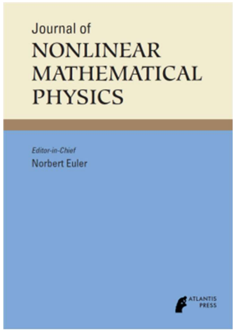

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INCOMPLETE q-GAMMA FUNCTION AND TRICOMI EXPANSION

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In this paper, we introduce a q-analogue of the Tricomi expansion for the incomplete q-gamma function. A general method is described for converting a power series into an expansion in incomplete q-gamma function. Also, we use the q-Tricomi expansion for giving a formal proof of the relation between the incomplete gamma function and the exponential integral. Finally, we formally deduce the q-Tricomi expansion via the q-Taylor expansion.

Keywords: Incomplete q-gamma function; exponential integral; the q-Tricomi expansion; basic hypergeometric function; q-binomial theorem; q-Taylor formula.

Mathematics Subject Classification 2000: 33B20, 33E20, 33D15.

1. Introduction

The incomplete gamma function is given by [3, 12]:

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt = \alpha^{-1} x^{\alpha} {}_1F_1\left(\begin{matrix} \alpha \\ \alpha+1 \end{matrix} \middle| -x \right). \tag{1.1}$$

In 1950, Francessco G. Tricomi [11] stated without proof the following expansion:

$$\gamma(\alpha, \omega x) = \omega^{\alpha} \sum_{n=0}^{\infty} \frac{\gamma(\alpha+n, x)}{n!} (1-\omega)^n, \tag{1.2}$$

which is a special case of the multiplication theorem [3, 6.14(1)]

$${}_1F_1\left(\begin{matrix} a \\ b \end{matrix} \middle| \omega x \right) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \, {}_1F_1\left(\begin{matrix} a+n \\ b+n \end{matrix} \middle| x \right) \frac{[-(1-\omega)x]^n}{n!} \tag{1.3}$$

for the confluent hypergeometric functions. In a search for better methods of evaluating the exponential integral

$$E_1(x) = \int_x^{\infty} t^{-1} e^{-t} dt \tag{1.4}$$

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(which occurs widely in applications, most notably in quantum-mechanical electronic structure calculations), Gautschi *et al.* [6] discovered the following equation

$$E_1(x) = -\gamma - \ln x + \sum_{n=1}^{\infty} \frac{\gamma(n,x)}{n!} \tag{1.5}$$

from the Tricomi expansion as a limiting case, where γ denotes the Euler constant. Also, Shy-Der Lin *et al.* [10] presented a rather elementary demonstration of Eq. (1.5) without using the Tricomi expansion.

Gupta [7] defined a q-analogue of the incomplete gamma function by

$$\Gamma_q(\alpha, x) = \frac{1}{(1-q)^\alpha} \int_0^x t^{\alpha-1} (tq;q)_\infty d_q t \tag{1.6}$$

and studied its important analytical properties and gave an application of it in statistical distribution theory.

The organization of this paper is as follows. In Sec. 2, we give some q-notations. In Sec. 3, we introduce a q-analogue of the incomplete gamma function, essentially the same as (1.6), and some of its identities. Some of the properties obtained in this section were already obtained in [7] with different proofs. Section 4 is devoted to proving a q-analogue of the Tricomi expansion. In Sec. 5, we give a general method to express any power series in a specific form as a series in the incomplete q-gamma function. In Sec. 6, we present a q-analogue of an important relation of the incomplete gamma function. Finally, Sec. 7 presents another proof of the q-Tricomi expansion by the q-Taylor expansion.

2. Preliminaries

In this paper we will always assume that $0 < q < 1$. The basic hypergeometric function is defined by [9]

$$ {}_r\varphi_s \left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \middle| q; z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} ((-1)^k q^{\frac{k}{2}(k-1)})^{1+s-r} \frac{z^k}{(q;q)_k}, \tag{2.1}$$

where the q-shifted factorials are defined by [9]

$$\begin{aligned} &(a;q)_0 = 1, \\ &(a;q)_k = \prod_{j=o}^{k-1} (1 - aq^j); \qquad k = 1, 2, \ldots, \\ &(a_1, \ldots, a_r; q)_k = \prod_{i=1}^{r} (a_i, q)_k, \\ &(a;q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i). \end{aligned}$$

The classical exponential function e^x has two different natural q-extensions denoted by $e_q(x)$ and $E_q(x)$ and defined by [9]

$$e_q(x) = {}_1\varphi_0 \left(\begin{matrix} 0 \\ - \end{matrix} \middle| q; x \right) = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty}; \quad |x| < 1 \tag{2.2}$$

and

$$E_q(x) = {}_0\varphi_0 \left(\begin{matrix} - \\ - \end{matrix} \middle| q; -x \right) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{(q;q)_k} = (-x;q)_\infty, \tag{2.3}$$

where $x \in \mathbb{C}$.

The q-difference operator D_q is defined by [9]

$$D_q f(x) = \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)x}, & x \neq 0 \\ \dfrac{df(0)}{dx}, & x = 0 \end{cases} \tag{2.4}$$

where

$$\lim_{q \to 1} D_q f(x) = \frac{df(x)}{dx}.$$

The q-integral on $(0, x)$ is defined by [9]

$$\int_0^x f(t) d_q t = (1-q)x \sum_{k=0}^{\infty} q^k f(xq^k). \tag{2.5}$$

3. A *q*-Analogue of the Incomplete Gamma Function

Definition 1. The incomplete q-gamma function is defined by

$$\gamma_q(\alpha, x) = \int_0^x t^{\alpha-1} E_q(-(1-q)t) d_q t, \quad Re(\alpha) > 0. \tag{3.1}$$

As a special case of (3.1), we have

$$\gamma_q\left(\alpha, \frac{q}{1-q}\right) = q^{\alpha} \Gamma_q(\alpha),$$

where

$$\Gamma_q(\alpha) = \int_0^{\frac{1}{1-q}} t^{\alpha-1} E_q(-(1-q)qt) d_q t = \frac{(q;q)_\infty}{(q^\alpha;q)_\infty}(1-q)^{1-\alpha}.$$

The relation between the two formulas (1.6) and (3.1) is that

$$\Gamma_q(\alpha, x) = q^{-\alpha} \gamma_q\left(\alpha, \frac{qx}{1-q}\right), \quad 0 \le x \le 1.$$

Lemma 1.

$$\gamma_q(\alpha, x) = \frac{x^\alpha}{[\alpha]_q} \, {}_1\varphi_1\left(\begin{matrix} q^\alpha \\ q^{\alpha+1} \end{matrix} \middle| q; -(q-1)x \right), \tag{3.2}$$

where the quantum number $[\alpha]_q = \frac{1-q^\alpha}{1-q}$.

Proof.

$$\begin{aligned} \gamma_q(\alpha, x) &= \int_0^x t^{\alpha-1} E_q(-(1-q)t) d_q t \\ &= (1-q)x \sum_{k=0}^{\infty} q^k (xq^k)^{\alpha-1} E_q(-(1-q)xq^k) \\ &= (1-q)x^\alpha ((1-q)x;q)_\infty \sum_{k=0}^{\infty} \frac{q^{\alpha k}}{((1-q)x;q)_k}. \end{aligned}$$

Then

$$\gamma_q(\alpha, x) = (1-q)x^\alpha((1-q)x;q)_\infty \ {}_2\varphi_1\left(\begin{matrix} q, 0 \\ (1-q)x \end{matrix}\middle| q; q^\alpha\right). \tag{3.3}$$

By using the relation [9]

$${}_2\varphi_1\left(\begin{matrix} a, 0 \\ c \end{matrix}\middle| q; z\right) = \frac{(az;q)_\infty}{(c,z;q)_\infty} \ {}_1\varphi_1\left(\begin{matrix} z \\ az \end{matrix}\middle| q; c\right),$$

we get

$$\begin{aligned}\gamma_q(\alpha, x) &= \frac{(1-q)x^\alpha(q^{\alpha+1};q)_\infty}{(q^\alpha;q)_\infty} \ {}_1\varphi_1\left(\begin{matrix} q^\alpha \\ q^{\alpha+1} \end{matrix}\middle| q; -(q-1)x\right) \\ &= \frac{x^\alpha}{[\alpha]_q} \ {}_1\varphi_1\left(\begin{matrix} q^\alpha \\ q^{\alpha+1} \end{matrix}\middle| q; -(q-1)x\right). \qquad \square\end{aligned}$$

If we take the limit as $q \to 1$, then we get the second equality in (1.1).

Lemma 2.

$$D_q\gamma_q(\alpha, x) = x^{\alpha-1}E_q(-(1-q)x). \tag{3.4}$$

Proof. We get the proof immediately from the observation that $(D_qF)(x) = f(x)$ if $F(x) = \int_0^x f(t)\, d_qt$. □

If we take the limit as $q \to 1$, then we obtain [3, 9.2(8)]

$$\frac{d}{dx}\gamma(\alpha, x) = x^{\alpha-1}e^{-x}.$$

Lemma 3.

$$D_q(x^{-\alpha}\gamma_q(\alpha, x)) = -(qx)^{-\alpha-1}\gamma_q(\alpha+1, qx). \tag{3.5}$$

Proof.

$$\begin{aligned}D_q(x^{-\alpha}\gamma_q(\alpha, x)) &= \frac{1}{[\alpha]_q(1-q)x}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(q^\alpha;q)_k((1-q)x)^k}{(q;q)_k(q^{\alpha+1};q)_k}(1-q^k) \\ &= \frac{-1}{[\alpha]_q}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}q^{\binom{k-1}{2}}(q^{\alpha+1};q)_{k-1}((1-q)qx)^{k-1}}{(q;q)_{k-1}(q^{\alpha+2};q)_{k-1}}\frac{1-q^\alpha}{1-q^{\alpha+1}} \\ &= \frac{-1}{[\alpha]_q}\frac{1-q^\alpha}{1-q^{\alpha+1}} \ {}_1\varphi_1\left(\begin{matrix} q^{\alpha+1} \\ q^{\alpha+2} \end{matrix}\middle| q; -(q-1)qx\right) \\ &= -(qx)^{-\alpha-1}\gamma_q(\alpha+1, qx). \qquad \square\end{aligned}$$

Also, we can prove the following lemma by induction:

Lemma 4.

$$D_q^n(x^{-\alpha}\gamma_q(\alpha, x)) = (-1)^n q^{\binom{n}{2}}(q^n x)^{-\alpha-n}\gamma_q(\alpha+n, q^n x). \tag{3.6}$$

If we take the limit as $q \to 1$, then we have [3, 9.2(9)]

$$\frac{d^n}{dx^n}(x^{-\alpha}\gamma(\alpha,x)) = (-1)^n x^{-\alpha-n}\gamma(\alpha+n,x).$$

Lemma 5.

$$\gamma_q(\alpha+1,qx) = -q^{\alpha+1}[-\alpha]_q\gamma_q(\alpha,x) - qx^\alpha E_q(-(1-q)x). \tag{3.7}$$

Proof. By using Eq. (3.5), we have

$$\gamma_q(\alpha+1,qx) = -(qx)^{\alpha+1}D_q(x^{-\alpha}\gamma_q(\alpha,x)).$$

By using the q-Leibniz' rule [9]

$$(D_q(fg))(x) = (D_qf)(x)g(x) + f(qx)(D_qg)(x) \tag{3.8}$$

and Eq. (3.4), we obtain

$$\begin{aligned}\gamma_q(\alpha+1,qx) &= -(qx)^{\alpha+1}([-\alpha]_q x^{-\alpha-1}\gamma_q(\alpha,x) + (qx)^{-\alpha}x^{\alpha-1}E_q(-(1-q)x))\\ &= -q^{\alpha+1}[-\alpha]_q\gamma_q(\alpha,x) - qx^\alpha E_q(-(1-q)x).\end{aligned}$$

□

If we take the limit as $q \to 1$, we get [2]

$$\gamma(\alpha+1,x) = \alpha\gamma(\alpha,x) - x^\alpha e^{-x}.$$

Lemma 6.

$$D_q(e_q((1-q)x)\gamma_q(\alpha,x)) = e_q((1-q)x)\gamma_q(\alpha,x) + x^\alpha[x^{-1} - 1 + q]. \tag{3.9}$$

Proof.

$$\begin{aligned}D_q(e_q((1-q)x)\gamma_q(\alpha,x)) = \frac{1}{x(1-q)}\{&e_q((1-q)x)\gamma_q(\alpha,x) - [1-(1-q)x]e_q((1-q)x)[\gamma_q(\alpha,x)\\ &- (1-q)x^\alpha E_q(-(1-q)x)]\}\end{aligned}$$

and by using the relation [9]

$$e_q(z)E_q(-z) = 1, \tag{3.10}$$

then

$$D_q(e_q((1-q)x)\gamma_q(\alpha,x)) = e_q((1-q)x)\gamma_q(\alpha,x) + x^\alpha[x^{-1} - 1 + q].$$

□

Also, we can prove the following lemma by induction:

Lemma 7.

$$D_q^n(e_q((1-q)x)\gamma_q(\alpha,x)) = e_q((1-q)x)\gamma_q(\alpha,x) + \sum_{i=0}^{n-1} D_q^i(x^\alpha[x^{-1} - 1 + q]) \tag{3.11}$$

$$= e_q((1-q)x)\gamma_q(\alpha,x) + x^{\alpha-1} - (1-q)x^\alpha + \sum_{i=1}^{n-1}\frac{x^{\alpha-i}(q^\alpha;q^{-1})_i}{(1-q)^{i-1}}\left[\frac{1-q^{\alpha-i}}{x(1-q)(1-q^\alpha)} - 1\right]. \tag{3.12}$$

If we take the limit as $q \to 1$, then we obtain

$$\frac{d^n}{dx^n}(e^x\gamma(\alpha,x)) = e^x\gamma(\alpha,x) + \sum_{i=0}^{n-1} x^{\alpha-1-i}(\alpha-1)_{-i},$$

where $(a)_{-m} = a(a-1)(a-2)\cdots(a-m+1)$.

By using Eq. (3.4), we can prove the following lemma:

Lemma 8. *The q-difference equation of the function $\gamma_q(\alpha,x)$ is given by:*

$$[(1-q)x][1+(1-q)x]D_q^2\gamma_q(\alpha,x) - [1-(1-q)x-q^{\alpha-1}]D_q\gamma_q(\alpha,x) = 0. \tag{3.13}$$

If we take the limit as $q \to 1$, then we have

$$x\gamma''(\alpha,x) + (x-\alpha+1)\gamma'(\alpha,x) = 0,$$

which is the differential equation of the function $\gamma(\alpha,x)$.

4. *q*-Tricomi Expansion

In this section, we will give a q-analogue of the Tricomi expansion.

Theorem 1. (q-*Tricomi expansion*)

$$\gamma_q(\alpha,\omega x) = \omega^\alpha \sum_{n=0}^{\infty} \frac{\gamma_q(\alpha+n,x)}{[n]_q!}(\omega;q)_n; \quad 0 \le x < \frac{1}{1-q}; \quad 0 \le \omega \le 1; \quad \alpha > 0, \tag{4.1}$$

where $[n]_q! = \frac{(q;q)_n}{(1-q)^n}$.

Proof. Consider the following double series

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_{k,n} = (1-q)(x\omega)^\alpha((1-q)x;q)_\infty \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{q^{\alpha k}}{((1-q)x;q)_k}\frac{(\omega;q)_n}{(q;q)_n}[(1-q)xq^k]^n, \quad \alpha > 0. \tag{4.2}$$

This double series will have positive terms if $x \ge 0$, $(1-q)xq^i < 1\ \forall i = 0,1,2,\ldots$, $\omega \ge 0$ and $\omega q^i \le 1\ \forall i = 0,1,2,\ldots$. But $\frac{1}{1-q} \le \frac{1}{q^i(1-q)}$ and $1 \le \frac{1}{q^i}\ \forall i = 0,1,2,\ldots$. Then, the double series (4.2) is of positive terms if $0 \le x < \frac{1}{1-q}$ and $0 \le \omega \le 1$.

The sum by rows of the double series (4.2) is given by

$$\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{k,n}\right) = (1-q)(x\omega)^\alpha((1-q)x;q)_\infty \sum_{k=0}^{\infty} \frac{q^{\alpha k}}{((1-q)x;q)_k}\left(\sum_{n=0}^{\infty}\frac{(\omega;q)_n}{(q;q)_n}[(1-q)xq^k]^n\right). \tag{4.3}$$

By using the q-binomial theorem [9]

$$\sum_{r=0}^{\infty} \frac{(a;q)_r}{(q;q)_r} z^r = \frac{(az;q)_\infty}{(z;q)_\infty} \quad |z| < 1, \tag{4.4}$$

then we get

$$\sum_{n=0}^{\infty} \frac{(\omega;q)_n}{(q;q)_n}[(1-q)zq^k]^n = \frac{(q^k(1-q)\omega z;q)_\infty}{(q^k(1-q)z;q)_\infty} \quad |z| < \frac{1}{1-q}.$$

Then

$$\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{k,n}\right) = (1-q)(x\omega)^\alpha((1-q)x;q)_\infty \sum_{k=0}^{\infty} \frac{q^{\alpha k}}{((1-q)x;q)_k}\frac{(q^k(1-q)\omega x;q)_\infty}{(q^k(1-q)x;q)_\infty}.$$

By using the relation [9]

$$\frac{(a;q)_\infty}{(a;q)_m} = (aq^m;q)_\infty, \quad m \in \mathbb{C},$$

then we get

$$\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{k,n}\right) = (1-q)(x\omega)^\alpha \sum_{k=0}^{\infty}((1-q)x\omega q^k;q)_\infty q^{\alpha k} = \gamma_q(\alpha,\omega x). \tag{4.5}$$

Then, the series $\sum_{k=0}^{\infty}(\sum_{n=0}^{\infty} a_{k,n})$ converges.

Now, the double series (4.2) is of positive terms and one of its repeated series is convergent, so also is the other and also the double series; and the three sums are the same. Also, the interchanging of summation order is always true [1]. Then

$$\begin{aligned}\gamma_q(\alpha,\omega x) &= \omega^\alpha \sum_{n=0}^{\infty} \frac{(\omega;q)_n}{[n]_q!}\left[(1-q)x^{\alpha+n}((1-q)x;q)_\infty \sum_{k=0}^{\infty}\frac{q^{(\alpha+n)k}}{((1-q)x;q)_k}\right] \\ &= \omega^\alpha \sum_{n=0}^{\infty} \frac{(\omega;q)_n}{[n]_q!}\gamma_q(\alpha+n,x).\end{aligned}$$

□

Now, in Theorem 1 if we take the limit as $q \to 1$ we can easily prove in a formal way that

$$\gamma(\alpha,\omega x) = \omega^\alpha \sum_{n=0}^{\infty} \frac{\gamma(\alpha+n,x)}{n!}(1-\omega)^n, \tag{4.6}$$

which is the Tricomi expansion in the case of the incomplete gamma function.

By using (4.1) and (3.2) with x replaced by $\frac{x}{1-q}$, we obtain

Lemma 9.

$${}_1\varphi_1\left(\begin{matrix} q^\alpha \\ q^{\alpha+1}\end{matrix}\middle| q;\omega x\right) = [\alpha]_q \sum_{n=0}^{\infty} \frac{(\omega,q)_n}{[\alpha+n]_q}\, {}_1\varphi_1\left(\begin{matrix} q^{\alpha+n} \\ q^{\alpha+n+1}\end{matrix}\middle| q;x\right)\frac{x^n}{(q,q)_n}; \quad 0 \le x < 1, \quad 0 \le \omega \le 1. \tag{4.7}$$

Replace x by $(q-1)x$ and take the limit when $q \to 1$, then we have formally

$${}_1F_1\left(\begin{matrix} \alpha \\ \alpha+1\end{matrix}\middle| \omega x\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\alpha+1)_n}\, {}_1F_1\left(\begin{matrix} \alpha+n \\ \alpha+n+1\end{matrix}\middle| x\right)\frac{[-(1-\omega)x]^n}{n!}, \tag{4.8}$$

which is a special case of Eq. (1.3) (Erdélyi multiplication formula).

5. Some Expansions in $\gamma_q(\alpha, x)$

We can write the q-Tricomi expansion in the form

$$\frac{\gamma_q(\alpha,\omega x)}{\omega^\alpha} - \gamma_q(\alpha,x) = \sum_{n=1}^{\infty}\frac{\gamma_q(\alpha+n,x)}{[n]_q!}(\omega,q)_n$$

and by using Eq. (3.2), we get

$$(1-q)x^\alpha((1-q)\omega x;q)_\infty \sum_{k=0}^{\infty}\frac{q^{\alpha k}}{((1-q)\omega x;q)_k} - \gamma_q(\alpha,x) = \sum_{n=1}^{\infty}\frac{\gamma_q(\alpha+n,x)}{[n]_q!}(\omega,q)_n.$$

When $\omega \to 0$, we have

$$\frac{x^\alpha}{[\alpha]_q} - \gamma_q(\alpha,x) = \sum_{n=1}^{\infty}\frac{\gamma_q(\alpha+n,x)}{[n]_q!}, \tag{5.1}$$

or

$$\frac{x^\alpha}{[\alpha]_q} = \sum_{n=0}^{\infty} \frac{\gamma_q(\alpha+n,x)}{[n]_q!}, \quad 0 \le x < \frac{1}{1-q}. \tag{5.2}$$

Then

$$\frac{x^{\alpha+k}}{[\alpha+k]_q} = \sum_{n=k}^{\infty} \frac{\gamma_q(\alpha+n,x)}{[n-k]_q!}; \quad k \ge 0. \tag{5.3}$$

We can choose arbitrary d_k, subject to convergence, to form the following series

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{x^{\alpha+k}}{[\alpha+k]_q[k]_q!} d_k &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{\gamma_q(\alpha+n,x)}{[k]_q![n-k]_q!} d_k \\ &= \sum_{n=0}^{\infty} \frac{\gamma_q(\alpha+n,x)}{[n]_q!} \sum_{k=0}^{n} \left[{n \atop k}\right]_q d_k,\end{aligned}$$

where $\left[{n \atop k}\right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$ (q-binomial). Then

$$\sum_{k=0}^{\infty} \frac{x^{\alpha+k}}{[\alpha+k]_q[k]_q!} d_k = \sum_{n=0}^{\infty} c_{n,q} \frac{\gamma_q(\alpha+n,x)}{[n]_q!}, \tag{5.4}$$

where $c_{n,q} = \sum_{k=0}^{n} \left[{n \atop k}\right]_q d_k$. So, the power series which take the form given in the left side of Eq. (5.4) can therefore be written as a series in incomplete q-gamma functions.

For example, if we take $d_k = (-1)^k q^{\frac{k(k-1)}{2}} \omega^{\alpha+k}$, then

$$c_{n,q} = \omega^\alpha \sum_{k=0}^{n} \left[{n \atop k}\right]_q (-1)^k q^{\frac{k(k-1)}{2}} \omega^k.$$

By using (4.4) if we replace a by q^{-n} and z by $-\frac{aq^n}{x}$, we get

$$x^n(-a/x;q)_n = \sum_{k=0}^{n} \left[{n \atop k}\right]_q q^{\frac{k(k-1)}{2}} a^k x^{n-k}, \tag{5.5}$$

which is called Gauss's q-binomial formula. Replace x by $1/\omega$ and put $a = -1$, then

$$(\omega;q)_n = \sum_{k=0}^{n} \left[{n \atop k}\right]_q q^{\frac{k(k-1)}{2}} (-1)^k \omega^k. \tag{5.6}$$

Hence,

$$c_{n,q} = \omega^\alpha(\omega;q)_n,$$

and

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k-1)}{2}} (\omega x)^{\alpha+k}}{[\alpha+k]_q[k]_q!} = \sum_{n=0}^{\infty} \frac{\gamma_q(\alpha+n,x)}{[n]_q!} \omega^\alpha(\omega;q)_n. \tag{5.7}$$

In view of (3.2), (4.1) is the special case of $d_k = (-1)^k q^{\frac{k(k-1)}{2}} \omega^{\alpha+k}$ of (5.4).

6. A Formal Proof of the Expansion Formula in Eq. (1.5)

It is well known that [3, 9.7(5)]

$$E_1(x) = -\gamma - \ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n \cdot n!}, \tag{6.1}$$

so it would suffice to prove that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n.n!} = \sum_{n=1}^{\infty} \frac{\gamma(n,x)}{n!}$.

Now, in (5.4) with $\alpha > 0$ if we take

$$d_k = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{[k+1]_q},$$

then

$$\begin{aligned} c_{n,q} &= \sum_{k=0}^{n} \left[{n \atop k}\right]_q \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{[k+1]_q} \\ &= \frac{-1}{[n+1]_q} \sum_{k=1}^{n+1} \left[{n+1 \atop k}\right]_q (-1)^k q^{\frac{k(k-1)}{2}} q^{1-k}. \end{aligned}$$

By using Eq. (5.5) at $x = q$ and $a = -1$, we get

$$\begin{aligned} c_{n,q} &= \frac{-q^{-n}}{[n+1]_q} \{q^{n+1}(1/q;q)_{n+1} - q^{n+1}\} \\ &= \frac{q}{[n+1]_q}, \quad n > 0 \end{aligned}$$

and $c_{o,q} = 1$

By using Eq. (5.4), we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k-1)}{2}} x^{\alpha+k}}{[\alpha+k]_q [k+1]_q!} = q \sum_{n=1}^{\infty} \frac{\gamma_q(\alpha+n,x)}{[n+1]_q!} + \gamma_q(\alpha,x).$$

Put $\alpha = 1$ and take the limit when $q \to 1$, then formally

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k.k!} = \sum_{n=1}^{\infty} \frac{\gamma(n,x)}{n!}, \tag{6.2}$$

which proves the expansion formula in Eq. (1.5).

7. *q*-Tricomi Expansion and *q*-Taylor Expansion

In this section, we will formally deduce the q-Tricomi expansion via the q-Taylor expansion. Let $f(z)$ be a continuous function on some interval (a,b) and $c \in [a,b]$. Then the q-Taylor expansion ([4, 8] and [5]) is given by the formal series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n (c/z,q)_n}{[n]_q!} (D_q^n f)(c), \quad z \in (a,b). \tag{7.1}$$

By using the substitution $q \to q^{-1}$, we get the formal series

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n (c^{-1}z,q)_n}{q^{\frac{n(n-1)}{2}} [n]_q!} (D_q^n f)(q^{-n}c).$$

Then

$$f(\omega x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n (\omega,q)_n}{q^{\frac{n(n-1)}{2}} [n]_q!} (D_q^n f)(q^{-n}x).$$

Now, let $f(x) = x^{-\alpha}\gamma_q(\alpha,x)$ and by using the relation (3.6), we obtain

$$(\omega x)^{-\alpha}\gamma_q(\alpha,\omega x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n (\omega,q)_n}{q^{\frac{n(n-1)}{2}} [n]_q!} (-1)^n q^{\frac{n(n-1)}{2}} x^{-\alpha-n} \gamma_q(\alpha,x)$$

thereby recovering Eq. (4.1).

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