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SOLUTIONS OF THE $(2 + 1)$ -DIMENSIONAL KP, SK AND KK EQUATIONS GENERATED BY GAUGE TRANSFORMATIONS FROM NONZERO SEEDS

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By using gauge transformations, we manage to obtain new solutions of $(2 + 1)$ -dimensional Kadomtsev–Petviashvili (KP), Kaup–Kuperschmidt (KK) and Sawada–Kotera (SK) equations from nonzero seeds. For each of the preceding equations, a Galilean type transformation between these solutions u_2 and the previously known solutions u'_2 generated from zero seed is given. We present several explicit formulas of the single-soliton solutions for u_2 and u'_2 , and further point out the two main differences of them under the same value of parameters, i.e., height and location of peak line, which are demonstrated visibly in three figures.

Keywords: Gauge transformation; soliton; KP equation; BKP equation; CKP equation.

1. Introduction

In the 1980s, Sato and his colleagues brought us the famous Sato theory [3, 29]. Since then, the pseudo-differential operator has been playing an important role in the research of the Kadomtsev–Petviashvili (KP) hierarchy [5], which can yield many important nonlinear partial differential equations, such as the generalized nonlinear Schrödinger equation, the KdV equation, the Sine–Gordon equation and the famous KP equation. To be self-consistent, we would like to give a brief review of the KP hierarchy [3, 5, 13, 29].

Let

$$L = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + \cdots, \quad (1.1)$$

be a pseudo-differential operator (Ψ DO), here $\{u_i\}$, $u_i = u_i(t_1, t_2, t_3, \dots)$ serve as generators of a differential algebra \mathcal{A} . The corresponding generalized Lax equations are defined as

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad n = 1, 2, 3, \dots, \quad (1.2)$$

which give rise to infinite number of partial differential equations of the KP hierarchy, B_n is defined as $B_n = [L^n]_+$. It can be easily showed that Eq. (1.2) is equivalent to the so-called Zakharov–Shabat (ZS) Eq. [34]

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_m, B_n] = 0, \quad (m, n = 2, 3, \dots). \quad (1.3)$$

The eigenfunction ϕ and conjugate eigenfunction ψ corresponding to L are defined by

$$\frac{\partial \phi}{\partial t_n} = B_n \phi, \quad (1.4)$$

$$\frac{\partial \psi}{\partial t_n} = -B_n^* \psi. \quad (1.5)$$

The first non-trivial example is the KP equation given by the t_2 -flow and t_3 -flow of the KP hierarchy

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} = 0, \quad (1.6)$$

in which $u = u_2$, $x = t_1$, $y = t_2$ and $t = t_3$.

Suppose L given by Eq. (1.1) and L^* defined by

$$L^* = -\partial + \sum_{i=1}^{\infty} (-1)^i \partial^{-i} u_{i+1}.$$

If L satisfies $L^* + L = 0$, then L is called the Lax operator of the CKP hierarchy [2, 13], and the corresponding flow equations of the CKP hierarchy are described by

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad n = 1, 3, 5, \dots \quad (1.7)$$

The first non-trivial example is the CKP equation [13, 23]

$$u_t = \frac{5}{9} \left(\partial_x^{-1} u_{yy} + 3u_x \partial_x^{-1} u_y - \frac{1}{5} u_{xxxxx} - 3uu_{xxx} - \frac{15}{2} u_x u_{xx} - 9u^2 u_x + u_{xxy} + 3uu_y \right) \quad (1.8)$$

which is generated by t_3 -flow and t_5 -flow and also called the $(2+1)$ -dimensional Kaup–Kuperschmidt (KK) equation [18]. Here $u = u_2$, $x = t_1$, $y = t_3$ and $t = t_5$. Moreover, L is called the Lax operator of the BKP hierarchy [3, 4] if it satisfies $L^* = -\partial L \partial^{-1}$, and the flow equations of the BKP hierarchy associated with it are also described by Eq. (1.7). The first non-trivial example is the BKP equation [22, 24]

$$u_t = \frac{5}{9} \left(\partial_x^{-1} u_{yy} + 3u_x \partial_x^{-1} u_y - \frac{1}{5} u_{xxxxx} - 3uu_{xxx} - 3u_x u_{xx} - 9u^2 u_x + u_{xxy} + 3uu_y \right), \quad (1.9)$$

which is generated by t_3 -flow and t_5 -flow and also called the $(2+1)$ -dimensional Sawada–Kotera (SK) equation [18]. Here, $u = u_2$, $x = t_1$, $y = t_3$ and $t = t_5$.

If we find a set of functions u_2, u_3, \dots which makes the corresponding pseudo-differential operator L satisfies Eq. (1.3), then we have a solution of the KP hierarchy. It is a well-known result that this set of solutions can be generated from one single function $\tau(x)$ as the following way

$$u_2 = \frac{\partial^2}{\partial t_1^2} \log \tau, \quad (1.10)$$

$$u_3 = \frac{1}{2} \left[\frac{\partial^2}{\partial t_1 \partial t_2} - \frac{\partial^3}{\partial t_1^3} \right] \log \tau. \quad (1.11)$$

During the last two decades, in order to solve the KP hierarchy, the gauge transformation was formally introduced in Ref. [21]. The basic idea behind gauge transformation is to find a transformation for the initial Lax operator $L^{(0)}$ of the KP hierarchy after which the new operator $L^{(1)}$ and

$B_n^{(1)}$ still satisfies Lax equation Eq. (1.2) and Eq. (1.3) respectively. Here

$$L^{(1)} = T \circ L^{(0)} \circ T^{-1}, \quad B_n^{(1)} = (L^{(1)})_+^n, \quad (1.12)$$

T is a suitable pseudo-differential operator. There exist two kinds of gauge transformation operators [21]

$$T_D(\phi^{(0)}) = \phi^{(0)} \partial (\phi^{(0)})^{-1}, \quad (1.13)$$

$$T_I(\psi^{(0)}) = (\psi^{(0)})^{-1} \partial^{-1} \psi^{(0)}, \quad (1.14)$$

in which $\phi^{(0)}$, $\psi^{(0)}$ are eigenfunction and conjugate eigenfunction of $L^{(0)}$ respectively and they are also called the generating functions of the gauge transformation. T_D is called differential type of gauge transformation, T_I is called integral type of gauge transformation. After one gauge transformation T_D , the new τ -function

$$\tau^{(1)} = \phi^{(0)} \tau^{(0)}, \quad (1.15)$$

is transformed from an initial τ -function $\tau^{(0)}$ associated with the initial Lax operator $L^{(0)}$. A similar result can be formulated for the case of T_I

$$\tau^{(1)} = \psi^{(0)} \tau^{(0)}. \quad (1.16)$$

With the help of formulas Eq. (1.10), Eq. (1.11), Eq. (1.15) and Eq. (1.16), we can obtain new solutions $\{u_i^{(1)}\}$ from the known seed solutions $\{u_i^{(0)}\}$ in the $L^{(0)}$. For example, $u_2^{(1)} = u_2^{(0)} + (\log \phi^{(0)})_{xx}$ by the gauge transformation in Eq. (1.15). By a successive application of gauge transformations, the determinant representation of $\tau^{(n+k)}$ is given in [14] and further more $u_2^{(n+k)}$ can be deduced by using Eq. (1.10).

In the last decade, the method of gauge transformation has been developed by several researchers. The original form of this transformation proposed in Ref. [21] cannot be applied directly to the sub-hierarchies of the KP hierarchy. So in [16, 26, 27], an improvement was made which makes it applicable to the BKP and CKP hierarchies, and in [1, 15, 17, 20, 28, 32] another improvement was made so that the gauge transformation can be used on the constrained KP hierarchy. Besides gauge transformation, some other methods have been used to solve the KP, BKP, CKP equations. In [10], Hirota method was considered on the KP equation. Darboux transformation was applied on this equation in Chap. 3 of [25]. N-soliton solutions of the BKP equation was obtained through Hirota method in [11, 12], lump solutions was obtained through this method in [9], the same method was applied to the $(2+1)$ -dimensional KK equations in [33] and 3-soliton solutions were obtained explicitly. Darboux transformation was applied to $(2+1)$ -dimensional KK, SK equations in [19]. In [6], $\bar{\partial}$ -dressing method was used on the $(2+1)$ -dimensional KK, SK equations and line solitons and line rational lumps were obtained. It is easy to recognize that all these known solutions are corresponding to the solutions given by gauge transformation from zero seed. However, solving the soliton equations starting from a “nonzero seed” has not attracted enough attention. There are very few works on the KPI and KP II equations with a non-decay initial background [7, 8, 31] by dressing method and classical inverse scattering method. On the other hand, gauge transformation from nonzero seeds was not considered before to our knowledge. One possible reason is that in the case of the KdV equation, solutions obtained by gauge transformation from zero seed can be transformed to those solutions from nonzero seeds by a Galilean transformation [30]. So far, we have not seen any similar discussions on solutions of $(2+1)$ -dimensional KP, KK, SK equations. Therefore, in this paper, we solve these equations by gauge transformation from nonzero seeds and manage to find out the relations between new solutions and those from zero seed.

The organization of this paper is as follows. In Sec. 2 we consider the KP equation. In Sec. 3 and Sec. 4, we discuss $(2+1)$ -dimensional KK and SK equations respectively. Section 5 is devoted to the conclusions and discussions. The notations we use in this paper is the same as in [20].

2. Successive Gauge Transformation for KP Equation

It is a natural thought to consider successive application of gauge transformation for KP hierarchy. In [14, 21], a very useful theorem was introduced about the result after successive gauge transformations.

Lemma 1 [14, 21]. *After n times T_D and k times T_I transformations ($n \geq k$), we have:*

$$\begin{aligned}\tau^{(k+n)} &= \psi_k^{(k-1+n)} \cdot \psi_{k-1}^{(k-2+n)} \dots \psi_1^{(n)} \cdot \tau^{(n)} \\ &= \text{IW}_{k,n}(\psi_k^{(0)}, \psi_{k-1}^{(0)}, \dots, \psi_1^{(0)}; \phi_1^{(0)}, \phi_2^{(0)}, \dots, \phi_n^{(0)}) \cdot \tau^{(0)},\end{aligned}\quad (2.1)$$

in which $\text{IW}_{k,n}(\psi_k^{(0)}, \psi_{k-1}^{(0)}, \dots, \psi_1^{(0)}; \phi_1^{(0)}, \phi_2^{(0)}, \dots, \phi_n^{(0)})$ stands for

$$\text{IW}_{k,n} = \begin{vmatrix} \int \phi_1^{(0)} \cdot \psi_k^{(0)} & \int \phi_2^{(0)} \cdot \psi_k^{(0)} & \dots & \int \phi_n^{(0)} \cdot \psi_k^{(0)} \\ \int \phi_1^{(0)} \cdot \psi_{k-1}^{(0)} & \int \phi_2^{(0)} \cdot \psi_{k-1}^{(0)} & \dots & \int \phi_n^{(0)} \cdot \psi_{k-1}^{(0)} \\ \vdots & \vdots & \dots & \vdots \\ \int \phi_1^{(0)} \cdot \psi_1^{(0)} & \int \phi_2^{(0)} \cdot \psi_1^{(0)} & \dots & \int \phi_n^{(0)} \cdot \psi_1^{(0)} \\ \phi_1^{(0)} & \phi_2^{(0)} & \dots & \phi_n^{(0)} \\ \phi_{1,x}^{(0)} & \phi_{2,x}^{(0)} & \dots & \phi_{n,x}^{(0)} \\ \vdots & \vdots & \dots & \vdots \\ (\phi_1^{(0)})^{(n-k-1)} & (\phi_2^{(0)})^{(n-k-1)} & \dots & (\phi_n^{(0)})^{(n-k-1)} \end{vmatrix}$$

$\phi_i^{(0)}$ and $\psi_i^{(0)}$ are solutions of Eq. (1.4) and Eq. (1.5) associated with the initial value $\tau^{(0)}$, further we have

$$u_2^{(k+n)} = (\log \text{IW}_{k,n})_{x,x} + u_2^{(0)}. \quad (2.2)$$

By using the above theorem, we now start to construct the new solutions of the KP equation in Eq. (1.6) from nonzero seeds. To the end, we choose the initial Lax operator of the KP hierarchy to be

$$L^{(0)} = \partial + \partial^{-1} + \partial^{-2} + \partial^{-3} + \dots,$$

such that all $u_i^{(0)} = 1$ and then the seed solution of the KP equation is $u^{(0)} = u_2^{(0)} = 1$. We know that the KP equation is generated by t_2 -flow and t_3 -flow of the KP hierarchy, so the generating functions $\phi_i^{(0)}$ and $\psi_i^{(0)}$ for the gauge transformation satisfy

$$\begin{cases} \phi_{i,t_2}^{(0)} = B_2^{(0)} \phi_i^{(0)} = (\partial^2 + 2) \phi_i^{(0)}, & B_2^{(0)} = (L^{(0)})_+^2 \\ \phi_{i,t_3}^{(0)} = B_3^{(0)} \phi_i^{(0)} = (\partial^3 + 3\partial + 3) \phi_i^{(0)}, & B_3^{(0)} = (L^{(0)})_+^3 \end{cases} \quad (2.3)$$

$$\begin{cases} \psi_{i,t_2}^{(0)} = -(B_2^{(0)})^* \psi_i^{(0)} = -(\partial^2 + 2) \psi_i^{(0)}, \\ \psi_{i,t_3}^{(0)} = -(B_3^{(0)})^* \psi_i^{(0)} = (\partial^3 + 3\partial - 3) \psi_i^{(0)}. \end{cases} \quad (2.4)$$

Lemma 2. *The solutions of Eq. (2.3), Eq. (2.4) are in form of*

$$\phi_i^{(0)} = \sum_{j=1}^n k_j e^{\frac{\beta_j-3}{\alpha_j+1}x + \alpha_j y + \beta_j t}, \quad \beta_j = \beta_j(\alpha_j), \quad (2.5)$$

$$\psi_i^{(0)} = \sum_{j=1}^m \tilde{k}_j e^{\frac{\tilde{\beta}_j+3}{-\tilde{\alpha}_j+1}x + \tilde{\alpha}_j y + \tilde{\beta}_j t}, \quad \tilde{\beta}_j = \tilde{\beta}_j(\tilde{\alpha}_j). \quad (2.6)$$

Here $\alpha_j, \beta_j, \widetilde{\alpha}_j, \widetilde{\beta}_j$ should satisfy the following relations

$$(\beta_j - 3)^2 = (\alpha_j + 1)^2(\alpha_j - 2), \quad (2.7)$$

$$(\widetilde{\beta}_j + 3)^2 = (-\widetilde{\alpha}_j + 1)^2(-\widetilde{\alpha}_j - 2). \quad (2.8)$$

Proof. We assume the solutions of Eq. (2.3) have the form $\widehat{\phi} = X(x)Y(y)T(t)$, then Eq. (2.3) is equivalent to

$$\begin{cases} \frac{Y_y}{Y} = \frac{X_{xx}}{X} + 2, \\ \frac{T_t}{T} = \frac{X_{xxx}}{X} + 3\frac{X_x}{X} + 3. \end{cases} \quad (2.9)$$

Let

$$\frac{Y_y}{Y} = \alpha, \quad \frac{T_t}{T} = \beta, \quad (2.10)$$

where α and β are constants, we have

$$\begin{cases} (\alpha - 2)X = X_{xx}, \\ (\beta - 3)X = X_{xxx} + 3X_x, \end{cases} \quad (2.11)$$

which can be reduced to

$$\begin{cases} X_x = \frac{(\alpha + 1)(\alpha - 2)}{\beta - 3}X, \\ X_x = \frac{\beta - 3}{\alpha + 1}X. \end{cases} \quad (2.12)$$

Under the consistency condition $(\beta - 3)^2 = (\alpha + 1)^2(\alpha - 2)$ we can obtain

$$X(x) = c_1 e^{\frac{\beta-3}{\alpha+1}x}. \quad (2.13)$$

From Eq. (2.10), we have

$$Y(y) = c_2 e^{\alpha y}, \quad T(t) = c_3 e^{\beta t},$$

which infer the solutions of Eq. (2.3)

$$\widehat{\phi} = k e^{\frac{\beta-3}{\alpha+1}x + \alpha y + \beta t}, \quad (2.14)$$

with the help of Eq. (2.13), where $k = c_1 c_2 c_3$. By linear superposition, the linear combination of $\widehat{\phi}$ in Eq. (2.14) with respect to different α and β is still a solution of Eq. (2.3), that is

$$\phi_i^{(0)} = \sum_{j=1}^n k_j \widehat{\phi}_j = \sum_{j=1}^n k_j e^{\frac{\beta_j-3}{\alpha_j+1}x + \alpha_j y + \beta_j t}. \quad (2.15)$$

A similar procedure can be applied to $\psi_i^{(0)}$ which yields Eq. (2.6). \square

Having these results, it's sufficient to perform gauge transformation on $L^{(0)}$. But according to Lemma 1, the transformed τ -function may not be satisfactory, since it may vanish on some point. To rule out this situation, we need the following theorem.

Theorem 1. Let the generating functions of n -steps T_D be $\phi_m^{(0)}$ ($m = 1, 2, \dots, n$) in Eq. (2.5) and rewritten as $\phi_m^{(0)} = \sum_{i=1}^{p_m} k_{m,i} \exp^{a_{m,i}x + \alpha_{m,i}y + \beta_{m,i}t}$ for simplicity, then the new τ -function

$$\tau^{(n)} = \text{IW}_{0,n} \cdot \tau^{(0)} = W_n(\phi_1^{(0)}, \phi_2^{(0)}, \dots, \phi_n^{(0)}) \cdot \tau^{(0)}, \quad (2.16)$$

and $W_n(\phi_1^{(0)}, \phi_2^{(0)}, \dots, \phi_n^{(0)}) > 0$ if $k_{m,i} > 0$, $a_{m,i} < a_{m',j}$ for all $m < m'$ and $\forall i, j$. The transformed solution $u_2^{(n)}$ of KP equation is

$$u_2^{(n)} = 1 + (\log(W_n(\phi_1^{(0)}, \phi_2^{(0)}, \dots, \phi_n^{(0)})))_{xx} \quad (2.17)$$

Proof. First, W_n takes the following form

$$W_n = \begin{vmatrix} \phi_1^{(0)} & \phi_2^{(0)} & \cdots & \phi_n^{(0)} \\ \frac{\partial}{\partial x} \phi_1^{(0)} & \frac{\partial}{\partial x} \phi_2^{(0)} & \cdots & \frac{\partial}{\partial x} \phi_n^{(0)} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^{n-1}}{\partial x^{n-1}} \phi_1^{(0)} & \frac{\partial^{n-1}}{\partial x^{n-1}} \phi_2^{(0)} & \cdots & \frac{\partial^{n-1}}{\partial x^{n-1}} \phi_n^{(0)} \end{vmatrix}_{n \times n}$$

then we expand the determinant with respect to columns using the equation

$$\phi_m^{(0)} = \sum_{i=1}^{p_m} k_{m,i} e^{a_{m,i}x + \alpha_{m,i}y + \beta_{m,i}t}, \quad m = 1 \cdots n.$$

Then we have:

$$W_n = \sum_{1 \leq i_q \leq p_q, q=1 \dots n} \prod_{j=1}^n k_{j,i_j} e^{a_{j,i_j}x + \alpha_{j,i_j}y + \beta_{j,i_j}t} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_{1,i_1} & a_{2,i_2} & \cdots & a_{n,i_n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1,i_1}^{n-1} & a_{2,i_2}^{n-1} & \cdots & a_{n,i_n}^{n-1} \end{vmatrix} \quad (2.18)$$

Notice the Vendermonde determinants in the above equation. Since $k_{m,i} > 0$, the coefficients of these Vendermonde determinants are positive. Using $a_{m,i} < a_{m',j}$ for all $m < m'$ and $\forall i, j$, it is easy to prove that all Vendermonde determinants in the above equation are positive, so $W_n > 0$. Using Eq. (2.16), Eq. (2.2) and $u_2^{(0)} = 1$, we can obtain Eq. (2.17). \square

Next, we give single-soliton solutions of the KP equation from a zero seed and a nonzero seed respectively. Notations with prime are corresponding to the results of gauge transformation from a zero seed. The generating functions are

$$(\phi_1^{(0)})' = k' e^{\xi'_1} + k' e^{\xi'_2}, \quad (2.19)$$

$$\phi_1^{(0)} = k e^{\xi_1} + k e^{\xi_2}, \quad (2.20)$$

where

$$\xi'_1 = \frac{\beta'_1}{\alpha'_1} x + \alpha'_1 y + \beta'_1 t, \quad (2.21)$$

$$\xi'_2 = \frac{\beta'_2}{\alpha'_2} x + \alpha'_2 y + \beta'_2 t, \quad (2.22)$$

$$\xi_1 = \frac{\beta_1 - 3}{\alpha_1 + 1} x + \alpha_1 y + \beta_1 t, \quad (2.23)$$

$$\xi_2 = \frac{\beta_2 - 3}{\alpha_2 + 1} x + \alpha_2 y + \beta_2 t, \quad (2.24)$$

and $(\alpha'_i)^3 = (\beta'_i)^2$, $(\beta_i - 3)^2 = (\alpha_i + 1)^2 (\alpha_i - 2)$, $i = 1, 2$. The two single-solitons of the KP equation can be written as

$$(u_2^{(1)})' = \frac{1}{4} \left(\frac{\beta'_1}{\alpha'_1} - \frac{\beta'_2}{\alpha'_2} \right)^2 \operatorname{sech}^2 \left(\frac{\xi'_2 - \xi'_1}{2} \right), \quad (2.25)$$

$$u_2^{(1)} = 1 + \frac{1}{4} \left(\frac{\beta_1 - 3}{\alpha_1 + 1} - \frac{\beta_2 - 3}{\alpha_2 + 1} \right)^2 \operatorname{sech}^2 \left(\frac{\xi_1 - \xi_2}{2} \right). \quad (2.26)$$

There are two differences between u_2 and u'_2 under the same parameters α : (1) the height of solitons, (2) the location of the peak line of the solitons, which are demonstrated visibly in Fig. 1. In Fig. 2,

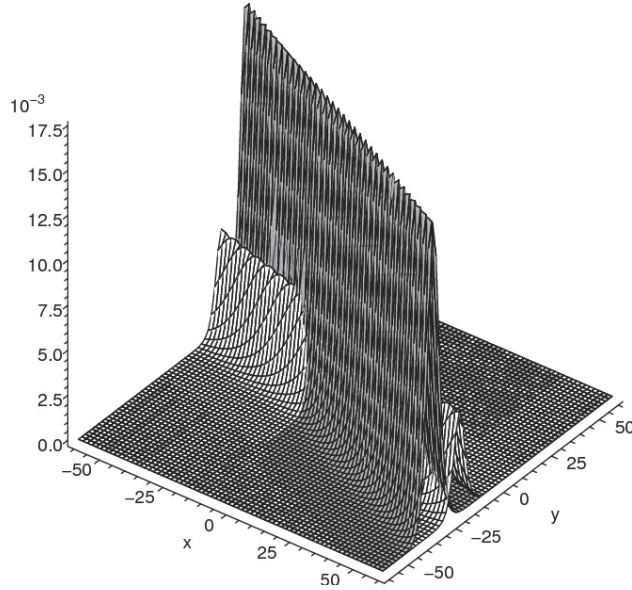


Fig. 1. Single-soliton solutions at $t = 1$ of the KP equation. The lower one is $(u_2^{(1)})'$ with $k' = 1$, $\alpha'_1 = 2.7225$ and $\alpha'_2 = 3.24$; the higher one is $(u_2^{(1)} - 1)$ with parameters $k = 1$, $\alpha_1 = 2.7225$ and $\alpha_2 = 3.24$.

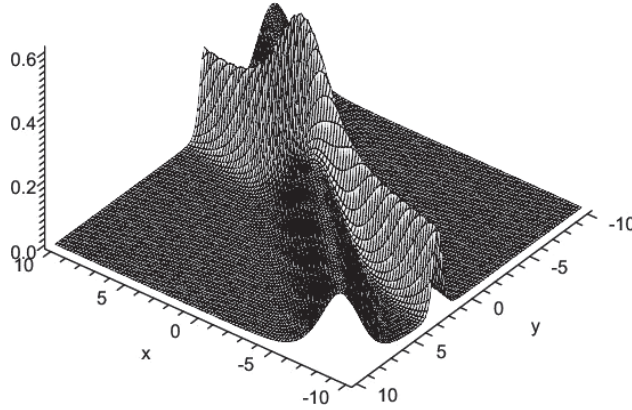


Fig. 2. Two-soliton solution at $t = 0$ of the KP equation.

we demonstrate the solution obtained by a two-step gauge transformation by using Eq. (2.17) and

$$\phi_1^{(0)} = e^{2y+3t} + e^{x+3y+7t}, \quad (2.27)$$

$$\phi_2^{(0)} = e^{\sqrt{2}x+4y+(3+5\sqrt{2})t} + e^{\sqrt{6}x+8y+(3+9\sqrt{6})t}. \quad (2.28)$$

Corollary 1. *There exists a Galilean type transformation*

$$u'_2 \mapsto u_2(x, y, t) = 1 + u'_2(x + 3t, y, t). \quad (2.29)$$

between u'_2 in Eq. (2.25) and u_2 in Eq. (2.26).

Obviously, this result is consistent with the Galilean transformation [30] of the KdV equation by a dimensional reduction.

3. Gauge Transformation for (2 + 1)-Dimensional KK Equation

Gauge transformation of the CKP hierarchy is somewhat different from that of the KP hierarchy, because a transformed Lax operator $L^{(1)}$ by one-step gauge transformation has to satisfy $(L^{(1)})^* + L^{(1)} = 0$. To meet this requirement, we introduce the following lemma.

Lemma 3 ([16]). (1) *The appropriate gauge transformation T_{n+k} is given by $n = k$ and generating functions $\psi_i^{(0)} = \phi_i^{(0)}$ for $i = 1, 2, \dots, n$.*

(2) *The τ -function $\tau_{CKP}^{(n+n)}$ of the CKP hierarchy has the form*

$$\begin{aligned} \tau_{CKP}^{(n+n)} &= \text{IW}_{n,n}(\phi_n^{(0)}, \phi_{n-1}^{(0)}, \dots, \phi_1^{(0)}; \phi_1^{(0)}, \phi_2^{(0)}, \dots, \phi_n^{(0)}) \cdot \tau_{CKP}^{(0)} \\ &= \begin{vmatrix} \int \phi_n^{(0)} \cdot \phi_1^{(0)} & \cdots & \int \phi_n^{(0)} \cdot \phi_n^{(0)} \\ \vdots & \cdots & \vdots \\ \int \phi_1^{(0)} \cdot \phi_1^{(0)} & \cdots & \int \phi_1^{(0)} \cdot \phi_n^{(0)} \end{vmatrix} \cdot \tau_{CKP}^{(0)}, \end{aligned} \quad (3.1)$$

and further we have

$$u_2^{(n+n)} = u_2^{(0)} + (\log \text{IW}_{n,n})_{xx}. \quad (3.2)$$

To solve the (2 + 1)-dimensional KK equation from nonzero seed solution, we choose a initial Lax operator $L^{(0)}$ of the CKP hierarchy to be

$$L^{(0)} = \partial + \partial^{-1} + \partial^{-3} + \partial^{-5} + \dots.$$

Since the (2 + 1)-dimensional KK equation is generated by t_3 -flow and t_5 -flow of the CKP hierarchy, we solve

$$\begin{cases} \phi_{i,t_3}^{(0)} = B_3^{(0)} \phi_i^{(0)} = (\partial^3 + 3\partial) \phi_i^{(0)}, & B_3^{(0)} = (L^{(0)})_+^3, \\ \phi_{i,t_5}^{(0)} = B_5^{(0)} \phi_i^{(0)} = (\partial^5 + 5\partial^3 + 15\partial) \phi_i^{(0)}, & B_5^{(0)} = (L^{(0)})_+^5, \end{cases} \quad (3.3)$$

in order to obtain the eigenfunctions.

Lemma 4. *The solutions of Eq. (3.3) are*

$$\phi_i^{(0)} = \sum_{j=1}^n k_j e^{\frac{\alpha_j^3 - 18\alpha_j + 9\beta_j}{\alpha_j^2 + \alpha_j\beta_j + 81}x + \alpha_j y + \beta_j t}, \quad \beta_j = \beta_j(\alpha_j), \quad (3.4)$$

here α_j, β_j should satisfy the relation

$$\alpha_j^5 - 25\alpha_j^3 + 30\beta_j\alpha_j^2 + 1215\alpha_j - \beta_j^3 - 243\beta_j = 0. \quad (3.5)$$

Proof. First, we assume the solution of Eq. (3.3) has the form $\widehat{\phi} = X(x)Y(y)T(t)$ then we have

$$\begin{cases} \frac{Y_y}{Y} = \frac{X_{xxx}}{X} + 3\frac{X_x}{X}, \\ \frac{T_t}{T} = \frac{X_{xxxxx}}{X} + 5\frac{X_{xxx}}{X} + 15\frac{X_x}{X}. \end{cases} \quad (3.6)$$

Let

$$\frac{Y_y}{Y} = \alpha, \quad \frac{T_t}{T} = \beta, \quad (3.7)$$

where α and β are constants, Eq. (3.6) become

$$\begin{cases} X_{xxx} = \alpha X - 3X_x, \\ X_{xxxxx} = \beta X - 15X_x - 5X_{xxx}, \end{cases} \quad (3.8)$$

which can be further reduced to

$$\begin{cases} 9X_{xx} - (\alpha + \beta)X_x + \alpha^2 X = 0, \\ \alpha X_{xx} + 9X_x + (2\alpha - \beta)X = 0. \end{cases} \quad (3.9)$$

Combining the two equations in Eq. (3.9) together, we have

$$(\alpha^2 + \alpha\beta + 81)X_x = (\alpha^3 - 18\alpha + 9\beta)X. \quad (3.10)$$

The solution of Eq. (3.10)

$$X(x) = c_1 e^{\frac{\alpha^3 - 18\alpha + 9\beta}{\alpha^2 + \alpha\beta + 81}x}. \quad (3.11)$$

By substituting Eq. (3.11) back into Eq. (3.8), we have

$$\alpha^5 - 25\alpha^3 + 30\beta\alpha^2 + 1215\alpha - \beta^3 - 243\beta = 0, \quad (3.12)$$

that means if α and β satisfy Eq. (3.12), then Eq. (3.11) is the solution of Eq. (3.8). From Eq. (3.7), we have

$$Y(y) = c_2 e^{\alpha y}, \quad T(t) = c_3 e^{\beta t},$$

together with Eq. (3.11) we have

$$\widehat{\phi} = k e^{\frac{\alpha^3 - 18\alpha + 9\beta}{\alpha^2 + \alpha\beta + 81}x + \alpha y + \beta t}, \quad (3.13)$$

where $k = c_1 c_2 c_3$. Using the linear superposition as we did in Lemma 2, we can obtain

$$\phi_i^{(0)} = \sum_{j=1}^n k_j \widehat{\phi}_j = \sum_{j=1}^n k_j e^{\frac{\alpha_j^3 - 18\alpha_j + 9\beta_j}{\alpha_j^2 + \alpha_j\beta_j + 81}x + \alpha_j y + \beta_j t}. \quad (3.14) \quad \square$$

Similar to the previous section about KP equation, we need the following theorem to assure that the solutions we get are without singularities.

Theorem 2. Let eigenfunctions $\phi_m^{(0)}$ take the form as in Lemma 4

$$\phi_m^{(0)} = \sum_{i=1}^n k_{m,i} e^{a_{m,i}x + \alpha_{m,i}y + \beta_{m,i}t}, \quad (3.15)$$

where $m = 1, 2$, if $k_{m,i} > 0$, $a_{1,i} < a_{2,j}$, then $\text{IW}_{2,2}(\phi_2^{(0)}, \phi_1^{(0)}; \phi_1^{(0)}, \phi_2^{(0)}) < 0$. The solution of the $(2+1)$ -dimensional KK equation can be written as

$$u_2^{(2+2)} = 1 + (\log \text{IW}_{2,2})_{xx}. \quad (3.16)$$

Proof. We rewrite $\phi_1^{(0)}$ and $\phi_2^{(0)}$ in Eq. (3.15) as

$$\begin{cases} \phi_1^{(0)} = \sum_{i=1}^n R_i e^{a_i x}, \\ \phi_2^{(0)} = \sum_{i=1}^n S_i e^{b_i x}. \end{cases}$$

Here the values of R_i and S_i are greater than zero. Then we have

$$\int (\phi_1^{(0)})^2 = \sum_{i,j=1}^n R_i R_j \frac{e^{(a_i+a_j)x}}{a_i + a_j}, \quad (3.17)$$

$$\int (\phi_2^{(0)})^2 = \sum_{i,j=1}^n S_i S_j \frac{e^{(b_i+b_j)x}}{b_i + b_j}, \quad (3.18)$$

$$\int \phi_1^{(0)} \phi_2^{(0)} = \sum_{i,j=1}^n R_i S_j \frac{e^{(a_i+b_j)x}}{a_i + b_j}. \quad (3.19)$$

Since $a_i < b_j$ for $i, j = 1 \cdots n$, it is easy prove the following inequality

$$R_i R_j \frac{e^{(a_i+a_j)x}}{a_i + a_j} S_k S_l \frac{e^{(b_k+b_l)x}}{b_k + b_l} > R_i S_k R_j S_l \frac{e^{(a_i+b_k)x}}{a_i + b_k} \frac{e^{(a_j+b_l)x}}{a_j + b_l}, \quad (3.20)$$

where $1 \leq i, j, k, l \leq n$, then

$$\left| \begin{array}{cc} \int \phi_1^{(0)} \phi_2^{(0)} & \int (\phi_1^{(0)})^2 \\ \int (\phi_2^{(0)})^2 & \int \phi_1^{(0)} \phi_2^{(0)} \end{array} \right| = \left(\int \phi_1^{(0)} \phi_2^{(0)} \right)^2 - \int (\phi_1^{(0)})^2 \int (\phi_2^{(0)})^2 < 0. \quad (3.21)$$

can be directly verified by using Eq. (3.17), Eq. (3.18), Eq. (3.19). Equation (3.16) can be obtained by Eq. (3.2) and $u_2^{(0)} = 1$. \square

Remark 1. For $T_{(1+1)} = T_I T_D$, with the generating function $\phi_1^{(0)}$ as in Eq. (3.15), it is easy to show that

$$\tau^{(1+1)} = \left(\int (\phi_1^{(0)})^2 \right) \tau^{(0)} \quad (3.22)$$

is positive. The corresponding new solution of the $(2+1)$ -dimensional KK equation can be represented as

$$u_2^{(1+1)} = 1 + \left(\log \int (\phi_1^{(0)})^2 \right)_{xx} \quad (3.23)$$

Here, we give the single-soliton solution of the $(2+1)$ -dimensional KK equation from the generating function

$$\phi_1^{(0)} = e^{\xi_1} + e^{\xi_2}, \quad (3.24)$$

where $\xi_i = \frac{\alpha_i^3 - 18\alpha_i + 9\beta_i}{\alpha_i^2 + \alpha_i\beta_i + 81} x + \alpha_i y + \beta_i t$, the solution is

$$u_2^{(1+1)} = 1 + \frac{(a_1 - a_2)^2}{a_1 + a_2} \frac{\left(e^{\frac{\xi_1 - \xi_2}{2}} + e^{\frac{\xi_2 - \xi_1}{2}} \right) \left(e^{\frac{\xi_1 - \xi_2}{2}} + e^{\frac{\xi_2 - \xi_1}{2}} \right)}{\left(\frac{e^{\xi_1 - \xi_2}}{2a_1} + \frac{e^{\xi_2 - \xi_1}}{2a_2} + \frac{2}{a_1 + a_2} \right)^2}, \quad (3.25)$$

where $a_i = \frac{\alpha_i^3 - 18\alpha_i + 9\beta_i}{\alpha_i^2 + \alpha_i\beta_i + 81}$. The solution $(u_2^{(1+1)})'$ generated from zero seed have the form

$$(u_2^{(1+1)})' = \frac{(a_1' - a_2')^2}{a_1' + a_2'} \frac{\left(e^{\frac{\xi_1' - \xi_2'}{2}} + e^{\frac{\xi_2' - \xi_1'}{2}} \right) \left(e^{\frac{\xi_1' - \xi_2'}{2}} + e^{\frac{\xi_2' - \xi_1'}{2}} \right)}{\left(e^{\frac{\xi_1' - \xi_2'}{2}} + e^{\frac{\xi_2' - \xi_1'}{2}} + \frac{2}{a_1' + a_2'} \right)^2}, \quad (3.26)$$

where $\xi_i' = \frac{(\alpha_i')^2}{\beta_i'}x + \alpha_i'y + \beta_i't$, $a_i' = \frac{(\alpha_i')^2}{\beta_i'}$ and $(\alpha_i')^5 = (\beta_i')^3$. The differences between $u_2^{(1+1)}$ and $(u_2^{(1+1)})'$ under the same value of parameters are showed in Fig. 3. By taking

$$\begin{aligned} \phi_1^{(0)} = & e^{0.0001999999974x + 0.0006y + 0.003t} + e^{0.000666666665679x + 0.002y + 0.01t} \\ & + e^{0.0033333320988x + 0.01y + 0.05t} + e^{0.00666666567904x + 0.02y + 0.1t}, \end{aligned} \quad (3.27)$$

$$\begin{aligned} \phi_2^{(0)} = & e^{1.218304787x + 5.463203409y + 30t} + e^{0.4917724251x + 1.594247576y + 8t} \\ & + e^{0.6835764081x + 2.370148557y + 12t} + e^{0.970831384x + 3.827515914y + 20t}. \end{aligned} \quad (3.28)$$

in Eq. (3.16), we can obtain solution of the $(2+1)$ -dimensional KK equation which is plotted in Fig. 4.

4. Gauge Transformation for $(2+1)$ -Dimensional SK Equation

The procedure of this section is mostly the same as the previous section except that the transformed Lax operator $L^{(1)}$ by one-step gauge transformation should satisfy $(L^{(1)})^* = -\partial L^{(1)} \partial^{-1}$, so we need Lemma 5 about gauge transformation for BKP hierarchy.

Lemma 5 ([16]). (1) *The appropriate gauge transformation T_{n+k} is given by $n = k$ and generating functions $\psi_i^{(0)} = \phi_{i,x}^{(0)}$ for $i = 1, 2, \dots, n$.*

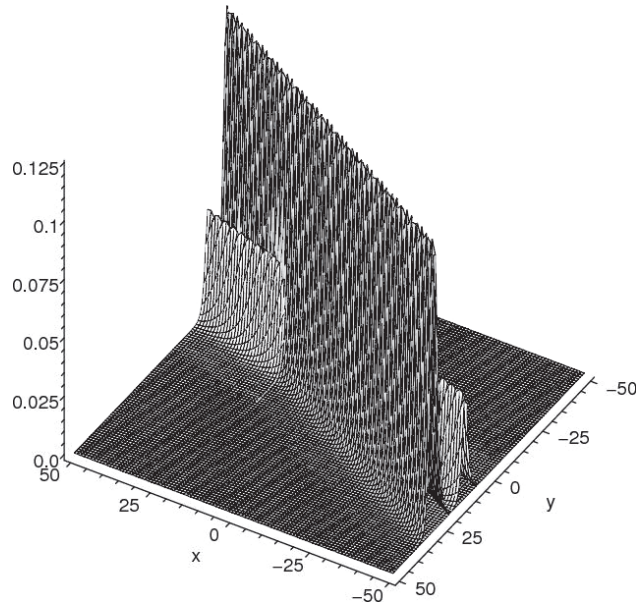
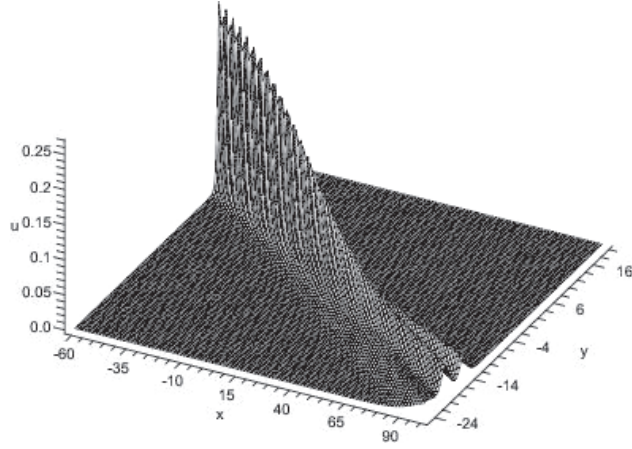


Fig. 3. Single-soliton solutions at $t = 1$ of the $(2+1)$ -dimensional KK equation. The higher one is $(u_2^{(1+1)})'$ with $\alpha_1' = 0.970299$ and $\alpha_2' = 0.075$; the lower one is $(u_2^{(1+1)} - 1)$ with parameters $\alpha_1 = 0.970299$ and $\alpha_2 = 0.075$.

Fig. 4. Solution at $t = 0$ of the $(2 + 1)$ -dimensional KK equation.

(2) The τ -function $\tau_{\text{BKP}}^{(n+n)}$ of the BKP hierarchy has the form

$$\begin{aligned} \tau_{\text{BKP}}^{(n+n)} &= \text{IW}_{n,n}(\phi_{n,x}^{(0)}, \phi_{n-1,x}^{(0)}, \dots, \phi_{1,x}^{(0)}; \phi_1^{(0)}, \phi_2^{(0)}, \dots, \phi_n^{(0)}) \cdot \tau_{\text{BKP}}^{(0)} \\ &= \begin{vmatrix} \int \phi_{n,x}^{(0)} \cdot \phi_1^{(0)} & \cdots & \int \phi_{n,x}^{(0)} \cdot \phi_n^{(0)} \\ \vdots & \cdots & \vdots \\ \int \phi_{1,x}^{(0)} \cdot \phi_1^{(0)} & \cdots & \int \phi_{1,x}^{(0)} \cdot \phi_n^{(0)} \end{vmatrix} \cdot \tau_{\text{BKP}}^{(0)} \end{aligned} \quad (4.1)$$

and we have

$$u_2^{(n+n)} = u_2^{(0)} + (\log \text{IW}_{n,n})_{xx}. \quad (4.2)$$

With this theorem, we can write down the solutions of the $(2 + 1)$ -dimensional SK equation explicitly after successive application of gauge transformations. We take the initial Lax operator $L^{(0)}$ of the BKP hierarchy as

$$L^{(0)} = \partial + \partial^{-1} + \partial^{-3} + \partial^{-5} + \cdots.$$

The corresponding eigenfunction $\phi_i^{(0)}$ and conjugate eigenfunction $\psi_i^{(0)} = \phi_{i,x}^{(0)}$ are given by Lemma 4 and Lemma 5, i.e.

$$\phi_i^{(0)} = \sum_{j=1}^n k_j e^{\frac{\alpha_j^3 - 18\alpha_j + 9\beta_j}{\alpha_j^2 + \alpha_j\beta_j + 81}x + \alpha_j y + \beta_j t}, \quad (4.3)$$

$$\psi_i^{(0)} = \sum_{j=1}^n k_j \frac{\alpha_j^3 - 18\alpha_j + 9\beta_j}{\alpha_j^2 + \alpha_j\beta_j + 81} e^{\frac{\alpha_j^3 - 18\alpha_j + 9\beta_j}{\alpha_j^2 + \alpha_j\beta_j + 81}x + \alpha_j y + \beta_j t}, \quad \beta_j = \beta_j(\alpha_j). \quad (4.4)$$

Similar as Sec. 2 and Sec. 3, we need the following theorem to assure that the new τ -function we get after gauge transformations will not vanish at any point.

Theorem 3. Let eigenfunction $\phi_m^{(0)}$ take the form as in Eq. (4.3)

$$\sum_{i=1}^n k_{m,i} e^{a_{m,i}x + \alpha_{m,i}y + \beta_{m,i}t}, \quad m = 1, 2, \quad \text{if } 0 < 3 \cdot a_{1,i} < a_{2,j},$$

then we have $\text{IW}_{2,2}(\phi_{2,x}^{(0)}, \phi_{1,x}^{(0)}; \phi_1^{(0)}, \phi_2^{(0)}) < 0$. The solution can be written as

$$u_2^{(2+2)} = 1 + (\log \text{IW}_{2,2})_{xx}. \quad (4.5)$$

Proof. $\phi_1^{(0)}$ and $\phi_2^{(0)}$ can be rewritten as

$$\begin{cases} \phi_1^{(0)} = \sum_{i=1}^n R_i e^{a_i x}, \\ \phi_2^{(0)} = \sum_{i=1}^n S_i e^{b_i x}, \end{cases}$$

where the value of R_i and S_i are greater than zero, then we have

$$\frac{(\phi_1^{(0)})^2}{2} = \frac{1}{2} \sum_{i,j=1}^n R_i R_j e^{(a_i+a_j)x}, \quad (4.6)$$

$$\frac{(\phi_2^{(0)})^2}{2} = \frac{1}{2} \sum_{i,j=1}^n S_i S_j e^{(b_i+b_j)x}, \quad (4.7)$$

$$\int \phi_{1,x}^{(0)} \phi_2^{(0)} = \sum_{i,j=1}^n R_i S_j \frac{a_i}{a_i + b_j} e^{(a_i+b_j)x}, \quad (4.8)$$

$$\int \phi_{2,x}^{(0)} \phi_1^{(0)} = \sum_{i,j=1}^n R_j S_i \frac{b_i}{a_j + b_i} e^{(a_j+b_i)x}. \quad (4.9)$$

The following inequality

$$(a_i + b_k)(a_j + b_l) > 4a_i b_l,$$

is trivial if we use $0 < 3 \cdot a_{1,i} < a_{2,j}$ which means $0 < 3 \cdot a_i < b_k$, together with Eq. (4.6)–(4.9), we can prove

$$\left| \begin{array}{cc} \int \phi_1^{(0)} \phi_{2,x}^{(0)} & \frac{(\phi_2^{(0)})^2}{2} \\ \frac{(\phi_1^{(0)})^2}{2} & \int \phi_{1,x}^{(0)} \phi_2^{(0)} \end{array} \right| = \left(\int \phi_{1,x}^{(0)} \phi_2^{(0)} \right) \left(\int \phi_{2,x}^{(0)} \phi_1^{(0)} \right) - \frac{(\phi_1^{(0)})^2 (\phi_2^{(0)})^2}{4} < 0, \quad (4.10)$$

by a direct calculation. Equation (4.5) can be obtained by Eq. (4.2) and $u_2^{(0)} = 1$. \square

Remark 2. For $T_{1+1} = T_I T_D$, with the generating function $\phi_1^{(0)}$ as in Eq. (4.3), it is easy to show that

$$\tau^{(1+1)} = \frac{(\phi_1^{(0)})^2}{2} \tau^{(0)} \quad (4.11)$$

is positive. The corresponding new solution of the $(2+1)$ -dimensional SK equation can be represented as

$$u_2^{(1+1)} = 1 + \left(\log \left(\frac{(\phi_1^{(0)})^2}{2} \right) \right)_{xx} \quad (4.12)$$

To obtain a single-soliton solution of the $(2+1)$ -dimensional SK equation, we start from a generating function

$$\phi_1^{(0)} = e^\xi + e^{-\xi}, \quad (4.13)$$

and the solution is

$$u_2^{(1+1)} = 1 + 2a^2 \text{sech}^2(\xi), \quad (4.14)$$

here $\xi = \frac{\alpha^3 - 18\alpha + 9\beta}{\alpha^2 + \alpha\beta + 81}x + \alpha y + \beta t$ and $a = \frac{\alpha^3 - 18\alpha + 9\beta}{\alpha^2 + \alpha\beta + 81}$. A solution generated from zero seed is

$$(u_2^{(1+1)})' = 2(a')^2 \text{sech}^2(\xi'), \quad (4.15)$$

in which $\xi' = \frac{(\alpha')^2}{\beta'}x + \alpha'y + \beta't$, $(\alpha')^5 = (\beta')^3$ and $a' = \frac{(\alpha')^2}{\beta'}$. The differences between $u_2^{(1+1)}$ and $(u_2^{(1+1)})'$ are showed in Fig. 5. In Fig. 6, we plot the solution of the $(2+1)$ -dimensional SK equation by taking

$$\begin{aligned} \phi_1^{(0)} = & e^{0.009999666694x+0.02999999998y+0.15t} + e^{0.01333254332x+0.03999999992y+0.2t} \\ & + e^{0.006666567904x+0.02y+0.1t}, \end{aligned} \quad (4.16)$$

$$\begin{aligned} \phi_2^{(0)} = & e^{0.5924749002x+1.985399095y+10t} + e^{0.06656825084x+0.1999997386y+t} \\ & + e^{1.218304787x+5.463203409y+30t}, \end{aligned} \quad (4.17)$$

in Eq. (4.5).

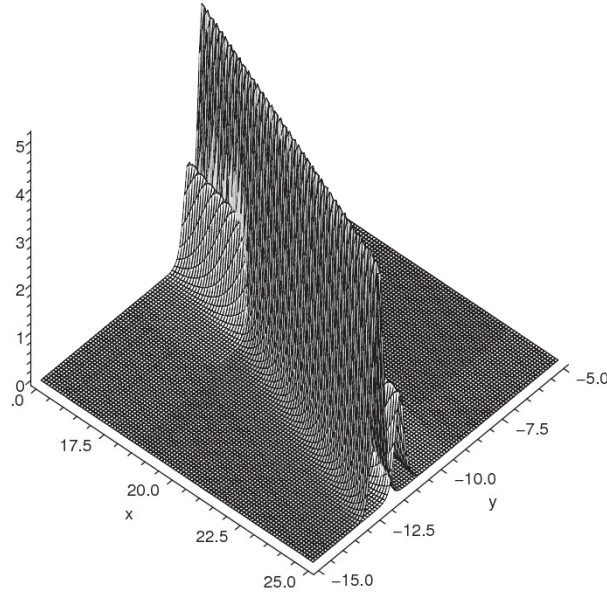


Fig. 5. Single-soliton solutions at $t = 1$ of $(2+1)$ -dimensional SK equation. The higher one is $(u_2^{(1+1)})'$ with $\alpha' = 4.096$, the lower one is $(u_2^{(1+1)} - 1)$ with parameters $\alpha = 4.096$.

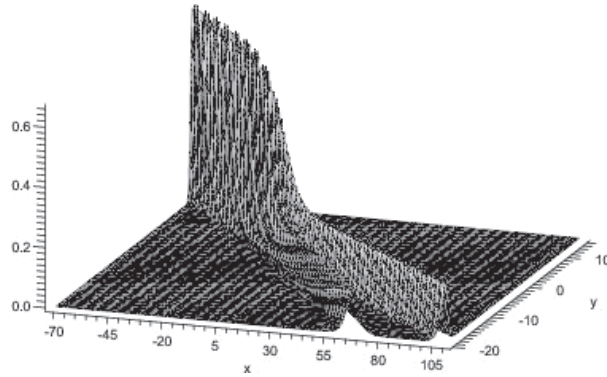


Fig. 6. Solution at $t = 0$ of $(2+1)$ -dimensional SK equation.

Corollary 2. For the $(2+1)$ -dimensional KK equation and $(2+1)$ -dimensional SK equation, there exist a common Galilean type transformation between $(u_2^{(1+1)})'$ (generated from zero seed) and $u_2^{(1+1)}$ (generated from nonzero seed), i.e.

$$u_2'(x, y, t) \mapsto u_2(x, y, t) = 1 + u_2'(x + 3y + 15t, y + 5t, t). \quad (4.18)$$

5. Conclusions and Discussions

By now we have obtained new solutions $u_2^{(n)}$ in Theorem 1 for KP equation, $u_2^{(2+2)}$ in Theorem 2 for $(2+1)$ -dimensional KK equation and $u_2^{(2+2)}$ in Theorem 3 for $(2+1)$ -dimensional SK equation by using the gauge transformations of the KP hierarchy, CKP hierarchy and BKP hierarchy respectively. The corresponding generating functions of the gauge transformations previously mentioned are explicitly expressed in Lemma 2 and Lemma 4. For these three equations, the single-soliton $u_2^{(1)}$ (or $u_2^{(1+1)}$) generated from nonzero seeds and $(u_2^{(1)})'$ (or $(u_2^{(1+1)})'$) generated from zero seed are constructed. The main differences between the u_2 and $(u_2)'$ are height and locations of the peak line under the same value of parameters, which are demonstrated visibly in Figs. 1, 2 and 3. We also found a Galilean type transformation in Eq. (2.29) between $(u_2^{(1)})'$ and $u_2^{(1)}$ for the KP equation, and another one in Eq. (4.18) between $(u_2^{(1+1)})'$ and $u_2^{(1+1)}$ for the $(2+1)$ -dimensional KK and SK equations. To guarantee the new solutions u_2 generated by gauge transformations is smooth, in other words, the transformed τ -function does not vanish at any point, we only consider the W_n in Theorem 1 and $IW_{2,2}$ in Theorem 2 and theorem 3.

The Corollary 1 and Corollary 2 show that we can establish a one-parameter transformation group (specifically, Galilean type transformation) of the solutions of these three equations by setting the seeds $u_2^{(0)} = \epsilon$ (arbitrary constant) instead of $u_2^{(0)} = 1$. The advantage of this new method to find one-parameter group is to avoid solving the characteristic line equation, which is not easy to solve, as usual approach of Lie point transformation. We will try to do this in the future. On the other hand, if we can choose some more complicated initial Lax operator $L^{(0)}$ in which $\{u_i^{(0)}\}$ are not constants and we are able to solve the corresponding generating functions, then we can get some other new solutions. Of course, the calculation is much tedious although the idea is straightforward. The present work is the first step to this difficult purpose.

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