^{(N_1, N_4, S)} \\ &= q^{-N_1} a_{k+1,r}^{(N_1, N_4, S)} [S+2k+2](q - q^{-1})^2 \quad \text{for } 0 \leq r \leq k. \end{aligned}$$

The constants $c_{k,r}^{(N_2, N_4, S)}$ and $d_{k,r}^{(N_2, N_4, S)}$ fulfil a similar system of equations.

The system of equations for $a_{k,r}$ and $b_{k,r}$ (for simplicity we omit the upper indices) can be solved in the following way:

Let $a_{0,0} = 1$. If we know any $a_{s,t}$ for $0 \leq t \leq s \leq k$ and $b_{s,t}$ for $0 \leq t \leq s \leq k-1$, we find $b_{k,r}$ from the equations

$$\begin{aligned} [2k+1][2S+2k]b_{k,k} &= q^{-N_1}(q - q^{-1})^2[S+2k]a_{k,k}, \\ [2k+1][2S+2k]b_{k,r} &= q^{-N_1}(q - q^{-1})^2[S+2k]a_{k,r} - q^{-2N_1}b_{k-1,r} \end{aligned} \quad (4.2)$$

for $0 \leq r \leq k-1$.

If we know any $a_{s,t}$ and $b_{s,t}$ for $0 \leq t \leq s \leq k$, we find $a_{k+1,r}$ from the equations

$$\begin{aligned} [2k+2][2S+2k+1]a_{k+1,0} &= -q^{-2N_1}a_{k,0}, \\ [2k+2][2S+2k+1]a_{k+1,r} &= q^{-N_1}(q - q^{-1})^2[S+2k+1]b_{k,r-1} - q^{-2N_1}a_{k,r} \end{aligned} \quad (4.3)$$

for $1 \leq r \leq k$,

$$[2k+2][2S+2k+1]a_{k+1,k+1} = q^{-N_1}(q - q^{-1})^2[S+2k+1]b_{k,k}.$$

As far as $S > 0$, $S + k \neq 0$ for any k and from Eqs. (4.2) and (4.3) we can evaluate $a_{k,r}$ and $b_{k,r}$ with the help of $a_{0,0} = 1$.

For $S = -L \leq 0$ we obtain zero in (4.2) for $k = L$ and we are not able to find $b_{L,r}$. For $L = 0$ and $k = 0$ Eq. (4.2) is trivially valid and gives no other condition for $b_{0,0}$. For $L \geq 1$ the systems (4.2) and (4.3) look like

$$\begin{aligned} [2k+1][2L-2k]b_{k,k} &= q^{-N_1}(q-q^{-1})^2[L-2k]a_{k,k}, \\ [2k+1][2L-2k]b_{k,r} &= q^{-N_1}(q-q^{-1})^2[L-2k]a_{k,r} + q^{-2N_1}b_{k-1,r} \\ &\quad \text{for } 0 \leq r \leq k-1, \\ [2k+2][2L-2k-1]a_{k+1,0} &= q^{-2N_1}a_{k,0}, \\ [2k+2][2L-2k-1]a_{k+1,r} &= q^{-N_1}(q-q^{-1})^2[L-2k-1]b_{k,r-1} + q^{-2N_1}a_{k,r} \\ &\quad \text{for } 1 \leq r \leq k, \\ [2k+2][2L-2k-1]a_{k+1,k+1} &= q^{-N_1}(q-q^{-1})^2[L-2k-1]b_{k,k}. \end{aligned} \quad (4.4)$$

For $0 \leq k \leq L-1$ and $k = L$ the system (4.2) is reduced to

$$a_{L,L} = 0, \quad b_{L-1,r} = q^{N_1}(q-q^{-1})^2[L]a_{L,r} \quad \text{for } 0 \leq r \leq L-1. \quad (4.5)$$

In the appendix we show that the system (4.4) has a solution for which $a_{0,0} = 1$ and the relations (4.5) are a consequence of it.

We make the following choice. The constants $b_{L,r}$, $0 \leq r \leq L$ which are not determined by the system are set to zero. The constants $a_{k,r}$ and $b_{k,r}$ for $k \geq L$ are uniquely determined by equations (4.3) for $k \geq L$ and by equations (4.2) for $k \geq L+1$. Just this element is denoted by $|N_1, 0, N_4, S\rangle$ for $S \leq 0$.

Analogously, we define the elements $|0, N_2, N_4, S\rangle$.

4.2.2. The decomposition of the adjoint representation in $\widehat{\mathcal{W}}$

Next we introduce the space $\widehat{\mathcal{W}} \subset \overline{\mathcal{W}}$ which is generated by the elements $|N_1, 0, N_4, S\rangle$ and $|0, N_2, N_4, S\rangle$. Now we find the action of the adjoint representation on these vectors. We obtain

$$\begin{aligned} \text{ad}_E|N_1, 0, N_4, S\rangle &= q^{-N_1-S}(q-q^{-1})X^{N_1+1} \\ &\quad \times \sum_{k=0}^{\infty} \sum_{r=0}^k (q^{-2k}a_{k,r}^{(N_1, N_4, S)}T_{N_1+S+2k+1}C^{N_4+2r} \\ &\quad + q^{-2k-1}b_{k,r}^{(N_1, N_4, S)}T_{N_1+S+2k+2}C^{N_4+2r+1}). \end{aligned}$$

If we denote

$$a_{k,r} = q^{-2k}a_{k,r}^{(N_1, N_4, S)}, \quad b_{k,r} = q^{-2k-1}b_{k,r}^{(N_1, N_4, S)},$$

we can easily show that if $a_{k,r}^{(N_1, N_4, S)}$ and $b_{k,r}^{(N_1, N_4, S)}$ fulfil Eqs. (4.2) and (4.3) for N_1 and S , then $a_{k,r}$, $b_{k,r}$ fulfil these systems for $N_1 + 1$ and S . In addition for $S = -L \leq 0$ we have $b_{L,r} = 0$. Since $a_{0,0} = a_{0,0}^{(N_1, N_4, S)} = 1$,

$$\text{ad}_E|N_1, 0, N_4, S\rangle = (q-q^{-1})q^{-N_1-S}|N_1+1, 0, N_4, S\rangle. \quad (4.6)$$

Similarly, we show

$$\text{ad}_F|0, N_2, N_4, S\rangle = -(q-q^{-1})q^{-N_2-S}|0, N_2+1, N_4, S\rangle. \quad (4.7)$$

If we apply ad_F on $|N_1, 0, N_4, S\rangle$ where $N_1 \geq 1$, similar but the more cumbersome calculations gives

$$\text{ad}_F|N_1, 0, N_4, S\rangle = -\frac{q^{N_1+S-1}[N_1-S][S+N_1-1]}{q-q^{-1}}|N_1-1, 0, N_4, S\rangle \quad (4.8)$$

and for action ad_E on $|0, N_2, N_4, S\rangle$ we can show that for $N_2 \geq 1$

$$\text{ad}_E|0, N_2, N_4, S\rangle = \frac{q^{N_2+S-1}[N_2-S][S+N_2-1]}{q-q^{-1}}|0, N_2-1, 0, N_4, S\rangle. \quad (4.9)$$

Moreover, we have

$$\begin{aligned} \text{ad}_K|N_1, 0, N_4, S\rangle &= q^{2N_1}|N_1, 0, N_4, S\rangle, \\ \text{ad}_K|0, N_2, N_4, S\rangle &= q^{-2N_2}|0, N_2, N_4, S\rangle. \end{aligned}$$

The representation preserves the value of N_4 and S but changes N_1 , and N_2 . Nevertheless, we see that the lowest and highest weight vectors are

$$\begin{aligned} \text{ad}_F|N_1, 0, N_4, S\rangle &= 0 \quad \text{when } S = N_1 \geq 1 \text{ or } S = 1 - N_1 \leq 0, \\ \text{ad}_E|0, N_2, N_4, S\rangle &= 0 \quad \text{when } S = N_2 \geq 1 \text{ or } S = 1 - N_2 \leq 0. \end{aligned}$$

From the computations above it follows that for $N_4 \geq 0$ the subspaces $\widehat{\mathcal{W}}$ defined as

$$\begin{aligned} \widehat{\mathcal{W}}_{(S,0,N_4,S)} &= \text{ad}_{U_q(\mathfrak{sl}(2))}|S, 0, N_4, S\rangle = \{|N_1, 0, N_4, S\rangle; N_1 \geq S \geq 1\} \\ &\quad \text{for } S \geq 1, \\ \widehat{\mathcal{W}}_{(0,S,N_4,S)} &= \text{ad}_{U_q(\mathfrak{sl}(2))}|0, S, N_4, S\rangle = \{|0, N_2, N_4, S\rangle; N_2 \geq S \geq 1\} \\ &\quad \text{for } S \geq 1, \\ \widehat{\mathcal{W}}_{(1-S,0,N_4,S)} &= \text{ad}_{U_q(\mathfrak{sl}(2))}|1-S, 0, N_4, S\rangle = \{|N_1, 0, N_4, S\rangle; N_1 \geq 1-S\} \\ &\quad \text{for } S \leq 0, \\ \widehat{\mathcal{W}}_{(0,1-S,N_4,S)} &= \text{ad}_{U_q(\mathfrak{sl}(2))}|0, 1-S, N_4, S\rangle = \{|0, N_2, N_4, S\rangle; N_2 \geq 1-S\} \\ &\quad \text{for } S \leq 0 \end{aligned} \quad (4.10)$$

are invariant.

Except the elements from the subspaces (4.10) in the space $\widehat{\mathcal{W}}$ there are the elements of the form $|N_1, 0, N_4, S\rangle$ with $0 \leq N_1 < S$ and $|0, N_2, N_4, S\rangle$, where $0 \leq N_2 < S$. For the vectors $|0, 0, N_4, S\rangle$, where $N_4 \geq 0$ and $S \geq 1$ we have from relations (4.6) and (4.7)

$$\begin{aligned} \text{ad}_{E^k}|0, 0, N_4, S\rangle &= (q-q^{-1})^k q^{-S-k(k-1)/2}|k, 0, N_4, S\rangle, \\ \text{ad}_{F^k}|0, 0, N_4, S\rangle &= (q^{-1}-q)^k q^{-S-k(k-1)/2}|0, k, N_4, S\rangle. \end{aligned}$$

For $N_4 \geq 0$ and $S \geq 1$ we consider the invariant subspace

$$\widehat{\mathcal{W}}_{(0,0,N_4,S)} = \text{ad}_{U_q(\mathfrak{sl}(2))}|0, 0, N_4, S\rangle.$$

This subspace contains all vectors $|N_1, 0, N_4, S\rangle$ and $|0, N_2, N_4, S\rangle$ and is reducible. It is clear that its irreducible components are $\widehat{\mathcal{W}}_{(S,0,N_4,S)}$ and $\widehat{\mathcal{W}}_{(0,S,N_4,S)}$.

If we bring together the previous results, we obtain:

Proposition. *The space $\widehat{\mathcal{W}}$ can be expressed as a direct sum of the invariant subspaces in the form*

$$\widehat{\mathcal{W}} = \bigoplus_{N_4=0}^{\infty} \left(\bigoplus_{N_1=1}^{\infty} (\widehat{\mathcal{W}}_{(N_1,0,N_4,1-N_1)} \oplus \widehat{\mathcal{W}}_{(0,N_1,N_4,1-N_1)}) \oplus \bigoplus_{S=1}^{\infty} \widehat{\mathcal{W}}_{(0,0,N_4,S)} \right).$$

The subspaces $\widehat{\mathcal{W}}_{(0,0,N_4,S)}$ are reducible, their irreducible components are $\widehat{\mathcal{W}}_{(S,0,N_4,S)}$ and $\widehat{\mathcal{W}}_{(0,S,N_4,S)}$, and the dimension of the factor-space $\widehat{\mathcal{W}}_{(0,0,N_4,S)}/(\widehat{\mathcal{W}}_{(S,0,N_4,S)} + \widehat{\mathcal{W}}_{(0,S,N_4,S)})$ is $2S-1$.

4.2.3. *The infinite dimensional representation for the adjoint representation of $U_q(\mathfrak{sl}(2))$*

In Subsec. 4.2.1, we introduced the factor-space \mathcal{W} , its modification $\widehat{\mathcal{W}}$ and factor representation on the space $\widehat{\mathcal{W}}$. In this section we return to the adjoint representation on the $U_q(\mathfrak{sl}(2))$. This representation acts on the vectors $X^{n_1}T_{n_5}C^{n_4}$ and $T_{n_5}Y^{n_2}C^{n_4}$ for $n_5 > 1$, as was described in 4.2.1, but for $n_5 = 1$ we obtain

$$\begin{aligned} \text{ad}_F(X^{n_1}T_1C^{n_4}) &= \frac{q^{-2n_1+2}}{q-q^{-1}}X^{n_1-1}T_2C^{n_4} \\ &\quad + q^{-n_1+1}[n_1-1](q-q^{-1})X^{n_1-1}T_1C^{n_4+1} - \frac{[2n_1-1]}{q-q^{-1}}X^{n_1-1}C^{n_4}, \\ \text{ad}_E(T_1Y^{n_2}C^{n_4}) &= -\frac{q^{-2n_2+2}}{q-q^{-1}}T_2Y^{n_2-1}C^{n_4} \\ &\quad - q^{-n_2+1}[n_2-1](q-q^{-1})T_1Y^{n_2-1}C^{n_4+1} + \frac{[2n_2-1]}{q-q^{-1}}Y^{n_2-1}C^{n_4}. \end{aligned}$$

To get the eigenvectors and eigenvalues of the operator of ad_C back in the game, we modify $U_q(\mathfrak{sl}(2))$ as we did in the case of the space \mathcal{W} . This space will be denoted by $\widehat{U}_q(\mathfrak{sl}(2))$.

It is easy to show that for $N_1 + S > 1$ the representation is defined by relations (4.6)–(4.10), but for $N_1 + S = 1$ we have relations (4.7), (4.9) and

$$\begin{aligned} \text{ad}_F|N_1, 0, N_4, 1 - N_1\rangle &= -\frac{[2N_1-1]}{q-q^{-1}}X^{N_1-1}C^{N_4}, \\ \text{ad}_E|0, N_2, N_4, 1 - N_2\rangle &= \frac{[2N_2-1]}{q-q^{-1}}Y^{N_2-1}C^{N_4}. \end{aligned} \tag{4.11}$$

Consequently, the spaces

$$\begin{aligned} \mathcal{V}_{(N_1, 0, N_4, 1-N_1)} &= \text{ad}_{U_q(\mathfrak{sl}(2))}|N_1, 0, N_4, 1 - N_1\rangle, \\ \mathcal{V}_{(0, N_2, N_4, 1-N_2)} &= \text{ad}_{U_q(\mathfrak{sl}(2))}|0, N_2, N_4, 1 - N_2\rangle \end{aligned}$$

are no longer irreducible but contain the finite dimensional invariant subspaces

$$\text{ad}_{U_q(\mathfrak{sl}(2))}(X^{N_1-1}C^{N_4}) \quad \text{and} \quad \text{ad}_{U_q(\mathfrak{sl}(2))}(Y^{N_2-1}C^{N_4}).$$

If we now define for any $N_1 \geq 1$, $S \geq 1$ and $N_4 \geq 0$ the invariant subspaces of $\widehat{U}_q(\mathfrak{sl}(2))$

$$\begin{aligned} \mathcal{V}_{(N_1-1, N_4)} &= \text{ad}_{U_q(\mathfrak{sl}(2))}(X^{N_1-1}C^{N_4}), \\ \mathcal{V}_{(N_1, 0, N_4, 1-N_1)} &= \text{ad}_{U_q(\mathfrak{sl}(2))}|N_1, 0, N_4, 1 - N_1\rangle, \\ \mathcal{V}_{(0, N_1, N_4, 1-N_1)} &= \text{ad}_{U_q(\mathfrak{sl}(2))}|0, N_1, N_4, 1 - N_1\rangle, \\ \mathcal{V}_{(S, 0, N_4, S)} &= \text{ad}_{U_q(\mathfrak{sl}(2))}|S, 0, N_4, S\rangle, \\ \mathcal{V}_{(0, S, N_4, S)} &= \text{ad}_{U_q(\mathfrak{sl}(2))}|0, S, N_4, S\rangle, \\ \mathcal{V}_{(0, 0, N_4, S)} &= \text{ad}_{U_q(\mathfrak{sl}(2))}|0, 0, N_4, S\rangle, \end{aligned}$$

we can formulate the following theorem.

Theorem. *The space $\widehat{U}_q(\mathfrak{sl}(2))$ can be decomposed to*

$$\widehat{U}_q(\mathfrak{sl}(2)) = \bigoplus_{N_4=0}^{\infty} \left(\bigoplus_{N_1=1}^{\infty} (\mathcal{V}_{(N_1, 0, N_4, 1-N_1)} + \mathcal{V}_{(0, N_1, N_4, 1-N_1)}) \oplus \bigoplus_{S=1}^{\infty} \mathcal{V}_{(0, 0, N_4, S)} \right).$$

Moreover, $\mathcal{V}_{(N_1-1, N_4)} = \mathcal{V}_{(N_1, 0, N_4, 1-N_1)} \cap \mathcal{V}_{(0, N_1, N_4, 1-N_1)}$ is the irreducible subspace of dimension $2N_1 - 1$, and the subspace $\mathcal{V}_{(0, 0, N_4, S)}$ has the invariant infinite subspaces $\mathcal{V}_{(S, 0, N_4, S)}$ and $\mathcal{V}_{(0, S, N_4, S)}$, and the factor-space $\mathcal{V}_{(0, 0, N_4, S)}/(\mathcal{V}_{(S, 0, N_4, S)} + \mathcal{V}_{(0, S, N_4, S)})$ has the dimension $2S - 1$.

5. Conclusion

In the presented paper, we have studied the decomposition of the adjoint representation of the quantum group $U_q(\mathfrak{sl}(2))$. By construction, we modified this quantum group $U_q(\mathfrak{sl}(2))$ by adding some formal series. Let us mention that this deformation can be obtained from $U_h(\mathfrak{sl}(2))$ that contains the formal series. The quantum group $U_q(\mathfrak{sl}(2))$ is a subgroup in $U_h(\mathfrak{sl}(2))$ where we restrict ourselves to the special formal series, especially to the formal series for K , K^{-1} and $[E, F] = \frac{K-K^{-1}}{q-q^{-1}}$.

We showed that the decomposition of the quantum algebra $\widehat{U}_q(\mathfrak{sl}(2))$ is more complicated than the non-deformed case. The modified space $\widehat{U}_q(\mathfrak{sl}(2))$ contains finite and infinite dimensional invariant subspaces and, in addition, this space is not fully reducible, i.e., we cannot write it as a direct sum of invariant irreducible subspaces.

Of much interest is a duality between the subspaces $\widehat{U}_q(\mathfrak{sl}(2))$ with $S > 0$ and $S \leq 0$. It would be useful to study whether this duality is a property of $U_q(\mathfrak{sl}(2))$ only or it is a general property for all quantum groups $U_q(\mathfrak{g})$.

Appendix A.

In this appendix we show that the system (4.4) has a solution for which $a_{0,0} = 1$ and that the solution fulfils relation (4.5). First, we define the new variables $\alpha_{k,r}$ and $\beta_{k,r}$ by the relations

$$a_{k,r} = q^{-2kN_1}(q - q^{-1})^{2r} \alpha_{k,r}, \quad b_{k,r} = q^{-(2k+1)N_1}(q - q^{-1})^{2r+2} \beta_{k,r}.$$

With respect to these new variables the system (4.4) has the form

$$\begin{aligned} [2k+1][2L-2k]\beta_{k,k} &= [L-2k]\alpha_{k,k}, \\ [2k+1][2L-2k]\beta_{k,r} &= [L-2k]\alpha_{k,r} + \beta_{k-1,r}, \quad 0 \leq r \leq k-1, \\ [2k+2][2L-2k-1]\alpha_{k+1,0} &= \alpha_{k,0}, \\ [2k+2][2L-2k-1]\alpha_{k+1,r} &= [L-2k-1]\beta_{k,r-1} + \alpha_{k,r}, \quad 1 \leq r \leq k, \\ [2k+2][2L-2k-1]\alpha_{k+1,k+1} &= [L-2k-1]\beta_{k,k}, \end{aligned} \tag{A.1}$$

which is valid for $0 \leq k \leq L-1$, and the conditions (4.5) are of the form

$$\alpha_{L,L} = 0, \quad \beta_{L-1,r} = [L]\alpha_{L,r} \quad \text{for } 0 \leq r \leq L-1. \tag{A.2}$$

For $k \leq L-1$ we obtain from the first and last equations of (A.1)

$$\begin{aligned} \beta_{k,k} &= \frac{[L-2k]}{[2k+1] \cdot [2L-2k]} \alpha_{k,k}, \\ \alpha_{k+1,k+1} &= \frac{[L-2k-1][L-2k]}{[2k+1][2k+2] \cdot [2L-2k-1][2L-2k]} \alpha_{k,k} \end{aligned}$$

which results in

$$\begin{aligned} \alpha_{k,k} &= \frac{1}{[2k]!} \cdot \frac{[2L-2k]!}{[2L]!} \cdot \frac{[L]!}{[L-2k]!}, \\ \beta_{k,k} &= \frac{1}{[2k+1]!} \cdot \frac{[2L-2k-1]!}{[2L]!} \cdot \frac{[L]!}{[L-2k-1]!}. \end{aligned} \tag{A.3}$$

In particular, from (A.3) we obtain $\alpha_{k,k} = 0$ for $k > \frac{1}{2}L$ and $\beta_{k,k} = 0$ for $k > \frac{1}{2}(L-1)$, and thus $\alpha_{L,L} = 0$.

Next for $k \geq r$ we put

$$\alpha_{k,r} = \frac{[2r]!!}{[2k]!!} \cdot \frac{[2L-2k-1]!!}{[2L-2r-1]!!} R_{k,r}, \quad \beta_{k,r} = \frac{[2r+1]!!}{[2k+1]!!} \cdot \frac{[2L-2k-2]!!}{[2L-2r-2]!!} S_{k,r},$$

(and we define $[0]!! = [-1]!! = 1$), we obtain from (A.1) the system

$$\begin{aligned} R_{k+1,0} - R_{k,0} &= 0, \\ R_{k+1,r} - R_{k,r} &= \frac{[2k]!!}{[2k+1]!!} \cdot \frac{[2r-1]!!}{[2r]!!} \cdot \frac{[2L-2k-2]!!}{[2L-2k-1]!!} \\ &\quad \times \frac{[2L-2r-1]!!}{[2L-2r]!!} [L-2k-1] S_{k,r-1}, \quad r \geq 1, \\ S_{k+1,r} - S_{k,r} &= \frac{[2k+1]!!}{[2k+2]!!} \cdot \frac{[2r]!!}{[2r+1]!!} \cdot \frac{[2L-2k-3]!!}{[2L-2k-2]!!} \\ &\quad \times \frac{[2L-2r-2]!!}{[2L-2r-1]!!} [L-2k-2] R_{k+1,r}, \quad r \geq 0. \end{aligned}$$

This systems results in

$$\begin{aligned} R_{k,0} &= 1, \\ R_{k,r} &= \frac{[2r-1]!!}{[2r]!!} \cdot \frac{[2L-2r-1]!!}{[2L-2r]!!} \sum_{s=r-1}^{k-1} \frac{[2s]!!}{[2s+1]!!} \\ &\quad \times \frac{[2L-2s-2]!!}{[2L-2s-1]!!} \cdot [L-2s-1] S_{s,r-1}, \quad r \geq 1, \\ S_{k,r} &= \frac{[2r]!!}{[2r+1]!!} \cdot \frac{[2L-2r-2]!!}{[2L-2r-1]!!} \\ &\quad \times \sum_{s=r}^k \frac{[2s-1]!!}{[2s]!!} \cdot \frac{[2L-2s-1]!!}{[2L-2s]!!} \cdot [L-2s] R_{s,r}, \quad r \geq 0 \end{aligned} \tag{A.4}$$

and the following relations:

$$\begin{aligned} R_{k_0,r} = 0 \quad \text{and} \quad S_{k',r-1} = 0 \quad \text{for all } k' \geq k_0 &\Rightarrow R_{k,r} = 0 \quad \text{for all } k > k_0, \\ S_{k_0,r} = 0 \quad \text{and} \quad R_{k',r} = 0 \quad \text{for all } k' > k_0 &\Rightarrow S_{k,r} = 0 \quad \text{for all } k > k_0. \end{aligned}$$

To prove (A.2) it suffices to show the relation

$$S_{L-1,0} = \frac{[L]}{[2L]} R_{L,0} = \frac{[L]}{[2L]} \tag{A.5}$$

and that for any r , $1 \leq r \leq \frac{1}{2}L$ we have

$$R_{L-r+1,r} = S_{L-r,r} = 0. \tag{A.6}$$

First, we prove (A.5). In accordance with (A.4) we have

$$S_{L-1,0} = \frac{[L]}{[2L]} + \frac{[2L-2]!!}{[2L-1]!!} \sum_{s=1}^{L-1} \frac{[2s-1]!!}{[2s]!!} \cdot \frac{[2L-2s-1]!!}{[2L-2s]!!} \cdot [L-2s].$$

If we in substitute $s \rightarrow L-s$ in the sum, we will discover that Eq. (A.5) holds.

The conditions $R_{L-r+1,r} = S_{L-1,r} = 0$ for $1 \leq r \leq \frac{1}{2}L$ are equivalent to the equations

$$\begin{aligned} \sum_{s=r-1}^{L-r} \frac{[2s]!!}{[2s+1]!!} \cdot \frac{[2L-2s-2]!!}{[2L-2s-1]!!} \cdot [L-2s-1]S_{s,r-1} &= 0, \\ \sum_{s=r}^{L-r} \frac{[2s-1]!!}{[2s]!!} \cdot \frac{[2L-2s-1]!!}{[2L-2s]!!} \cdot [L-2s]R_{s,r} &= 0. \end{aligned} \quad (\text{A.7})$$

To prove these equations, we use (A.4). After an appropriate arrangement of the terms in the sums we can deduce that (A.7) is valid if the following statement holds:

Let $1 \leq r \leq \frac{1}{2}L$. Then for $L = 2J$ we have

$$\begin{aligned} S_{J+s,r-1} - S_{J-s-1,r-1} &= 0 \quad \text{for } 0 \leq s \leq J-r, \\ R_{J+s,r} - R_{J-s,r} &= 0 \quad \text{for } 1 \leq s \leq J-r, \end{aligned} \quad (\text{A.8})$$

and for $L = 2J+1$ is

$$\begin{aligned} S_{J+s,r-1} - S_{J-s,r-1} &= 0 \quad \text{for } 1 \leq s \leq J-r+1, \\ R_{J+s+1,r} - R_{J-s,r} &= 0 \quad \text{for } 0 \leq s \leq J-r. \end{aligned} \quad (\text{A.9})$$

We will study the first case (A.8). Let $1 \leq r \leq J$. The relation $R_{J+s,r} - R_{J-s,r} = 0$ for $1 \leq s \leq J-r$ with the use of (A.4) is equivalent to the equation

$$\sum_{t=J-s}^{J+s-1} \frac{[2t]!!}{[2t+1]!!} \cdot \frac{[4J-2t-2]!!}{[4J-2t-1]!!} \cdot [2J-2t-1](S_{t,r-1} - S_{2J-t-1,r-1}) = 0,$$

which is the consequence of the relation $S_{J+s,r-1} - S_{J-s-1,r-1} = 0$ for $0 \leq s \leq J-r$. On the other hand, for $2 \leq r$ the equality $S_{J+s,r-1} - S_{J-s-1,r-1} = 0$ for $0 \leq s \leq J-r$ is equivalent to the equation

$$\sum_{t=J-s}^{J+s} \frac{[2t-1]!!}{[2t]!!} \cdot \frac{[4J-2t-1]!!}{[4J-2t]!!} \cdot [2J-2t](R_{t,r-1} - R_{2J-t,r-1}) = 0$$

and this is the consequence of the relation $R_{J+s,r-1} - R_{J-s,r-1} = 0$ for $1 \leq s \leq J-r+1$.

It follows from this consideration that for $L = 2J$ all relations (A.8) are result of the relations $S_{J+s,0} - S_{J-s-1,0} = 0$ for $0 \leq s \leq J-1$.

Similarly, we can see, that for $L = 2J+1$ the validity (A.8) follows from $S_{J+s,0} - S_{J-s,0} = 0$ for $1 \leq s \leq J$.

For $L = 2J$ with the use of (A.4) the condition $S_{J+s,0} - S_{J-s-1,0} = 0$ is equivalent to the equation

$$\sum_{r=J-s}^{J+s} \frac{[2r-1]!!}{[2r]!!} \cdot \frac{[4J-2r-1]!!}{[4J-2r]!!} \cdot [2J-2r] = 0,$$

which we can easily prove by substitution $r \rightarrow 2J-r$, and for $L = 2J+1$ the condition $S_{J+s,0} - S_{J-s,0} = 0$ is equivalent to the equation

$$\sum_{r=J-s+1}^{J+s} \frac{[2r-1]!!}{[2r]!!} \cdot \frac{[4J-2r+1]!!}{[4J-2r+2]!!} \cdot [2J-2r+1] = 0,$$

which is easily seen to hold after substitution $r \rightarrow 2J-r+1$.

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