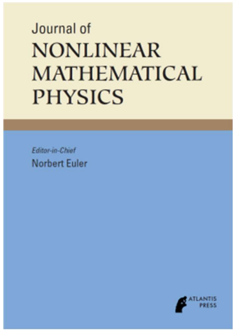

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ON GEODESIC COMPLETENESS OF NONDEGENERATE SUBMANIFOLDS IN SEMI-EUCLIDEAN SPACES

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In this paper, we study the geodesic completeness of nondegenerate submanifolds in semi-Euclidean spaces by extending the study of Beem and Ehrlich [1] to semi-Euclidean spaces. From the physical point of view, this extend may have a significance that a semi-Euclidean space contains more variety of Lorentzian submanifolds rather than those of Lorentzian hypersurfaces in a Minkowski space as in [1]. From mathematical point of view, since there is no distinction in the analysis of geodesic completeness of Lorentzian submanifolds and nondegenerate submanifolds in a semi-Euclidean space, we treat the mathematically more general case of nondegenerate submanifolds in a semi-Euclidean space. The new ideas leading to this generalization are the sufficient conditions for algorithms in the proofs of the results in [1]. Indeed these sufficient conditions for the algorithms also work well for the nondegenerate submanifolds in a semi-Euclidean space.

Keywords: Nondegenerate submanifold; semi-Euclidean spaces; geodesic completeness; affine growth condition; second fundamental form tensor; nonacute geodesic; nonobtuse geodesic; totally umbilic.

Mathematics Subject Classification 2000: 53C50

1. Introduction

Unlike in Riemannian Geometry, a closed imbedded submanifold of a complete semi-Riemannian manifold may not be geodesically complete (cf. [7] and [9]). Cheng and Yau [6] have shown that a closed imbedded spacelike hypersurface in Minkowski space is complete in the induced metric if it has constant mean curvature. Harris [7] has shown that the assumption of constant mean curvature in the above result can be replaced by the assumption of bounded principle curvatures. Beem and Ehrlich [1] have obtained sufficient conditions for the geodesic completeness of a Lorentzian hypersurface M in a Minkowski space by assuming that a unit normal vector field on M satisfies an affine growth condition on M. In this paper, we study the geodesic completeness of nondegenerate submanifolds in semi-Euclidean spaces by extending the study of Beem and Ehrlich [1] to semi-Euclidean spaces. From the physical point of view, this extend may have a significance that a semi-Euclidean space contains more variety of Lorentzian submanifolds rather than those of Lorentzian hypersurfaces in a Minkowski space as in [1]. From mathematical point of view, since there is no distinction in the analysis of geodesic completeness of Lorentzian submanifolds and nondegenerate submanifolds in a semi-Euclidean space, we treat the mathematically more general case of nondegenerate submanifolds in a semi-Euclidean space. We first state a slightly more general form of the subaffine growth condition in [1]. Then, we show that, if the Euclidean norm of the second fundamental tensor of a properly immersed nondegenerate submanifold $M \in \mathbb{R}^n_\nu$ satisfies an affine growth condition along an inextendible geodesic c of M then c is complete. In particular, we show that every properly immersed totally umbilic nondegenerate submanifold $M \in \mathbb{R}^n_\nu$ with $\dim M \geq 2$ is geodesically complete. We call an inextendible geodesic $c : (a, b) \to M$ of M nonobtuse (respectively, nonacute) with respect to a normal vector field Z on M if $\dot{c}(t)$ makes a nonobtuse (respectively, nonacute) angle with Z with respect to the auxiliary Euclidean metric on $\mathbb{R}^n_\nu$. Then, we show that, if an inextendible geodesic $c : (a, b) \to M$ of a properly immersed nondegenerate submanifold $M \in \mathbb{R}^n_\nu$ is nonacute with respect to the second fundamental form tensor along itself then $b = \infty$. For a holomorphic point of view about geodesic completeness see [5].

The new ideas leading to this generalization are the sufficient conditions for algorithms in the proofs of the results in [1]. Indeed these sufficient conditions for the algorithms also work well for the nondegenerate submanifolds in a semi-Euclidean space. For examples, Definition 3.1, that is the geodesic-wise version of the subaffine growth condition in [1], is sufficient for the algorithm showing the geodesic completeness of nondegenerate submanifolds in a semi-Euclidean space. Also, in the case of a Lorentzian hypersurface M in a Minkowski space, timelike geodesics of M makes either nonobtuse or nonacute angle with a normal vector field of M, and hence with the second fundamental form tensor of M along timelike geodesics of M. But the algorithm of its proof only depends on the nonobtuse/nonacute angle it makes with the second fundamental form tensor along these geodesics. Hence, by this observation, it can be generalized to nondegenerate submanifolds of a semi-Euclidean space.

2. Preliminaries

Here, we briefly state the main concepts and definitions used throughout this paper.

$\mathbb{R}^n_\nu$ denote n-dimensional semi-Euclidean space with standard metric

$$g = -\sum_{i=1}^{\nu} dx^i \otimes dx^i + \sum_{i=\nu+1}^{n} dx^i \otimes dx^i$$

and let

$$g_E = \sum_{i=1}^{n} dx^i \otimes dx^i$$

be the associated auxiliary Euclidean metric on $\mathbb{R}^n_\nu$, where $1 \leq \nu < n$. (Notice that both metrics have the same Levi–Civita connection ∇). A vector $0 \neq v \in \mathbb{R}^n_\nu$ is called timelike (respectively, null or spacelike) if $g(v, v) < 0$ (respectively, $g(v, v) = 0$ or $g(v, v) > 0$). An immersion $f : M \to \mathbb{R}^n_\nu$ of a manifold M is called nondegenerate if f^*g is a (nonsingular) metric on M. Without loss of generality, we shall always assume that f^*g is not negative definite. By a geodesic of M, we shall always mean a geodesic of (M, f^*g). Let $f : M \to \mathbb{R}^n_\nu$ be a nondegenerate immersion of a manifold M. For each $p \in M$, we identify T_pM with $f_*T_pM \subset T_{f(p)}M$ and denote $(f_*T_pM)^\perp$ by $T_pM^\perp$. We also identify $T_p\mathbb{R}^n_\nu$ with $\mathbb{R}^n_\nu$ for each $p \in \mathbb{R}^n_\nu$. The second fundamental form tensor $\mathrm{II} : T_pM \times T_pM \to T_pM^\perp$ of a nondegenerate submanifold in a semi-Riemannian manifold $\mathbb{R}^n_\nu$ is defined by $\mathrm{II}(x, y) = \nabla_x Y^\perp$, where $Y \in \Gamma TM$ with $Y_p = y$, ∇ is the Levi–Civita connection on $\mathbb{R}^n_\nu$ and $\nabla_x Y^\perp$ is the component of $\nabla_x Y$ normal to M. The second fundamental form operator $L_z : T_pM \to T_pM$ in the direction of $z \in T_pM^\perp$ is defined by $g(L_z x, y) = g(\mathrm{II}(x, y), z)$, where $x, y \in T_pM$. (Thus, $L_z x = -(\nabla_x Z)^T$, where Z is a normal extension of z to a neighborhood of $p \in M$ and $(\nabla_x Z)^T$ is the component of $\nabla_x Z$ tangent to M). The second fundamental form II_z in the direction $z \in T_pM^\perp$ is defined by $\mathrm{II}_z(x, y) = g(L_z x, y)$, where $x, y \in T_pM$ (cf. [8, pp. 97–108]). We denote the Euclidean norm of a vector $v \in \mathbb{R}^n_\nu$ by $\|v\|_E$ (that is $\|v\| = [g_E(v, v)]^{\frac{1}{2}}$).

Remark 2.1. Although, in general, the induced metric on M^k may have any signature $(p, q), p+q = k, p < \nu$ and cannot be made not negatively defined i.e., $(k, 0)$; indeed we may assume f^*g on M

is nonnegative definite because in case it is negative definite, we multiply g with -1 and obtain metric f^*g on M positive definite without changing any concept in the paper. That is we consider the immersion into $\mathbb{R}^n_{n-\nu}$ instead of $\mathbb{R}^n_\nu$.

3. Geodesic Completeness of Nondegenerate Submanifolds

Let $f : M \to \mathbb{R}^n_\nu$ be an immersion of a manifold M. Let d_M and $L_M(c)$ respectively denote the distance function and the length of a curve c on M in the Riemannian structure of (M, f^*g_E). In [1], a normal vector field Z on a nondegenerate submanifold M in (Minkowski space) $\mathbb{R}^n_1$ is said to satisfy the subaffine growth condition on M if there exist $p_0 \in M$ and positive constants A, B such that $\|Z(p)\|_E \leq A + Bd_M(p_0, p)$ for all $p \in M$ (cf. [1, Definition 3.3]). We now state a slightly more general form of this definition.

Definition 3.1. Let $f : M \to \mathbb{R}^n_\nu$ be an immersion of a manifold M and let $c : I \to M$ be a curve. A function φ along c is said to satisfy the affine growth condition along c if there exist a point $p_0 \in M$ and positive constants A, B (which may depend on c) such that $|\varphi(t)| \leq A + Bd_M(p_0, c(t))$ for all $t \in I$. A function φ on M is said to satisfy the affine growth condition on M if it satisfies the affine growth condition along each curve $c : I \to M$. A function φ on M is said to satisfy the subaffine growth condition on M if there exist $p_0 \in M$ and positive constants A, B such that $|\varphi(p)| \leq A + Bd_M(p_0, p)$ for all $p \in M$.

Remark 3.2. Notice that, if a function φ on M satisfies the subaffine growth condition on M then φ satisfies the affine growth condition on M.

Recall that if $c : (a, b) \to M$ is a geodesic of a nondegenerate submanifold in $\mathbb{R}^n_\nu$ then $\nabla_{\dot{c}}\dot{c} = \text{II}(\dot{c}, \dot{c})$ (cf. [8, p. 103]). Let $c : (a, b) \to M$ be a geodesic of M, $t_0 \in (a, b)$ and $s = s(t)$ be arc length of a geodesic segment $c_{|[t_0,t]}$ in (M, f^*g_E). Then, since g and g_E have the same Levi–Civita connection, it follows from $\frac{ds}{dt} = \|\dot{c}\| = [g_E(\dot{c}, \dot{c})]^{\frac{1}{2}}$ that $\frac{d^2s}{dt^2} = g_E(\text{II}(\dot{c}, \dot{c}), \dot{c})[g_E(\dot{c}, \dot{c})]^{-\frac{1}{2}} = g_E(\text{II}(\dot{c}, \dot{c}), \frac{\dot{c}}{\|\dot{c}\|_E})$.

Remark 3.3. The arc length s of the geodesic c of M which means in (M, g_M) is taken with respect to induced Euclidean metric on M as indicated in the above paragraph since a geodesic with respect to one metric is still a curve and has an arc length as a curve with respect to the other. Hence $\frac{ds}{dt} = \|\dot{c}\|$. So, in the light of the above point the arc length should not be taken with respect to induced semi-Euclidean metric on M which is nonsense in semi-Riemannian geometry since null geodesics have zero arc length function. Consequently there is no confusion of orthogonal space of T_pM with respect to g and g_E. In fact, orthogonal space to T_pM with respect to g_E is never used.

Theorem 3.4. *Let $f : M \to \mathbb{R}^n_\nu$ be a nondegenerate proper immersion of a manifold M and let $c : (a, b) \to M$ be an inextendible geodesic of M. If $\|\text{II}(\dot{c}, \dot{c})\|_E$ satisfies the affine growth condition along c then c is complete.*

Proof. (Following [1]). From Schwartz inequality,

$$\begin{aligned}
\frac{d^2s}{dt^2} &= g_E\left(\text{II}(\dot{c}(t), \dot{c}(t)), \frac{\dot{c}(t)}{\|\dot{c}(t)\|_E}\right) \\
&\leq \left| g_E\left(\text{II}(\dot{c}(t), \dot{c}(t)), \frac{\dot{c}(t)}{\|\dot{c}(t)\|_E}\right)\right| \\
&\leq \|\text{II}(\dot{c}(t), \dot{c}(t))\|_E \\
&\leq A + Bd_M(p_0, c(t)) \\
&\leq A + Bd_M(p_0, c(t_0)) + Bd_M(c(t_0), c(t)) \\
&\leq A' + Bs,
\end{aligned}$$

where $p_0 \in M, t_0 \in (a,b), s(t) = L_M(c\mid_{[t_0,t]})$ and $A' = A + Bd_M(p_o, c(t_0))$. Hence, multiplying this inequality with $\frac{ds}{dt}(=\|\dot{c}\|_E > 0)$ and integrating, we obtain $[\frac{ds}{dt}]^2 \leq \lambda + 2A's + Bs^2$, where $\lambda = [\frac{ds}{dt}(t_0)]^2$. Thus, $\frac{ds}{dt} \leq [\lambda + 2A's + Bs^2]^{1/2}$, and therefore

$$\int_{s_0}^{s(b^-)} [\lambda + 2A's + Bs^2]^{-1/2} ds \leq \int_{t_0}^{b^-} dt = b^- - t_0,$$

where $s_0 = s(t_0)$. On the other hand, since c is inextendible in M and f is proper,

$$\lim_{t \to b^-} s(t) = \infty$$

(cf. [2, p. 64, Lemma 2.52] or [3, p. 102, Lemma 3.65]). Hence, it suffices to show that

$$\int_{s_0}^{\infty} [\lambda + 2A's + Bs^2]^{-1/2} ds = \infty.$$

It is easy to see that there exist $S \in (s_0, \infty)$ such that $[\lambda + 2A's + Bs^2]^{1/2} \leq 2\sqrt{B}s$ for all $s \geq S$. Thus,

$$\infty = \int_S^{\infty} (1/(2\sqrt{B}s))ds \leq \int_S^{\infty} [\lambda + 2A's + Bs^2]^{-1/2} ds.$$

A similar argument shows that $a = \infty$. □

As a straightforward consequence of the above theorem, we have the following corollary.

Corollary 3.5. *Let $f : M \to \mathbb{R}^n_\nu$ be a nondegenerate proper immersion of a manifold M. If $\|\mathrm{II}(\dot{c}, \dot{c})\|_E$ satisfies the affine growth condition along every inextendable geodesic $c : (a,b) \to M$ then M is geodesically complete.*

We recall that a nondegenerate submanifold M is called umbilic at $p \in M$ if there exist a normal vector z to M at p such that $\mathrm{II}(x,y) = g(x,y)z$. M is called totally umbilic if M is umbilic at each $p \in M$. Thus, if M is totally umbilic then there is a normal vector field Z on M such that $\mathrm{II}(X,Y) = g(X,Y)Z$, and Z is called the normal curvature vector field, where X, Y are vector fields tangent to M (cf. [8, p. 105]).

Corollary 3.6. *Let $f : M \to \mathbb{R}^n_\nu$ be a proper immersion of a connected manifold M as a nondegenerate totally umbilic submanifold, where* $\dim M \geq 2$. *Then M is geodesically complete.*

Proof. Let $c : (a,b) \to M$ be an inextendible geodesic of M. Then, since M is totally umbilic, $\mathrm{II}(\dot{c}, \dot{c}) = g(\dot{c}, \dot{c})Z$, where Z is the normal curvature vector field. Thus, it suffices to show that $\|Z\|_E$ satisfies the subaffine growth condition on M. Then, it follows from Theorem 3.4 that M is geodesically complete. Let L_Z be the second fundamental form operator in the direction Z. Then, for $X, Y \in \Gamma TM$, $g(L_Z X, Y) = g(\mathrm{II}(X,Y), Z) = g(X,Y)g(Z,Z)$. Thus, $L_Z = g(Z,Z)id$ on TM, where $id : TM \to TM$ is the identity homeomorphism. Also, since $\mathbb{R}^n_\nu$ is of constant curvature, it follows that Z is normal parallel, and hence $k = g(Z,Z)$ is constant on M (cf. [8, p. 124, Exercise 6]). Now, let $p_0 \in M$ and $p \in M$ be any point. Since, f is a proper immersion, $(M, f^* g_E)$ is a complete connected Riemannian manifold, and therefore there exist a geodesic $\gamma : [c,d] \to M$ of $(M, f^* g_E)$

with $\gamma(c) = p_0, \gamma(d) = p$ and $L_M(\gamma) = d_M(p_0, p_0)$. Then, since Z is normal parallel, at $\gamma(t) \in M$,

$$\begin{aligned}\frac{d}{dt} g_E(Z,Z) &= 2g_E(\nabla\gamma' Z, Z) \\ &= 2g_E([\nabla_{\gamma'} Z]^T, Z) \\ &= -2g_E(L_Z\gamma', Z) \\ &= -2kg_E(\gamma', Z) \\ &\leq 2|k||g_E(\gamma', Z)| \\ &\leq 2|k|\|\gamma'\|_E\|Z\|_E,\end{aligned}$$

from Schwarz inequality, where $[\nabla_{\gamma'} Z]^T$ is the component of $\nabla_{\gamma'} Z$ tangent to M with respect to metric g. On the other hand, since $\frac{d}{dt} g_E(Z,Z) = 2\|Z\|_E(\frac{d}{dt}\|Z\|_E)$, it follows that $\frac{d}{dt}\|Z\|_E \leq |k|\|\gamma'\|_E$. Thus,

$$\begin{aligned}\|Z(p)\|_E &\leq \|Z(p_0)\|_E + |k| \int_c^d \|\gamma'\|_E dt \\ &= \|Z(p_0)\|_E + |k| d_M(p_0, p).\end{aligned}$$

Therefore, $\|Z\|_E$ satisfies the subaffine growth condition on M. □

Theorem 3.7. *If $f : M \to \mathbb{R}^n_\nu$ be a proper immersion of a connected manifold M as a nondegenerate totally umbilic submanifold, where* $\dim M \geq 2$, *then M is a space form. As a partial converse, every constant curvature hypersurface M in a semi-Euclidean space satisfies affine growth condition.*

Proof. M is geodesically complete by Corollary 3.6. Since M is totally umbilic then is of constant curvature. Hence M is a space form.

Conversely, let $c : [0, a) \to M$ be an inextendable geodesic in M. Then, since M is a totally umbilic hypersurface, $\text{II}(\dot{c}, \dot{c}) = g(\dot{c}, \dot{c})Z$, where Z is the normal curvature vector field of M. Thus it follows as in the above corollary that, there exist constants A and B such that $\|\text{II}(\dot{c}(t), \dot{c}(t))\|_E \leq A + Bd_M(c(0), c(t))$ for all $t \in [0, a)$. □

Remark 3.8. Also note that, if $f : M \to \mathbb{R}^n_\nu$ is a proper immersion of a connected manifold M of dimension ≥ 3 as a nondegenerate nonzero constant curvature hypersurface then, since M is totally umbilic (cf. [8, p. 117] — which mentions only hypersurfaces, as opposed to the general codimension of this paper), M is complete from Corollary 3.6.

It is also shown in [1] that if M is a properly immersed spacelike submanifold in $\mathbb{R}^n_1$ which has a unit normal timelike vector field satisfying the subaffine growth condition on M then M is complete (cf. [1, Theorem 3.4]). Proof of this result uses the fact that, if $z, n \in \mathbb{R}^n_1$ with $g(z,z) = -1, g(n,n) = 1$, and $g(z,n) = 0$ then $\|z\|_E \geq \|n\|_E$. However, if $\nu \geq 2$ then $z, n \in \mathbb{R}^n_\nu$ with $g(z,z) = -1, g(n,n) = 1$, and $g(z,n) = 0$ implies $\|z\|_E \geq \|n\|_E$ if z^- and n^- are linearly dependent, where z^- and n^- are, respectively, the components of z and n in $\text{span}\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^\nu}\}$ (notice that, this is the case when $\nu = 1$). Hence, [1, Theorem 3.4] can also be extended to semi-Riemannian submanifolds in semi-Euclidean spaces with an additional (but highly restrictive) assumption. That is; if $c : (a, b) \to M$ is a spacelike inextendible geodesic of a properly immersed nondegenerate submanifold M in $\mathbb{R}^n_\nu$ then, c is complete if there exist a unit normal timelike vector field Z to M along c such that Z satisfies the affine growth condition along c and $Z^- = h(t)\dot{c}^-$ along c, where h is a function along c.

Definition 3.9. Let M be a nondegenerate submanifold of $\mathbb{R}^n_\nu$, $c : (a, b) \to M$ be an inextendible geodesic of M and Z be a normal vector field to M along c. c is called nonacute (respectively, nonobtuse) geodesic with respect to Z if $g_E(Z(t), \dot{c}(t)) \leq 0$ (respectively, $g_E(Z(t), \dot{c}(t)) \geq 0$) for all $t \geq t_0$ for some $t_0 \in (a, b)$.

Examples of nonacute and nonobtuse geodesic can be obtained by intersecting hyperquadrics in $\mathbb{R}^n_\nu$ by the planes through the origin (cf. [8, pp. 108–114]).

Remark 3.10. Note that, if c is a nonspacelike geodesic of a timelike hypersurface M in $\mathbb{R}^n_1$ then c is either nonobtuse or a nonacute geodesic with respect to unit normal on M (cf. [1, Lemma 4.4]). However, a spacelike geodesic of M may be not either nonobtuse or nonacute with respect to Z. Also, a geodesic of a nondegenerate submanifold M of $\operatorname{codim}(M) \geq 2$ in $\mathbb{R}^n_1$ may be not either nonobtuse or nonacute.

Theorem 3.11. *Let $f : M \to \mathbb{R}^n_\nu$ a nondegenerate proper immersion of M. If $c : (a, b) \to M$ is a nonacute inextendable geodesic with respect to $\mathrm{II}(\dot{c}, \dot{c})$ then $b = \infty$.*

Proof. (Following [1]). Since, $\frac{d^2s}{dt^2} = g_E(\mathrm{II}(\dot{c}(t), \dot{c}(t)), \dot{c}(t)) \leq 0$ for all $t \geq t_0$ for some $t_0 \in (a, b)$, $\frac{ds}{dt} = \|\dot{c}\|_E$ is bounded on $[t_0, b)$. Thus,

$$\int_{s_0}^{s} ds = \int_{t_0}^{t} \|\dot{c}\|_E \leq \Big(\sup_{[t_0, b)} \|\dot{c}\|_E \Big)(t - t_0),$$

where $s = s(t)$ and $s_0 = s(t_0)$. On the other hand, since c is inextendible in M and f is proper,

$$\lim_{t \to t_0} s(t) = \infty,$$

and therefore $b = \infty$. □

The sign ϵ of a nondegenerate hypersurface M in a semi-Riemannian manifold (M', g) is defined by $\epsilon = g(Z, Z)$, where Z is a unit normal vector field to M (cf. [8, p. 106]).

Corollary 3.12. *Let $f : M \to \mathbb{R}^n_\nu$ be a proper immersion of a manifold M as a nondegenerate hypersurface with sign $\epsilon = -1$ (respectively, $\epsilon = 1$) and let $c : (a, b) \to M$ be an inextendible geodesic of M. If unit normal vector field Z is in $\operatorname{span}\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^\nu}\}$ (respectively, in $\operatorname{span}\{\frac{\partial}{\partial x^{\nu+1}}, \ldots, \frac{\partial}{\partial x^n}\}$) along c then c is complete.*

Proof. Assume $\epsilon = -1$. Let, $Z = \sum_{i=1}^{\nu} a_i(\frac{\partial}{\partial x^i})$ and $\dot{c} = \sum_{i=1}^{n} b_i(\frac{\partial}{\partial x^i})$ along c, where $a_i (i = 1, \ldots, \nu)$ and $b_i (i = 1, \ldots, n)$ are some functions along c. Then, since $g(Z, \dot{c}) = 0$, $\sum_{i=1}^{\nu} a_i b_i = 0$ along c. Thus, $g_E(Z, \dot{c}) = 0$, and it follows from Theorem 3.11 that c is complete since c is a nonacute geodesic with respect to $\mathrm{II}(\dot{c}, \dot{c})$. Proof of $\epsilon = 1$ is similar. □

Corollary 3.13. *Let $f : M \to \mathbb{R}^n_\nu$ be a proper immersion of a manifold M as a nondegenerate hypersurface and let Z be a unit normal vector field to M. If $\mathrm{II}_Z(\dot{c}, \dot{c}) \geq 0$ (respectively, $\mathrm{II}_Z(\dot{c}, \dot{c}) \leq 0$) then every nonacute (respectively, nonobtuse) inextendible geodesic $c : (a, b) \to M$ with respect to Z has domain (a, ∞).*

Proof. Immediate from Theorem 3.11 since $\mathrm{II}(\dot{c}, \dot{c}) = \mathrm{II}_Z(\dot{c}, \dot{c})Z$. □

Remark 3.14. Let M be a nondegenerate hypersurface with sign ϵ in $\mathbb{R}^n_\nu$, Z be a unit normal vector field to M and let $\wp$ be a nondegenerate plane to M. Then, from Gauss equation, the sectional curvature $K(\wp)$ of $\wp$ is given by

$$K(\wp) = \frac{\epsilon[\mathrm{II}_Z(x, x)\, \mathrm{II}_Z(y, y) - (\mathrm{II}_Z(x, y))^2]}{g(x, x)g(y, y) - (g(x, y))^2},$$

where $\wp = \operatorname{span}\{x, y\}$. Then, if $\mathrm{II}_Z(x, x) \geq 0$ (respectively, if $\mathrm{II}_Z(x, x) \leq 0$) for every $x \in \Gamma TM$ then $\mathrm{II}_Z(x+ty, x+ty) = \mathrm{II}_Z(x, x) + 2t\,\mathrm{II}_Z(x, y) + t^2\,\mathrm{II}_Z(y, y) \geq 0$ (respectively, ≤ 0) for all $t \in \mathbb{R}$. Thus, in either case, $\mathrm{II}_Z(x,x)+2t\mathrm{II}_Z(x,y)+t^2\mathrm{II}_Z(y,y)\geq 0$ for every $\{x,y\}$ which span a nondegenerate plane tangent to M. Hence, if $\epsilon=1$ (respectively, $\epsilon=-1$) then the sectional curvatures of nondegenerate planes tangent to M with signature $(+,-)$ are nonpositive (respectively, nonnegative) and the sectional curvatures of nondegenerate planes tangent to M with signature $(+,+)$ or $(-,-)$ are nonpositive (respectively, nonnegative). Thus, if M is of constant curvature C then $C=0$. Note also that, if M is of constant curvature $(C=0)$ then, M is totally geodesic iff M is totally umbilic. (Also see the Remark below Corollary 3.6).

A nondegenerate hypersurface M in $\mathbb{R}^n_\nu$ is called diagonal if its second fundamental form operator is diagonalizable with respect to an orthonormal basis at each point $p\in M$

Remark 3.15. Notice that every spacelike hypersurface M in $\mathbb{R}^n_\nu$ is diagonal. Moreover, if principal curvatures of M are bounded then the second fundamental form of M is bounded along the geodesics of M. However, if M is a semi-Riemannian hypersurface in $\mathbb{R}^n_\nu$ then, even if M is diagonal, the fundamental form of M may not be bounded (cf. [1]).

We finally state a special case of Theorem 3.11 for diagonal semi-Riemannian hypersurfaces in $\mathbb{R}^n_\nu$ by extending [1, Theorem 4.5] in semi-Euclidean spaces. Let M be a diagonal nondegenerate hypersurface. Then, since the second fundamental form of M is diagonalizable, let $\{e_1,\ldots,e_{\nu'},e_{\nu'+1},\ldots,e_{n'}\}$ be the principle directions with corresponding principle curvatures $\{k_1,\ldots,k_{\nu'},k_{\nu'+1},\ldots,k_{n'}\}$ with respect to a unit normal vector field Z at each point $p\in M$, where

$$\nu'=\begin{cases}\nu & \text{if } \epsilon=1\\ \nu-1 & \text{if } \epsilon=-1\end{cases},\quad n'=n-1$$

and

$$g(e_i,e_i)=\begin{cases}-1 & \text{for } 1\leq i\leq \nu'\\ 1 & \text{for } \nu'+1\leq i\leq n'\end{cases}.$$

Corollary 3.16. *Let $f:M\to\mathbb{R}^n_\nu$ be a proper immersion of a manifold M as a diagonal semi-Riemannian hypersurface of sign ϵ and let Z be a unit normal vector field on M, where $n\geq 2$. If the principle curvatures of M with respect to Z satisfy $\min\{k_1,\ldots,k_{\nu'}\}\geq 0\geq \max\{k_{\nu'+1},\ldots,k_{n'}\}$ (respectively, $\min\{k_{\nu'+1},\ldots,k_{n'}\}\geq 0\geq\max\{k_1,\ldots,k_{\nu'}\}$) at each point $p\in M$ then every nonobtuse timelike (respectively, nonacute spacelike) inextendible geodesic $c:(a,b)\to M$ with respect to Z has domain (a,∞).*

Proof. We shall only prove the case when c is nonspacelike. Proof of the case when c is spacelike is similar. Let $k_0=\min\{k_1,\ldots,k_{\nu'}\}$ and $K_0=\max\{k_{\nu'+1},\ldots,kn'\}$ at $p\in M$. Then, for all nonspacelike $u\in T_pM$

$$\begin{aligned}\mathrm{II}_Z(u,u)&=-\sum_{i=1}^{\nu'}k_i(g(u,e_i))^2+\sum_{i=\nu'+1}^{n'}k_i(g(u,e_i))^2\\&\leq -k_0\sum_{i=1}^{\nu'}(g(u,e_i))^2+K_0\sum_{i=\nu'+1}^{n'}(g(u,e_i))^2\\&\leq k_0\left[-\sum_{i=1}^{\nu'}(g(u,e_i))^2+\sum_{i=\nu'+1}^{n'}(g(u,e_i))^2\right]\\&=k_0g(u,u)\\&\leq 0.\end{aligned}$$

Then, from Theorem 3.11, every inextendible nonobtuse nonspacelike geodesic of M with respect to Z has domain(a, ∞). □

Note that, if M is a Lorentzian hypersurface in $\mathbb{R}^n_\nu$ with sign ϵ then, $\epsilon = -1$ iff $\nu = 1$, and $\epsilon = 1$ iff $\nu = 2$. In [1] and [4], Beem and Ehrlich studied the nonspacelike geodesic completeness of spacetimes which can be isometrically immersed in $\mathbb{R}^n_1$ as diagonal Lorentzian hypersurfaces with nonvanishing second fundamental form. In the remarks below, we discuss the difference between the Lorentzian hypersurfaces in $\mathbb{R}^n_1$ and $\mathbb{R}^n_2$.

4. Remarks

1. Every Lorentzian hypersurface in $\mathbb{R}^n_1$ is necessarily stably causal (cf. [1]). However, a Lorentzian hypersurface in $\mathbb{R}^n_2$ may contain closed timelike curves, for example, the pseudohyperbolic space $\mathbb{H}^n_1$ contains closed timelike curves (cf. [8, p. 229]).
2. If M is a Lorentzian hypersurface in $\mathbb{R}^n_1$ then a unit normal vector field can be chosen so that every future directed (respectively, past directed) nonspacelike geodesic of M is nonobtuse (respectively, nonacute) (cf. [1]). However, if M is a Lorentzian hypersurface in $\mathbb{R}^n_2$ then a future directed nonspacelike geodesic may be neither nonobtuse nor nonacute with respect to a unit normal vector field. (For example, elliptic timelike geodesics of pseudohyperbolic space $\mathbb{H}^n_1$ are not either nonabtuse or nonacute).

An example of a spacetime which can be isometrically immersed in $\mathbb{R}^n_2$ as a Lorentzian hypersurface is the universal anti-de Sitter spacetime $\tilde{\mathbb{H}}^n_1$ which is the universal covering space of the pseudo hyperbolic space $\mathbb{H}^n_1$. $\tilde{\mathbb{H}}^n_1$ is a causally simple (but not globally hyperbolic) spacetime with constant curvature $C < 0$ (cf. [8, p. 229]).

References

[1] J. K. Beem and P. E. Ehrlich, Geodesic completeness of submanifolds in Minkowski space, *Geom. Dedicata* **18** (1985) 213–226.
[2] J. K. Beem and P. E. Ehrlich, *Global Lorentzian Geometry* (Marcel Dekker, New York, 1981).
[3] J. K. Beem, P. E. Ehrlich and K. L. Easley, *Global Lorentzian Geometry*, 2nd edn. (Marcel Dekker, New York, 1996).
[4] J. K. Beem and P. E. Ehrlich, Incompleteness of timelike submanifolds with nonvanishing second fundamental form, *Gen. Rel. Grav.* **17** (1985) 293–300.
[5] C. Meneghini, A holomorphic point of view about geodesic completeness, *J. Nonlinear Math. Phys.* **11**(3) (2004) 297–324.
[6] S.-Y. Cheng and S.-T. Yau, Maximal spacelike hypersurfaces in the Lorenz Minkowski spaces, *Math. Ann.* **104** (1976) 407–419.
[7] S. G. Harris, Closed and complete spacelike hypersurfaces in Minkowski space, *Class. Quantum Grav.* **5** (1988) 111–119.
[8] B. O'Neill, *Semi-Riemannian Geometry* (Academic Press, New York, 1983).
[9] R. Schneider, Zur affinen Differentialgeometrie im Grossen I, *Math Z.* **101** (1967) 375–406.