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Article

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COMMUTATIVITY OF PFAFFIANIZATION AND BÄCKLUND TRANSFORMATIONS: THE LEZNOV LATTICE

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In this paper, we first obtain Wronskian solutions to the Bäcklund transformation of the Leznov lattice and then derive the coupled system for the Bäcklund transformation through Pfaffianization. It is shown the coupled system is nothing but the Bäcklund transformation for the coupled Leznov lattice introduced by J. Zhao etc. [1]. This implies that Pfaffianization and Bäcklund transformation is commutative for the Leznov lattice. Moreover, since the two-dimensional Toda lattice constitutes the Leznov lattice, it is obvious that the commutativity is also valid for it.

Keywords: The coupled Leznov lattice; Pfaffianization; Bäcklund transformation; Wronskian; pfaffian.

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1. Introduction

Bäcklund transformations play a very important role and have made great contributions in the development of soliton theory [2, 3]. As the transformation between a solution of a given linear or nonlinear differential equation and another solution to another same or different differential equation, a Bäcklund transformation provides us a powerful tool to construct solutions. Moreover, its bilinear form not only generate Lax pairs used in the inverse scattering method in a standard way, but also new soliton equations and Miura transformations. A typical example is the Bäcklund transformation for the KdV equation which can be used to derive Lax pair for the KdV equation, generate the modified KdV equation (mKdV) and the Miura transformation between the KdV equation and the mKdV equation [4, 5].

Pfaffians are more general than determinants and have richer structures. Many interesting features of pfaffians have been discovered through research into soliton equations. The pfaffianization procedure, introduced by Hirota and Ohta in 1991 [6], is a fairly effective technique to generalize soliton equations with determinant solutions into their coupled systems with pfaffian solutions. Recently, this procedure has been successfully applied to several important equations [7–13]. Besides, Gilson has also generalized all the equations in bilinear KP hierarchy to their pfaffian forms [14].

So far, the solutions to bilinear equations are expressed by determinants or pfaffians. Determinants are for the KP hierarchy. The pfaffian representations are for the B-type KP hierarchy and for the coupled KP hierarchy. As the Bäcklund transformation for the KP equation, the modified KP equation also has determinant solutions and thus can be pfaffianized. Based on these facts, Hu etc. proposed the idea of commutativity of pfaffianization and Bäcklund transformation for the KP equation in [15]. They successfully derived the Bäcklund transformation for the coupled KP equation [6], which to a certain point, solved the open problem posed by Hirota in his book [4]: What kind of soliton equations are generated from Bäcklund transformation formulae of the coupled KP equation? The main idea of commutativity of pfaffianization and BT (CPBT) may be explained as follows. Given a general soliton equation, say Σ, suppose that the coupled Σ system is generated through pfaffianization of Σ and the modified Σ ($\mathrm{m}\Sigma$) system also serves as a BT for Σ, then the system derived by pfaffianizing $\mathrm{m}\Sigma$ should provide us with a Bäcklund transformation for the coupled Σ if the CPBT is valid for Σ (see Fig. 1).

In the present paper, we show that the commutativity of pfaffianization and BT, is also valid for the two-dimensional Leznov lattice. It is worth to point out that the commutativity is also valid for the two-dimensional Toda lattice equation as proved in [23], noting that the bilinear system of the Leznov lattice is nothing but the bilinear form of the two-dimensional Toda lattice plus one more equation.

The paper is organized as follows. In Sec. 2, we present Wronskian solutions to the modified Leznov lattice. In Sec. 3, we derive the pfaffianized system to the modified Leznov lattice. In Sec. 4, we show that the pfaffianized system constitutes a Bäcklund transformation for the coupled Leznov lattice. Further discussions are given in Sec. 5.

2. Wronskian Solutions to the Modified Leznov Lattice

The Leznov lattice under consideration is given by [16]

$$\frac{\partial^2}{\partial x \partial y} \ln \theta_n = \theta_{n+1} P_{n+1} - 2\theta_n P_n + \theta_{n-1} P_{n-1}, \tag{2.1}$$

$$\frac{\partial P_n}{\partial y} = \theta_{n+1} - \theta_{n-1}, \tag{2.2}$$

$$\begin{tikzcd}
\mathrm{m}\Sigma \arrow[r, "\text{Pfaffianization}"] & \text{Pfaffianized } \mathrm{m}\Sigma \\
\Sigma \arrow[u, "\text{BT}"] \arrow[r, "\text{Pfaffianization}"] & \text{coupled } \Sigma \arrow[u, "\text{BT}"']
\end{tikzcd}$$

Fig. 1.

which occurs as a special case of the so-called U-Toda system $UT(m_1, m_2)$ with $m_1 = 1$ and $m_2 = 2$. By introducing an additional variable z and the following dependent variable transformations:

$$\theta_n = \frac{f_{n+1}f_{n-1}}{f_n^2}, \quad P_n = \frac{1}{2}\frac{D_xD_yf_n \cdot f_n}{f_{n+1}f_{n-1}},$$

we obtain the bilinear form of the Leznov lattice (2.1)–(2.2) [17]

$$[D_yD_z - 2(e^{D_n} - 1)]f_n \cdot f_n = 0, \tag{2.3}$$

$$(D_yD_x - 2D_ze^{D_n})f_n \cdot f_n = 0. \tag{2.4}$$

where the bilinear operators D_xD_s and $\exp(D_n)$ are defined by [4, 18]

$$D_xD_sa \cdot b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial s'}\right)a(x,s)b(x',s')|_{x'=x,s'=s}$$

and

$$\exp(D_n)a_n \cdot b_n \equiv a_{n+1}b_{n-1}.$$

It has the following Bäcklund Transformation [17]

$$(D_z - \lambda^{-1}e^{D_n} - \mu)f_n \cdot f'_n = 0, \tag{2.5}$$

$$(D_ye^{\frac{D_n}{2}} + \lambda e^{\frac{-D_n}{2}} - \gamma e^{\frac{D_n}{2}})f_n \cdot f'_n = 0, \tag{2.6}$$

$$(D_x - \lambda^{-1}D_ze^{D_n} - \lambda^{-1}\mu e^{D_n} + k)f_n \cdot f'_n = 0. \tag{2.7}$$

Considering the special case $\lambda = \gamma = k = 1$ and $\mu = -1$, we have the modified Leznov lattice

$$(D_z - e^{D_n} + 1)f_n \cdot f'_n = 0, \tag{2.8}$$

$$(D_ye^{\frac{D_n}{2}} + e^{\frac{-D_n}{2}} - e^{\frac{D_n}{2}})f_n \cdot f'_n = 0, \tag{2.9}$$

$$(D_x - D_ze^{D_n} - e^{D_n} + 1)f_n \cdot f'_n = 0. \tag{2.10}$$

Similar to the 2D modified Toda equation [4], we can easily show that the soliton solutions to (2.8)–(2.10) can be expressed as determinants

$$f_n = \det(\phi_i(n+j-1))_{1\le i,j\le N}, \quad f'_n = \det(\widehat{\phi}_i(n+j-1))_{1\le i,j\le N}, \tag{2.11}$$

where $\phi_i(m)$ and $\widehat{\phi}_i(m)$ satisfy the following relations:

$$\widehat{\phi}_i(m) = \phi_i(m+1) - \phi_i(m), \quad \frac{\partial}{\partial x}\phi_i(m) = -\phi_i(m-2), \tag{2.12}$$

$$\frac{\partial}{\partial y}\phi_i(m) = \phi_i(m+1), \quad \frac{\partial}{\partial z}\phi_i(m) = -\phi_i(m-1), \tag{2.13}$$

for $i = 1, 2, \ldots, N$.

In fact, we can express f_n and f'_n in a compact form [19, 20]

$$f'_n = |\widehat{0}, \ldots, \widehat{N-1}|, \quad f'_{n+1} = |\widehat{1}, \ldots, \widehat{N}|, \tag{2.14}$$

$$f_n = |0, \widehat{0}, \ldots, \widehat{N-2}| = |\widehat{-1}, \widehat{0}, \ldots, \widehat{N-2}| + |-1, \widehat{0}, \ldots, \widehat{N-2}|, \tag{2.15}$$

where $j-1$ and $\widehat{j-1}$ denotes the column vector $(\phi_1(n+j-1), \phi_2(n+j-1), \ldots, \phi_n(n+j-1))^T$ and $(\widehat{\phi}_1(n+j-1), \widehat{\phi}_2(n+j-1), \ldots, \widehat{\phi}_n(n+j-1))^T$ respectively. By using the dispersion relation

(2.12) and (2.13), we obtain the differential and difference formulae

$$
\begin{aligned}
f'_{n-1} &= |\widehat{-1},\ldots,\widehat{N-2}|, \quad \frac{\partial}{\partial y} f'_n = |\widehat{0},\ldots,\widehat{N-2},\widehat{N}|,\\
\frac{\partial}{\partial x} f'_n &= |\widehat{-1},\widehat{0},\widehat{2},\ldots,\widehat{N-1}| - |\widehat{-2},\widehat{1},\ldots,\widehat{N-1}|,\\
\frac{\partial}{\partial z} f'_n &= -|\widehat{-1},\widehat{1},\ldots,\widehat{N-1}| \quad \frac{\partial}{\partial z} f'_{n-1} = -|\widehat{-2},\widehat{0},\ldots,\widehat{N-2}|,\\
f_{n+1} &= |1,\widehat{1},\ldots,\widehat{N-1}| = |\widehat{0},\ldots,\widehat{N-1}| + |0,\widehat{1},\ldots,\widehat{N-1}|,\\
\frac{\partial}{\partial x} f_n &= -|-2,\widehat{0},\ldots,\widehat{N-2}| - |0,\widehat{-2},\widehat{1},\ldots,\widehat{N-2}| + |0,\widehat{-1},\widehat{0},\widehat{2},\ldots,\widehat{N-2}|,\\
\frac{\partial}{\partial y} f_{n+1} &= |\widehat{0},\ldots,\widehat{N-2},\widehat{N}| + |1,\widehat{1},\ldots,\widehat{N-1}| + |0,\widehat{1},\ldots,\widehat{N-2},\widehat{N}|,\\
\frac{\partial}{\partial z} f_n &= -|-1,\widehat{0},\ldots,\widehat{N-2}| - |0,\widehat{-1},\widehat{1},\ldots,\widehat{N-2}|\\
&= -|-1,\widehat{0},\ldots,\widehat{N-2}| - |-1,\widehat{-1},\widehat{1},\ldots,\widehat{N-2}|,\\
\frac{\partial}{\partial z} f_{n+1} &= -|0,\widehat{1},\ldots,\widehat{N-1}| - |1,\widehat{0},\widehat{2},\ldots,\widehat{N-1}|\\
&= -|0,\widehat{1},\ldots,\widehat{N-1}| - |0,\widehat{0},\widehat{2},\ldots,\widehat{N-1}|.
\end{aligned}
$$

Substitution of the above expressions into the modified Leznov lattice will lead to the following Plücker relation respectively

$$
\begin{aligned}
&(D_z + e^{-D_n} - 1) f'_n \cdot f_n = f_n \frac{\partial}{\partial z} f'_n - f'_n \frac{\partial}{\partial z} f_n + f'_{n-1} f_{n+1} - f'_n f_n\\
&\quad = |\widehat{-1},\ldots,\widehat{N-2}| \times |0,\widehat{1},\ldots,\widehat{N-1}| - |\widehat{-1},\widehat{1},\ldots,\widehat{N-1}| \times |0,\widehat{0},\ldots,\widehat{N-2}|\\
&\qquad + |\widehat{0},\ldots,\widehat{N-1}| \times |0,\widehat{-1},\widehat{1},\ldots,\widehat{N-2}| \equiv 0,
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
&(D_y e^{-\frac{D_n}{2}} - e^{\frac{D_n}{2}} + e^{-\frac{D_n}{2}}) f'_n \cdot f_n = f_{n+1} \frac{\partial}{\partial y} f'_n - f'_n \frac{\partial}{\partial y} f_{n+1} - f'_{n+1} f_n + f'_n f_{n+1}\\
&\quad = |\widehat{0},\ldots,\widehat{N-2},\widehat{N}| \times |0,\widehat{1},\ldots,\widehat{N-1}| - |\widehat{0},\ldots,\widehat{N-1}| \times |0,\widehat{1},\ldots,\widehat{N-2},\widehat{N}|\\
&\qquad - |\widehat{1},\ldots,\widehat{N}| \times |0,\widehat{0},\ldots,\widehat{N-2}| \equiv 0,
\end{aligned} \tag{2.17}
$$

$$
\begin{aligned}
&(D_z e^{-D_n} - e^{-D_n} - D_x + 1) f'_n \cdot f_n\\
&\quad = f_{n+1} \frac{\partial}{\partial z} f'_{n-1} - f'_{n-1} \frac{\partial}{\partial z} f_{n+1} - f_n \frac{\partial}{\partial x} f'_n + f'_n \frac{\partial}{\partial x} f_n - f'_{n-1} f_{n+1} + f_n f'_n\\
&\quad = |0,\widehat{0},\ldots,\widehat{N-2}| \times |\widehat{-2},\widehat{1},\ldots,\widehat{N-1}| - |\widehat{-2},\widehat{0},\ldots,\widehat{N-2}| \times |0,\widehat{1},\ldots,\widehat{N-1}|\\
&\qquad - |\widehat{0},\ldots,\widehat{N-1}| \times |0,\widehat{-2},\widehat{1},\ldots,\widehat{N-2}| + |\widehat{0},\ldots,\widehat{N-1}| \times |0,\widehat{-1},\widehat{0},\widehat{2},\ldots,\widehat{N-2}|\\
&\qquad - |0,\widehat{0},\ldots,\widehat{N-2}| \times |\widehat{-1},\widehat{0},\widehat{2},\ldots,\widehat{N-1}| - |\widehat{-1},\ldots,\widehat{N-2}| \times |0,\widehat{0},\widehat{2},\ldots,\widehat{N-1}| \equiv 0.
\end{aligned} \tag{2.18}
$$

Therefore, the modified Leznov lattice has Wronskian solutions f_n and f'_n given by (2.11).

3. Pfaffianization of the Modified Leznov Lattice

Generally speaking, for high-dimensional soliton equations with Wronskian solutions, we can always obtain their coupled systems through pfaffianization. In this section, we will consider the coupled system for the modified Leznov lattice (2.8)–(2.10) in the same way. In order to do this, we replace f_n and f'_n expressed as determinants with those expressed as pfaffians

$$f_n = \mathrm{pf}(1,2,\ldots,N)_n, \quad f'_n = \mathrm{pf}(1,2,\ldots,N,N+1,c)_n, \quad N \text{ is even}, \tag{3.1}$$

whose entries satisfy

$$\begin{aligned}
&\frac{\partial}{\partial x}\mathrm{pf}(i,j)_n = -\mathrm{pf}(i-2,j)_n - \mathrm{pf}(i,j-2)_n, \quad \frac{\partial}{\partial y}\mathrm{pf}(i,j)_n = \mathrm{pf}(i+1,j)_n + \mathrm{pf}(i,j+1)_n,\\
&\frac{\partial}{\partial z}\mathrm{pf}(i,j)_n = -\mathrm{pf}(i-1,j)_n - \mathrm{pf}(i,j-1)_n,\\
&\mathrm{pf}(i,j)_{n+1} = \mathrm{pf}(i+1,j+1)_n, \quad \mathrm{pf}(i,c)_n = 1.
\end{aligned}$$

Then we can calculate that

$$f_{n,x} = \mathrm{pf}(0,1,3,\ldots,N)_n - \mathrm{pf}(-1,2,\ldots,N)_n, \quad f_{n,z} = -\mathrm{pf}(0,2,\ldots,N)_n, \tag{3.2}$$

$$f_{n+1} = \mathrm{pf}(2,3,\ldots,N,N+1)_n, \quad f_{n+1,y} = \mathrm{pf}(2,\ldots,N,N+2)_n, \tag{3.3}$$

$$f'_{n+1} = \mathrm{pf}(2,\ldots,N+2,c)_n, \quad f'_{n-1} = \mathrm{pf}(0,\ldots,N,c)_n, \tag{3.4}$$

$$f'_{n,x} = \mathrm{pf}(1,\ldots,N+1,c)_n - \mathrm{pf}(-1,2,\ldots,N+1,c)_n + \mathrm{pf}(0,1,3,\ldots,N+1,c)_n, \tag{3.5}$$

$$f'_{n,y} = \mathrm{pf}(1,\ldots,N,N+2,c)_n - \mathrm{pf}(1,\ldots,N+1,c)_n, \tag{3.6}$$

$$f'_{n,z} = \mathrm{pf}(1,\ldots,N+1,c)_n - \mathrm{pf}(0,2,\ldots,N+1,c)_n. \tag{3.7}$$

Following Hirota and Ohta's procedure, we now introduce four new functions defined by

$$\sigma_n = \mathrm{pf}(0,1,\ldots,N,N+1)_n, \quad g'_n = \mathrm{pf}(0,1,\ldots,N+1,N+2,c)_n, \tag{3.8}$$

$$g_n = \mathrm{pf}(2,3,\ldots,N-1)_n, \quad \sigma'_n = \mathrm{pf}(2,3,\ldots,N-1,N,c)_n, \tag{3.9}$$

Then we can show that $f_n, f'_n, g_n, g'_n, \sigma_n$, and σ'_n so defined satisfy the following bilinear equations

$$(D_z + e^{-D_n} - 1)f'_n \cdot f_n - \sigma_n\sigma'_n = 0, \tag{3.10}$$

$$(D_y e^{-\frac{D_n}{2}} - e^{\frac{D_n}{2}} + e^{-\frac{D_n}{2}})f'_n \cdot f_n + e^{\frac{D_n}{2}}\sigma_n\sigma'_n = 0, \tag{3.11}$$

$$(D_z e^{-D_n} - e^{-D_n} - D_x + 1)f'_n \cdot f_n - \sigma_n\sigma'_n - D_z\sigma_n \cdot \sigma'_n = 0, \tag{3.12}$$

$$D_z f_n \cdot \sigma'_{n-1} + f_{n-1}\sigma'_n + f_n\sigma'_{n-1} - f'_{n-1}g_n = 0, \tag{3.13}$$

$$D_z\sigma_{n+1} \cdot f'_n + \sigma_n f'_{n+1} + \sigma_{n+1}f'_n - f_{n+1}g'_n = 0, \tag{3.14}$$

$$D_y f_n \cdot \sigma'_n - f_{n+1}\sigma'_{n-1} - f_n\sigma'_n + g_n f'_n = 0, \tag{3.15}$$

$$D_y\sigma_n \cdot f'_n - \sigma_{n+1}f'_{n-1} - \sigma_n f'_n + f_n g'_n = 0, \tag{3.16}$$

$$D_z f'_{n+1} \cdot \sigma_n - D_x f'_n \cdot \sigma_{n+1} - f'_{n+1}\sigma_n + f'_n\sigma_{n+1} + D_z g'_n \cdot f_{n+1} - g'_n f_{n+1} = 0, \tag{3.17}$$

$$D_z\sigma'_n \cdot f_{n-1} - D_x\sigma'_{n-1} \cdot f_n - \sigma'_n f_{n-1} + \sigma'_{n-1}f_n - D_z g_n \cdot f'_{n-1} - g_n f'_{n-1} = 0. \tag{3.18}$$

In fact, substitution of (3.1)–(3.9) into (3.10) will lead to the following pfaffian algebraic identity [5, 6]

$$\begin{aligned}
&\mathrm{pf}(a_1,a_2,\ldots,a_{N-1},\alpha,\beta,\gamma)_n\mathrm{pf}(a_1,a_2,\ldots,a_{N-1},\delta)_n\\
&\quad - \mathrm{pf}(a_1,a_2,\ldots,a_{N-1},\alpha,\beta,\delta)_n\mathrm{pf}(a_1,a_2,\ldots,a_{N-1},\gamma)_n\\
&\quad + \mathrm{pf}(a_1,a_2,\ldots,a_{N-1},\alpha,\gamma,\delta)_n\mathrm{pf}(a_1,a_2,\ldots,a_{N-1},\beta)_n\\
&\quad - \mathrm{pf}(a_1,a_2,\ldots,a_{N-1},\beta,\gamma,\delta)_n\mathrm{pf}(a_1,a_2,\ldots,a_{N-1},,\alpha)_n = 0.
\end{aligned} \tag{3.19}$$

where the list $\{a_1, a_2, \ldots, a_{N-1}\}$ represents $\{2, 3, \ldots, N\}$ and the list $\{\alpha, \beta, \gamma, \delta\}$ is chosen to be $\{0, 1, N+1, c\}$. Thus, (3.10) holds. Similarly, we can prove that (3.11) and (3.12) also hold. Moreover, with the help of another pfaffian identity

$$\begin{aligned}
&\mathrm{pf}(a_1, a_2, \ldots, a_{N-2}, \alpha, \beta, \gamma, \delta)_n \mathrm{pf}(a_1, a_2, \ldots, a_{N-2})_n \\
&\quad - \mathrm{pf}(a_1, a_2, \ldots, a_{N-2}, \alpha, \beta)_n \mathrm{pf}(a_1, a_2, \ldots, a_{N-2}, \gamma, \delta)_n \\
&\quad + \mathrm{pf}(a_1, a_2, \ldots, a_{N-2}, \alpha, \gamma)_n \mathrm{pf}(a_1, a_2, \ldots, a_{N-2}, \beta, \delta)_n \\
&\quad - \mathrm{pf}(a_1, a_2, \ldots, a_{N-2}, \alpha, \delta)_n \mathrm{pf}(a_1, a_2, \ldots, a_{N-2}, \beta, \gamma)_n = 0,
\end{aligned} \tag{3.20}$$

we can prove that (3.13)–(3.18) hold. Therefore Eqs. (3.10)–(3.18) constitute a pfaffianized version of the modified Leznov lattice (2.8)–(2.10).

4. Bäcklund Transformation for the Pfaffianized Leznov Lattice

Recall that f_n, σ_n and g_n given by (3.1), (3.8) and (3.9) are solutions to the following coupled Leznov lattice [1]

$$[D_y D_z - 2(e^{D_n} - 1)] f_n \cdot f_n + 2\sigma_n g_n = 0, \tag{4.1}$$
$$(D_x D_y - 2D_z e^{D_n}) f_n \cdot f_n = 2D_z \sigma_n \cdot g_n, \tag{4.2}$$
$$D_y e^{-\frac{1}{2}D_n} \sigma_n \cdot f_n = -D_z e^{\frac{1}{2}D_n} \sigma_n \cdot f_n, \tag{4.3}$$
$$D_y e^{-\frac{1}{2}D_n} f_n \cdot g_n = -D_z e^{\frac{1}{2}D_n} f_n \cdot g_n, \tag{4.4}$$

which is generated through pfaffianization of the Leznov lattice (2.3) and (2.4). Then we have

Proposition 4.1. *The pfaffianized version* (3.10)–(3.18) *of the modified Leznov lattice* (2.8)–(2.10) *serves as a BT for the coupled Leznov lattice* (4.1)–(4.3).

Proof. For the sake of convenience, we introduce an additional discrete variable m and set

$$f_n = f_n(m), \quad \sigma_n = f_n(m+1), \quad g_n = f_n(m-1), \tag{4.5}$$
$$f'_n = f'_n(m), \quad g'_n = f'_n(m+1), \quad \sigma'_n = f'_n(m-1). \tag{4.6}$$

Then Eqs. (4.1)–(4.3) are reduced to

$$[D_y D_z - 2(e^{D_n} - 1)] f_n \cdot f_n + 2 f_{m+1} f_{m-1} = 0, \tag{4.7}$$
$$(D_x D_y - 2D_z e^{D_n} - 2D_z e^{D_m}) f_n \cdot f_n = 0, \tag{4.8}$$
$$D_y e^{-\frac{1}{2}D_n + \frac{D_m}{2}} f_n \cdot f_n = -D_z e^{\frac{1}{2}D_n + \frac{D_m}{2}} f_n \cdot f_n. \tag{4.9}$$

In this case, (3.10)–(3.18) are transformed into

$$(D_y e^{-\frac{1}{2}D_n} - e^{\frac{1}{2}D_n} + e^{-\frac{1}{2}D_n} + e^{-D_m - \frac{1}{2}D_n}) f'_n(m) \cdot f_n(m) = 0, \tag{4.10}$$
$$(D_z - e^{D_n} + e^{D_m} + 1) f_n(m) \cdot f'_n(m) = 0, \tag{4.11}$$
$$(D_y e^{-\frac{1}{2}D_m} + e^{-D_n - \frac{1}{2}D_m} + e^{-\frac{1}{2}D_m} - e^{\frac{1}{2}D_m}) f'_n(m) \cdot f_n(m) = 0, \tag{4.12}$$
$$(D_z e^{\frac{1}{2}D_n + \frac{1}{2}D_m} + e^{-\frac{1}{2}D_n + \frac{1}{2}D_m} + e^{\frac{1}{2}D_n + \frac{1}{2}D_m} - e^{\frac{1}{2}D_n - \frac{1}{2}D_m}) f_n(m) \cdot f'_n(m) = 0, \tag{4.13}$$
$$(D_z e^{-D_n} - e^{-D_n} - D_x + 1 - e^{-D_m} + D_z e^{-D_m}) f'_n(m) \cdot f_n(m) = 0, \tag{4.14}$$
$$\begin{aligned}
&(D_z e^{\frac{D_n}{2} - \frac{D_m}{2}} - D_x e^{-\frac{D_n}{2} - \frac{D_m}{2}} - e^{\frac{D_n}{2} - \frac{D_m}{2}} \\
&\quad + e^{-\frac{D_n}{2} - \frac{D_m}{2}} + D_z e^{\frac{D_m}{2} - \frac{D_n}{2}} - e^{\frac{D_m}{2} - \frac{D_n}{2}}) f'_n(m) \cdot f_n(m) = 0.
\end{aligned} \tag{4.15}$$

Therefore what we need to do is to prove that (4.10)–(4.15) are a bilinear BT for (4.7)–(4.9), i.e. the following equations hold,

$$\begin{aligned}
&P1 \equiv [D_yD_z - 2(e^{D_n}-1) + 2e^{D_m}]f'_n(m)\cdot f'_n(m) = 0,\\
&P2 \equiv (D_y e^{\frac{1}{2}D_m-\frac{1}{2}D_n} + D_z e^{\frac{1}{2}D_m+\frac{1}{2}D_n})f'_n(m)\cdot f'_n(m) = 0,\\
&P3 = (D_xD_y - 2D_ze^{D_n} - 2D_ze^{D_m})f'_n(m)\cdot f'_n(m) = 0.
\end{aligned}$$

In fact, by using (4.10)–(4.15) and the bilinear identities in Appendix, we can precisely show that

$$\begin{aligned}
-f_n(m)^2P1 &\equiv [D_yD_zf_n(m)\cdot f_n(m) - 2f_{n+1}(m)f_{n-1}(m) + 2f_n(m)^2\\
&\quad + 2f_n(m+1)f_n(m-1)]f'_n(m)^2 - [D_yD_zf'_n(m)\cdot f'_n(m) - 2f'_{n+1}(m)f'_{n-1}(m)\\
&\quad + 2f'_n(m)^2 + 2f'_n(m+1)f'_n(m-1)]f_n(m)^2\\
&= 2D_y[D_zf_n(m)\cdot f'_n(m) - f_{n+1}(m)f'_{n-1}(m) + f_n(m+1)f'_n(m-1)\\
&\quad + f_n(m)f'_n(m)]\cdot(f_n(m)f'_n(m)) + 2[(D_yf_{n+1}(m)\cdot f'_n(m) + f_n(m)f'_{n+1}(m)\\
&\quad - f_{n+1}(m)f'_n(m) - f_{n+1}(m+1)f'_n(m-1))]f_n(m)f'_{n-1}(m) - 2[(D_yf_n(m)\cdot f'_{n-1}(m)\\
&\quad + f_{n-1}(m)f'_n(m) - f_n(m)f'_{n-1}(m) - f_n(m+1)f'_{n-1}(m-1))]f_{n+1}(m)f'_n(m)\\
&\quad - 2[(D_yf_n(m+1)\cdot f'_n(m) - f_{n+1}(m+1)f'_{n-1}(m) - f_n(m+1)f'_n(m)\\
&\quad + f_n(m)f'_n(m+1))]f_n(m)f'_n(m-1) - 2[(D_yf_n(m)\cdot f'_n(m-1)\\
&\quad - f_{n+1}(m)f'_{n-1}(m-1) - f_n(m)f'_n(m-1) + f_n(m-1)f'_n(m))]f_n(m+1)f'_n(m)\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
&-[e^{\frac{1}{2}D_m+\frac{1}{2}D_n}f_n(m)\cdot f_n(m)]P2\\
&\quad\equiv [(D_ye^{\frac{1}{2}D_m-\frac{1}{2}D_n} + D_ze^{\frac{1}{2}D_m+\frac{1}{2}D_n})f_n(m)\cdot f_n(m)][e^{\frac{1}{2}D_m+\frac{1}{2}D_n}f'_n(m)\cdot f'_n(m)]\\
&\qquad - [(D_ye^{\frac{1}{2}D_m-\frac{1}{2}D_n} + D_ze^{\frac{1}{2}D_m+\frac{1}{2}D_n})f'_n(m)\cdot f'_n(m)][e^{\frac{1}{2}D_m+\frac{1}{2}D_n}f_n(m)\cdot f_n(m)]\\
&\quad= 2\sinh\left(\frac{1}{2}D_m\right)(D_ye^{\frac{1}{2}D_n}f_n(m)\cdot f'_n(m))\cdot(e^{-\frac{1}{2}D_n}f_n(m)\cdot f'_n(m))\\
&\qquad - 2\sinh\left(\frac{1}{2}D_n\right)(D_ye^{\frac{1}{2}D_m}f_n(m)\cdot f'_n(m))\cdot(e^{-\frac{1}{2}D_m}f_n(m)\cdot f'_n(m))\\
&\qquad + 2\sinh\left(\frac{1}{2}D_m+\frac{1}{2}D_n\right)(D_zf_n(m)\cdot f'_n(m))\cdot f_n(m)f'_n(m)\\
&\quad= 2\sinh\left(\frac{1}{2}D_m\right)[(e^{\frac{1}{2}D_n} + e^{\frac{1}{2}D_n+D_m})f_n(m)\cdot f'_n(m)]\cdot(e^{-\frac{1}{2}D_n}f_n(m)\cdot f'_n(m))\\
&\qquad - 2\sinh\left(\frac{1}{2}D_n\right)[(e^{\frac{1}{2}D_m} + e^{D_n+\frac{1}{2}D_m})f_n(m)\cdot f'_n(m)]\cdot(e^{-\frac{1}{2}D_m}f_n(m)\cdot f'_n(m))\\
&\qquad + 2\sinh\left(\frac{1}{2}D_m+\frac{1}{2}D_n\right)[(e^{D_n} - e^{D_m})f_n(m)\cdot f'_n(m)]\cdot(f_n(m)f'_n(m))\\
&\quad= 0,
\end{aligned}$$

$$\begin{aligned}
f_n^2(m)P3 &\equiv f_n^2(m)(D_xD_y - 2D_ze^{D_n} - 2D_ze^{D_m})f'_n(m)\cdot f'_n(m)\\
&\quad - f'^2_n(m)(D_xD_y - 2D_ze^{D_n} - 2D_ze^{D_m})f_n(m)\cdot f_n(m)
\end{aligned}$$

$$
\begin{aligned}
&= 2D_y(D_x f_n'(m)\cdot f_n(m))\cdot f_n'(m)f_n(m)\\
&\quad -4D_z\cosh\frac{D_n}{2}(e^{\frac{D_n}{2}}f_n'(m)\cdot f_n(m))\cdot(e^{-\frac{D_n}{2}}f_n'(m)\cdot f_n(m))\\
&\quad -4D_z\cosh\frac{D_m}{2}(e^{\frac{D_m}{2}}f_n'(m)\cdot f_n(m))\cdot(e^{-\frac{D_m}{2}}f_n'(m)\cdot f_n(m))\\
&= 2D_y(D_x f_n'(m)\cdot f_n(m))\cdot f_n'(m)f_n(m)\\
&\quad -4D_z\cosh\frac{D_n}{2}[(D_y e^{-\frac{D_n}{2}}+e^{-D_m-\frac{D_n}{2}})f_n'(m)\cdot f_n(m)]\cdot(e^{-\frac{D_n}{2}}f_n'(m)\cdot f_n(m))\\
&\quad -4D_z\cosh\frac{D_m}{2}[(D_y e^{-\frac{D_m}{2}}+e^{-D_n-\frac{D_m}{2}})f_n'(m)\cdot f_n(m)]\cdot(e^{-\frac{D_m}{2}}f_n'(m)\cdot f_n(m))\\
&= 2D_y(D_x f_n'(m)\cdot f_n(m))\cdot f_n'(m)f_n(m)\\
&\quad -2D_y[(D_z f_n'(m)\cdot f_n(m))\cdot(e^{-D_n}f_n'(m)\cdot f_n(m))\\
&\quad -f_n'(m)f_n(m)\cdot(D_z e^{-D_n}f_n'(m)\cdot f_n(m))]\\
&\quad -2D_y[(D_z f_n'(m)\cdot f_n(m))\cdot(e^{-D_m}f_n'(m)\cdot f_n(m))\\
&\quad -f_n(m)'f_n(m)\cdot(D_z e^{-D_m}f_n'(m)\cdot f_n(m))]\\
&\quad -4D_z\cosh\frac{D_n}{2}[e^{-D_m-\frac{D_n}{2}}f_n'(m)\cdot f_n(m)]\cdot(e^{-\frac{D_n}{2}}f_n'(m)\cdot f_n(m))\\
&\quad -4D_z\cosh\frac{D_m}{2}[e^{-D_n-\frac{D_m}{2}}f_n'(m)\cdot f_n(m)]\cdot(e^{-\frac{D_m}{2}}f_n'(m)\cdot f_n(m))\\
&= 2D_y f_n'(m)f_n(m)\cdot[(-D_x+D_z e^{-D_n}+D_z e^{-D_m}-e^{-D_n}-e^{-D_m})f_n'(m)\cdot f_n(m)]\\
&\quad +4[D_z f_n'(m)\cdot f_{n-1}'(m-1)+D_y f_{n-1}'(m)\cdot f_n'(m-1)]f_n(m+1)f_{n+1}(m)\\
&\quad +4[D_z f_n(m)\cdot f_{n+1}(m+1)+D_y f_{n+1}(m)\cdot f_n(m+1)]f_{n-1}'(m)f_n'(m-1)\\
&= 0.
\end{aligned}
$$

In this way, we have completed the proof of Proposition 1. □

To sum up, we have proved that the CPBT is valid for the Leznov lattice.

5. Conclusion

In this paper, we present Wronskian solutions (2.11) to the modified Leznov lattice (2.8)–(2.10) and derive its coupled system (3.10)–(3.18) by pfaffianization. We further show that the coupled system for the modified Leznov lattice (3.10)–(3.18) is nothing but the Bäcklund transformation for the coupled system of the Leznov lattice (4.1)–(4.3). This means that the CPBT is also valid for the Leznov lattice besides the KP equation. Moreover, we proved that the CPBT is applicable to a special lattice proposed by Blaszk and Szum [21, 22].

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Appendix

The following bilinear identities hold for arbitrary functions a, b, c and d:

$$(D_x D_s a \cdot a)b^2 - a^2(D_x D_s b \cdot b) = 2D_x(D_s a \cdot b) \cdot (ab) \tag{A.1}$$

$$\begin{aligned}&(D_x e^{\frac{1}{2}D_m - \frac{1}{2}D_n} a \cdot a)(e^{\frac{1}{2}D_m + \frac{1}{2}D_n} b \cdot b) - (D_x e^{\frac{1}{2}D_m - \frac{1}{2}D_n} b \cdot b)(e^{\frac{1}{2}D_m + \frac{1}{2}D_n} a \cdot a)\\ &\quad = 2\sinh\left(\frac{1}{2}D_m\right)(D_x e^{\frac{1}{2}D_n} a \cdot b) \cdot (e^{-\frac{1}{2}D_n} a \cdot b) - 2\sinh\left(\frac{1}{2}D_n\right)(D_x e^{\frac{1}{2}D_m} a \cdot b) \cdot (e^{-\frac{1}{2}D_m} a \cdot b)\end{aligned} \tag{A.2}$$

$$(D_x e^{D_n} a \cdot a)b^2 - (D_x e^{D_n} b \cdot b)a^2 = 2D_x \cosh\left(\frac{D_n}{2}\right)(e^{\frac{D_n}{2}} a \cdot b) \cdot (e^{-\frac{D_n}{2}} a \cdot b) \tag{A.3}$$

$$\begin{aligned}&2D_z \cosh\left(\frac{D_n}{2}\right)(D_y e^{-(1/2)D_n} a \cdot b) \cdot (e^{-(1/2)D_n} a \cdot b)\\ &\quad = D_y[(D_z a \cdot b) \cdot (e^{-D_n} a \cdot b) - ab \cdot (D_z e^{-D_n} a \cdot b)].\end{aligned} \tag{A.4}$$

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