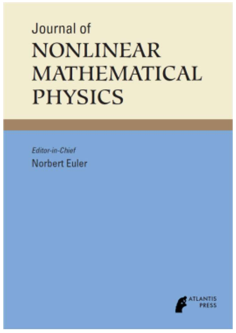


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# DARBOUX POLYNOMIALS FOR LOTKA–VOLTERRA SYSTEMS IN THREE DIMENSIONS

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We consider Lotka–Volterra systems in three dimensions depending on three real parameters. By using elementary algebraic methods we classify the Darboux polynomials (also known as second integrals) for such systems for various values of the parameters, and give the explicit form of the corresponding cofactors. More precisely, we show that a Darboux polynomial of degree greater than one is reducible. In fact, it is a product of linear Darboux polynomials and first integrals.

*Keywords*: Lotka–Volterra model; integrability; Darboux polynomials.

## 1. Introduction

The Lotka–Volterra model is a basic model of predator-prey interactions. The model was developed independently by Alfred Lotka (1925), and Vito Volterra (1926). It forms the basis for many models used today in the analysis of population dynamics. It has other applications in Physics, e.g. laser Physics, plasma Physics (as an approximation to the Vlasov–Poisson equation), and neural networks. In three dimensions it describes the dynamics of a biological system where three species interact.

The most general form of Lotka–Volterra equations is

$$\dot{x}_i = \varepsilon_i x_i + \sum_{j=1}^{n} a_{ij} x_i x_j, \quad i = 1, 2, \ldots, n.$$

We consider Lotka–Volterra equations without linear terms ($\varepsilon_i = 0$), and where the matrix of interaction coefficients $A = (a_{ij})$ is skew-symmetric. This is a natural assumption related to the principle that crowding inhibits growth.

The most famous special case of Lotka–Volterra system is the KM system (also known as the Volterra system) defined by

$$\dot{x}_i = x_i(x_{i+1} - x_{i-1}) \quad i = 1, 2, \ldots, n, \tag{1.1}$$

where $x_0 = x_{n+1} = 0$. It was first solved by Kac and van-Moerbeke in [13], using a discrete version of inverse scattering due to Flaschka [10]. In [16] Moser gave a solution of the system using the method of continued fractions, and in the process he constructed action-angle coordinates. Equations (1.1) can be considered as a finite-dimensional approximation of the Korteweg-de Vries (KdV) equation.

The variables $x_i$ are an intermediate step in the construction of the action-angle variables for the Liouville model on the lattice. This system has a close connection with the Toda lattice,

$$\dot{a}_i = a_i(b_{i+1} - b_i) \quad i = 1, \dots, n-1$$
$$\dot{b}_i = 2(a_i^2 - a_{i-1}^2) \quad i = 1, \dots, n.$$

In fact, a transformation of Hénon connects the two systems:

$$a_i = -\frac{1}{2}\sqrt{x_{2i}x_{2i-1}} \quad i = 1, \dots, n-1$$
$$b_i = \frac{1}{2}(x_{2i-1} + x_{2i-2}) \quad i = 1, \dots, n.$$

The systems which we consider are all integrable in the sense of Liouville. In other words, there are enough integrals in involution to ensure the complete integrability of the system.

Any constant value of a first integral defines a submanifold which is invariant under the flow of the Hamiltonian vector field. A second integral is a function which is constant on a specific level set. While a first integral satisfies $\dot{f} = 0$, a second integral is characterized by the property $\dot{f} = \lambda f$, for some function $\lambda$ which is called the cofactor of $f$. Second integrals are also called special functions, stationary solutions, and in the case of polynomials, eigenpolynomials, or, more frequently, Darboux polynomials. In systems which have a Lie theoretic origin (e.g. the full Kostant Toda lattice), they arise from semi-invariants of group actions. The importance of Darboux polynomials lies in the following simple fact. If $f$ and $g$ are relatively prime Darboux polynomials, with the same cofactor, then their quotient is a first integral. We propose to understand the behavior of a system based on the algebraic properties of its Darboux polynomials.

As a starting point we consider the system

$$\begin{aligned}\dot{x}_1 &= x_1(rx_2 + sx_3)\\ \dot{x}_2 &= x_2(-rx_1 + tx_3)\\ \dot{x}_3 &= x_3(-sx_1 - tx_2)\end{aligned} \tag{1.2}$$

where $r$, $s$, $t \in \mathbb{R}$.

Our main result is the following:

**Theorem 1.** *An arbitrary Darboux polynomial of the system* (1.2) *is reducible. In fact, it is a product of linear Darboux polynomials.*

The method of proof that we use follows the approach of Labrunie in [14] for the so called ABC system.

The system (1.2) is Hamiltonian. We define the following quadratic Poisson bracket in $\mathbb{R}^3$ by the formula

$$\pi = rx_1x_2\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + sx_1x_3\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} + tx_2x_3\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}. \tag{1.3}$$

Generically, the rank of this Poisson bracket is 2 and it possesses a Casimir given by $F = x_1^t x_2^{-s} x_3^r$. The function $H = x_1 + x_2 + x_3$ is always a constant of motion. In fact, taking $H$ as the Hamiltonian and using the Poisson bracket (1.3) we obtain Eq. (1.2).

Lotka–Volterra systems have been studied extensively, see e.g. [4, 12, 19]. The Darboux method of finding integrals of finite dimensional vector fields and especially for various types of Lotka–Volterra systems has been used by several authors, e.g. [2, 3, 5–7, 14, 15, 17, 18].

The paper is organized as follows. In Sec. 2, we recall a few basic facts about Darboux polynomials. In Sec. 3 we prove Theorem 1 under general conditions for $r$, $s$, $t$, and we also give the explicit form of the cofactors. Section 4 deals with the case $s = t$. We did not examine other such cases since the method of proof is identical with these two cases. Finally in Sec. 5 we present in detail three examples which include the open and periodic KM-system in three dimensions.

## 2. Darboux Polynomial Preliminaries

Consider a system of ordinary differential equations

$$\frac{dx_i}{dt} = v_i(x_1(t), \dots, x_n(t)), \quad i = 1, \dots, n, \tag{2.1}$$

where the functions $v_i$ are smooth on a domain $U \subset \mathbb{K}^n$. Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and we denote by $\mathbb{K}[\mathbf{x}]$, $\mathbf{x} = (x_1, \dots, x_n)$, the ring of polynomials in $n$ variables over $\mathbb{K}$. Let $\phi : I \to U$ be a solution of (2.1) defined on an open non-empty interval $I$ of the real axis. A continuous function $F : U \to \mathbb{R}$ is called a *first integral* of system (2.1) if it is constant along its solution, i.e. if the function $F \circ \phi$ is constant on its domain of definition for arbitrary solution $\phi$ of (2.1). When $F$ is differentiable, it is a first integral of system (2.1) if

$$L_{\mathbf{v}}(F) = \sum_{i=1}^{n} v_i(\mathbf{x}) \frac{\partial F}{\partial x_i}(\mathbf{x}) = 0, \tag{2.2}$$

where $L_{\mathbf{v}}$ is the *Lie derivative* along the vector field $\mathbf{v} = (v_1, \dots, v_n)$. If $A$ is any function of $\mathbf{x}$, then the Lie derivative of $A$ is the *time derivative* of $A$, i.e. $\dot{A} = \frac{dA}{dt} = L_{\mathbf{v}}(A)$. The vector field generates a *flow* $\phi_t$ that maps a subset $U$ of $\mathbb{K}^n$ to $\mathbb{K}^n$ in such a way that a point in $U$ follows the solution of the differential equation. That is, $\dot{\phi}(\mathbf{x})(t) = \mathbf{v}(\phi(\mathbf{x})) \ \forall \ \mathbf{x} \in U$. The time derivative is also called the *derivative along the flow* since it describes the variation of a function of $\mathbf{x}$ with respect to $t$ as $\mathbf{x}$ evolves according to the differential system.

Many first integral search techniques, such as the Prelle–Singer procedure, are based on the Darboux polynomials. A polynomial $f \in \mathbb{K}[\mathbf{x}]$ is called a *Darboux polynomial* of system (2.1) if

$$L_{\mathbf{v}}(f) = \lambda f, \tag{2.3}$$

for some polynomial $\lambda \in \mathbb{K}[\mathbf{x}]$, which is called the cofactor of $f$. When $\lambda = 0$, the Darboux polynomial is a first integral; $f$ is said to be a proper Darboux polynomial if $\lambda \neq 0$. Let $f_1$, $f_2$ be Darboux polynomials with cofactors $\lambda_1$, $\lambda_2$, respectively. It is easy to verify that:

(i) The product $f_1 f_2$ is also a Darboux polynomial, with cofactor $\lambda_1 + \lambda_2$, and
(ii) If $\lambda_1 = \lambda_2 = \lambda$ then the sum $f_1 + f_2$ is also a Darboux polynomial, with cofactor $\lambda$.

The following propositions ([11]) give some more elementary but important properties of Darboux polynomials.

**Proposition 1.** *Let* $f$, $g \in \mathbb{K}[\mathbf{x}]$ *be nonzero and coprime* (i.e. *they do not have common divisors different from constants*). *Then,* $f \backslash g$ *is a rational first integral if and only if* $f$ *and* $g$ *are Darboux polynomials with the same cofactor* $\lambda \in \mathbb{K}[\mathbf{x}]$.

**Proposition 2.** (i) *All irreducible factors of a Darboux polynomial are Darboux polynomials,*
(ii) *Suppose that the system* (2.1) *is homogeneous of degree* $m$, i.e. *all* $v_i$ *are homogeneous of degree* $m$, *and let* $f$ *be an arbitrary Darboux polynomial of* (2.1) *with cofactor* $\lambda$. *Then* $\lambda$ *is homogeneous of degree* $m - 1$, *and all homogeneous components of* $f$ *are Darboux polynomials of* (2.1) *with cofactor* $\lambda$.

Thus, the search for Darboux polynomials can be restricted to irreducible polynomials, and, if the system is homogeneous, to homogeneous polynomials. Since the dynamical system (1.2) is homogeneous of degree 2, the cofactor of any Darboux polynomial of the system will be a linear combination of the variables $x_1$, $x_2$, $x_3$. It follows that any Darboux polynomial $f$ of system (1.2) will satisfy

$$\begin{aligned} L(f) &= x_1(rx_2+sx_3)\frac{\partial f}{\partial x_1} + x_2(-rx_1+tx_3)\frac{\partial f}{\partial x_2} + x_3(-sx_1-tx_2)\frac{\partial f}{\partial x_3} \\ &= (\alpha x_1 + \beta x_2 + \gamma x_3)f, \end{aligned} \tag{2.4}$$

where $\alpha$, $\beta$, $\gamma$ are constants. If it is not clear from the context, we shall denote these constants by $\alpha(f)$, $\beta(f)$, $\gamma(f)$ respectively.

## 3. Darboux Polynomials of the Lotka–Volterra System

We carry out our analysis aiming at maximum generality, that is, imposing as few conditions on the parameters $r$, $s$, $t$ as possible. In this section we make such assumptions in Propositions 9, 10 and Theorem 2, however, as we note in Remark 3, one can obtain the results by making assumptions about the cofactor of the Darboux polynomial instead of the parameters. An important role in this work plays the homogeneity property, as can be seen in the following two propositions.

**Proposition 3.** *Let $f$ be a homogeneous Darboux polynomial of system* (1.2) *of degree $m$. If $\gamma(f) \neq 0$, then $f$ has no $x_3^m$ term so that $f(x_1,x_2,x_3) = x_1\phi(x_1,x_2,x_3) + x_2\psi(x_1,x_2,x_3)$.*

**Proof.** Since the polynomial $f$ is homogeneous, we use Euler's identity

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + x_3\frac{\partial f}{\partial x_3} = mf. \tag{3.1}$$

Using Eq. (3.1) we substitute for $x_1\frac{\partial f}{\partial x_1}$ in Eq. (2.4) to obtain

$$\begin{aligned} x_2(-rx_1+tx_3-rx_2-sx_3)\frac{\partial f}{\partial x_2} &+ x_3(-sx_1-tx_2-rx_2-sx_3)\frac{\partial f}{\partial x_3} \\ &= (\alpha x_1 + (\beta - mr)x_2 + (\gamma - ms)x_3)f. \end{aligned}$$

Setting $x_1 = 0$, $x_2 = 0$, and letting $F(x_3) = f(0,0,x_3)$ we have

$$-sx_3^2F'(x_3) = (\gamma - ms)x_3F(x_3). \tag{3.2}$$

If $s = 0$, $\gamma \neq 0$, Eq. (3.2) implies that $F = 0$. Otherwise, if $s \neq 0$ we have $F(x_3) = \kappa x_3^{m-\gamma/s}$, for some constant $\kappa$. Since $f$ is homogeneous of degree $m$, the only term containing only $x_3$ is necessarily $x_3^m$. Thus, if $\gamma \neq 0$ we must have $F = 0$ also in this case, and the proposition is proved. □

We shall use the following notation: for a polynomial $f = f(x_1,x_2,x_3)$ we denote $\hat{f} = f|_{x_1=0}$, $\bar{f} = f|_{x_2=0}$, $\breve{f} = f|_{x_3=0}$. We denote $N_m = \{1,2,\ldots,m\}$, $\mathcal{N}_m = N_m \cup \{0\}$, and for any number $r$, $N_m r = \{nr : n \in N_m\}$.

**Proposition 4.** *Let $f$ be a homogeneous Darboux polynomial of degree $m$. If $\gamma(f) \neq 0$, then $s = 0 \Rightarrow x_2|f$, $t = 0 \Rightarrow x_1|f$, $s \neq 0$ and $\gamma \notin N_m s \Rightarrow x_2|f$, $t \neq 0$ and $\gamma \notin N_m t \Rightarrow x_1|f$.*

*We also have the following statements for $\alpha$ and $\beta$:*

*If $\beta(f) \neq 0$, then $r = 0 \Rightarrow x_3|f$, $t = 0 \Rightarrow x_1|f$, $r \neq 0$ and $\beta \notin N_m r \Rightarrow x_3|f$, $t \neq 0$ and $\beta \notin -N_m t \Rightarrow x_1|f$.*

*If $\alpha(f) \neq 0$, then $r = 0 \Rightarrow x_3|f$, $s = 0 \Rightarrow x_2|f$, $r \neq 0$ and $\alpha \notin -N_m r \Rightarrow x_3|f$, $s \neq 0$ and $\alpha \neq -N_m s \Rightarrow x_2|f$.*

**Proof.** We prove the statements for $\gamma$. The proof of the statements for $\alpha$ and $\beta$ is similar. Assume that $\gamma \neq 0$. Then, it follows from Proposition 3 that $f = x_1\phi_1 + x_2\psi_1$, where $\phi_1 = \phi_1(x_1,x_2,x_3)$,

$\psi_1 = \psi_1(x_1, x_2, x_3)$ are either homogeneous polynomials of degree $m-1$, or zero (but they are not both zero). Setting this in Eq. (2.4) yields

$$\begin{aligned} x_1 L(\phi_1) + x_2 L(\psi_1) &= (\alpha x_1 + \beta x_2 + \gamma x_3 - r x_2 - s x_3) x_1 \phi_1 \\ &\quad + (\alpha x_1 + \beta x_2 + \gamma x_3 + r x_1 - t x_3) x_2 \psi_1. \end{aligned} \tag{3.3}$$

Setting $x_2 = 0$ in Eq. (3.3) we have

$$x_1 \overline{L(\phi_1)} = (\alpha x_1 + (\gamma - s) x_3) x_1 \bar{\phi}_1.$$

The operator $\phi_1 \to \bar{\phi}_1$ commutes with the derivations with respect to $x_1$ and $x_3$, and therefore we obtain

$$s x_1 x_3 \left( \frac{\partial \bar{\phi}_1}{\partial x_1} - \frac{\partial \bar{\phi}_1}{\partial x_3} \right) = (\alpha x_1 + (\gamma - s) x_3) \bar{\phi}_1. \tag{3.4}$$

If $s = 0$ then $\bar{\phi}_1 = 0$, which implies that $\phi_1$ is divisible by $x_2$ and that $f = x_1\phi_1 + x_2\psi_1$ is divisible by $x_2$. Suppose now that $s \neq 0$, $\deg \bar{\phi}_1 = \deg \phi_1 = m - 1$, and that $\gamma \neq ns$, $n \in N_m$. The r.h.s. of (3.4) is divisible by $x_1$, and since $\gamma - s \neq 0$, it follows that $x_1 | \bar{\phi}_1$. Let $\bar{\phi}_1 = x_1 \phi_2$, where $\phi_2$ is a homogeneous polynomial of degree $m-2$. Then, we have

$$\frac{\partial \bar{\phi}_1}{\partial x_1} = x_1 \frac{\partial \phi_2}{\partial x_1} + \phi_2, \qquad \frac{\partial \bar{\phi}_1}{\partial x_3} = x_1 \frac{\partial \phi_2}{\partial x_3},$$

and from (3.4) we obtain

$$s x_1 x_3 \left( \frac{\partial \phi_2}{\partial x_1} - \frac{\partial \phi_2}{\partial x_3} \right) = (\alpha x_1 + (\gamma - 2s) x_3) \phi_2.$$

Since $\gamma - 2s \neq 0$, $\phi_2$ is divisible by $x_1$. Continuing in the same way we obtain

$$s x_1 x_3 \left( \frac{\partial \phi_{m-1}}{\partial x_1} - \frac{\partial \phi_{m-1}}{\partial x_3} \right) = (\alpha x_1 + (\gamma - (m-1)s) x_3) \phi_{m-1},$$

where $\deg \phi_{m-1} = 1$, and $x_1 | \phi_{m-1}$. Thus, $\phi_{m-1} = \text{const.}\ x_1$, and from the above equation we have $s x_3 = \alpha x_1 + (\gamma - (m-1)s) x_3$. By equating coefficients we obtain $\gamma = ms$, which is a contradiction. Therefore, we must have $\bar{\phi}_1 = 0$, which implies that $f$ is divisible by $x_2$.

Setting $x_1 = 0$ in (3.3) and using (2.4) we obtain

$$t x_2 x_3 \left( \frac{\partial \widehat{\psi}_1}{\partial x_2} - \frac{\partial \widehat{\psi}_1}{\partial x_3} \right) = (\beta x_2 + (\gamma - t) x_3) \widehat{\psi}_1.$$

If $t = 0$ then $\widehat{\psi}_1 = 0$, hence $\psi_1$ is divisible by $x_1$ and so $f$ is divisible by $x_1$. Suppose that $t \neq 0$, $\deg \widehat{\psi}_1 = \deg \psi_1 = m - 1$, and $\gamma \neq nt$, $n \in N_m$. Then it can be shown in a similar way as above that $\psi_1$ is divisible by $x_1$, which implies that $f$ is divisible by $x_1$, and the proposition is proved. □

This leads to the characterization of the cofactors of Darboux polynomials of system (1.2), as follows.

**Proposition 5.** *Let $f$ be a homogeneous Darboux polynomial of degree $m$. We have either $\gamma(f) = 0$, or $\gamma(f) = \gamma_1 s$, $\gamma_1 \in N_m$, or $\gamma(f) = \gamma_2 t$, $\gamma_2 \in N_m$, or $\gamma(f) = \gamma_1 s + \gamma_2 t$, $\gamma_2 \in \{1, 2, \ldots, m-1\}$, $\gamma_1 \in N_{m-\gamma_2}$.*

**Proof.** Since $f$ is a Darboux polynomial it satisfies $L(f) = (\alpha x_1 + \beta x_2 + \gamma x_3) f$. Suppose that $\gamma \neq 0$ and $\gamma \neq ns$, $n \in N_m$. Then by Proposition 4 $f$ is divisible by $x_2$, that is $f = x_2 f_1$ for some homogeneous polynomial $f_1$ of degree $m-1$ and we have

$$L(f_1) = ((\alpha + r) x_1 + \beta x_2 + (\gamma - t) x_3) f_1.$$

Suppose that $\gamma(f_1) \neq 0$, i.e. $\gamma \neq t$, and that $\gamma(f_1) \neq ns$, $n \in N_{m-1}$, that is $\gamma \neq ns + t$, $n \in N_{m-1}$. Then, again by Proposition 4 it follows that $f_1$ is divisible by $x_2$, and writing $f_1 = x_2 f_2$ we obtain

$$L(f_2) = ((\alpha + 2r)x_1 + \beta x_2 + (\gamma - 2t)x_3)f_2.$$

If $\gamma \neq 2t$, and $\gamma \neq ns + 2t$, $n \in N_{m-2}$, then $f_2$ is divisible by $x_2$. Continuing in the same way, after $m-1$ steps we obtain

$$L(f_{m-1}) = ((\alpha + (m-1)r)x_1 + \beta x_2 + (\gamma - (m-1)t)x_3)f_{m-1}, \tag{3.5}$$

where $\deg f_{m-1} = 1$. If $\gamma \neq (m-1)t$ and $\gamma \neq s + (m-1)t$, then $x_2 | f_{m-1}$, and thus $f_{m-1} = \text{const } x_2$. From Eq. (3.5) we then have $-rx_1 + tx_3 = (\alpha + (m-1)r)x_1 + \beta x_2 + (\gamma - (m-1)t)x_3$, and by equating coefficients we obtain $\gamma = mt$. We therefore conclude that we have either $\gamma = 0$, or $\gamma = ns$, or $\gamma = nt$, $n \in N_m$, or $\gamma = \gamma_1 s + \gamma_2 t$, $\gamma_2 = 1, 2, \ldots, m-1$, $\gamma_1 \in N_{m-\gamma_2}$, and the proposition is proved. □

**Remark 1.** We note that in the proof of Proposition 5 we can make the successive assumptions $\gamma(f) \neq nt$ $(n \in N_m)$, $\gamma(f_1) \neq nt$, $(n \in N_{m-1}), \ldots, \gamma(f_{m-1}) \neq nt$, $(n \in N_1)$, which imply that the respective functions are divisible by $x_1$. We obtain the same result also in this case, in particular the relation $\gamma_1 s + \gamma_2 t$ with the conditions $\gamma_1 = 1, 2, \ldots, m-1$, $\gamma_2 \in N_{m-\gamma_1}$, which are the same with the conditions stated in the proposition.

**Proposition 6.** *Let $f$ be a homogeneous Darboux polynomial of degree $m$. We have*:

(a) $\alpha(f) = 0$, *or* $\alpha(f) = -\alpha_1 r$, $\alpha_1 \in N_m$, *or* $\alpha(f) = -\alpha_2 s$, $\alpha_2 \in N_m$, *or* $\alpha(f) = -\alpha_1 r - \alpha_2 s$, $\alpha_2 = 1, 2, \ldots, m-1$, $\alpha_1 \in N_{m-\alpha_2}$.
(b) $\beta(f) = 0$, *or* $\beta(f) = \beta_1 r$, $\beta_1 \in N_m$, *or* $\beta(f) = -\beta_2 t$, $\beta_2 \in N_m$, *or* $\beta(f) = \beta_1 r - \beta_2 t$, $\beta_2 = 1, 2, \ldots, m-1$, $\beta_1 \in N_{m-\beta_2}$.

**Proof.** The proof is similar to the proof of Proposition 5. □

The following propositions give further analysis on the cofactors, and their relation with the parameters and the form of the Darboux polynomials.

**Proposition 7.** *Let $r$, $s$, $t$ be nonzero, $r \backslash s = q_1$, $r \backslash t = q_2$, and $s \backslash t = q_3$. Let $f$ be a homogeneous Darboux polynomial of degree $m$, and $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, $\gamma_1$, $\gamma_2$ the integers which appear in Propositions* 5 *and* 6.

(a) *If* $\alpha_1 + (\alpha_2 - j)\frac{1}{q_1} \notin \mathcal{N}_{m-j}$ *and* $\beta_1 - (\beta_2 - j)\frac{1}{q_2} \notin \mathcal{N}_{m-j}$, *for* $j = 0, 1, 2, \ldots, m-1$, *then* $\alpha_2 = \beta_2$.
(b) *If* $(\alpha_1 - j)q_1 + \alpha_2 \notin \mathcal{N}_{m-j}$ *and* $\gamma_1 + (\gamma_2 - j)\frac{1}{q_3} \notin \mathcal{N}_{m-j}$, *for* $j = 0, 1, 2, \ldots, m-1$, *then* $\alpha_1 = \gamma_2$.
(c) *If* $-(\beta_1 - j)q_2 + \beta_2 \notin \mathcal{N}_{m-j}$ *and* $(\gamma_1 - j)q_3 + \gamma_2 \notin \mathcal{N}_{m-j}$, *for* $j = 0, 1, 2, \ldots, m-1$, *then* $\beta_1 = \gamma_1$.

**Proof.** We prove statement (a). The proof of statements (b) and (c) is similar. If $\alpha_2$ or $\beta_2$ is nonzero, then by hypothesis we have $\alpha(f) = -(\alpha_1 + \alpha_2 \frac{1}{q_1})r \neq 0$ and $\alpha(f) \neq -nr$, $n \in N_m$, or $\beta(f) = (\beta_1 - \beta_2 \frac{1}{q_2})r \neq 0$ and $\beta(f) \neq nr$, $n \in N_m$, respectively. In either case, it follows from Proposition 4 that $f$ is divisible by $x_3$. We can write $f = x_3 f_1$, for some homogeneous polynomial $f_1$ of degree $m-1$, and we have

$$\begin{aligned} L(f_1) &= ((\alpha + s)x_1 + (\beta + t)x_2 + \gamma x_3)f_1 \\ &= ((-\alpha_1 r - (\alpha_2 - 1)s)x_1 + (\beta_1 r - (\beta_2 - 1)t)x_2 + \gamma x_3)f_1. \end{aligned}$$

By the same argument as above, if we do not have $\alpha_2(f_1) = \beta_2(f_1) = 0$, i.e. if we do not have $\alpha_2 = \beta_2 = 1$, then we have either $\alpha(f_1) = -(\alpha_1 + (\alpha_2 - 1)\frac{1}{q_1})r \neq 0$ and $\alpha(f_1) \neq -nr$, $n \in N_{m-1}$, or

$\beta(f_1) = (\beta_1 - (\beta_2 - 1)\frac{1}{q_2})r \neq 0$ and $\beta(f_1) \neq nr$, $n \in N_{m-1}$, and $f_1$ is divisible by $x_3$. Continuing in the same way, after $m-1$ steps we obtain

$$L(f_{m-1}) = ((-\alpha_1 r - (\alpha_2 - (m-1))s)x_1 + (\beta_1 r - (\beta_2 - (m-1))t)x_2 + \gamma x_3)f_{m-1}, \tag{3.6}$$

where $\deg f_{m-1} = 1$. If we do not have $\alpha_2 = \beta_2 = m-1$, then it follows by our assumptions that $x_3 | f_{m-1}$, which implies that $f_{m-1} = \text{const } x_3$ and $f = \text{const } x_3^m$. However, $\alpha_2(x_3) = \beta_2(x_3) = 1$, and by simple properties of Darboux polynomials it follows that $\alpha_2(f) = \beta_2(f) = m$. Therefore, we must have $\alpha_2 = \beta_2 = n$, for some integer $n \in \{0, 1, 2, \ldots, m\}$, and the proposition is proved. $\square$

**Proposition 8.** *Let $r$, $s$, $t$ be nonzero, $r \backslash s = q_1$, and $r \backslash t = q_2$. Let $f$ be a proper Darboux polynomial, homogeneous of degree $m$, with $\gamma(f) = 0$, and let $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ be the integers which appear in Proposition 6.*

(a) *If $\alpha_1 \neq 0$ and $\alpha_1 q_1 + \alpha_2 \notin \mathcal{N}_m$, or $\beta_1 \neq 0$ and $-(\beta_1 q_2 - \beta_2) \notin \mathcal{N}_m$, then $s = -pt$, for some positive rational number $p$.*
(b) *If $\alpha_1 = \beta_1 = 0$, $(\alpha_2 - j)\frac{1}{q_1} \notin N_{m-j}$ and $-(\beta_2 - j)\frac{1}{q_2} \notin N_{m-j}$, for $j = 0, 1, 2, \ldots, m-1$, then we have $f = x_3^{\alpha_2} I$ where $I$ is a first integral.*

**Proof.** (a) Suppose $\alpha_1 \neq 0$ and $\alpha_1 q_1 + \alpha_2 \notin \mathcal{N}_m$. The other case is similar. Then we have $\alpha(f) = -(\alpha_1 q_1 + \alpha_2)s \neq 0$ and $\alpha(f) \neq -ns$, $n \in N_m$. From Proposition 4 it follows that $f$ is divisible by $x_2$, so that $f = x_2 f_1$ for some homogeneous polynomial $f_1$ of degree $m-1$, and we have

$$L(f_1) = ((\alpha + r)x_1 + \beta x_2 - t x_3)f_1. \tag{3.7}$$

Equation (3.7) shows that $f_1$ is a Darboux polynomial with $\gamma(f_1) = -t$. However, from Proposition 5 we have $\gamma(f_1) = \gamma_1 s + \gamma_2 t$ for non-negative integers $\gamma_1, \gamma_2 \in \{0, 1, 2, \ldots, m-1\}$. Therefore, $\gamma_1 s + \gamma_2 t = -t$, which is possible only if $\gamma_1 \neq 0$, in which case $s = -\frac{(1+\gamma_2)}{\gamma_1} t$, and the statement is proved with $p = \frac{1+\gamma_2}{\gamma_1}$.

(b) Suppose that $\alpha_1 = \beta_1 = 0$. Since $f$ is a proper Darboux polynomial with $\gamma(f) = 0$ we must have $\alpha_2 \neq 0$ or $\beta_2 \neq 0$, and our assumptions imply that in fact $\alpha_2 = \beta_2$ (see Proposition 7). We have $\alpha(f) = -\alpha_2 \frac{1}{q_1} r \neq 0$ and $\alpha(f) \neq -nr$, $n \in N_m$. It follows from Proposition 4 that $f$ is divisible by $x_3$. So $f = x_3 f_1'$ for some polynomial $f_1'$ of degree $m-1$, and we have

$$L(f_1') = (-(\alpha_2 - 1)s x_1 - (\beta_2 - 1)t x_2)f_1'.$$

By the same argument, if $\alpha_2 - 1 = \beta_2 - 1 \neq 0$, then $f_1'$ is divisible by $x_3$. Continuing in the same way, we find that $f = x_3^{\alpha_2} I$ for some first integral $I$ ($I \equiv 1$ if $\alpha_2 = \beta_2 = m$), and the proposition is proved. $\square$

These results allow us to characterize the Darboux polynomials of system (1.2).

**Proposition 9.** *Let $f$ be a Darboux polynomial of system* (1.2), *homogeneous of degree $m$. If $s = 0$ then*

$$f = x_2^{\gamma_2} f_1, \tag{3.8}$$

*where $f_1$ is a Darboux polynomial with $\gamma(f_1) = 0$. If $s$, $t$ are nonzero and $N_m s \cap N_m t = \emptyset$, then we have*

$$f = x_1^{\gamma_1} x_2^{\gamma_2} f_2, \tag{3.9}$$

*where $f_2$ is a Darboux polynomial with $\gamma(f_2) = 0$. Here, the non-negative integers $\gamma_1$, $\gamma_2$ are such that $\gamma(f) = \gamma_1 s + \gamma_2 t$.*

**Proof.** If $\gamma(f) = 0$, then the result in each case follows by setting $\gamma_1 = \gamma_2 = 0$, $f_1 = f_2 = f$. Suppose that $\gamma(f) \neq 0$ and $s = 0$. Then, by Proposition 4 $f$ is divisible by $x_2$, and writing $f = x_2 f_1'$ we have

$$L(f_1') = ((\alpha + r)x_1 + \beta x_2 + (\gamma - t)x_3)f_1'.$$

Let this procedure be repeated as many times as it can, and let $\gamma_2$ be the number of times that it can. We have $f = x_2^{\gamma_2} f_1$, where $f_1$ is a Darboux polynomial with $\gamma(f_1) = \gamma - \gamma_2 t = 0$ since we had to stop the division procedure by $x_2$, and Eq. (3.8) is proved. Suppose now that $\gamma(f) \neq 0$, $s$, $t$ are nonzero and $N_m s \cap N_m t = \emptyset$. Thus $\gamma \notin N_m s$ or $\gamma \notin N_m t$. Let us consider the case $\gamma \notin N_m s$. The case $\gamma \notin N_m t$ is similar. Then $f$ is divisible by $x_2$ and as before we have $f = x_2^{\gamma_2} f_2'$, where $f_2'$ is a Darboux polynomial with $\gamma(f_2') = \gamma - \gamma_2 t$. Since we had to stop the division procedure by $x_2$, we must have either $\gamma(f_2') = 0$, in which case Eq. (3.9) is satisfied with $\gamma_1 = 0$ and $f_2 = f_2'$, or $\gamma(f_2') = \gamma_1 s$, for some $\gamma_1 \in N_m$. In the latter case $\gamma(f_2') \notin N_m t$ and $f_2'$ is divisible $\gamma_1$ times by $x_1$, that is, $f_2' = x_1^{\gamma_1} f_2$, $\gamma(f_2) = 0$, and Eq. (3.9) follows. □

**Remark 2.** The condition $N_m s \cap N_m t = \emptyset$ implies that there do not exist integers $n_1$, $n_2 \in N_m$ such that $s = \frac{n_2}{n_1} t$. This condition is satisfied in each of the following cases:

(a) one of $s$, $t$ is positive and the other is negative,
(b) $s$, $t$ have the same sign but one is rational and the other irrational,
(c) $s$, $t$ have the same sign, they are both irrational, and their ratio is irrational,
(d) $s$, $t$ have the same sign, they are both rational, and $s/t < 1/m$ or $s/t > m$,
(e) $s$, $t$ have the same sign, they are both irrational, their ratio is rational, and $s/t < 1/m$ or $s/t > m$.

**Remark 3.** In Proposition 9, instead of the condition $N_m s \cap N_m t = \emptyset$, we can make an alternative assumption as follows. First let $s \backslash t = q_3$, and let $f_k$, $k = 0, 1, 2, \ldots, f = f_0$, be a sequence of Darboux polynomials as we describe below. We denote $\gamma(f_k) = \gamma_1(f_k)s + \gamma_2(f_k)t$, $\gamma_1 = \gamma_1(f)$, $\gamma_2 = \gamma_2(f)$. For $k = 0, 1, 2, \ldots, \gamma_1 + \gamma_2 - 1$, we suppose that

$$\text{(i)}\ \ \gamma_1(f_k) + \gamma_2(f_k)\frac{1}{q_3} \notin \mathcal{N}_{m-k} \quad \text{or} \quad \text{(ii)}\ \ \gamma_1(f_k)q_3 + \gamma_2(f_k) \notin \mathcal{N}_{m-k}. \tag{3.10}$$

In particular, if $\gamma_1(f_k) = 0$ then we require condition (i) to hold, whereas if $\gamma_2(f_k) = 0$ then we require condition (ii) to hold (if $\gamma_1(f_k) \neq 0$ and $\gamma_2(f_k) \neq 0$ then we can have either condition (i) or (ii)). If condition (i) holds, then $\gamma(f_k) \neq 0$ and $\gamma(f_k) \neq ns$, $n \in N_{m-k}$, which implies that $f_k$ is divisible by $x_2$. Thus $f_k = x_2 f_{k+1}$, and $\gamma_1(f_{k+1}) = \gamma_1(f_k)$, $\gamma_2(f_{k+1}) = \gamma_2(f_k) - 1$. If condition (ii) holds, then $\gamma(f_k) \neq 0$ and $\gamma(f_k) \neq nt$, $n \in N_{m-k}$, which implies that $x_1 | f_k$. In this case we have $f_k = x_1 f_{k+1}$, $\gamma_1(f_{k+1}) = \gamma_1(f_k) - 1$, $\gamma_2(f_{k+1}) = \gamma_2(f_k)$. Following this procedure, after $\gamma_1 + \gamma_2$ steps we obtain $f = x_1^{\gamma_1} x_2^{\gamma_2} f'$, where $\gamma(f') = 0$.

The following proposition states similar results in terms of the constants $\alpha$ and $\beta$. The proof is similar to the proof of Proposition 9.

**Proposition 10.** *Let $f$ be a homogeneous Darboux polynomial of degree $m$, and let $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ be the integers which appear in Proposition* 6.

(i) *If $s = 0$ then $f = x_2^{\alpha_1} f_1$, where $f_1$ is a Darboux polynomial with $\alpha(f_1) = 0$.*
(ii) *If $r$, $s$ are nonzero and $-N_m r \cap (-N_m s) = \emptyset$ then $f = x_2^{\alpha_1} x_3^{\alpha_2} f_2$, where $f_2$ is a Darboux polynomial with $\alpha(f_2) = 0$.*

(iii) *If* $r, t$ *are nonzero and* $N_m r \cap (-N_m t) = \emptyset$ *then* $f = x_1^{\beta_1} x_3^{\beta_2} f_3$, *where* $f_3$ *is a Darboux polynomial with* $\beta(f_3) = 0$.

We are now ready to prove the main result of this section.

**Theorem 2.** *Let* $f$ *be a Darboux polynomial of system* (1.2), *homogeneous of degree* $m$. *Suppose that either*: (i) $s = 0$ *and* $N_m r \cap (-N_m t) = \emptyset$, *or* (ii) $r$, $s$, $t$ *are nonzero*, $N_m r \cap (-N_m t) = \emptyset$, $N_m s \cap N_m t = \emptyset$, *and* $(-N_m r) \cap (-N_m s) = \emptyset$. (*In particular, condition* (ii) *is satisfied, for example, when* $r > 0$, $t > 0$ *and* $s < 0$, *or* $r < 0$, $t < 0$ *and* $s > 0$). *Then, there exist three non-negative integers* $i$, $j$, $k$ *and a polynomial first integral* $I$ *— which may be trivial — such that*

$$f = x_1^i x_2^j x_3^k I \tag{3.11}$$

*and*

$$\alpha(f) = -rj - sk, \quad \beta(f) = ri - tk, \quad \gamma(f) = si + tj. \tag{3.12}$$

**Proof.** Consider the case $s = 0$ and $N_m r \cap (-N_m t) = \emptyset$. The other case is similar. We use Eq. (3.8) of Proposition 9 and the equations in statements (i) and (iii) of Proposition 10 in the following algorithm.

(1) Set $n = 0$ and $f_n = f$.
(2) Applying Proposition 10 for $\alpha$ (statement (i)) yields

$$f_n = x_2^{\alpha_1} f_{n+1}, \quad \alpha(f_{n+1}) = 0.$$

If $f_{n+1}$ is a first integral, go to the final step, else increment $n$ by one.
(3) Applying Proposition 10 for $\beta$ (statement (iii)) yields

$$f_n = x_1^{\beta_1} x_3^{\beta_2} f_{n+1}, \quad \beta(f_{n+1}) = 0.$$

If $f_{n+1}$ is a first integral, go to the final step, else increment $n$ by one.
(4) Applying Proposition 9 for $\gamma$ (Eq. (3.8)) yields

$$f_n = x_2^{\gamma_2} f_{n+1}, \quad \gamma(f_{n+1}) = 0.$$

If $f_{n+1}$ is a first integral, go to the final step, else increment $n$ by one and return to step 2.
(5) (Final step) Set $I = f_{n+1}$ and using the sequence of equations linking $f_l$ to $f_{l+1}$, $l = 1, \ldots, n$ given by the algorithm determine the exponents $i$, $j$, $k$ in Eq. (3.11).

At every step one has $\deg f_{l+1} \leq \deg f_l$; when three consecutive terms of the sequence are of the same degree, they are equal and $\alpha(f_l) = \beta(f_l) = \gamma(f_l) = 0$, so $f_l$ is a first integral. Thus the algorithm converges in a finite number of steps. Equation (3.12) follows from simple properties of Darboux polynomials.

If condition (ii) holds, then the proof is the same but now in steps 2 and 4 of the algorithm we use the equation in statement (ii) of Proposition 10, and Eq. (3.9) of Proposition 9, respectively. □

## 4. The Case $s = t$

In this section we study the case $s = t$, which is not covered by Theorem 2 in the previous section. It can be seen that in this case $x_1 + x_2$ is an additional linear Darboux polynomial of system (1.2), with cofactor $sx_3$. Therefore, polynomials of the form $f = x_1^i x_2^j x_3^k (x_1 + x_2)^l$, where $i$, $j$, $k$, $l$ are non-negative integers, are Darboux polynomials. We show that a Darboux polynomial will have this form with $l > 0$, provided its cofactor satisfies some conditions which depend on the ratio $r \backslash s$.

**Proposition 11.** *Suppose that $r$, $s$, $t$ are nonzero, $s = t$, and let $r \backslash s = q_1$. Let $f$ be a homogeneous Darboux polynomial of degree $m$ which does not have the form* (3.11), *and let $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, $\gamma_1$, $\gamma_2$ be the integers which appear in Propositions* 5 *and* 6. *For $j = 0, 1, 2, \ldots, m-1$, suppose that $\alpha_1 + (\alpha_2 - j)\frac{1}{q_1} \notin \mathcal{N}_{m-j}$, $\beta_1 - (\beta_2 - j)\frac{1}{q_1} \notin \mathcal{N}_{m-j}$, $(\alpha_1 - j)q_1 + \alpha_2 \notin \mathcal{N}_{m-j}$, and $-((\beta_1 - j)q_1 - \beta_2) \notin \mathcal{N}_{m-j}$. Then, we have* (i) $\alpha_2 = \beta_2$ *and* (ii) $\alpha_1 + \beta_1 < \gamma_1 + \gamma_2$.

**Proof.** Relation (i) is statement (a) of Proposition 7. We prove the inequality (ii). Suppose on the contrary that $\alpha_1 + \beta_1 > \gamma_1 + \gamma_2$. By arguments that we have used repeatedly in this paper (for example see Proposition 8), $f$ is divisible $\alpha_1$ times by $x_2$ and $\beta_1$ times by $x_1$. Thus we have $f = x_1^{\beta_1} x_2^{\alpha_1} f'$, where $f'$ is a Darboux polynomial of degree $m - (\alpha_1 + \beta_1)$ such that

$$L(f') = (-\alpha_2 s x_1 - \beta_2 s x_2 + (\gamma - (\alpha_1 + \beta_1)s)x_3)f'.$$

By Proposition 5 there exist non-negative integers $\gamma_1'$, $\gamma_2' \in \{0, 1, \ldots, m - (\alpha_1 + \beta_1)\}$ such that $\gamma(f') = \gamma_1' s + \gamma_2' t = (\gamma_1' + \gamma_2')s$. This implies that $\gamma_1' + \gamma_2' = \gamma_1 + \gamma_2 - \alpha_1 - \beta_1 < 0$, a contradiction.

If $\alpha_1 + \beta_1 = \gamma_1 + \gamma_2$, then from the equation above we have $L(f') = (-\alpha_2 s x_1 - \beta_2 s x_2)f'$, and our assumptions imply that $f'$ is divisible $\alpha_2$ times by $x_3$ (Proposition 8). So we have $f' = x_3^{\alpha_2} I$, and therefore $f = x_1^{\beta_1} x_2^{\alpha_1} x_3^{\alpha_2} I$, where $I$ is a first integral. Since we assume that $f$ does not have the form (3.11) we may exclude this possibility, and the proof is completed. □

**Proposition 12.** *Suppose that $r$, $s$, $t$ are nonzero, $s = t$, and let $r \backslash s = q_1$. Let $f$ be a homogeneous Darboux polynomial of degree $m$ which does not have the form* (3.11). *With the same assumptions as in Proposition* 11 *we have $f = (x_1 + x_2)f_1$, for some polynomial $f_1$.*

**Proof.** From Proposition 8 it follows that we may assume $\gamma(f) \neq 0$. By Proposition 3 $f$ does not have an $x_3^m$ term and we can write $f = x_1\phi_1 + x_2\psi_1$, for some polynomials $\phi_1$, $\psi_1$. For a polynomial $f = f(x_1, x_2, x_3)$ we denote by $\widetilde{f}$ the polynomial obtained from $f$ by setting $x_2 = -x_1$, that is $\widetilde{f} = \widetilde{f}(x_1, x_3) = f|_{x_2=-x_1}$. So, $\widetilde{f} = x_1(\widetilde{\phi_1} - \widetilde{\psi_1})$, and letting $h_1 = \phi_1 - \psi_1$ we have $\widetilde{f} = x_1\widetilde{h_1}$. Setting $s = t$ and $x_2 = -x_1$ in Eq. (3.3) we obtain

$$x_1\widetilde{L(\phi_1)} - x_1\widetilde{L(\psi_1)} = ((\alpha - \beta + r)x_1 + (\gamma - s)x_3)x_1(\widetilde{\phi_1} - \widetilde{\psi_1})$$

or

$$\widetilde{L(h_1)} = ((\alpha - \beta + r)x_1 + (\gamma - s)x_3)\widetilde{h_1}. \tag{4.1}$$

Setting $s = t$ and $x_2 = -x_1$ in Eq. (2.4) we obtain

$$\widetilde{L(h_1)} = -x_1(rx_1 - sx_3)\left(\widetilde{\frac{\partial h_1}{\partial x_1}} - \widetilde{\frac{\partial h_1}{\partial x_2}}\right). \tag{4.2}$$

Combining Eqs. (4.1) and (4.2), and noting that $\alpha_2 = \beta_2$ (Proposition 11), we obtain

$$-x_1(rx_1 - sx_3)\left(\widetilde{\frac{\partial h_1}{\partial x_1}} - \widetilde{\frac{\partial h_1}{\partial x_2}}\right) = (-(\alpha_1 + \beta_1 - 1)rx_1 + (\gamma_1 + \gamma_2 - 1)sx_3)\widetilde{h_1}. \tag{4.3}$$

From Proposition 11 we also have $\alpha_1 + \beta_1 < \gamma_1 + \gamma_2$, which implies that the term $-(\alpha_1 + \beta_1 - 1)rx_1 + (\gamma_1 + \gamma_2 - 1)sx_3$ is not a constant multiple of $(rx_1 - sx_3)$. Since $(rx_1 - sx_3)$ divides the right-hand side of Eq. (4.3), it divides $\widetilde{h}_1$. Therefore we have

$$h_1 = (rx_1 - sx_3)\rho_1 + (-rx_2 - sx_3)\chi_1,$$

for some polynomials $\rho_1$, $\chi_1$. Let $h_2 = \rho_1 + \chi_1$. Then, $\widetilde{h}_1 = (rx_1 - sx_3)\widetilde{h}_2$ and $\widetilde{f} = x_1(rx_1 - sx_3)\widetilde{h}_2$. We have

$$\widetilde{\frac{\partial h_1}{\partial x_1}} = (rx_1 - sx_3)\widetilde{\frac{\partial \rho_1}{\partial x_1}} + r\widetilde{\rho}_1 + (rx_1 - sx_3)\widetilde{\frac{\partial \chi_1}{\partial x_1}}, \tag{4.4}$$

$$\widetilde{\frac{\partial h_1}{\partial x_2}} = (rx_1 - sx_3)\widetilde{\frac{\partial \rho_1}{\partial x_2}} + (rx_1 - sx_3)\widetilde{\frac{\partial \chi_1}{\partial \chi_2}} - r\widetilde{\chi}_1. \tag{4.5}$$

Substituting for $\widetilde{\frac{\partial h_1}{\partial x_1}}$, $\widetilde{\frac{\partial h_1}{\partial x_2}}$ from Eqs. (4.4) and (4.5) respectively in Eq. (4.3) we obtain

$$\begin{aligned}
&-x_1(rx_1 - sx_3)\left((rx_1 - sx_3)\left(\widetilde{\frac{\partial \rho_1}{\partial x_1}} + \widetilde{\frac{\partial \chi_1}{\partial x_1}}\right) - (rx_1 - sx_3)\left(\widetilde{\frac{\partial \rho_1}{\partial x_2}} + \widetilde{\frac{\partial \chi_1}{\partial x_2}}\right) + r(\widetilde{\rho}_1 + \widetilde{\chi}_1)\right)\\
&\quad = (-(\alpha_1 + \beta_1 - 1)rx_1 + (\gamma_1 + \gamma_2 - 1)sx_3)\widetilde{h}_1
\end{aligned}$$

and simplifying further we have

$$-x_1(rx_1 - sx_3)\left(\widetilde{\frac{\partial h_2}{\partial x_1}} - \widetilde{\frac{\partial h_2}{\partial x_2}}\right) = (-(\alpha_1 + \beta_1 - 2)rx_1 + (\gamma_1 + \gamma_2 - 1)sx_3)\widetilde{h}_2. \tag{4.6}$$

The term $-(\alpha_1 + \beta_1 - 2)rx_1 + (\gamma_1 + \gamma_2 - 1)sx_3$ is not a constant multiple of $(rx_1 - sx_3)$, and so $(rx_1 - sx_3)|\widetilde{h}_2$. Continuing in the same way we find that $\widetilde{f}$ is divisible by an infinity of powers of $(rx_1 - sx_3)$, which is a contradiction. Therefore we must have $\widetilde{f} = 0$. This implies that $f = (x_1 + x_2)f_1$, for some polynomial $f_1$, and the proof of the proposition is completed. □

**Corollary 1.** *Suppose that $r$, $s$, $t$ are nonzero, $s = t$, and let $r\backslash s = q_1$. Let $f$ be a homogeneous Darboux polynomial of degree $m$. With the same assumptions as in Proposition* 11 *we have*

$$f = x_1^i x_2^j x_3^k (x_1 + x_2)^l I, \tag{4.7}$$

*where $I$ is a first integral and $i$, $j$, $k$, $l$ are non-negative integers.*

**Proof.** Note first that if $\gamma(f) = 0$ then by Proposition 8 it follows that we must have $\alpha_1 = \beta_1 = 0$ and $f = x_3^{\alpha_2} I$. If $\gamma(f) \neq 0$ and $f$ does not have the form (3.11) (which is (4.7) with $l = 0$), then by Proposition 12 we have $f = (x_1 + x_2)f_1$ for some polynomial $f_1$, and

$$L(f_1) = (\alpha x_1 + \beta x_2 + (\gamma - s)x_3)f_1.$$

Repeating this procedure a finite number of steps, we find that $f$ has the form (4.7). □

**Remark 4.** Similar results hold when $r = s$ and $r = -t$. It can be seen that if $r = s$ then $x_2 + x_3$ is a linear Darboux polynomial with cofactor $-rx_1$. Under conditions analogous to the ones we have used in this section, we have $f = x_1^i x_2^j x_3^k (x_2 + x_3)^l I$. Similarly, if $r = -t$ then $x_1 + x_3$ is a linear Darboux polynomial with cofactor $rx_2$, and we have $f = x_1^i x_2^j x_3^k (x_1 + x_3)^k I$.

## 5. Examples

### 5.1. *The KM-system*

We give a complete description of Darboux polynomials for the case of the KM system ($r = 1$, $s = 0$, $t = 1$):

$$\begin{aligned}\dot{x}_1 &= x_1 x_2\\ \dot{x}_2 &= -x_1 x_2 + x_2 x_3\\ \dot{x}_3 &= -x_2 x_3.\end{aligned} \tag{5.1}$$

The Hamiltonian description of system (1.1) can be found in [9] and [8]. We will follow [8] and use the Lax pair of that reference. The Lax pair in the case $n = 3$ is given by

$$\dot{L} = [B, L],$$

where

$$L = \begin{pmatrix} x_1 & 0 & \sqrt{x_1 x_2} & 0 \\ 0 & x_1 + x_2 & 0 & \sqrt{x_2 x_3} \\ \sqrt{x_1 x_2} & 0 & x_2 + x_3 & 0 \\ 0 & \sqrt{x_2 x_3} & 0 & x_3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & \frac{1}{2}\sqrt{x_1 x_2} & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{x_2 x_3} \\ -\frac{1}{2}\sqrt{x_1 x_2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\sqrt{x_2 x_3} & 0 & 0 \end{pmatrix}.$$

This is an example of an isospectral deformation; the entries of $L$ vary over time but the eigenvalues remain constant. It follows that the functions $H_i = \frac{1}{i}\operatorname{Tr} L^i$ are constants of motion. We note that

$$H_1 = 2(x_1 + x_2 + x_3)$$

corresponds to the total momentum and

$$H_2 = \sum_{i=1}^{3} x_i^2 + 2\sum_{i=1}^{2} x_i x_{i+1}.$$

Using (1.3) we define the following quadratic Poisson bracket, $\{x_i, x_{i+1}\} = x_i x_{i+1}, i = 1, 2$, and $\{x_1, x_3\} = 0$. For this bracket $\det L = x_1^2 x_3^2$ is a Casimir and the eigenvalues of $L$ are in involution. Taking the function $H_1 = x_1 + x_2 + x_3$ as the Hamiltonian we obtain Eqs. (5.1). Therefore the system has a Casimir given by $F = x_1 x_3$ and a constant of motion $x_1 + x_2 + x_3$ corresponding to the Hamiltonian. Note that $H_2 = H_1^2 - 2F$.

In the following Tables 1–3, we present all Darboux polynomials of degree $\leq 3$ and the corresponding cofactors.

Table 1. Linear Darboux polynomials and corresponding cofactors.

| | Darboux polynomial | Cofactor |
|---|---|---|
| 1 | $x_1$ | $x_2$ |
| 2 | $x_2$ | $-x_1 + x_3$ |
| 3 | $x_3$ | $-x_2$ |
| 4 | $x_1 + x_2 + x_3$ | 0 |

Table 2. Quadratic Darboux polynomials and corresponding cofactors. Note that (10) is a sum of two first integrals, and thus a first integral; $c_1$, $c_2$ are constants.

| | Darboux polynomial | Cofactor | | Darboux polynomial | Cofactor |
|---|---|---|---|---|---|
| 1 | $x_1^2$ | $2x_2$ | 6 | $x_2x_3$ | $-x_1 - x_2 + x_3$ |
| 2 | $x_2^2$ | $-2x_1 + 2x_3$ | 7 | $x_1(x_1 + x_2 + x_3)$ | $x_2$ |
| 3 | $x_3^2$ | $-x_2$ | 8 | $x_2(x_1 + x_2 + x_3)$ | $-x_1 + x_3$ |
| 4 | $x_1x_2$ | $-x_1 + x_2 + x_3$ | 9 | $x_3(x_1 + x_2 + x_3)$ | $-x_2$ |
| 5 | $x_1x_3$ | 0 | 10 | $c_1(x_1^2 + x_2^2 + x_3^2 + 2x_1x_2$ $+ 2x_2x_3) + c_2x_1x_3$ | 0 |

Table 3. Cubic Darboux polynomials and corresponding cofactors; $c_1, \ldots, c_6$ are constants.

| | Darboux polynomial | Cofactor | | Darboux polynomial | Cofactor |
|---|---|---|---|---|---|
| 1 | $x_1^3$ | $3x_2$ | 9 | $x_2^2(x_1 + x_2 + x_3)$ | $-2x_1 + 2x_3$ |
| 2 | $x_2^3$ | $-3x_1 + 3x_3$ | 10 | $x_3^2(x_1 + x_2 + x_3)$ | $-2x_2$ |
| 3 | $x_3^3$ | $-3x_2$ | 11 | $x_1x_2(x_1 + x_2 + x_3)$ | $-x_1 + x_2 + x_3$ |
| 4 | $x_1^2x_2$ | $-x_1 + 2x_2 + x_3$ | 12 | $x_2x_3(x_1 + x_2 + x_3)$ | $-x_1 - x_2 + x_3$ |
| 5 | $x_2^2x_1$ | $-2x_1 + x_2 + 2x_3$ | 13 | $c_1x_1(x_1^2 + x_2^2 + x_3^2$ $+ 2x_1x_2) + c_2x_1^2x_3$ | $x_2$ |
| 6 | $x_2^2x_3$ | $-2x_1 - x_2 + 2x_3$ | 14 | $c_3x_2(x_1^2 + x_2^2 + x_3^2 + 2x_1x_2$ $+ 2x_2x_3) + c_4x_1x_2x_3$ | $-x_1 + x_3$ |
| 7 | $x_3^2x_2$ | $-x_1 - 2x_2 + x_3$ | 15 | $c_5x_3(x_1^2 + x_2^2 + x_3^2 + 2x_1x_2$ $+ 2x_2x_3) + c_6x_3^2x_1$ | $-x_2$ |
| 8 | $x_1^2(x_1 + x_2 + x_3)$ | $2x_2$ | | | |

### 5.2. ***Periodic KM-system***

The periodic KM-system ($r = 1$, $s = -1$, $t = 1$) is given with the same equations (1.1) plus a periodicity condition $x_i = x_{i+n}$. In the case $n = 3$ we obtain:

$$\dot{x}_1 = x_1x_2 - x_1x_3$$
$$\dot{x}_2 = -x_1x_2 + x_2x_3$$
$$\dot{x}_3 = x_1x_3 - x_2x_3.$$

We give a different type of Lax pair for this system from [1].

$$L = \begin{pmatrix} 0 & x_1 & 1 \\ 1 & 0 & x_2 \\ x_3 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & x_1x_2 \\ x_2x_3 & 0 & 0 \\ 0 & x_3x_1 & 0 \end{pmatrix}.$$

It follows that the functions $H_i = \frac{1}{i}\mathrm{Tr}\, L^i$ are constants of motion. We note that $H_1 = 0$, $H_2 = x_1 + x_2 + x_3$ and $H_3 = 1 + x_1x_2x_3$. As expected the function $H_2 = x_1 + x_2 + x_3$ plays the role of the Hamiltonian with respect to the Poisson bracket (1.3) while $F = x_1x_2x_3$ is a Casimir.

In the following Tables 4–6, we present all Darboux polynomials of degree $\leq 3$ and the corresponding cofactors.

Table 4. Linear Darboux polynomials and corresponding cofactors.

| | Darboux polynomial | Cofactor |
|---|---|---|
| 1 | $x_1$ | $x_2 - x_3$ |
| 2 | $x_2$ | $-x_1 + x_3$ |
| 3 | $x_3$ | $x_1 - x_2$ |
| 4 | $x_1 + x_2 + x_3$ | 0 |

Table 5. Quadratic Darboux polynomials and corresponding cofactors.

| | Darboux polynomial | Cofactor | | Darboux polynomial | Cofactor |
|---|---|---|---|---|---|
| 1 | $x_1^2$ | $2x_2 - 2x_3$ | 6 | $x_2x_3$ | $-x_2 + x_3$ |
| 2 | $x_2^2$ | $-2x_1 + 2x_3$ | 7 | $x_1(x_1 + x_2 + x_3)$ | $x_2 - x_3$ |
| 3 | $x_3^2$ | $2x_1 - 2x_2$ | 8 | $x_2(x_1 + x_2 + x_3)$ | $-x_1 + x_3$ |
| 4 | $x_1x_2$ | $-x_1 + x_2$ | 9 | $x_3(x_1 + x_2 + x_3)$ | $x_1 - x_2$ |
| 5 | $x_1x_3$ | $x_1 - x_3$ | 10 | $(x_1 + x_2 + x_3)^2$ | 0 |

Table 6. Cubic Darboux polynomials and corresponding cofactors; $c_1$, $c_2$ are constants.

| | Darboux polynomial | Cofactor | | Darboux polynomial | Cofactor |
|---|---|---|---|---|---|
| 1 | $x_1^3$ | $3x_2 - 3x_3$ | 11 | $x_2(x_1 + x_2 + x_3)^2$ | $-x_1 + x_3$ |
| 2 | $x_2^3$ | $-3x_1 + 3x_3$ | 12 | $x_3(x_1 + x_2 + x_3)^2$ | $x_1 - x_2$ |
| 3 | $x_3^3$ | $3x_1 - 3x_2$ | 13 | $x_1^2(x_1 + x_2 + x_3)$ | $2x_2 - 2x_3$ |
| 4 | $x_1^2x_2$ | $-x_1 + 2x_2 - x_3$ | 14 | $x_2^2(x_1 + x_2 + x_3)$ | $-2x_1 + 2x_3$ |
| 5 | $x_1^2x_3$ | $x_1 + x_2 - 2x_3$ | 15 | $x_3^2(x_1 + x_2 + x_3)$ | $2x_1 - 2x_2$ |
| 6 | $x_2^2x_1$ | $-2x_1 + x_2 + x_3$ | 16 | $x_1x_2(x_1 + x_2 + x_3)$ | $-x_1 + x_2$ |
| 7 | $x_2^2x_3$ | $-x_1 - x_2 + 2x_3$ | 17 | $x_1x_3(x_1 + x_2 + x_3)$ | $x_1 - x_3$ |
| 8 | $x_3^2x_1$ | $2x_1 - x_2 - x_3$ | 18 | $x_2x_3(x_1 + x_2 + x_3)$ | $-x_2 + x_3$ |
| 9 | $x_3^2x_2$ | $x_1 - 2x_2 + x_3$ | 19 | $x_1x_2x_3$ | 0 |
| 10 | $x_1(x_1 + x_2 + x_3)^2$ | $x_2 - x_3$ | 20 | $c_1(x_1^3 + x_2^3 + x_3^3 + 3x_1^2x_2 + 3x_1^2x_3$ $+ 3x_2^2x_1 + 3x_2^2x_3 + 3x_3^2x_1$ $+ 3x_3^2x_2) + c_2x_1x_2x_3$ | 0 |

### 5.3. *The case $s = t$ $(r = 5, s = t = 1)$*

$$\begin{aligned} \dot{x}_1 &= 5x_1x_2 + x_1x_3 \\ \dot{x}_2 &= -5x_1x_2 + x_2x_3 \\ \dot{x}_3 &= -x_1x_3 - x_2x_3. \end{aligned}$$

Table 7. Linear Darboux polynomials and corresponding cofactors.

| | Darboux polynomial | Cofactor |
|---|---|---|
| 1 | $x_1 + x_2$ | $x_3$ |

Table 8. Quadratic Darboux polynomials and corresponding cofactors.

| | Darboux polynomial | Cofactor | | Darboux polynomial | Cofactor |
|---|---|---|---|---|---|
| 1 | $x_1(x_1 + x_2)$ | $5x_2 + 2x_3$ | 4 | $(x_1 + x_2)^2$ | $2x_3$ |
| 2 | $x_2(x_1 + x_2)$ | $-5x_1 + 2x_3$ | 5 | $(x_1 + x_2)(x_1 + x_2 + x_3)$ | $x_3$ |
| 3 | $x_3(x_1 + x_2)$ | $-x_1 - x_2 + x_3$ | — | — | — |

Table 9. Cubic Darboux polynomials and corresponding cofactors.

| | Darboux polynomial | Cofactor | | Darboux polynomial | Cofactor |
|---|---|---|---|---|---|
| 1 | $(x_1 + x_2)^3$ | $3x_3$ | 9 | $x_1x_3(x_1 + x_2)$ | $-x_1 + 4x_2 + 2x_3$ |
| 2 | $x_1(x_1 + x_2)^2$ | $5x_2 + 3x_3$ | 10 | $x_2x_3(x_1 + x_2)$ | $-6x_1 - x_2 + 2x_3$ |
| 3 | $x_2(x_1 + x_2)^2$ | $-5x_1 + 3x_3$ | 11 | $x_1(x_1 + x_2)(x_1 + x_2 + x_3)$ | $5x_2 + 2x_3$ |
| 4 | $x_3(x_1 + x_2)^2$ | $-x_1 - x_2 + 2x_3$ | 12 | $x_2(x_1 + x_2)(x_1 + x_2 + x_3)$ | $-5x_1 + 2x_3$ |
| 5 | $x_1^2(x_1 + x_2)$ | $10x_2 + 3x_3$ | 13 | $x_3(x_1 + x_2)(x_1 + x_2 + x_3)$ | $-x_1 - x_2 + x_3$ |
| 6 | $x_2^2(x_1 + x_2)$ | $-10x_1 + 3x_3$ | 14 | $(x_1 + x_2)(x_1 + x_2 + x_3)^2$ | $x_3$ |
| 7 | $x_3^2(x_1 + x_2)$ | $-2x_1 - 2x_2 + x_3$ | 15 | $(x_1 + x_2)^2(x_1 + x_2 + x_3)$ | $2x_3$ |
| 8 | $x_1x_2(x_1 + x_2)$ | $-5x_1 + 5x_2 + 3x_3$ | | | |

We list in Tables 7–9 all linear, quadratic, and cubic Darboux polynomials of the above system which do not have the form (3.11), and their corresponding cofactors.

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