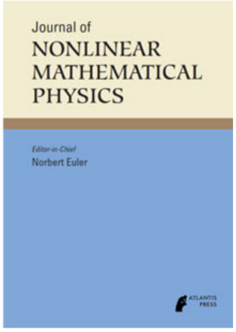

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GEOMETRIZATION OF THE LEADING TERM IN ACOUSTIC GAUSSIAN BEAMS

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We study Gaussian beams for the wave equation on a Riemannian manifold. For the transport equation we geometrize the leading term at the center of the Gaussian beam. More precisely, if

$$u(x,t) = e^{iP\theta(x,t)}\left(u_0(x,t) + \frac{u_1(x,t)}{iP} + \frac{u_2(x,t)}{(iP)^2} + \cdots\right)$$

is a Gaussian beam propagating along a geodesic c, then we show that

$$u_0(c(t),t) = C\frac{1}{(\det Y(t))^{1/2}}$$

where C is a constant and Y is a complex Jacobi tensor. Using a constant of motion for the non-linear Riccati equation related to the Jacobi equation, we prove that asymptotically the leading term of the energy carries constant energy.

Keywords: Gaussian beams; wave packets; asymptotic analysis; wave equation; conservation of energy; leading amplitude term.

0. Introduction

A *Gaussian beam* is an asymptotic solution to a linear hyperbolic equations that have a very characteristic feature. Namely, at each time instant, the entire energy of the solution is concentrated around one point in space. When time moves forward, the beam propagates along a curve, but always retains its shape of a Gaussian bell curve. Electromagnetic Gaussian beams are also known as quasi-photons. See [3,12–14]. For the wave equation, see [15,16,19,20], and for the elastic equation, see [17]. Historical accounts on Gaussian beams can be found in [1,19,20].

Gaussian beams are governed by two sets of equations. The *Hamilton–Jacobi equation* determines the phase function, and a set of *transport equations* determine the amplitude functions for a Gaussian beam (Sec. 1). A central property for Gaussian beams is that the Hamilton–Jacobi equation geometrize. For example, in suitable media, Gaussian beams propagate along geodesics of a Riemannian metric determined by the media. (This media may be inhomogeneous and anisotropic.) As the geodesic equation is an ODE, this means that Gaussian beams are much easier to propagate than the original equation, say, Maxwell's equations in the time domain. Therefore Gaussian beams have been used in the *traveltime problem*: how long does it take for a signal to travel between two points in space. This geometrization result is common to all Gaussian beams regardless of their physical setting (acoustic, electric, or seismic). The reason for this is that the Hamilton–Jacobi equation has the same form in all settings. On the other hand, each physical setting gives rise to its own transport equations. For example, in acoustics the transport equations are scalar equations whereas in electromagnetism they are vector equations. Although these have been studied, and solution methods are known in local coordinates [15, 12], there does not seem to exist many geometric treatments on these equations. An exception seem to be [21], which derives a Rytov's law in electromagnetism in an essentially isotropic media.

Theorems 3 and 4 are the main results of the present paper. These show how the leading amplitude term evaluated at the center of an acoustic Gaussian beam can be solved from the complex Riccati equation or the complex Jacobi tensor (Subsec. 1.2). Essentially, a Jacobi tensor is a tensor analogue to a Jacobi field in Riemannian geometry [6–8]. See also [10, 11] for a connection to Raychaudhuri's equation studied in general relativity. Apart from the Riemann geometric interpretation, Theorems 4 and 3 are well known. For example, see [23] or [15, Sec. 2.4.19]. Assuming that the underlying manifold is $\mathbb{R}^n$ (but has an arbitrary Riemannian metric) we also show that asymptotically, the leading term in the Gaussian beam carries constant energy. The proof of this is based on a constant of motion for solutions to the non-linear complex Riccati equation (Theorem 2).

We assume that M is a n-dimensional manifold, and by a manifold we mean a second countable, Hausdorff, topological manifold equipped with smooth transition maps in $\mathbb{R}^n$. All objects are assumed to be smooth, and by default, all functions are assumed to be complex valued. The Einstein summing convention is used throughout. By TM and $T(M,\mathbb{R})$ we mean the tangent bundles for M over $\mathbb{C}$ and $\mathbb{R}$, respectively. We always assume that I is an open interval containing 0.

1. Asymptotic Expansion of u

On the Riemann manifold, the wave equation reads

$$u_{tt} - \Delta u = 0.$$

Here Δ is the *Laplace–Beltrami operator* $\Delta = \nabla \cdot \nabla$ with respect to a metric g that encodes the media properties. To define a Gaussian beam, let us briefly review asymptotic solutions to the wave equation. For detailed accounts on this topic, see [15, 16].

Suppose $u(x,t)$ is a function of the form

$$u(x,t) = e^{iP\theta(x,t)} \sum_{k=0}^{N} \frac{u_k(x,t)}{(iP)^k}, \tag{1.1}$$

where $N \geq 0$, $P > 0$ is a large constant, and θ, u_k are complex functions. Function θ is the *phase function* and $u_0, u_1, \ldots$ are *amplitude functions.* In order for u to be stable in the limit $P \to \infty$, we assume that $\operatorname{Im}\theta \geq 0$. Inserting u into the wave equation and equating separate powers of (iP) to zero yields

$$(\theta_t)^2 - g(\nabla\theta, \nabla\theta) = 0, \tag{1.2}$$

$$\mathscr{L}_\theta(u_0) = 0, \tag{1.3}$$

$$\mathscr{L}_\theta(u_{k+1}) = (\partial_t^2 - \Delta)u_k, \quad k = 0, \ldots, N-1. \tag{1.4}$$

Equation (1.2) is the *Hamilton–Jacobi equation*, and Eqs. (1.3)–(1.4) are the *transport equations.* In these, the *transport operator* is given by

$$\mathscr{L}_\theta(u) = 2\theta_t u_t - 2g(\nabla\theta, \nabla u) + \left(\left(\partial_t^2 - \Delta\right)\theta\right) \cdot u, \quad u \in C^\infty(M \times I).$$

Here $g(u,v)$ is the metric tensor on M and it does not involve a complex conjugate.

1.1. *Gaussian beams*

Let us construct a Gaussian beam propagating along a smooth curve $c\colon I \to M$ that, for simplicity, is covered by local coordinates x^i. Furthermore, let

$$\phi\colon I \to \mathbb{C}, \quad p\colon I \to \mathbb{C}^n, \quad H\colon I \to \mathbb{C}^{n\times n}$$

be the first three coefficients in the Taylor expansion of θ evaluated on $c(t)$;

$$\phi(t) = \theta(c(t),t), \quad p_j(t) = \frac{\partial \theta}{\partial x^j}(c(t),t), \quad H_{jk}(t) = \frac{\partial^2 \theta}{\partial x^j\, \partial x^k}(c(t),t).$$

Then u is a *Gaussian beam* on c provided that for all $t \in I$:

(C1) $p(t) = (p_i(t))_i$ is nonzero.
(C2) $\phi(t)$ and $p(t)$ are real.
(C3) $H(t) = (H_{ij}(t))_{ij}$ is symmetric and its imaginary part is positive definite.

These conditions do not depend on local coordinates, and using the chain rule,

$$\theta(x,t) = \phi_0(t) + p_i(t)z^i + \frac{1}{2}H_{ij}(t)z^i z^j + O(|z|^3), \tag{1.5}$$

where $z^i = z^i(x,t) = x^i - c^i(t)$. Therefore,

$$|\exp(iP\theta(x,t))| \approx \exp\left(-\frac{P}{2} z^i \operatorname{Im} G_{ij} z^j\right)$$

since $\operatorname{Im} G = \operatorname{Im} H$. That is, at time t, u is concentrated around $c(t)$. Next, we can solve ϕ, c, and H from the Hamilton–Jacobi equation (1.2). This is done by expanding the Hamilton–Jacobi equation into a Taylor series centered at $c(t)$ and equating the first three Taylor coefficients. This gives:

(1) ϕ_0 is constant.
(2) c is a geodesic with respect to g, and p is determined

$$p_i(t) = g_{ij}\dot{c}^j.$$

With no loss in generality, we assume that c is pathlength parameterized so that $g(\dot{c},\dot{c}) = 1$.

(3) In local coordinates, H is determined by a complex matrix Riccati equation. Unfortunately, H is not a tensor. However, by slightly perturbing H, one gets a tensor G. Let $\Lambda(t) = (\Gamma^m_{ij} p_m)_{ij}$, where Γ^i_{jk} are the *Christoffel symbols*, and let

$$G(t) = G_{ij}(t) dx^i \otimes dx^j|_{c(t)}, \quad G_{ij} = (H - \Lambda)_{ij}.$$

Then G is the 2-tensor on c determined by the *complex tensor Riccati equation*

$$G' + GCG - R = 0. \tag{1.6}$$

Here G' is the covariant derivative of G along c, and $C = C^{ij}(t) dx^i \otimes dx^j|_{c(t)}$ and $R = R_{ij}(t) dx^i \otimes dx^j|_{c(t)}$ are tensors on c determined by

$$C^{ij} = g^{ij} - \dot{c}^i \dot{c}^j, \quad R_{ij} = p_m R^m_{ijk} \dot{c}^k,$$

and R^m_{ijk} are components of the *Riemann curvature tensor*

$$R^m_{ijk} = \frac{\partial \Gamma^m_{ik}}{\partial x^j} - \frac{\partial \Gamma^m_{ij}}{\partial x^k} + \Gamma^s_{ik}\Gamma^m_{js} - \Gamma^s_{ij}\Gamma^m_{ks}.$$

Assuming G satisfies (C3) at $t = 0$, this condition holds for all t.

It should be emphasized that the above equations for c, p and G are independent of local coordinates. That is, these objects geometrize using Riemannian geometry.

1.2. *The transverse Riccati equation*

In this section we review the transverse Riccati equation. It is the projection of the Riccati equation (1.6) onto the orthogonal complement of the underlying geodesic. We also show how this transverse Riccati equation in related to transverse Jacobi equation.

Let $\Pi = \Pi_i^j(t)\frac{\partial}{\partial x^j} \otimes dx^i|_{c(t)}$ be the tensor determined by

$$\Pi_i^j = g_{ia}C^{aj} = \delta_i^j - p_i \dot{c}^j,$$

where δ_i^j is the *Kronecker delta symbol.* Then Π is a projection that maps $T_{c(t)}M$ onto the *orthogonal complement* $c^\perp$ of $\dot{c}(t)$,

$$c^\perp|_t = \{a \in T_{c(t)}M : g(a, \dot{c}(t)) = 0\}.$$

A tensor L is *transversal* if L is a $(1,1)$-tensor on c and $\Pi L \Pi = L$. We identify such tensors with pointwise linear maps $\dot{c}^\perp \to \dot{c}^\perp$.

When working with $\dot{c}^\perp$, it is convenient to use *Fermi coordinates* [18, 2, 4]. These are local coordinates on a Riemannian manifold adapted to a fixed geodesic. Suppose $c\colon (a', b') \to M$ is a geodesic that can be covered with one coordinate chart, and suppose that $a' < a < b < b'$. Then there exists local coordinates $\{\tilde{x}^i\}$ defined in the tube $\tilde{x}^1 \in (a,b), (\tilde{x}^2)^2 + \cdots + (\tilde{x}^n)^2 < \varepsilon$, and for $t \in (a,b)$ these satisfy:

$$(t, 0, 0) \text{ represents } c(t), \quad \tilde{g}_{ij}(c(t)) = \delta_{ij}, \quad \tilde{\Gamma}^i_{jk}(c(t)) = 0.$$

The last property implies that in Fermi coordinates, the covariant derivative of a transverse tensor coincides with the usual derivative. In consequence, the derivative of a composition of two $(1,1)$-tensors satisfies Leibniz' rule. It also follows that the derivative of a transverse tensor is again a transverse tensor.

Let E be the $(1,1)$-tensor

$$E = E^i_j \frac{\partial}{\partial x^i} \otimes dx^j, \quad E^i_j = g^{ia}G_{aj}. \tag{1.7}$$

Then E is a solution to the Riccati equation

$$E' + E\Pi E - K = 0 \tag{1.8}$$

where K is the curvature tensor

$$K = K^i_j \frac{\partial}{\partial x^i} \otimes dx^j, \quad K^i_j = g^{ia}R_{aj}.$$

To verify Eq. (1.8) it suffices to write out the equation in Fermi coordinates whence it reduces to the Riccati equation for G.

Let F be the projection of E onto $\dot{c}^\perp$. That is, the transverse $(1,1)$-tensor

$$F = \Pi \cdot E \cdot \Pi,$$

whence F satisfies the Riccati equation,

$$F' + F^2 - K = 0. \tag{1.9}$$

To verify Eq. (1.9), we have $\Pi' = 0$, so $F' = \Pi E' \Pi$, and Eq. (1.9) follows from Eq. (1.8) since K is a transverse tensor [4]. To formulate the next theorem, we need the *transpose* L^T and *complex transpose* L^* of a $(1,1)$-tensor L on c. The transpose L^T is determined by

$$g(L^T a, b) = g(a, Lb), \quad a, b \in T_{c(t)}M,$$

and the *complex transpose* $L^* = \bar{L}^T$, where $\bar{L}$ is the (componentwise) *complex conjugate* of L. If $L = L^i_j dx^j \otimes \frac{\partial}{\partial x^i}$ is a (possibly transverse) tensor on c, then components for L^T and L^* are

$$(L^T)^i_j = g^{ir} L^s_r g_{sj}, \quad (L^*)^i_j = g^{ir} \overline{L}^s_r g_{sj},$$

so if we represent L as a matrix $(L^i_j)_{ij}$ in Fermi coordinates, then matrices for L^T and L^* are just the transpose and Hermitian transpose of L_{ij}. Also, using Fermi coordinates, we have $(L^\tau)' = (L')^\tau$ for $\tau = *, T$.

The Siegel upper half plane can be seen as a generalization of the upper half plane of the complex plane. For an introduction, see [9].

Definition 1 (Siegel upper half plane). A transverse tensor $L\colon \dot{c}^\perp|_{c(t)} \to \dot{c}^\perp|_{c(t)}$ is in the *Siegel upper half plane* provided that

(i) L is symmetric ($L = L^T$),
(ii) $v \mapsto g(\operatorname{Im} L \cdot v, \overline{v})$ is positive definite for $v \in \dot{c}^\perp|_{c(t)}$.

Theorem 1 (Complex Jacobi tensor). *Suppose A_0, F_0 are $(1,1)$-tensors at $c(0)$. Then there exists a unique $(1,1)$-tensor Y on c determined by*

$$\begin{aligned} Y'' - K \cdot Y &= 0, \\ Y(0) = A_0, \quad Y'(0) &= F_0. \end{aligned} \tag{1.10}$$

This Y satisfies the following properties:

(1) *If A_0, F_0 are transverse, then Y is transverse.*
(2) *For any initial values, $\Pi Y \Pi$ is a transverse solution from transverse initial values $\Pi A_0 \Pi$, $\Pi F_0 \Pi$. Also, if Y and $\tilde{Y}$ are solutions from initial values A_0, F_0, and $\Pi A_0 \Pi$, $\Pi F_0 \Pi$, respectively, then $\tilde{Y} = \Pi Y \Pi$.*
(3) *If Y_1 and Y_2 are two solutions to the complex Jacobi equation, then*

$$((Y_1')^\tau \cdot Y_2 - Y_1^\tau \cdot Y_2')' = 0, \quad \tau = T, *.$$

(4) *Suppose $A_0 = \operatorname{Id}|_{\dot{c}^\perp}$, and F_0 is a transverse tensor in the Siegel upper half plane. Then Y is invertible (as a map $\dot{c}^\perp \to \dot{c}^\perp$) for all t.*

Proof. We may write Eq. (1.10) as a first order linear equation,

$$\begin{aligned} \begin{pmatrix} Y \\ Z \end{pmatrix}' &= \begin{pmatrix} 0 & I \\ K & 0 \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix}, \\ \begin{pmatrix} Y \\ Z \end{pmatrix}\Big|_{t=0} &= \begin{pmatrix} A_0 \\ F_0 \end{pmatrix}. \end{aligned} \tag{1.11}$$

Therefore a unique Y exists for all $t \in I$. Property 1 follows by writing out these equations in Fermi coordinates; if the first row and column are zero at $t = 0$, they are zero for all t. Property 2 follows since $(\Pi L \Pi)' = \Pi L' \Pi$, and since K is transversal [4]. Property 3 follows by a short calculation using Eq. (1.10) and that K is symmetric [4]. For Property 4, suppose that $Y \cdot \eta = 0$ for some nonzero $\eta \in \dot{c}^\perp|_s$. By a parallel transport, we may extend η into a vector field on c. (Since this is just the Euclidean parallel transport in Fermi coordinates, we have $\eta \in \dot{c}^\perp$ for all t. In particular, $\eta(0) \neq 0$.) Let

$$f(t) = g(((Y')^* \cdot Y - Y^* \cdot Y') \cdot \eta, \overline{\eta}), \quad t \in I.$$

For a symmetric tensor we have $\operatorname{Im} S = \frac{-1}{2i}(S^* - S)$. Hence $f(0) \neq 0$. On the other hand, $f(s) = 0$ and $f' = 0$, so Y must be invertible. $\square$

1.3. *Connections between the Riccati and Jacobi equation*

The complex Riccati equation and the complex Jacobi equation (1.10) are closely related. To see this, let us assume that F_0 is a transverse tensor in the Siegel upper half plane. Furthermore, let F be the solution to Eq. (1.9) from $F(0) = F_0$, and let Y be the solution to Eq. (1.10) from $Y(0) = \operatorname{Id}|_{\dot{c}^\perp}, Y'(0) = F_0$. Then $F = Y' \cdot Y^{-1}$ and Y is given by

$$Y' = F \cdot Y,$$
$$Y(0) = \operatorname{Id}|_{\dot{c}^\perp}, \quad Y'(0) = F(0).$$

Hereafter, we always assume that F_0, F and Y are related in this way.

The next proposition can be seen as a coordinate-independent reformulation of Lemma 2.58 in [15].

Proposition 1. *Tensors F, Y' are invertible, and:*

(1) *F is in the Siegel upper half plane for all t.*
(2) $(\det Y)' = \operatorname{trace} F \cdot (\det Y)$.
(3) *If we denote by $\operatorname{Im} F_0$ also the parallel transport of $\operatorname{Im} F_0$ along c, then*

$$\operatorname{Im} F = (Y^{-1})^* \cdot \operatorname{Im} F_0 \cdot Y^{-1},$$
$$\operatorname{Im}(F^{-1}) = -(Y')^{-*} \cdot \operatorname{Im} F_0 \cdot (Y')^{-1}.$$

Proof. That F is invertible is proven in [4], and Y' is invertible as $Y' = F \cdot Y$. By Theorem 1.3, F is symmetric. To see that F is positive definite let us first note that in Fermi coordinates the parallel transport is just the Euclidean transport. Therefore, the spectrum of a $(1,1)$-tensor is preserved under a parallel transport. (Alternatively, one can write down a Lax-pair type equation in arbitrary local coordinates.) Hence the spectrum of the parallel transport of $\operatorname{Im} F_0$ remains constant along c, and the result follows from Property 3 proven below. Property 2 follows from the matrix identity

$$(\det A)' = \operatorname{trace}(A'A^{-1}) \cdot \det A.$$

For Property 3, we have

$$\frac{-1}{2i}(Y'^*Y - Y^*Y') = \operatorname{Im} F_0$$

with given notation. The first claim follows since

$$\begin{aligned}\operatorname{Im} F &= \frac{-1}{2i}(F^* - F) \\ &= \frac{-1}{2i} Y^{-*}(Y'^*Y - Y^*Y')Y^{-1}.\end{aligned}$$

The proof of the latter claim is analogous (F^{-1} is also symmetric). □

Theorem 2 (A constant of motion for the Riccati equation). *Let E is the tensor defined in Eq. (1.7), and let F be its projection onto $\{\dot{c}\}^\perp$. Then*

$$\frac{\det \operatorname{Im} E}{\det \operatorname{Im} F} = \text{constant}.$$

Proof. In Fermi coordinates let us partition the symmetric tensor E as

$$E = \left(\begin{array}{c|c} \lambda & u^T \\ \hline u & F \end{array}\right),$$

where $\lambda\colon I \to \mathbb{C}$, $u\colon I \to \mathbb{C}^{(n-1)\times 1}$, and $F\colon I \to \mathbb{C}^{(n-1)\times(n-1)}$ are functions with initial values λ_0, u_0, F_0, respectively. Here F is also the local representation of the projection of E onto $\dot{c}^\perp$.

Let us also denote the real and imaginary parts by sub-indices r and i, respectively. Say, $\lambda = \lambda_r + i\lambda_i$. Using a Shur complement on $\operatorname{Im} E$, we have

$$\frac{\det \operatorname{Im} E}{\det \operatorname{Im} F} = \lambda_i - u_i^T \cdot (F_i)^{-1} \cdot u_i. \tag{1.12}$$

Since E is a solution to the Riccati equation (1.8), λ, u, and F satisfy

$$\begin{aligned} \lambda' &= -u^T \cdot u, \\ u' &= -F \cdot u, \\ F' + F^2 &= K, \end{aligned}$$

where K is a matrix representing the transverse curvature tensor K in Fermi coordinates. Taking imaginary parts gives

$$\begin{aligned} \lambda_i' &= -2u_r^T \cdot u_i, \\ u_i' &= -F_r \cdot u_i - F_i \cdot u_r, \\ F_i' &= -F_r \cdot F_i - F_i \cdot F_r. \end{aligned}$$

Differentiating the right-hand side in Eq. (1.12) gives the result. □

2. Equation for Leading Term at $(c(t), t)$

At the center of a Gaussian beam (that is, at $z = 0$), we have $e^{iP\theta} = 1$, and assuming that P^k grows much faster than $u_k(c(t), t)$, we have

$$u(c(t), t) \approx u_0(c(t), t).$$

It is therefore motivated to study the function

$$a(t) = u_0(c(t), t).$$

We already know that this function is determined by Eq. (1.3). In this section we show how a is related to both the Riccati equation and the Jacobi equation.

Let us point out that the equation for $a(t)$ in local coordinates is well known. For example, see [23] or Sec. 2.4.19 in [15]. Thus the novelty of the below equations for $a(t)$ is that they give a geometric description for $a(t)$ that does not depend on local coordinates.

Theorem 3. *Function $a(t)$ satisfy*

$$\frac{d}{dt} a(t) = -\frac{1}{2} \operatorname{trace} F(t)\, a(t). \tag{2.1}$$

Proof. The proof is based on the following identities:

$$\theta_t(c(t), t) = -1, \tag{2.2}$$

$$\nabla\theta(c(t), t) = \dot{c}(t), \tag{2.3}$$

$$\theta_{tt}(c(t), t) = G_{ij}\dot{c}^i\dot{c}^j, \tag{2.4}$$

$$\Delta\theta(c(t), t) = G_{ij}g^{ij}. \tag{2.5}$$

Equations (2.2)–(2.3) follow by differentiating (1.5). Equation (2.4) follows since $\dot{p}_l = \Gamma_{jkl}\dot{c}^j\dot{c}^k$, which in turn follows from the geodesic equation and the identity $\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ijk} + \Gamma_{jik}$. Equation (2.5) follows since

$$\Delta f = g^{ij}(\operatorname{Hess} f)_{ij},$$

where $\operatorname{Hess} f$ is the *Hessian* of a function f,

$$\operatorname{Hess} f = \left(\frac{\partial^2 f}{\partial x^i \, \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \right) dx^i \otimes dx^j.$$

For any $u \in C^\infty(M \times I)$, we have

$$\begin{aligned}\mathscr{L}_\theta(u)\,(c(t),t) &= -2\left(u_t + \dot{c}^i \frac{\partial u}{\partial x^i}\right) + (\partial_t^2 - \Delta)\theta \cdot u \\ &= -2\frac{d}{dt} u(c(t),t) - C^{ij} G_{ij} u,\end{aligned}$$

where all evaluations are at t or $(c(t),t)$. Since $\operatorname{trace} F = (\Pi E \Pi)^i_i = C^{ij} G_{ij}$, the result follows from Eq. (1.3). ☐

Property 2 in Lemma 1 now shows that we may write a in terms of Y:

Theorem 4. *The unique solution to Eq.* (2.1) *is*

$$a(t) = \frac{1}{(\det Y(t))^{1/2}} \cdot a_0,$$

where $a_0 \in \mathbb{C}$ is the initial value for a at $t = 0$.

3. Conservation of Energy

For a complex solution $u(x,t)$ to the wave equation, its *energy density* is defined as [15]

$$E_u(x,t) = \frac{1}{2}(|u_t|^2 + g(\nabla u, \overline{\nabla u})),$$

and the *energy* of u at time t is

$$E(t) = \int_M E_u(x,t) dV,$$

where $dV = \sqrt{\det g} dx$ is the Riemannian volume form of M. A key property of an exact solution to the wave equation is that its energy is constant (as long as the solution does not intersect a possible boundary). Next we derive a similar result for Gaussian beams.

The energy density of u (see Eq. (1.1)) is given by

$$E_u(x,t) = e_\theta |u_0|^2 e^{-P z^T \cdot \operatorname{Im} G \cdot z} P^2 + \text{lower order terms in } P,$$

where

$$e_\theta(x,t) = \frac{1}{2}(|\theta_t|^2 + g(\nabla\theta, \overline{\nabla\theta})).$$

In the previous section, we solved $u_0(c(t),t)$. By neglecting x-dependency in u_0, let us consider the approximate solution

$$v(x,t) = \frac{1}{(\det Y(t))^{1/2}} e^{iP\theta(x,t)} \chi(x,t),$$

where $\chi(x,t)$ is a smooth cut-off function that at time t equals 1 near $c(t)$ (see e.g. [15]), and

$$\theta(x,t) = \phi_0 + p_i(t) z^i + \frac{1}{2} H_{ij} z^i z^j.$$

The next theorem states that asymptotically the energy of the approximate solution v is constant. In other words, the leading term in Eq. (3.1) does not depend on time. The theorem also states that physically, the initial value $\det \operatorname{Im} E(0)$ describes the energy of the solution.

Theorem 5 (Conservation of energy). *Let $M = \mathbb{R}^n$ be equipped with an arbitrary Riemannian metric. Then for each $t \in I$, the energy for the approximate solution v is*

$$E_v(t) = \sqrt{\frac{\pi^n}{\det \operatorname{Im} E_0}}\frac{1}{P^{\frac{n}{2}-2}} + O\left(\frac{1}{P^{\frac{n}{2}-1}}\right), \quad P \to \infty. \tag{3.1}$$

Here E_0 is the initial value for tensor E defined in Eq. (1.7).

Proof. The energy density of v is

$$E_v(x,t) = e^{-Pz^T \cdot \operatorname{Im} G \cdot z}\left(e_\theta(x,t)\frac{1}{|\det Y(t)|}\chi^2 \; P^2 + A(x,t)P + B(x,t)\right),$$

where A, B are smooth functions with compact support for each t. Integrating each term over $\mathbb{R}^n$ and using Lemma 1 shows that the latter two terms contribute to $E_v(t)$ with an $O(1/P^{\frac{n}{2}-1})$-term. Since c is pathlength parameterized, we have $e_\theta(c(t),t) = 1$, and applying Lemma 1 to the first term gives

$$\sqrt{\frac{\pi^n \det g}{|\det Y|^2 \cdot \det \operatorname{Im} G}}\frac{1}{P^{\frac{n}{2}-2}} + O\left(\frac{1}{P^{\frac{n}{2}-1}}\right).$$

By Eq. (1.7) we have $\det \operatorname{Im} G = \det g \cdot \det \operatorname{Im} E$, and the result follows by Proposition 1.3 and Theorem 2. □

Appendix A. An Asymptotic Expansion

Lemma 1. *Suppose $f\colon \mathbb{R}^n \to \mathbb{C}$ is a rapidly decreasing function. If A is a real, positive definite symmetric matrix then*

$$\int_{\mathbb{R}^n} f(x)e^{-\frac{P}{2}x^T \cdot A \cdot x}dx = \frac{(2\pi)^{\frac{n}{2}}}{(\det A)^{\frac{1}{2}}}\frac{f(0)}{P^{\frac{n}{2}}} + O\left(\frac{1}{P^{\frac{n}{2}+1}}\right)$$

when $P \to \infty$.

The below proof follows [5] where a similar result is proven.

The space of *rapidly decreasing functions $\mathscr{S}$* consists of smooth functions $f\colon \mathbb{R}^n \to \mathbb{C}$ such that $x^\alpha \partial^\beta f$ is bounded in $\mathbb{R}^n$ for all multi-indices α, β. For such functions the *Fourier transform* $\mathscr{F}: \mathscr{S} \to \mathscr{S}$ and its inverse are

$$\mathscr{F}f(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx, \quad \mathscr{F}^{-1}g(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} g(\xi)e^{ix\cdot\xi}d\xi.$$

Proof of Lemma 1. If A is a real symmetric positive definite matrix, then

$$\mathscr{F}\left(e^{-\frac{1}{2}x^T \cdot A \cdot x}\right)(\xi) = \frac{(2\pi)^{n/2}}{(\det A)^{1/2}}e^{-\frac{1}{2}\xi^T \cdot A^{-1} \cdot \xi}, \quad \xi \in \mathbb{R}^n. \tag{A.1}$$

The case $A = I$ is proven in [22], and the general case follows by a coordinate change $y = S \cdot x$, where S is a symmetric positive square root of A. Let us also note that if $\Psi\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial function satisfying $\Psi \geq 0$, then

$$\int_{\mathbb{R}^n} f(x)e^{-\frac{1}{P}\Psi(x)}dx = \int_{\mathbb{R}^n} f(x)dx + O\left(\frac{1}{P}\right) \tag{A.2}$$

as $P \to \infty$. This follows from the Taylor series,

$$e^x = 1 + \theta(x)x, \quad x \in \mathbb{R},$$

where the Lagrange remainder term θ satisfies $|\theta(x)| \leq 1$ for $x \leq 0$. The inequality follows since $\theta(x) = e^\xi$ for some $\xi \in [0,x]$. For functions f, g in $\mathscr{S}$, we have $\int_{\mathbb{R}^n}\mathscr{F}f \cdot g dx = \int_{\mathbb{R}^n} f \cdot \mathscr{F}g dx$ [22]. Using the above observations, the left hand side in the claim can be written as

$$\int_{\mathbb{R}^n} \mathscr{F}^{-1} f \cdot \mathscr{F} e^{-\frac{P}{2} x^T \cdot A \cdot x} dx = B \int_{\mathbb{R}^n} \mathscr{F}^{-1} f(\xi) \cdot e^{-\frac{1}{2P} \xi^T \cdot A^{-1} \cdot \xi} d\xi, \quad B = \frac{1}{(\det A)^{1/2}} \left(\frac{2\pi}{P} \right)^{\frac{n}{2}}$$

$$= B \left(\int_{\mathbb{R}^n} \mathscr{F}^{-1} f d\xi + O\left(\frac{1}{P} \right) \right).$$

The result follows since $\int_{\mathbb{R}^n} \mathscr{F}^{-1} f dx = (\mathscr{F}\mathscr{F}^{-1} f)(0) = f(0)$. □

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