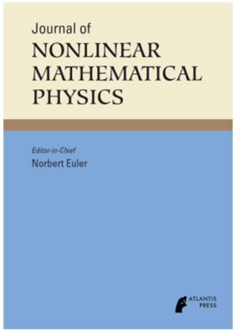

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LEPAGE EQUIVALENTS OF SECOND-ORDER EULER–LAGRANGE FORMS AND THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS

OLGA KRUPKOVÁ[*,†,‡] and DANA SMETANOVÁ[*,§]

[*]*Department of Algebra and Geometry*
Faculty of Science, Palacký University
Tomkova 40, 779 00 Olomouc, Czech Republic

[†]*Department of Mathematics La Trobe University*
Bundoora, Victoria 3086, Australia
[‡]*krupkova@inf.upol.cz*
[§]*Dana.Smetanova@upol.cz*

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In the calculus of variations, Lepage $(n+1)$-forms are *closed* differential forms, representing *Euler–Lagrange equations*. They are fundamental for investigation of variational equations by means of exterior differential systems methods, with important applications in Hamilton and Hamilton–Jacobi theory and theory of integration of variational equations. In this paper, Lepage equivalents of second-order Euler–Lagrange quasi-linear PDE's are characterised explicitly. A closed $(n+1)$-form uniquely determined by the Euler–Lagrange form is constructed, and used to find a geometric solution of the inverse problem of the calculus of variations.

Keywords: Second-order Euler–Lagrange equations; Euler–Lagrange form; Lepage form; Lepage equivalent of a Lagrangian; Lepage equivalent of an Euler–Lagrange form; inverse problem of the calculus of variations.

Mathematics Subject Classification 2000: 58E30, 35R30, 70S05

1. Introduction

The inverse problem of the calculus of variations is concerned with the question when a system of ordinary or partial differential equations of order r ($r \geq 1$) identifies with Euler–Lagrange equations, i.e., equations for extremals of a variational functional. This problem was first considered by Helmholtz in 1887, for a system of second order ordinary differential equations [11]. In his seminal paper, Helmholtz found necessary conditions for variationality, now called *Helmholtz conditions* (Mayer [23] later proved that the conditions are also sufficient). Since that time, questions on existence, multiplicity and construction of Lagrangians to differential equations have been investigated by many authors. A remarkable progress in the solution of the inverse variational problem was achieved around 1980 by methods of differential geometry and global analysis, in connection with new developments of the calculus of variations on fibred manifolds and the theory of variational bicomplexes: Helmholtz conditions were generalised to PDE's of an arbitrary order by Anderson and Duchamp [1] and Krupka [14], and the inverse problem was extended to study conditions for existence of a *global* Lagrangian. Nowadays, *geometric* and *global* rather than analytical aspects of the inverse variational problem are of main interest, and *relations between variationality and geometry of differential equations* are intensively studied and explored (see e.g. [4, 21, 22]).

One of the most significant advances in this direction was the discovery of an intimate relationship between *variational equations* and *closed forms*, i.e., between the *Euler–Lagrange operator* and the operator of the *exterior derivative of differential forms* (Crampin, Prince and Thompson [3], Dedecker and Tulczyjew [5], Krupka [14, 15], Tonti [25]). Due to this relationship, the local version of the inverse variational problem is transferred to an application of the Poincaré Lemma, and global existence results follow from De Rham Theorem. A direct geometric expression of this property is realised within the concept of *Lepage* $(n+1)$*-form*, where n is the number of independent variables (Krupková [17–19]).

Lepage $(n+1)$-forms are *closed differential forms*, exclusively representing *variational equations*; they are also called *Lepage equivalents* of variational equations. Besides questions connected with the inverse variational problem, they are used to study variational equations and their solutions with exterior differential systems methods. Most important applications arise in investigations of symmetries and conservation laws, Hamilton and Hamilton–Jacobi theory, and integration of variational equations. Techniques using Lepage forms also hold the promise of a natural extension of methods and results from the calculus of variations to the class of differential equations for which no Lagrangian exists.

For ordinary variational equations (in physical terminology "higher-order mechanics"), the theory of Lepage $(n+1)$-forms is well-established. For an exposition of results and applications with stress on the geometry of ordinary differential equations and the inverse problem of the calculus of variations we refer the reader to the book [20] and recent survey papers [21, 22]. On the other hand, for partial differential equations ("field theory") results achieved so far are by no means complete.

The aim of this paper is to investigate Lepage $(n+1)$-forms associated with *second-order Euler–Lagrange quasi-linear PDE's*. After a brief survey of the current status of the subject in Sec. 2, new results are reported in Sec. 3 and proved in Sec. 4.

While for *ordinary* differential equations there is a *one-to-one correspondence* between Lepage 2-forms and variational equations, for partial differential equations the situation is more complicated. We study the *structure* of Lepage $(n+1)$-forms, and provide an *explicit characterisation of all* corresponding *Lepage equivalents*. It turns out, however, that the class of Lepage equivalents contains a distinguished (local) "fundamental equivalent" uniquely determined by the Euler–Lagrange expressions. We discuss global existence of such an $(n+1)$-form, and find its relationship with the so called "fundamental Lepage equivalent of a Lagrangian" (Krupka n-form) [2, 13]. Finally, the closed $(n+1)$-form uniquely determined by the Euler–Lagrange form is used to obtain the *variationality conditions in an intrinsic form*.

Proofs of the theorems are quite straightforward, however, sometimes require long and difficult calculations. To make the article easily accessible to different readers, we decided to divide the exposition into two parts: Results are summarised in Sec. 3, and for interested readers, complete proofs are included in Sec. 4.

Finally, we note that analogous differential forms were considered on Grassmann bundles by Grigore and Popp, and Grigore [8, 9]. They introduced closed $(n+1)$-forms representing variational equations ("Lagrange–Souriau forms"), and used them to study Noether symmetries.

2. Lepage Forms

Throughout the paper, all manifolds and mappings are assumed smooth, and the summation over repeated indices is used whenever appropriate. The background for our considerations is the theory of jet bundles and the calculus of variations on fibred manifolds (see e.g. [16, 20, 24]). We consider a fibred manifold $\pi : Y \to X$, $\dim X = n$, $\dim Y = n+m$, where $n, m \geq 1$. For $r = 1, 2$ we denote by $\pi_r : J^rY \to X$ the r-jet prolongation of π, and by $\pi_{r,k} : J^rY \to J^kY$, $0 \leq k \leq r$, the canonical projections (here $J^0Y = Y$). A *section* of π is a mapping $\gamma : U \to Y$, where $U \subset X$ is an open set, such that $\pi \circ \gamma = \mathrm{id}_U$. The r-jet prolongation of γ is denoted by $J^r\gamma$; it is a section of the fibred manifold π_r.

On jet bundles it is convenient to use vector fields and differential forms adapted to the fibred structure [12]: A vector field ξ on J^rY is called *π_r-vertical* if it projects onto the zero vector field on X. A differential q-form η on J^rY is called *horizontal* (or, 0-*contact*) with respect to the projection π_r, if $i_\xi \eta = 0$ for every π_r-vertical vector field ξ on J^rY; η is called *contact* if $J^r\gamma^*\eta = 0$ for every section γ of π. A contact form is said to be 1-contact if for every vertical vector field ξ, the contraction $i_\xi\eta$ is horizontal. Recurrently, η is said to be k-contact if for every vertical vector field ξ, $i_\xi\eta$ is $(k-1)$-contact. We have a useful **Structure Theorem** due to Krupka [12], stating that *every q-form η on J^rY admits a unique decomposition as a sum of forms on $J^{r+1}Y$ as follows*:

$$\pi^*_{r+1,r}\eta = h\eta + \sum_{k=1}^{q} p_k\eta, \tag{2.1}$$

where $h\eta$ is a horizontal form (called the horizontal part of η); *and $p_k\eta$, $1 \le k \le q$, is a k-contact form* (called the k-contact part of η).

If (x^i, y^σ), $1 \le i \le n$, $1 \le \sigma \le m$, are fibred coordinates on Y, defined on an open set $V \subset Y$, we denote by $(x^i, y^\sigma, y^\sigma_j)$, and $(x^i, y^\sigma, y^\sigma_j, y^\sigma_{j_1 j_2})$, $j_1 \le j_2$, the associated coordinates on J^1Y and J^2Y, respectively. Next, we write

$$\omega_0 = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n, \quad \omega_{i_1 i_2 \cdots i_k} = i_{\partial/\partial x^{i_k}} \omega_{i_1 i_2 \cdots i_{k-1}}, \quad 1 \le k \le n,$$

for the local volume on X and its contractions, and

$$\omega^\sigma = dy^\sigma - y^\sigma_l dx^l, \quad \omega^\sigma_j = dy^\sigma_j - y^\sigma_{jl} dx^l,$$

for the associated basis of contact 1-forms on $\pi^{-1}_{2,0}(V) \subset J^2Y$.

A *dynamical form* E of order r is a 1-contact $(n+1)$-form on J^rY, horizontal with respect to the projection onto Y. In fibred coordinates,

$$E = E_\sigma \omega^\sigma \wedge \omega_0,$$

where E_σ are local functions on J^rY.

Let λ be a first-order *Lagrangian*, i.e., a horizontal n-form on J^1Y. A differential n-form η is called *Lepage equivalent* of λ (see [12]) if in the decomposition (2.1), $h\eta = \lambda$, and $p_1 d\eta$ is a dynamical form; the $(n+1)$-form $E_\lambda = p_1 d\eta$ is then called the *Euler–Lagrange form* of λ. For a first-order Lagrangian λ the Euler–Lagrange form is defined on J^2Y and in every fibred chart reads

$$E_\lambda = \left(\frac{\partial L}{\partial y^\sigma} - d_j \frac{\partial L}{\partial y^\sigma_j} \right) \omega^\sigma \wedge \omega_0.$$

Note that components E_σ of E_λ are functions *affine in the second derivatives*, since

$$\frac{\partial E_\sigma}{\partial y^\nu_{jl}} = -\frac{1}{2} \left(\frac{\partial^2 L}{\partial y^\sigma_j \partial y^\nu_l} + \frac{\partial^2 L}{\partial y^\sigma_l \partial y^\nu_j} \right)$$

are defined on an open subset of J^1Y.

As proved in [12], to every first-order Lagrangian a (global) Lepage equivalent exists and is non-unique. The family of Lepage equivalents of λ contains distinguished representatives that are completely determined by the Lagrangian: here we mention the famous *Poincaré–Cartan form* [6, 7, 12],

$$\Theta_\lambda = L\omega_0 + \frac{\partial L}{\partial y_j^\sigma}\omega^\sigma \wedge \omega_j,$$

and the *Krupka form* [13] (see also [2]),

$$\rho_\lambda = L\omega_0 + \sum_{k=1}^{n} \frac{1}{(k!)^2} \frac{\partial^k L}{\partial y_{j_1}^{\nu_1} \cdots \partial y_{j_k}^{\nu_k}} \omega^{\nu_1} \wedge \cdots \wedge \omega^{\nu_k} \wedge \omega_{j_1 \cdots j_k}. \tag{2.2}$$

The latter Lepage equivalent of λ has the following important property (not possessed by the Poincaré–Cartan form): $d\rho_\lambda = 0 \Leftrightarrow E_\lambda = 0$.

Consider a dynamical form E on J^2Y. E is said to be *locally variational* if to every point in J^2Y one has a neighbourhood U, and a Lagrangian λ on U, such that $E|_U = E_\lambda$. It is known that E is locally variational if and only if the components of E satisfy the following identities:

$$\frac{\partial E_\sigma}{\partial y_{jk}^\nu} - \frac{\partial E_\nu}{\partial y_{jk}^\sigma} = 0, \tag{2.3}$$

$$\frac{\partial E_\sigma}{\partial y_j^\nu} + \frac{\partial E_\nu}{\partial y_j^\sigma} - 2d_k \frac{\partial E_\nu}{\partial y_{jk}^\sigma} = 0, \tag{2.4}$$

$$\frac{\partial E_\sigma}{\partial y^\nu} - \frac{\partial E_\nu}{\partial y^\sigma} + d_j \frac{\partial E_\nu}{\partial y_j^\sigma} - d_j d_k \frac{\partial E_\nu}{\partial y_{jk}^\sigma} = 0. \tag{2.5}$$

We recall a fundamental theorem due to Krupka [15], relating locally variational forms with closed forms:

Theorem 1. *A dynamical form E is locally variational if and only if to every point in the domain of E there exists a neighbourhood W and an at least 2-contact form F_W on W such that the form $\alpha_W = E + F_W$ is closed.*

A $(n+1)$-form α is called *Lepage equivalent* of E [17] if $p_1\alpha = E$ and $d\alpha = 0$. One can see immediately that *if α is a Lepage equivalent of E then, around every point, $\alpha = d\eta$ where η is a Lepage equivalent of a local Lagrangian for E.*

The above theorem guarantees *local existence* of Lepage equivalents; it does not provide us with explicit formulas for α_W by means of the components of E.

The problem of (global) existence and multiplicity of Lepage equivalents has been completely solved for locally variational forms on J^1Y in [10]:

Theorem 2. *Every first-order locally variational form E has a unique Lepage equivalent defined on Y. It is denoted by α_E and takes the form*

$$\alpha_E = E_\sigma \omega^\sigma \wedge \omega_0 + \sum_{k=1}^{n} \frac{1}{k!(k+1)!} \frac{\partial^k E_\sigma}{\partial y_{j_1}^{\nu_1} \cdots \partial y_{j_k}^{\nu_k}} \omega^\sigma \wedge \omega^{\nu_1} \wedge \cdots \wedge \omega^{\nu_k} \wedge \omega_{j_1 \cdots j_k}. \tag{2.6}$$

Moreover, in a neighbourhood U of every point in Y,

$$\alpha_E|_U = d\rho_\lambda, \tag{2.7}$$

where λ is a local first order Lagrangian for E. All Lepage equivalents of E are then described by the formula $\alpha = \alpha_E + \phi$ where ϕ is an arbitrary closed at least 2-contact form defined on J^rY, $r \geq 1$.

In the next section we shall be interested in similar questions for *second-order* Euler–Lagrange equations. To this end we shall use the following result [19]:

Lemma 1. *Let α be a Lepage equivalent of a locally variational form E on J^2Y. Then*

$$p_2\alpha = \frac{1}{2}\frac{\partial E_\sigma}{\partial y^\nu_j}\omega^\sigma \wedge \omega^\nu \wedge \omega_j + \frac{\partial E_\sigma}{\partial y^\nu_{jp}}\omega^\sigma \wedge \omega^\nu_p \wedge \omega_j + p_2 d\phi, \tag{2.8}$$

where ϕ is a 2-contact n-form.

3. Lepage Equivalents of Second Order Euler–Lagrange Forms: Results

Let us consider a dynamical form E on J^2Y, denote as above, $E = E_\sigma\omega^\sigma \wedge \omega_0$. Assume that in every fibred chart

$$\frac{\partial^2 E_\sigma}{\partial y^\nu_{jk}\partial y^\rho_{pq}} = 0, \tag{3.1}$$

meaning that the components E_σ of E are affine functions in the second derivatives.

In what follows, we shall study the structure of Lepage equivalents of Euler–Lagrange forms the components of which are affine in the second derivatives. In this section we summarise the results of the paper, complete proofs are postponed to the next section.

The problem is to find all closed $(n+1)$-forms α such that $p_1\alpha = E$. The closedness condition on α means that at least some of the components of the higher-degree contact parts of α depend upon the Euler–Lagrange expressions E_σ, $1 \leq \sigma \leq m$. This means that α splits into a (not necessarily invariant) sum

$$\alpha = \alpha_E + \phi, \tag{3.2}$$

where α_E is completely determined by the Euler–Lagrange expressions, while ϕ does not depend upon E. Hence, the first step to solve the structure problem is to find the form α_E.

Theorem 3. *Let E be a locally variational form on J^2Y. If the condition* (3.1) *is satisfied then α_E is affine in the ω^ν_p's, and takes the form*

$$\begin{aligned}\alpha_E &= E_\sigma\omega^\sigma \wedge \omega_0 \\ &\quad + \sum_{k=1}^{n} \frac{1}{k!(k+1)!}\frac{\partial^k E_\sigma}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_k}_{j_k}}\omega^\sigma \wedge \omega^{\nu_1} \wedge \cdots \wedge \omega^{\nu_k} \wedge \omega_{j_1\cdots j_k} \\ &\quad + \sum_{k=1}^{n} \frac{1}{(k!)^2}\frac{\partial^k E_\sigma}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_{k-1}}_{j_{k-1}}\partial y^{\nu_k}_{j_k p}}\omega^\sigma \wedge \omega^{\nu_1} \wedge \cdots \wedge \omega^{\nu_{k-1}} \wedge \omega^{\nu_k}_p \wedge \omega_{j_1\cdots j_k}.\end{aligned} \tag{3.3}$$

Moreover, α_E is $\pi_{2,1}$-projectable.

In view of the preceding theorem we obtain the following solution to the problem of the structure of Lepage equivalents of "quasi-linear" second-order Euler–Lagrange equations:

Theorem 4. *Let E be a locally variational form on J^2Y satisfying* (3.1)*. Every Lepage equivalent α of E takes the form*

$$\alpha = \alpha_E + \phi, \tag{3.4}$$

where α_E is a closed (first-order) form, given by (3.3)*, and ϕ is a closed, at least 2-contact form.*

Corollary 1. *Let E be a locally variational form on J^2Y, satisfying condition* (3.1)*. Given a fibred chart (V, ψ) on Y with coordinates (x^i, y^σ), the $(n+1)$-form α_E determined by the Euler–Lagrange expressions of E and defined on $\pi^{-1}_{1,0}(V) \subset J^1Y$ by* (3.3) *is a Lepage equivalent of E.*

It remains to illuminate transformation properties of the form α_E with respect to fibred coordinates. Consider two overlapping charts (V,ψ), $\psi=(x^i,y^\sigma)$, and $(\bar V,\bar\psi)$, $\bar\psi=(\bar x^i,\bar y^\sigma)$ on Y. Using in the expression for α_E transformation formulas

$$\bar\omega_{j_i\cdots j_k}=\det\left(\frac{\partial\bar x}{\partial x}\right)\frac{\partial x^{p_1}}{\partial\bar x^{j_1}}\cdots\frac{\partial x^{p_k}}{\partial\bar x^{j_k}}\omega_{p_1\cdots p_k},$$

$$\bar\omega^\sigma=\frac{\partial y^\sigma}{\partial y^\nu}\omega^\nu,\quad \bar\omega^\sigma_j=\frac{\partial y^\sigma_j}{\partial y^\nu}\omega^\nu+\frac{\partial y^\sigma_j}{\partial y^\nu_p}\omega^\nu_p,$$

$$\bar y^\sigma_j=\frac{\partial x^k}{\partial\bar x^j}\left(\frac{\partial\bar y^\sigma}{\partial x^k}+\frac{\partial\bar y^\sigma}{\partial y^\rho}y^\rho_k\right),\quad \bar y^\sigma_{ji}=\frac{\partial x^k}{\partial\bar x^i}\left(\frac{\partial\bar y^\sigma_j}{\partial x^k}+\frac{\partial\bar y^\sigma_j}{\partial y^\rho}y^\rho_k+\frac{\partial\bar y^\sigma_j}{\partial y^\rho_p}y^\rho_{pk}\right),\tag{3.5}$$

$$\bar E_\sigma=\det\left(\frac{\partial x}{\partial\bar x}\right)\frac{\partial y^\nu}{\partial\bar y^\sigma}E_\nu,$$

and the relation

$$\frac{\partial y^\sigma_j}{\partial\bar y^\nu_k}\frac{\partial\bar y^\nu_k}{\partial y^\rho}=-\frac{\partial y^\sigma_j}{\partial\bar y^\nu}\frac{\partial\bar y^\nu}{\partial y^\rho},\tag{3.6}$$

we obtain the following result:

- *The at most* 2*-contact part* (*also called* principal part) *of* α_E

$$\hat\alpha_E=E_\sigma\omega^\sigma\wedge\omega_0+\frac{1}{2}\frac{\partial E_\sigma}{\partial y^\nu_j}\omega^\sigma\wedge\omega^\nu\wedge\omega_j+\frac{\partial E_\sigma}{\partial y^\nu_{jp}}\omega^\sigma\wedge\omega^\nu_p\wedge\omega_j\tag{3.7}$$

  is invariant with respect to fibred coordinate transformations. This means that formula (3.7) *defines a global differential form. The form* $\hat\alpha_E$, *however, is in general not closed.*
- *For* $k\geq 3$ *the form* $p_k\alpha_E$ *is generally not invariant. Consequently,* α_E *is not invariant, i.e., formula* (3.3) *does not define a global differential form.*

Remark. We have seen that the $(n+1)$-form α_E is global for $r=1$ but no longer for $r\geq 2$. We remind the reader that this situation is analogous to the case of the well-known Poincaré–Cartan form Θ_λ that is global for $r\leq 2$ but not for higher order Lagrangians (cf. e.g. [15]). An important case when α_E for second order E is global is when E arises from a global Lagrangian (see Theorem 6 below).

The next results clarify the meaning of the Lepage equivalent α_E of a locally variational form E.

Theorem 5. *Let* E *be a dynamical form on* J^2Y, *satisfying* (3.1). *The following conditions are equivalent*:

(1) *E is locally variational.*
(2) *α_E is closed.*
(3) $p_2 d\alpha_E=0$.
(4) *Components E_σ of E satisfy conditions* (4.23).

The above theorem provides us with a *geometric meaning of the variationality conditions*, as conditions, under which the $(n+1)$-form α_E is closed. Otherwise speaking,

$$p_2 d\alpha_E=0\tag{3.8}$$

is an *intrinsic expression of variationality conditions* (2.3)–(2.5) (respectively, (4.23) below).

Since the form α_E is a Lepage equivalent of a locally variational form E, around every point it equals $d\rho$, where ρ is a Lepage equivalent of a Lagrangian of E. We shall answer the question which of the Lepage equivalents of λ corresponds to α_E.

Theorem 6. *Let* λ *be a Lagrangian on* J^1Y*. Then*

$$d\rho_\lambda = \alpha_{E_\lambda},$$

where ρ_λ *is the Krupka form* (2.2) *of* λ.

4. Lepage Equivalents of Second Order Euler–Lagrange Forms: Proofs and Computations

Proof of Theorem 3. If α is an $(n+1)$-form such that $p_1\alpha = E$ then in fibred coordinates

$$\begin{aligned}\alpha &= E_\sigma \omega^\sigma \wedge \omega_0 \\ &\quad + \sum_{r+s=2}^{n+1} F^{j_1\cdots j_{r+s-1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_r,\rho_1\cdots\rho_s}\omega^{\sigma_1}\wedge\cdots\wedge\omega^{\sigma_r}\wedge\omega^{\rho_1}_{p_1}\wedge\cdots\wedge\omega^{\rho_s}_{p_s}\wedge\omega_{j_1\cdots j_{r+s-1}},\end{aligned} \tag{4.1}$$

where the components $F^{j_1\cdots j_{r+s-1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_r,\rho_1\cdots\rho_s}$ are completely skew-symmetric in lower indices $\sigma_1\cdots\sigma_r$, completely skew-symmetric in upper indices $j_1\cdots j_{r+s-1}$ and completely skew-symmetric in pairs of indices $(\rho_1 p_1)\cdots(\rho_s p_s)$. In what follows, let us denote the components of α_E, i.e., the part of the components of α, completely determined by E, by $\tilde{F}^{j_1\cdots j_{r+s-1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_r,\rho_1\cdots\rho_s}$.

Lemma 1 provides us with the following components of the $(n+1)$-form α_E:

$$\tilde{F}^j_{\sigma\nu} = \frac{1}{4}\left(\frac{\partial E_\sigma}{\partial y^\nu_j} - \frac{\partial E_\nu}{\partial y^\sigma_j}\right), \quad \tilde{F}^{j,p}_{\sigma,\rho} = F^{j,p}_{\sigma,\rho}|_{\mathrm{sym}\{jp\}} = \frac{\partial E_\sigma}{\partial y^\rho_{jp}}, \quad \tilde{F}^{j,\,p_1p_2}_{,\rho_1\rho_2} = 0. \tag{4.2}$$

Computing $d\alpha = 0$ we obtain the following relations:

(A) $r, s \geq 0$, $r+s = 2,\ldots,n$:

$$F^{j_1\cdots j_{r+s},p_1\cdots p_{s+1}}_{\sigma_1\cdots\sigma_r,\rho_1\cdots\rho_{s+1}}\Big|_{\mathrm{sym}\{j_{r+s}p_{s+1}\}} = \frac{1}{(s+1)(r+s)}\frac{\partial F^{j_1\cdots j_{r+s-1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_r,\rho_1\cdots\rho_s}}{\partial y^{\rho_{s+1}}_{p_{s+1}j_{r+s}}}, \tag{4.3}$$

and

$$\begin{aligned} &F^{j_1\cdots j_{r+s-1}p_{s+1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_r\rho_{s+1},\rho_1\cdots\rho_s}\Big|_{\mathrm{alt}\{(\rho_1p_1)\cdots(\rho_{s+1}p_{s+1})\}} \\ &\quad = (-1)^s\frac{1}{(r+1)}d_l F^{j_1\cdots j_{r+s-1}l,\,p_1\cdots p_{s+1}}_{\sigma_1\cdots\sigma_r,\,\rho_1\cdots\rho_{s+1}} \\ &\qquad + (-1)^s\frac{1}{(r+1)(r+s)}\frac{\partial F^{j_1\cdots j_{r+s-1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_r,\rho_1\cdots\rho_s}}{\partial y^{\rho_{s+1}}_{p_{s+1}}}\Bigg|_{\mathrm{alt}\{(\rho_1p_1)\cdots(\rho_{s+1}p_{s+1})\}} \\ &\qquad - \frac{1}{(r+1)(r+s)}\frac{\partial F^{j_1\cdots j_{r+s-1},p_1\cdots p_{s+1}}_{\sigma_1\cdots\sigma_{r-1},\rho_1\cdots\rho_{s+1}}}{\partial y^{\sigma_r}}\Bigg|_{\mathrm{alt}\{\sigma_1\cdots\sigma_r\}}.\end{aligned} \tag{4.4}$$

(B) $r, s \geq 0$, $r+s = n+1$:

$$(-1)^s\frac{\partial F^{j_1\cdots j_n,p_1\cdots p_s}_{\sigma_1\cdots\sigma_r,\rho_1\cdots\rho_s}}{\partial y^{\rho_{s+1}}_{p_{s+1}}}\Bigg|_{\mathrm{alt}\{(\rho_1p_1)\cdots(\rho_{s+1}p_{s+1})\}} - \frac{\partial F^{j_1\cdots j_n,p_1\cdots p_{s+1}}_{\sigma_1\cdots\sigma_{r-1},\rho_1\cdots\rho_{s+1}}}{\partial y^{\sigma_r}}\Bigg|_{\mathrm{alt}\{\sigma_1\cdots\sigma_r\}} = 0, \tag{4.5}$$

$$\frac{\partial F^{j_1\cdots j_n,p_1\cdots p_s}_{\sigma_1\cdots\sigma_r,\rho_1\cdots\rho_s}}{\partial y^\kappa_{l_1l_2}} = 0, \tag{4.6}$$

that yield, on one hand, *recurrence formulas for components of* α, and, on the other hand, *relations* between derivatives of the components of α. We note that if expressed by means of the components of the form α_E, these relations contain variationality conditions (2.3–2.5).

Above and in what follows, sym{} and alt{} means complete symmetrisation and skew-symmetrisation in the indicated indices (pairs of indices), respectively.

For the explicit computation of the form α_E we shall explore formulas (4.3) and (4.4). The desired recurrence formulas are obtained with help of the skew-symmetry conditions for the components of α following from (4.1). Using (4.3) we easily obtain

$$F^{j_1\cdots j_{r+s},p_1\cdots p_{s+1}}_{\sigma_1\cdots\sigma_r,\rho_1\cdots\rho_{s+1}}\Big|_{\mathrm{sym}\{j_{r+s}p_{s+1}\}} = \frac{1}{(s+1)(r+s)}\frac{\partial F^{j_1\cdots j_{r+s-1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_r,\rho_1\cdots\rho_s}}{\partial y^{\rho_{s+1}}_{p_{s+1}j_{r+s}}}\Bigg|_{\mathrm{alt}\{j_1\cdots j_{r+s}\},\mathrm{alt}\{(\rho_1p_1)\cdots(\rho_{s+1}p_{s+1})\}}. \tag{4.7}$$

Working with (4.4) we must be more careful: First of all, if $s=0$ we simply get for $r\geq 2$

$$F^{j_1\cdots j_r}_{\sigma_1\cdots\sigma_{r+1}} = \frac{1}{r(r+1)} \cdot\left(\frac{\partial F^{j_1\cdots j_{r-1},}_{\sigma_1\cdots\sigma_r,}}{\partial y^{\sigma_{r+1}}_{j_r}} - \frac{\partial F^{j_1\cdots j_{r-1},j_r}_{\sigma_1\cdots\sigma_{r-1},\sigma_{r+1}}}{\partial y^{\sigma_r}} + rd_lF^{j_1\cdots j_{r-1}l,j_r}_{\sigma_1\cdots\sigma_r,\sigma_{r+1}}\right)\Bigg|_{\mathrm{alt}\{\sigma_1\cdots\sigma_{r+1}\},\mathrm{alt}\{j_1\cdots j_r\}}. \tag{4.8}$$

Let $s\geq 1$. Accounting skew-symmetry conditions for the components of α we notice that

$$\begin{aligned}
&F^{j_1\cdots j_{r+s-1}p_{s+1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_r\rho_{s+1},\rho_1\cdots\rho_s}\Big|_{\mathrm{alt}\{(\rho_1p_1)\cdots(\rho_{s+1}p_{s+1})\}} = \frac{1}{(r+1)(r+s)}\\
&\cdot\Bigg((-1)^s\frac{\partial F^{j_1\cdots j_{r+s-1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_r,\rho_1\cdots\rho_s}}{\partial y^{\rho_{s+1}}_{p_{s+1}}}\Bigg|_{\mathrm{alt}\{(\rho_1p_1)\cdots(\rho_{s+1}p_{s+1})\},\mathrm{alt}\{\sigma_1\cdots\sigma_r\rho_{s+1}\},\mathrm{alt}\{j_1\cdots j_{r+s-1}p_{s+1}\}}\\
&-\frac{\partial F^{j_1\cdots j_{r+s-1},p_1\cdots p_{s+1}}_{\sigma_1\cdots\sigma_{r-1},\rho_1\cdots\rho_{s+1}}}{\partial y^{\sigma_r}}\Bigg|_{\mathrm{alt}\{\sigma_1\cdots\sigma_r\rho_1\cdots\rho_{s+1}\},\mathrm{alt}\{j_1\cdots j_{r+s-1}p_1\cdots p_{s+1}\}}\\
&+(-1)^s(r+s)d_lF^{j_1\cdots j_{r+s-1}l,\,p_1\cdots p_{s+1}}_{\sigma_1\cdots\sigma_r,\,\rho_1\cdots\rho_{s+1}}\Big|_{\mathrm{alt}\{\sigma_1\cdots\sigma_r\rho_1\cdots\rho_{s+1}\},\mathrm{alt}\{j_1\cdots j_{r+s-1}p_1\cdots p_{s+1}\}}\Bigg),
\end{aligned} \tag{4.9}$$

since

$$\frac{\partial F^{j_1\cdots j_{r+s-1},p_1\cdots p_{s+1}}_{\sigma_1\cdots\sigma_{r-1},\rho_1\cdots\rho_{s+1}}}{\partial y^{\sigma_r}}\Bigg|_{\mathrm{alt}\{\sigma_1\cdots\sigma_r\rho_{s+1}\},\mathrm{alt}\{j_1\cdots j_{r+s-1}p_{s+1}\}} = \frac{\partial F^{j_1\cdots j_{r+s-1},p_1\cdots p_{s+1}}_{\sigma_1\cdots\sigma_{r-1},\rho_1\cdots\rho_{s+1}}}{\partial y^{\sigma_r}}\Bigg|_{\mathrm{alt}\{\sigma_1\cdots\sigma_r\rho_1\cdots\rho_{s+1}\},\mathrm{alt}\{j_1\cdots j_{r+s-1}p_1\cdots p_{s+1}\}},$$

and similarly for $d_lF^{j_1\cdots j_{r+s-1}l,\,p_1\cdots p_{s+1}}_{\sigma_1\cdots\sigma_r,\,\rho_1\cdots\rho_{s+1}}$. This means, however, that formula (4.9) splits into two parts: one completely skew-symmetrised both in the lower indices $\sigma_1\cdots\sigma_r\rho_1\cdots\rho_{s+1}$ and the upper indices $j_1\cdots j_{r+s-1}p_1\cdots p_{s+1}$, and the complementary part. Applying the complete skew-symmetrisation, we notice that

$$F^{j_1\cdots j_{r+s-1}p_{s+1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_r\rho_{s+1},\rho_1\cdots\rho_s}\Big|_{\mathrm{alt}\{\sigma_1\cdots\sigma_r\rho_1\cdots\rho_{s+1}\},\mathrm{alt}\{j_1\cdots j_{r+s-1}p_1\cdots p_{s+1}\}} = 0,$$

being symmetric in the pairs of indices $(\rho_1p_1)\cdots(\rho_{s+1}p_{s+1})$, and at the same time by definition, skew-symmetric in the pairs of indices $(\rho_1p_1)\cdots(\rho_sp_s)$. Summarising, the completely skw-symmetrised part of the splitting of (4.9) provides only another relation between derivatives of components of α, while the recurrence formulas for the F's are provided by the complementary part of the splitting, and read as follows:

$$\begin{aligned} &F^{j_1\cdots j_{r+s-1}p_{s+1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_r\rho_{s+1},\rho_1\cdots\rho_s}\Big|_{\mathrm{alt}\{(\rho_1p_1)\cdots(\rho_{s+1}p_{s+1})\}} \\ &\quad = (-1)^s\frac{1}{(r+1)(r+s)} \\ &\quad\quad \cdot\frac{\partial F^{j_1\cdots j_{r+s-1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_r,\rho_1\cdots\rho_s}}{\partial y^{\rho_{s+1}}_{p_{s+1}}}\Bigg|_{\mathrm{alt}\{(\rho_1p_1)\cdots(\rho_{s+1}p_{s+1})\},\mathrm{alt}\{j_1\cdots j_{r+s-1}p_{s+1}\},\mathrm{alt}\{\sigma_1\cdots\sigma_r\rho_{s+1}\}} \\ &\quad = (-1)^s\frac{1}{(r+1)(r+s)} \\ &\quad\quad \cdot\frac{\partial F^{j_1\cdots j_{r+s-2}p_{s+1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_{r-1}\rho_{s+1},\rho_1\cdots\rho_s}}{\partial y^{\sigma_r}_{j_{r+s-1}}}\Bigg|_{\mathrm{alt}\{(\rho_1p_1)\cdots(\rho_{s+1}p_{s+1})\},\mathrm{alt}\{j_1\cdots j_{r+s-1}p_{s+1}\},\mathrm{alt}\{\sigma_1\cdots\sigma_r\rho_{s+1}\}}. \end{aligned} \tag{4.10}$$

Let us solve the recurrence formulas (4.7), (4.8) and (4.10) explicitly.

(i) Consider (4.8). First notice that the last two terms entering in this formula,

$$F^{j_1\cdots j_{r-1},\,j_r}_{\sigma_1\cdots\sigma_{r-1},\,\sigma_{r+1}}\Big|_{\mathrm{alt}\{\sigma_1\cdots\sigma_{r-1}\sigma_r\},\mathrm{alt}\{j_1\cdots j_r\}},\qquad F^{j_1\cdots j_{r-1}l,\,j_r}_{\sigma_1\cdots\sigma_r,\,\sigma_{r+1}}\Big|_{\mathrm{alt}\{\sigma_1\cdots\sigma_{r+1}\},\mathrm{alt}\{j_1\cdots j_r l\}}, \tag{4.11}$$

are completely symmetric in the pairs of indices $(\sigma_1j_1)\cdots(\sigma_{r-1}j_{r-1})(\sigma_{r+1}j_r)$ and $(\sigma_1j_1)\cdots(\sigma_{r-1}j_{r-1})(\sigma_r\,l)(\sigma_{r+1}j_r)$, respectively, and completely skew-symmetric in the upper indices. This means that these functions cannot be obtained from the recurrence formulas (4.7) and (4.10), i.e., in particular, they are independent upon a choice of E, and hence do not enter into α_E. In this way we obtain

$$\begin{aligned} \tilde F^{j_1\cdots j_r}_{\sigma_1\cdots\sigma_{r+1}} &= \frac{1}{r(r+1)}\frac{\partial\tilde F^{j_1\cdots j_{r-1}}_{\sigma_1\cdots\sigma_r}}{\partial y^{\sigma_{r+1}}_{j_r}}\Bigg|_{\mathrm{alt}\{\sigma_1\cdots\sigma_{r+1}\},\mathrm{alt}\{j_1\cdots j_r\}} \\ &= \frac{2}{r!(r+1)!}\,\frac{\partial^{r-1}\tilde F^{j_1}_{\sigma_1\sigma_2}}{\partial y^{\sigma_3}_{j_2}\cdots\partial y^{\sigma_{r+1}}_{j_r}}\Bigg|_{\mathrm{alt}\{\sigma_1\cdots\sigma_{r+1}\},\mathrm{alt}\{j_1\cdots j_r\}} \\ &= \frac{1}{r!(r+1)!}\,\frac{\partial^r E_{\sigma_1}}{\partial y^{\sigma_2}_{j_1}\partial y^{\sigma_3}_{j_2}\cdots\partial y^{\sigma_{r+1}}_{j_r}}\Bigg|_{\mathrm{alt}\{\sigma_1\cdots\sigma_{r+1}\},\mathrm{alt}\{j_1\cdots j_r\}}. \end{aligned} \tag{4.12}$$

(ii) Using (4.10) for $r-1$ and $s=1$ we can see that

$$\begin{aligned} &F^{j_1\cdots j_{r-1}p_2,p_1}_{\sigma_1\cdots\sigma_{r-1}\rho_2,\rho_1}\Big|_{\mathrm{alt}\{(\rho_1p_1)(\rho_2p_2)\}} \\ &\quad = -\frac{1}{r^2}\frac{\partial F^{j_1\cdots j_{r-2}p_2,p_1}_{\sigma_1\cdots\sigma_{r-2}\rho_2,\rho_1}}{\partial y^{\sigma_{r-1}}_{j_{r-1}}}\Bigg|_{\mathrm{alt}\{(\rho_1p_1)(\rho_2p_2)\},\mathrm{alt}\{j_1\cdots j_{r-1}p_2\},\mathrm{alt}\{\sigma_1\cdots\sigma_{r-1}\rho_2\}} \\ &\quad = -\frac{1}{r!^2}\frac{\partial^{r-1}F^{p_2,p_1}_{\rho_2,\rho_1}}{\partial y^{\sigma_1}_{j_1}\cdots\partial y^{\sigma_{r-1}}_{j_{r-1}}}\Bigg|_{\mathrm{alt}\{(\rho_1p_1)(\rho_2p_2)\},\mathrm{alt}\{j_1\cdots j_{r-1}p_2\},\mathrm{alt}\{\sigma_1\cdots\sigma_{r-1}\rho_2\}}, \end{aligned} \tag{4.13}$$

and since $F^{p_2,p_1}_{\rho_2,\rho_1}$ do not depend upon a choice of E, the F's above do not enter into α_E. In order to compute components at α_E, we have to use (4.7) for $s=0$. Then with help of (4.12) we get for $r \geq 2$

$$\tilde{F}^{j_1\cdots j_r,p_1}_{\sigma_1\cdots\sigma_r,\rho_1} = F^{j_1\cdots j_r,p_1}_{\sigma_1\cdots\sigma_r,\rho_1}\Big|_{\mathrm{sym}\{j_r p_1\}} = \frac{1}{r}\frac{\partial \tilde{F}^{j_1\cdots j_{r-1}}_{\sigma_1\cdots\sigma_r}}{\partial y^{\rho_1}_{p_1 j_r}}\Bigg|_{\mathrm{alt}\{j_1\cdots j_r\}}$$

$$= \frac{1}{r!^2}\frac{\partial^r E_{\sigma_1}}{\partial y^{\sigma_2}_{j_1}\partial y^{\sigma_3}_{j_2}\cdots\partial y^{\sigma_r}_{j_{r-1}}\partial y^{\rho_1}_{p_1 j_r}}\Bigg|_{\mathrm{alt}\{\sigma_1\cdots\sigma_r\},\mathrm{alt}\{j_1\cdots j_r\}}$$

$$= \frac{1}{r!^2}\frac{\partial^r E_{\rho_1}}{\partial y^{\sigma_2}_{j_1}\partial y^{\sigma_3}_{j_2}\cdots\partial y^{\sigma_r}_{j_{r-1}}\partial y^{\sigma_1}_{j_r p_1}}\Bigg|_{\mathrm{alt}\{\sigma_1\cdots\sigma_r\},\mathrm{alt}\{j_1\cdots j_r\}}, \tag{4.14}$$

since E is variational and satisfies (2.3).

(iii) Consider (4.7) for $r = 0$. It holds for $s \geq 2$

$$F^{j_1\cdots j_s,p_1\cdots p_{s+1}}_{,\rho_1\cdots\rho_{s+1}}\Big|_{\mathrm{sym}\{j_s p_{s+1}\}} = \frac{1}{s(s+1)}\frac{\partial F^{j_1\cdots j_{s-1},p_1\cdots p_s}_{,\rho_1\cdots\rho_s}}{\partial y^{\rho_{s+1}}_{p_{s+1}j_s}}\Bigg|_{\mathrm{alt}\{j_1\cdots j_s\},\mathrm{alt}\{(\rho_1 p_1)\cdots(\rho_{s+1}p_{s+1})\}}. \tag{4.15}$$

Since by Lemma 1 the functions $F^{j_1,\,p_1p_2}_{,\rho_1\rho_2}$ do not depend upon the Euler–Lagrange expressions, we get that in (4.1) the $F^{j_1\cdots j_s,p_1\cdots p_{s+1}}_{,\rho_1\cdots\rho_{s+1}}|_{\mathrm{sym}\{j_s p_{s+1}\}}$ are independent of a choice of E. Moreover, condition $d\alpha = 0$ gives no formulas for the skew-symmetric parts of these functions. Hence, all components of α_E for $r = 0$ are equal to zero. Note that this means that the form α_E belongs to the ideal generated by the one-forms ω^σ, $1 \leq \sigma \leq m$.

(iv) Finally, we shall show that the remaining components of α_E are equal to zero. To this end we first notice that (4.10) gives $F^{j_1\cdots j_{r+s-1}p_{s+1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_r\rho_{s+1},\rho_1\cdots\rho_s}|_{\mathrm{alt}\{(\rho_1p_1)(\rho_{s+1}p_{s+1})\}}$ expressed by means of derivatives of $F^{j_1\cdots j_{s-1}p_{s+1},p_1\cdots p_s}_{\rho_{s+1},\rho_1\cdots\rho_s}|_{\mathrm{alt}\{(\rho_1p_1)(\rho_{s+1}p_{s+1})\}}$. However, we have at disposal formulas for the symmetrised part of the latter functions in p_sp_{s+1}. The remaining parts are left arbitrary, i.e. do not contribute to α_E. Thus, next we have to consider (4.7) for $r = 1$. Substituting $\tilde{F}^{j,p}_{\sigma,\rho}$ from (4.2) and using assumption (3.1) we get for $s \geq 1$

$$\tilde{F}^{j_1\cdots j_{s+1},p_1\cdots p_{s+1}}_{\sigma,\rho_1\cdots\rho_{s+1}} = F^{j_1\cdots j_{s+1},p_1\cdots p_{s+1}}_{\sigma,\rho_1\cdots\rho_{s+1}}\Big|_{\mathrm{sym}\{j_{s+1}p_{s+1}\}}$$

$$= \frac{1}{(s+1)^2}\frac{\partial F^{j_1\cdots j_s,p_1\cdots p_s}_{\sigma,\rho_1\cdots\rho_s}}{\partial y^{\rho_{s+1}}_{p_{s+1}j_{s+1}}}\Bigg|_{\mathrm{alt}\{j_1\cdots j_{s+1}\},\mathrm{alt}\{(\rho_1p_1)\cdots(\rho_{s+1}p_{s+1})\}}$$

$$= \frac{1}{(s+1)!^2}\frac{\partial^{s+1}E_\sigma}{\partial y^{\rho_1}_{p_1j_1}\cdots\partial y^{\rho_{s+1}}_{p_{s+1}j_{s+1}}} = 0, \tag{4.16}$$

so that also $\tilde{F}^{j_1\cdots j_{r+s-1}p_{s+1},p_1\cdots p_s}_{\sigma_1\cdots\sigma_r\rho_{s+1},\rho_1\cdots\rho_s}|_{\mathrm{alt}\{(\rho_1p_1)(\rho_{s+1}p_{s+1})\}} = 0$. Hence

$$\tilde{F}^{j_1\cdots j_{r+s},p_1\cdots p_{s+1}}_{\sigma_1\cdots\sigma_r,\rho_1\cdots\rho_{s+1}} = F^{j_1\cdots j_{r+s},p_1\cdots p_{s+1}}_{\sigma_1\cdots\sigma_r,\rho_1\cdots\rho_{s+1}}\Big|_{\mathrm{sym}\{j_{s+1}p_{s+1}\}} = 0, \tag{4.17}$$

in view of (4.7), (4.14), and assumption (3.1).

It remains to prove that α_E is projectable onto an open subset of J^1Y. To this end let us write

$$E_\sigma = A_\sigma + B^{jk}_{\sigma\nu}y^\nu_{jk}, \tag{4.18}$$

where we may assume $B^{jk}_{\sigma\nu}$ symmetric in j,k. Then

$$\begin{aligned}\alpha_E &= \cdots + B^{pq}_{\sigma\nu} y^\nu_{pq}\, \omega^\sigma \wedge \omega_0 \\ &\quad + \sum_{k=1}^{n} \frac{1}{k!(k+1)!} \frac{\partial^k B^{pq}_{\sigma\rho}}{\partial y^{\nu_1}_{j_1} \cdots \partial y^{\nu_k}_{j_k}} y^\rho_{pq}\, \omega^\sigma \wedge \omega^{\nu_1} \wedge \cdots \wedge \omega^{\nu_k} \wedge \omega_{j_1 \cdots j_k} \\ &\quad - B^{pq}_{\sigma\nu} y^\nu_{pq}\, \omega^\sigma \wedge \omega_0 \\ &\quad - \sum_{k=2}^{n} \frac{1}{(k-1)!k!} \frac{\partial^{k-1} B^{j_k p}_{\sigma\nu_k}}{\partial y^{\nu_1}_{j_1} \cdots \partial y^{\nu_{k-1}}_{j_{k-1}}} y^{\nu_k}_{pj_k}\, \omega^\sigma \wedge \omega^{\nu_1} \wedge \cdots \wedge \omega^{\nu_{k-1}} \wedge \omega_{j_1 \cdots j_{k-1}} \\ &= \cdots + \frac{1}{n!(n+1)!} \frac{\partial^n B^{pq}_{\sigma\rho}}{\partial y^{\nu_1}_{j_1} \cdots \partial y^{\nu_n}_{j_n}} y^\rho_{pq}\, \omega^\sigma \wedge \omega^{\nu_1} \wedge \cdots \wedge \omega^{\nu_n} \wedge \omega_{j_1 \cdots j_n} \\ &= \cdots + \frac{1}{n!(n+1)!} \left.\frac{\partial^n B^{pq}_{\sigma\rho}}{\partial y^{\nu_1}_{j_1} \cdots \partial y^{\nu_n}_{j_n}}\right|_{\mathrm{alt}\{\sigma\nu_1 \cdots \nu_n\}, \mathrm{alt}\{j_1 \cdots j_n\}} y^\rho_{pq} \\ &\quad \cdot \omega^\sigma \wedge \omega^{\nu_1} \wedge \cdots \wedge \omega^{\nu_n} \wedge \omega_{j_1 \cdots j_n}, \end{aligned} \tag{4.19}$$

where the dots indicate first order terms. We shall show that the last term above vanishes. It is easy to see that variationality conditions (2.4) imply the following identity:

$$\left.\frac{\partial B^{pq}_{\sigma\nu}}{\partial y^\rho_r}\right|_{\mathrm{alt}\{\sigma\rho\}} = \left.\frac{\partial B^{pq}_{\sigma\nu}}{\partial y^\rho_r}\right|_{\mathrm{alt}\{qr\}}. \tag{4.20}$$

Now, however,

$$\left.\frac{\partial^n B^{pq}_{\sigma\rho}}{\partial y^{\nu_1}_{j_1} \cdots \partial y^{\nu_n}_{j_n}}\right|_{\mathrm{alt}\{\sigma\nu_1 \cdots \nu_n\}, \mathrm{alt}\{j_1 \cdots j_n\}} = \left.\frac{\partial^n B^{pq}_{\sigma\rho}}{\partial y^{\nu_1}_{j_1} \cdots \partial y^{\nu_n}_{j_n}}\right|_{\mathrm{alt}\{\nu_1 \cdots \nu_n\}, \mathrm{alt}\{qj_1 \cdots j_n\}} = 0, \tag{4.21}$$

as desired. □

Proof of Theorem 4. It is sufficient to show that α_E is closed. This means that components of α_E

$$\begin{aligned}\tilde{F}^{j_1 \cdots j_r}_{\sigma_1 \cdots \sigma_{r+1}} &= \frac{1}{r!(r+1)!} \left.\frac{\partial^r E_{\sigma_1}}{\partial y^{\sigma_2}_{j_1} \cdots \partial y^{\sigma_{r+1}}_{j_r}}\right|_{\mathrm{alt}\{\sigma_1 \cdots \sigma_{r+1}\}}, \quad 1 \le r \le n, \\ \tilde{F}^{j_1 \cdots j_r, p}_{\sigma_1 \cdots \sigma_r, \rho} &= \frac{1}{r!^2} \left.\frac{\partial^r E_{\sigma_1}}{\partial y^{\sigma_2}_{j_1} \cdots \partial y^{\sigma_r}_{j_{r-1}} \partial y^\rho_{j_r p}}\right|_{\mathrm{alt}\{\sigma_1 \cdots \sigma_r\}, \mathrm{alt}\{j_1 \cdots j_r\}}, \quad 1 \le r \le n \end{aligned} \tag{4.22}$$

have to satisfy conditions $p_2 d\alpha_E = 0$, that is,

$$\begin{aligned} \frac{\partial E_\sigma}{\partial y^\nu} - \frac{\partial E_\nu}{\partial y^\sigma} - \frac{1}{2} d_l \left(\frac{\partial E_\sigma}{\partial y^\nu_l} - \frac{\partial E_\nu}{\partial y^\sigma_l} \right) &= 0, \\ \frac{\partial E_\sigma}{\partial y^\nu_j} + \frac{\partial E_\nu}{\partial y^\sigma_j} - 2 d_l \frac{\partial E_\sigma}{\partial y^\nu_{jl}} &= 0, \\ \frac{\partial E_\sigma}{\partial y^\nu_{jp}} - \frac{\partial E_\nu}{\partial y^\sigma_{jp}} &= 0, \end{aligned} \tag{4.23}$$

and $p_k d\alpha_E = 0$, $k \ge 3$, i.e., conditions (4.3)–(4.6).

Conditions (4.23) are obviously equivalent with the *variationality conditions* (2.3–2.5); this means that they express the fact that E *is locally variational*.

Let us turn to relations (4.3–4.6).

Let $s = 0$, (4.3) read

$$\tilde{F}^{j_1 \cdots j_r,\, p}_{\sigma_1 \cdots \sigma_r,\, \rho} = \tilde{F}^{j_1 \cdots p,\, j_r}_{\sigma_1 \cdots \sigma_r,\, \rho} = \frac{1}{r} \frac{\partial \tilde{F}^{j_1 \cdots j_{r-1}}_{\sigma_1 \cdots \sigma_r}}{\partial y^\rho_{p j_r}}.$$

Accounting symmetries of the $\tilde{F}$'s, they split into two parts:

$$\tilde{F}^{j_1 \cdots j_r,\, p}_{\sigma_1 \cdots \sigma_r,\, \rho} = \tilde{F}^{j_1 \cdots p,\, j_r}_{\sigma_1 \cdots \sigma_r,\, \rho} = \frac{1}{r} \frac{\partial \tilde{F}^{j_1 \cdots j_{r-1}}_{\sigma_1 \cdots \sigma_r}}{\partial y^\rho_{p j_r}} \bigg|_{\mathrm{alt}\{j_1 \cdots j_r\}},$$

that are obviously satisfied with (4.22), and

$$\frac{\partial \tilde{F}^{j_1 \cdots j_{r-1}}_{\sigma_1 \cdots \sigma_r}}{\partial y^\rho_{p j_r}} \bigg|_{\mathrm{sym}\{j_{r-1} j_r\}} = 0, \quad \text{i.e.} \quad \frac{\partial^r E_{\sigma_1}}{\partial y^{\sigma_2}_{j_1} \cdots \partial y^{\sigma_r}_{j_{r-1}} \partial y^\rho_{p j_r}} \bigg|_{\mathrm{alt}\{\sigma_1 \cdots \sigma_r\}, \mathrm{sym}\{j_{r-1} j_r\}} = 0.$$

It is easy to show that due to the variationality conditions, the latter relations are identities: indeed, differentiating the first of (4.23) with respect to y^ρ_{pqj} we obtain

$$\frac{\partial^2 E_\sigma}{\partial y^\nu_j \partial y^\rho_{pq}} \bigg|_{\mathrm{alt}\{\sigma\nu\}, \mathrm{sym}\{pqj\}} = 0, \tag{4.24}$$

proving our assertion. Next, relations (4.4) for $s = 0$ give us the following conditions on components of α_E:

$$\tilde{F}^{j_1 \cdots j_{r-1} p,}_{\sigma_1 \cdots \sigma_r \rho,} = \frac{1}{(r+1)r} \frac{\partial \tilde{F}^{j_1 \cdots j_{r-1}}_{\sigma_1 \cdots \sigma_r}}{\partial y^\rho_p} \bigg|_{\mathrm{alt}\{j_1 \cdots j_{r-1} p\}, \mathrm{alt}\{\sigma_1 \cdots \sigma_r \rho\}},$$

$$\frac{\partial \tilde{F}^{j_1 \cdots j_{r-1}}_{\sigma_1 \cdots \sigma_r}}{\partial y^\rho_p} \bigg|_{\mathrm{sym}\{j_{r-1} p\}} - \frac{\partial \tilde{F}^{j_1 \cdots j_{r-1},\, p}_{\sigma_1 \cdots \sigma_{r-1},\, \rho}}{\partial y^{\sigma_r}} \bigg|_{\mathrm{alt}\{\sigma_1 \cdots \sigma_r\}} - r\, d_l \tilde{F}^{j_1 \cdots j_{r-2} l j_{r-1},\, p}_{\sigma_1 \cdots \sigma_r,\, \rho} = 0,$$

where $2 \le r \le n$. Substituting from (4.22), the former conditions are apparently satisfied. The latter ones become

$$\left(\frac{\partial^r E_{\sigma_1}}{\partial y^{\sigma_2}_{j_1} \cdots \partial y^{\sigma_{r-1}}_{j_{r-2}} \partial y^{\sigma_r}_{j_{r-1}} \partial y^\rho_p} \bigg|_{\mathrm{sym}\{j_{r-1} p\}} - r \frac{\partial^r E_{\sigma_1}}{\partial y^{\sigma_2}_{j_1} \cdots \partial y^{\sigma_{r-1}}_{j_{r-2}} \partial y^{\sigma_r} \partial y^\rho_{j_{r-1} p}} \right.$$
$$\left. - d_l \frac{\partial^r E_{\sigma_1}}{\partial y^{\sigma_2}_{j_1} \cdots \partial y^{\sigma_{r-1}}_{j_{r-2}} \partial y^{\sigma_r}_l \partial y^\rho_{j_{r-1} p}} \right) \bigg|_{\mathrm{alt}\{\sigma_1 \cdots \sigma_r\}} = 0, \tag{4.25}$$

where we have used that, in view of (4.24), the $\partial^2 E_\sigma / \partial y^\nu_l \partial y^\rho_{jp}$ are skew-symmetric in jl. We show that (4.25) are again identities as a consequence of the variationality conditions. Indeed, for $r = 2$ they read

$$\left(\frac{\partial^2 E_\sigma}{\partial y^\nu_j \partial y^\rho_p} \bigg|_{\mathrm{sym}\{jp\}} - 2 \frac{\partial^2 E_\sigma}{\partial y^\nu \partial y^\rho_{jp}} - d_l \frac{\partial^2 E_\sigma}{\partial y^\nu_l \partial y^\rho_{jp}} \right) \bigg|_{\mathrm{alt}\{\sigma\nu\}} = 0, \tag{4.26}$$

however, this is nothing but the derivative of the first of the variationality conditions (4.23) by y^ρ_{jp}. Relations (4.25) for $r = 3, \dots, n-1$ are then apparently obtained by consecutive differentiation of those for $r = 2$. It remains to check (4.5) and (4.6), which for $s = 0$ yield

$$\left.\frac{\partial \tilde{F}^{j_1 \cdots j_n}_{\sigma_1 \cdots \sigma_{n+1}}}{\partial y^\rho_p} - \frac{\partial \tilde{F}^{j_1 \cdots j_n,\, p}_{\sigma_1 \cdots \sigma_n, \rho}}{\partial y^{\sigma_{n+1}}}\right|_{\mathrm{alt}\{\sigma_1 \cdots \sigma_{n+1}\}} = 0, \quad \left.\frac{\partial \tilde{F}^{j_1 \cdots j_n}_{\sigma_1 \cdots \sigma_{n+1}}}{\partial y^\rho_p}\right|_{\mathrm{alt}\{j_n p\}} = 0, \tag{4.27}$$

and

$$\frac{\partial \tilde{F}^{j_1 \cdots j_n}_{\sigma_1 \cdots \sigma_{n+1}}}{\partial y^\kappa_{l_1 l_2}} = 0, \tag{4.28}$$

respectively. Substituting for the $\tilde{F}$'s and taking into account that $1 \le j_1, \dots, j_n \le n$, we can see immediately that the second of (4.27) are identities, and (4.28) are consequences of the variationality conditions, more precisely, of (4.24). Finally, we notice that the first set of relations in (4.27) arises in the same way as (4.25) by one more differentiation, hence these relations are identities due to the variationality conditions, as well.

To finish the proof we have to consider relations (4.3–4.6) for the case $s = 1$. In view of our assumption (3.1), (4.3) and (4.6) are satisfied trivially. (4.5) read

$$\left.\frac{\partial^{n+1} E_{\sigma_1}}{\partial y^{\sigma_2}_{j_1} \cdots \partial y^{\sigma_n}_{j_{n-1}} \partial y^{\rho_2}_{p_2} \partial y^{\rho_1}_{j_n p_1}}\right|_{\mathrm{alt}\{(\rho_1 p_1)(\rho_2 p_2)\}, \mathrm{alt}\{\sigma_1 \cdots \sigma_n\}, \mathrm{alt}\{j_1 \cdots j_n\}} = 0.$$

This is an identity: indeed, in the sum, terms where $p_2 = j_n$ are 0 due to (4.24), and terms where $p_2 = j_i \neq j_n$ vanish due to skew-symmetry in $\{(\rho_1 p_1)(\rho_2 p_2)\}$ and in $\{j_i j_n\}$. Finally, (4.4) split to

$$\tilde{F}^{j_1 \cdots j_r p_2,\, p_1}_{\sigma_1 \cdots \sigma_r \rho_2,\, \rho_1}\Big|_{\mathrm{alt}\{(\rho_1 p_1)(\rho_2 p_2)\}} = -\frac{1}{(r+1)^2} \left.\frac{\partial \tilde{F}^{j_1 \cdots j_r,\, p_1}_{\sigma_1 \cdots \sigma_r,\, \rho_1}}{\partial y^{\rho_2}_{p_2}}\right|_{\mathrm{alt}\{(\rho_1 p_1)(\rho_2 p_2)\}, \mathrm{sym}\{p_1 p_2\}}, \tag{4.29}$$

$1 \le r \le n-1$, which are identities due to (4.24), and

$$\left.\frac{\partial \tilde{F}^{j_1 \cdots j_r,\, p_1}_{\sigma_1 \cdots \sigma_r,\, \rho_1}}{\partial y^{\rho_2}_{p_2}}\right|_{\mathrm{alt}\{j_1 \cdots j_r p_1 p_2\}, \mathrm{alt}\{\sigma_1 \cdots \sigma_r \rho_1 \rho_2\}} = 0, \tag{4.30}$$

$1 \le r \le n-1$; the left-hand sides, however, vanish identically, since the $\tilde{F}$'s are symmetric in $j_r p_1$.

This completes the proof. □

Proof of Theorem 5. With help of Theorem 4, the proof is easy. (1) ⇒ (2) was proved above. (2) ⇒ (3) follows from the Structure Theorem (formula (2.1)). (3) ⇒ (4) was shown in the proof of Theorem 4. Finally, (4) ⇒ (1) was proved in [15]: one has to show that if E_σ satisfy (4.23) then

$$L = y^\sigma \int_0^1 E_\sigma(x^i, uy^\nu, uy^\nu_k, uy^\nu_{kl})\, du \tag{4.31}$$

is a local Lagrangian for E. This is done by a direct computation, showing that the Euler–Lagrange expressions of L are equal to the given functions E_σ. □

Proof of Theorem 6. Computing $d\rho_\lambda$ we get

$$
\begin{aligned}
d\rho_\lambda = E_\lambda &+ \sum_{k=2}^{n} (-1)^k \frac{1}{k!(k-1)!}\; d_{j_k}\left(\frac{\partial^k L}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_k}_{j_k}}\right)\omega^{\nu_1}\wedge\cdots\wedge\omega^{\nu_k}\wedge\omega_{j_1\cdots j_{k-1}} \\
&+ \sum_{k=1}^{n} \frac{1}{(k!)^2}\frac{\partial^{k+1} L}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_k}_{j_k}\partial y^\sigma}\;\omega^\sigma\wedge\omega^{\nu_1}\wedge\cdots\wedge\omega^{\nu_k}\wedge\omega_{j_1\cdots j_k} \\
&+ \sum_{k=1}^{n} (-1)^k \frac{1}{(k!)^2}\frac{\partial^{k+1} L}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_k}_{j_k}\partial y^\rho_p}\;\omega^{\nu_1}\wedge\cdots\wedge\omega^{\nu_k}\wedge\omega^\rho_p\wedge\omega_{j_1\cdots j_k} \\
&+ \sum_{k=2}^{n} (-1)^k \frac{1}{(k-1)!^2}\;\frac{\partial^k L}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_k}_{j_k}}\omega^{\nu_1}\wedge\cdots\wedge\omega^{\nu_{k-1}}\wedge\omega^{\nu_k}_{j_k}\wedge\omega_{j_1\cdots j_{k-1}}.
\end{aligned}
$$

Thus, components of $d\rho_\lambda$ take the following form:

$$
\begin{aligned}
F^{j_1\cdots j_k}_{\sigma\nu_1\cdots\nu_k} &= \frac{1}{k!(k+1)!}\left((k+1)\frac{\partial^{k+1} L}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_k}_{j_k}\partial y^\sigma} - d_l\,\frac{\partial^{k+1} L}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_k}_{j_k}\partial y^\sigma_l}\right)\Bigg|_{\mathrm{alt}\{\sigma\nu_1\cdots\nu_k\}} \\
&= \frac{1}{k!(k+1)!}\left((k+1)\frac{\partial^{k+1} L}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_k}_{j_k}\partial y^\sigma} + k\frac{\partial^{k+1} L}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_k}\partial y^\sigma_{j_k}}\right. \\
&\qquad \left.- \frac{\partial^k}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_k}_{j_k}}\left(d_l\frac{\partial L}{\partial y^\sigma_l}\right)\right)\Bigg|_{\mathrm{alt}\{\sigma\nu_1\cdots\nu_k\}} \\
&= \frac{1}{k!(k+1)!}\frac{\partial^k E_\sigma}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_k}_{j_k}}\Bigg|_{\mathrm{alt}\{\sigma\nu_1\cdots\nu_k\}} \\
&= \tilde{F}^{j_1\cdots j_k}_{\sigma\nu_1\cdots\nu_k}, \quad 1\le k\le n,
\end{aligned} \tag{4.32}
$$

and

$$
\begin{aligned}
&F^{j_1\cdots j_k,\,p}_{\sigma\nu_1\cdots\nu_{k-1},\,\rho} \\
&\quad = -\frac{1}{k!^2}\left(\frac{\partial^{k+1} L}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_{k-1}}_{j_{k-1}}\partial y^\sigma_{j_k}\partial y^\rho_p} - \frac{\partial^{k+1} L}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_{k-1}}_{j_{k-1}}\partial y^\sigma_{j_k}\partial y^\rho_p}\Bigg|_{\mathrm{alt}\{j_k p\}}\right)\Bigg|_{\mathrm{alt}\{\sigma\nu_1\cdots\nu_{k-1}\}} \\
&\quad = -\frac{1}{k!^2}\frac{\partial^{k+1} L}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_{k-1}}_{j_{k-1}}\partial y^\sigma_{j_k}\partial y^\rho_p}\Bigg|_{\mathrm{sym}\{j_k p\},\mathrm{alt}\{\sigma\nu_1\cdots\nu_{k-1}\}} \\
&\quad = \frac{1}{k!^2}\;\frac{\partial^k E_\sigma}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_{k-1}}_{j_{k-1}}\partial y^\rho_{j_k p}}\Bigg|_{\mathrm{alt}\{\sigma\nu_1\cdots\nu_{k-1}\},\mathrm{alt}\{j_1\cdots j_k\}} \\
&\quad = \tilde{F}^{j_1\cdots j_k,\,p}_{\sigma\nu_1\cdots\nu_{k-1},\,\rho}, \quad 1\le k\le n-1,
\end{aligned} \tag{4.33}
$$

$$
\begin{aligned}
F^{j_1\cdots j_n,\,p}_{\sigma\nu_1\cdots\nu_{n-1},\,\rho} &= -\frac{1}{(n!)^2}\frac{\partial^{n+1}L}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_{n-1}}_{j_{n-1}}\partial y^{\sigma}_{j_n}\,\partial y^{\rho}_{p}}\Bigg|_{\mathrm{alt}\{\sigma\nu_1\cdots\nu_{n-1}\},\mathrm{alt}\{j_1\cdots j_n\}} \\
&= -\frac{1}{(n!)^2}\frac{\partial^{n+1}L}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_{n-1}}_{j_{n-1}}\partial y^{\sigma}_{j_n}\,\partial y^{\rho}_{p}}\Bigg|_{\mathrm{sym}\{j_n p\},\mathrm{alt}\{\sigma\nu_1\cdots\nu_{n-1}\},\mathrm{alt}\{j_1\cdots j_n\}} \\
&= \frac{1}{n!^2}\frac{\partial^{n}E_\sigma}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_{n-1}}_{j_{n-1}}\partial y^{\rho}_{j_n p}}\Bigg|_{\mathrm{alt}\{\sigma\nu_1\cdots\nu_{n-1}\},\mathrm{alt}\{j_1\cdots j_n\}} \\
&= \tilde{F}^{j_1\cdots j_n,\,p}_{\sigma\nu_1\cdots\nu_{n-1},\,\rho}.
\end{aligned}
\tag{4.34}
$$

Hence, $d\rho_\lambda = \alpha_E$, as desired.

Note that in formula (4.32),

$$
d_l\frac{\partial^{n+1}L}{\partial y^{\nu_1}_{j_1}\cdots\partial y^{\nu_n}_{j_n}\partial y^{\sigma}_{l}}\Bigg|_{\mathrm{alt}\{\sigma\nu_1\cdots\nu_n\}} = 0. \tag{4.35}
$$

□

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