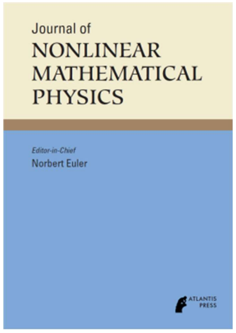


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# A NEW CLASS OF SYMMETRY REDUCTIONS FOR PARAMETER IDENTIFICATION PROBLEMS

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This paper introduces a new type of symmetry reductions called *extended nonclassical symmetries* that can be studied for parameter identification problems described by partial differential equations. Including the data function in the parameter space, we show that specific data and parameter classes that lead to a reduced dimension model can be found. More exactly, since the extended nonclassical symmetries relate the forward and inverse problems, the dimension of the studied equation may be reduced by expressing the data and parameter in terms of the group invariants. The main advantage of these new symmetries is that they may be incorporated into the boundary conditions as well, and, consequently, the dimension reduction problem can be analyzed on new types of domains. Special group-invariant solutions or additional information on the parameter can be obtained. Besides, in the case of the first-order partial differential equations, this symmetry reduction method might be an effective alternative tool for finding particular analytical solutions to the studied model, especially when the Maple subroutine `pdsolve` does not output satisfactory results. As an example, we consider the nonlinear stationary heat conduction equation. Our `MAPLE` routine `GENDEFNC` which uses the package `DESOLV` (authors Carminati and Vu) has been updated for this propose and its output is the nonlinear partial differential equation system of the determining equations of the extended nonclassical symmetries.

*Keywords*: Lie groups of transformations; classical symmetries; nonclassical symmetries; nonlinear partial differential equations; parameter identification problems.

## 1. Introduction

One of the fastest developing research fields in the last few years is the area of inverse problems. These problems occur frequently in engineering, mathematics, and physics. In particular, parameter identification problems deal with the identification of physical parameters from observations of the evolution of a system. They especially arise when the physical laws governing the processes are known, but the information about the parameters occurring in equations is needed. In general, these are ill-posed problems, in the sense that they do not fulfill Hadamard's postulates for all admissible data: a solution exists, the solution is unique, and the solution depends continuously on the given data. Arbitrary small changes in data may lead to arbitrary large changes in the solution (see, for example, [14] for more details). In this paper, we study the class of parameter identification problems modeled by partial differential equations (PDEs) of the form

$$F(x, w^{(m)}, E^{(n)}) = 0, \tag{1.1}$$

where the unknown function $E = E(x)$ called *parameter*, and the arbitrary function $w = w(x)$ called *data* are defined on a domain $\Omega \subset R^p$ (here $w^{(m)}$ denotes the function $w$ together with its partial derivatives up to order $m$. We shall assume that the parameter and data are analytical functions. The PDE (1.1) sometimes augmented with certain boundary conditions is called *the inverse problem* associated with a *direct* (*or forward*) *problem*. The direct problem is the same equation but the unknown function is the data (for which certain boundary conditions are imposed as well). In general, the parameter must be positive and satisfy additional conditions [14].

Symmetry analysis theory has been widely used to study nonlinear PDEs. A remarkable number of mathematical physics models have been successfully analyzed from this point of view. There is a considerable body of literature on this topic (see, for example, [1, 5–8, 12, 13, 17–19, 21–25, 27–29] and references from there). The notion of continuous groups of transformations (known today as *Lie groups of transformations*) was introduced by Sophus Lie [21] who also applied them to the study of differential equations. A *classical Lie symmetry* associated with a PDE is a (local) Lie group of transformations acting on the space of the independent and dependent variables of the equation with the property that it leaves the set of its solutions invariant. Additionally, the form or the equation remains unchanged and special types of group-invariant solutions can be obtained. Lie's method has been applied extensively to various mathematical models described by PDEs and it has been proven to be a powerful method for finding exact solutions to nonlinear PDEs. Subsequently, over the years, other methods for seeking explicit solutions to nonlinear PDEs that cannot be obtained by applying Lie's method have been developed. For instance, Bluman and Cole [5] introduced the nonclassical method which may lead to new classes of solutions but this depends on the differential structure of the equation. A *nonclassical symmetry* (or *conditional symmetry*) is a (local) group transformations that acts on the space of the independent and dependent variables of the equation with the property that it leaves only a subset of the set of all analytical solutions invariant. Any classical symmetry is a nonclassical symmetry but not conversely. The classical and nonclassical symmetries can be used to reduce the dimension of a PDE by rewriting it in terms of the group invariants. Therefore, special types of group-invariant solutions may be obtained. Sometimes the group-invariant solution obtained from the reduced equation cannot be found explicitly but, even in this case, one can obtain additional information on the studied model by applying symmetry reduction methods.

The aim of this paper is to introduce a class of symmetry reductions for parameter identification problems of the form (1.1) that lead to a reduced dimension model. These transformations, called *extended nonclassical symmetries*, apply to the case when the targeted data is known or belongs to a specific class, such as the parameter identification problem discussed in [3]. By including the data in the parameter space, we can find (or even predict) specific classes of data for which the dimension of the equation can be reduced. Moreover, this technique will allow us to incorporate noninvariant boundary conditions in invariant solutions extending the common case when the data and the bounded domain $\Omega$ are invariant under the same symmetry reduction. The idea of including the arbitrary functions in the set of the dependent variables (or in the set of independent variables if these are constant) appears in the literature, for instance, in [7] and [23]. Surprisingly, the extended classical symmetries have not been extensively exploited in the context of the symmetry reduction methods. In [3], the problem of finding different types of symmetry reductions for parameter identification problems of type (1.1) has been discussed in details and, in particular, the equivalence transformations associated with a mathematical model arising in car windshield design was analyzed (more details on equivalence transformations can be found in [22] and [25]). In this paper we will consider symmetry reductions associated with (1.1) regarded as a PDE in two unknown functions: the parameter and the data. Therefore, Eq. (1.1) becomes a nonlinear PDE in $w$ and $E$. The group transformations obtained by applying the classical Lie method to (1.1) in which both $w$ and $E$ are unknown functions will be called *full or extended classical symmetries* (we shall name them differently to distinguish them from the classical symmetries associated with (1.1) in which the unknown function is only $E$). Similarly, the symmetry reductions obtained by applying the nonclassical method to (1.1) in which both $w$ and $E$ are viewed as unknown functions will be called *extended nonclassical symmetries*. The latter group transformations also relate the forward and inverse problems as the extended classical symmetries do. Notice that the extended classical symmetries are solutions to an overdetermined linear PDE system while the extended nonclassical symmetries are found by solving an overdetermined nonlinear PDE system (with less equations that the first system). Any extended classical symmetry will be an extended nonclassical symmetry, but not conversely. To the best of our knowledge, these two types of transformations have not been studied so far in connection to parameter identification problems or PDEs depending on arbitrary functions. The extended classical symmetries and their relationship with the equivalence transformations have been recently studied in [4] and [23]. Since a large amount of calculations are required, we have updated our Maple package `GENDEFNC` [2] which uses the package `DESOLV` by Carminati and Vu [9]. The `GENDEFNC` output is the nonlinear PDE system of the determining equations of the extended nonclassical symmetries. Notice that `GENDEFNC` is based on a new method for finding nonclassical symmetries which has been extended recently by Bruzón and Gandarias [8].

To exemplify the new type of symmetry reductions, we shall consider a mathematical model arising in heat conduction, namely, the nonlinear stationary heat conduction equation given by

$$-\mathrm{div}(E(x,y)\nabla w(x,y)) = 1 \quad \text{in } \Omega, \tag{1.2}$$

where the unknown function $E = E(x,y)$ is the *parameter*, the arbitrary function $w = w(x,y)$ is the *data*, and $(x,y) \in \Omega$ with $\Omega \subset R^2$ a bounded domain (here $\nabla w = (w_x, w_y)$ is the gradient of $w$). The data function must also satisfy the Dirichlet boundary condition

$$w|_{\partial\Omega} = 0. \tag{1.3}$$

In 3D, the above problem is related to the heat conduction in a material occupying a domain $\Omega$ whose temperature is kept zero at the boundary [14]. After sufficiently long time, the temperature distribution $w$ can be modeled by

$$-\mathrm{div}(E(x,y,z)\nabla w(x,y,z)) = f(x,y,z) \quad \text{in } \Omega, \tag{1.4}$$

where $E$ is the heat conductivity and $f$ represents the heat sources. For given $E$ and $f$, the forward problem is to find the temperature distribution $w$ satisfying (1.4) and (1.3). Conversely, the inverse problem is to determine $E$ from (1.4) and (1.3) when $w$ is known. While the direct problem is an elliptic PDE for $w$, the inverse problem is a linear PDE (with variable coefficients) for $E$. On the other hand, in the inverse problems approach, if the solution of the forward problem is unique for each parameter $E$, the parameter-to-solution map associates with each parameter $E$ the forward problem solution. Since for the above problem the parameter-to-output map is nonlinear, (1.4) and (1.3) is a nonlinear problem from this point of view. In addition, when $w = w(E)$, a new dependent variable can be introduced by using the Kirchoff transformation [30, p. 113]; if the heat conductivity $E = 1$, then Eq. (1.4) becomes Poisson's equation [15, p. 316]. For simplicity, in this paper, we discuss the 2D case with the heat sources $f = 1$, i.e.,

$$w_x E_x + w_y E_y + E(w_{xx} + w_{yy}) = -1. \tag{1.5}$$

Notice that, even though (1.5) is a linear PDE for $E$, the solution of the equation cannot be found at a point $(x_0, y_0)$ at which the partial derivatives of $w$ are zero.

The goal of our work is to apply symmetry reductions to inverse problems. The connection of these two research fields is of current interest. In [3], a parameter identification problem arising in industrial mathematics has been analyzed from the point of view of symmetry reduction theory. Equation (1.5) that is considered in this paper merely serves as an example chosen to explain the new class of symmetry reductions. In fact, this equation can be solved theoretically by using the classical method of characteristics. However, solving the characteristic equations related to a quasilinear PDE is not an easy task, especially if these are given by nonlinear ODEs. In this latter case, the output of the `Maple` PDE-solver `pdsolve` — which is the general solution of the input PDE — may be almost unreadable or even null. As we explain in the last section, alternatively, this problem might be overcome by seeking particular solutions invariant with respect to this new kind of symmetry reductions related to (1.5). Nevertheless, it would be of further interest to apply these symmetry reductions to other similar parameter identification problems modeled by PDEs.

The paper is organized as follows. The extended classical symmetries related to (1.5) are presented in Sec. 2 and the extended nonclassical symmetries are discussed in Sec. 3. We show that the extended nonclassical symmetries are related to the Monge equation (3.19), the Monge-Ampère equation (3.22), and the Abel ordinary differential equations (ODEs) of second kind (3.29) and (3.37). The new symmetry reductions related to (1.5) are given by (3.27), (3.30), (3.35), (3.38), and, respectively, (3.31) and (3.39) excepting the case when $A(x,y) = (k_1 - k_3 y + k_4 x)/(k_2 + k_3 x + k_4 y)$. In Sec. 4, several domains and data for which the PDE (1.5) can be reduced are discussed along with a class of data that cannot be handled successfully by the `Maple` subroutine `pdsolve`.

## 2. Extended Classical Symmetries Related to (1.5)

Let us consider a one-parameter Lie group of transformations acting on an open set $\mathcal{D} \subset \Omega \times \mathcal{W} \times \mathcal{E}$, where $\mathcal{W}$ is the space of the data functions, and $\mathcal{E}$ is the space of the parameter functions, given by

$$\begin{cases} \tilde{x} = x + \varepsilon\Gamma(x,y,w,E) + \mathcal{O}(\varepsilon^2), \\ \tilde{y} = y + \varepsilon\Lambda(x,y,w,E) + \mathcal{O}(\varepsilon^2), \\ \tilde{w} = w + \varepsilon\Phi(x,y,w,E) + \mathcal{O}(\varepsilon^2), \\ \tilde{E} = E + \varepsilon\Psi(x,y,w,E) + \mathcal{O}(\varepsilon^2), \end{cases} \tag{2.1}$$

where $\varepsilon$ is the group parameter. Let

$$\mathcal{V} = \Gamma(x,y,w,E)\partial_x + \Lambda(x,y,w,E)\partial_y + \Phi(x,y,w,E)\partial_w + \Psi(x,y,w,E)\partial_E \tag{2.2}$$

be its associated general infinitesimal generator. Assume that $E = E(x,y)$ and $w = w(x,y)$ are both dependent variables in (1.5). The transformation (2.1) is called an *extended* (*or full*) *classical symmetry* related to the PDE (1.5) if it leaves Eq. (1.5) invariant, i.e., $\tilde{w}_{\tilde{x}}\tilde{E}_{\tilde{x}} + \tilde{w}_{\tilde{y}}\tilde{E}_{\tilde{y}} + \tilde{E}(\tilde{w}_{\tilde{x}\tilde{x}} + \tilde{w}_{\tilde{y}\tilde{y}}) = -1$. Note that the set of all analytical solutions of (1.5) will also be invariant. Thus, the criterion for infinitesimal invariance is given by

$$\mathrm{pr}^{(2)}\mathcal{V}(F)|_{F=0} = 0,$$

with $F(x,y,w^{(2)},E^{(1)}) = w_x E_x + w_y E_y + E(w_{xx} + w_{yy}) + 1$, where $\mathrm{pr}^{(2)}\mathcal{V}$ denotes the second order prolongation of the vector field $\mathcal{V}$ [24]. Observe that this prolongation is determined by taking into account that $E$ and $w$ are both dependent variables, exactly as one would proceed in finding the classical Lie symmetries for a PDE without arbitrary functions. The order of the prolongation of the vector field $\mathcal{V}$ is given by the highest leading derivative of the dependent variables. Applying the classical Lie method, we obtain the following infinitesimals

$$\begin{cases} \Gamma(x,y,w,E) = k_1 - k_3 y + k_4 x, \\ \Lambda(x,y,w,E) = k_2 + k_3 x + k_4 y, \\ \Phi(x,y,w,E) = \mu(w), \\ \Psi(x,y,w,E) = E(2k_4 - \mu'(w)), \end{cases} \tag{2.3}$$

where $k_i$, $i = 1, \ldots, 4$ are real constants and $\mu = \mu(w)$ is an arbitrary function. Hence, the infinitesimal generator (2.2) becomes

$$\mathcal{V} = \sum_{i=1}^{4} k_i \mathcal{V}_i + \mathcal{V}_\mu, \tag{2.4}$$

where

$$\mathcal{V}_1 = \partial_x, \quad \mathcal{V}_2 = \partial_y, \quad \mathcal{V}_3 = -y\partial_x + x\partial_y,$$
$$\mathcal{V}_4 = x\partial_x + y\partial_y + 2E\partial_E, \quad \mathcal{V}_\mu = \mu(w)\partial_w - E\mu'(w)\partial_E.$$

We obtain the following result.

**Proposition 1.** *There is an infinite dimensional Lie algebra of the extended classical symmetries related to* (1.5) *spanned by the infinitesimal generators* (2.4).

Therefore, the PDE (1.5) is invariant under translations in the $x$-space, $y$-space, rotations in the $(x, y)$-space, and, respectively, scaling transformations in the $(x, y, E)$-space. For instance, we find that (1.5) is invariant under translations in $w$-space if we choose $\mu =$ const., and that the equation remains unchanged under scaling transformations in $(w, E)$-space if we choose $\mu(w) = w$.

Furthermore, an extended classical symmetry (2.1) can be used to reduce the dimension of (1.5) by augmenting this equation with

$$\begin{cases} \Gamma(x,y,w,E)w_x + \Lambda(x,y,w,E)w_y - \Phi(x,y,w,E) = 0, \\ \Gamma(x,y,w,E)E_x + \Lambda(x,y,w,E)E_y - \Psi(x,y,w,E) = 0, \end{cases}$$

which is a first order PDE system defining the characteristics of the vector field (2.2). The above relations are also called *invariance surface conditions*.

## 3. Extended Nonclassical Symmetries Related to (1.5)

Consider a one-parameter Lie group of transformations acting on an open set $\mathcal{D} \subset \Omega \times \mathcal{W} \times \mathcal{E}$, where $\mathcal{W}$ is the space of the data functions, and $\mathcal{E}$ is the space of the parameter functions, given by

$$\begin{cases} \tilde{x} = x + \varepsilon\xi(x,y,w,E) + \mathcal{O}(\varepsilon^2), \\ \tilde{y} = y + \varepsilon\eta(x,y,w,E) + \mathcal{O}(\varepsilon^2), \\ \tilde{w} = w + \varepsilon\phi(x,y,w,E) + \mathcal{O}(\varepsilon^2), \\ \tilde{E} = E + \varepsilon\psi(x,y,w,E) + \mathcal{O}(\varepsilon^2), \end{cases} \tag{3.1}$$

where $\varepsilon$ is the group parameter. Let the following vector field

$$\mathcal{U} = \xi(x,y,w,E)\partial_x + \eta(x,y,w,E)\partial_y + \phi(x,y,w,E)\partial_w + \psi(x,y,w,E)\partial_E \tag{3.2}$$

be the general infinitesimal generator related to (3.1). The transformation (3.1) is called an *extended nonclassical symmetry* associated with the PDE (1.5) if this leaves the subset

$$S_{F,\phi_1,\phi_2} = \{F(x,y,w^{(2)},E^{(2)}) = 0, \quad \phi_1(x,y,w^{(1)},E^{(1)}) = 0, \quad \phi_2(x,y,w^{(1)},E^{(1)}) = 0\}$$

of the set of all analytical solutions invariant, where

$$\begin{cases} \phi_1 := \xi(x,y,w,E)w_x + \eta(x,y,w,E)w_y - \phi(x,y,w,E) = 0, \\ \phi_2 := \xi(x,y,w,E)E_x + \eta(x,y,w,E)E_y - \psi(x,y,w,E) = 0 \end{cases} \tag{3.3}$$

represents the characteristics of the vector field $\mathcal{U}$ (or the *invariant surface conditions*). Here the criterion for infinitesimal invariance is the following

$$\begin{cases} \mathrm{pr}^{(2)}\mathcal{U}(F)|_{F=0,\phi_1=0,\phi_2=0} = 0, \\ \mathrm{pr}^{(1)}\mathcal{U}(\phi_1)|_{F=0,\phi_1=0,\phi_2=0} = 0, \\ \mathrm{pr}^{(1)}\mathcal{U}(\phi_2)|_{F=0,\phi_1=0,\phi_2=0} = 0. \end{cases}$$

If $\eta \neq 0$, one can assume without loss of generality that $\eta = 1$ (the case $\eta = 0$ is not discussed in this paper), and, hence, (3.3) turns into

$$\begin{cases} w_y = \phi(x,y,w,E) - \xi(x,y,w,E)w_x, \\ E_y = \psi(x,y,w,E) - \xi(x,y,w,E)E_x. \end{cases} \tag{3.4}$$

At the first step, we augment the original PDE with (3.4) and eliminate all the partial derivatives of $w$ and $E$ with respect to $y$ occurring in (1.5). Hence, by using (3.4) and its differential consequences, we obtain

$$\mathcal{A}_1 w_{xx} + \mathcal{A}_2 w_x^2 + \mathcal{A}_3 w_x E_x + \mathcal{A}_4 w_x + \mathcal{A}_5 E_x + \mathcal{A}_6 = 0, \tag{3.5a}$$

where the coefficients $\mathcal{A}_i = \mathcal{A}_i(x,y,w,E)$, $i = 1\ldots 6$, are the following

$$\begin{aligned} \mathcal{A}_1 &= E(\xi^2 + 1), \\ \mathcal{A}_2 &= 2E\xi\xi_w, \\ \mathcal{A}_3 &= \xi^2 + 2E\xi\xi_E + 1, \\ \mathcal{A}_4 &= -\xi\psi + E(\xi\xi_x - \xi_y - \phi\xi_w - \psi\xi_E - 2\xi\phi_w), \\ \mathcal{A}_5 &= -\xi(\phi + 2E\phi_E), \\ \mathcal{A}_6 &= \phi\psi + 1 - E(\xi\phi_x - \phi_y - \phi\phi_w - \psi\phi_E). \end{aligned} \tag{3.5b}$$

Equation (3.5) has been obtained by using the `GENDEFNC` command

```
gendefnc(PDE, [w,E], [x,y], y, 3).
```

Since $\mathcal{A}_1 \neq 0$, (3.5) may be regarded as an ODE in the unknown functions $w$ and $E$ (with $y$ as a parameter). At the second step, by using the `GENDEFNC` command

```
gendefnc(PDE, [w,E], [x,y], y)
```

we obtain the *determining equations* of the extended nonclassical symmetries. This is an overdetermined nonlinear PDE system for the infinitesimals $\xi = \xi(x,y,w,E)$ and $\phi = \phi(x,y,w,E)$. Among these equations, we get $\xi_w = 0$, $\xi_E = 0$, and $\phi_E = 0$ which implies

$$\xi(x,y,w,E) = A(x,y), \tag{3.6}$$

and

$$\phi(x,y,w,E) = G(x,y,w), \tag{3.7}$$

where $A$ and $G$ are arbitrary functions. The substitution of the above functions into the remaining equations yields

$$\psi(x,y,w,E) = EF(x,y,w), \tag{3.8}$$

with $F$ an arbitrary function of its arguments. By using the above relations, the determining system is reduced to

$$F = -G_w + \frac{2(A_x - AA_y)}{A^2+1}, \tag{3.9}$$

$$G_x - AG_y - \frac{2AA_x - A_y(A^2-1)}{A^2+1}G = 0, \tag{3.10}$$

$$\begin{aligned} &G_{xx} + G_{yy} + F_yG + FG_y + 2GG_{yw} \\ &\quad + \frac{2(A_x - AA_y)}{A^2+1}(GG_w + G_y + FG) - \frac{2(AA_x + A_y)}{A^2+1}G_x = 0, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &(A^4-1)A_{xx} + 4A(A^2+1)A_{xy} - (A^4-1)A_{yy} - 2A(A^2-3)A_x^2 \\ &\quad - 4(3A^2-1)A_xA_y + 2A(A^2-3)A_y^2 = 0, \end{aligned} \tag{3.12}$$

where $A = A(x,y)$, $G = G(x,y,w)$, and $F = F(x,y,w)$ are the unknown functions.

**Remark.** The extended nonclassical symmetries do not leave the form of Eq. (1.5) invariant. Moreover, the nonclassical operators (3.2) do not form a vector space, still less a Lie algebra, as the symmetry operators do. Since every classical symmetry is a nonclassical symmetry but not conversely, there exists a set of common solutions of the determining system of the extended nonclassical symmetries and the determining system of the extended classical symmetries. This common solution is given by

$$\begin{cases} \xi(x,y,w,E) = \dfrac{\Gamma(x,y,w,E)}{\Lambda(x,y,w,E)} = \dfrac{k_1 - k_3y + k_4x}{k_2 + k_3x + k_4y}, \\ \phi(x,y,w,E) = \dfrac{\Phi(x,y,w,E)}{\Lambda(x,y,w,E)} = \dfrac{\mu(w)}{k_2 + k_3x + k_4y}, \\ \psi(x,y,w,E) = \dfrac{\Psi(x,y,w,E)}{\Lambda(x,y,w,E)} = \dfrac{E(2k_4 - \mu'(w))}{k_2 + k_3x + k_4y}, \end{cases} \tag{3.13}$$

where at least one of the constants $k_2$, $k_3$ or $k_4$ is nonzero.

To solve the determining equations (3.9)–(3.12), we proceed as follows: Eqs. (3.10) and (3.11) are analyzed in Subsec. 3.1, Eq. (3.12) is studied in Subsec. 3.2, and the solutions of the determining equations (3.9)–(3.12) are given in Subsec. 3.3.

### 3.1. *The infinitesimal* $\phi = G(x,y,w)$

First observe that (3.10) can be written in the following conservation form

$$\left(\frac{G}{A^2+1}\right)_x - \left(\frac{AG}{A^2+1}\right)_y = 0.$$

If $A \not\equiv 0$, then two cases may occur:

**Case G1.** Suppose $G \equiv 0$. In this case, (3.10) and (3.11) are both satisfied.

**Case G2.** If $G \not\equiv 0$, then there exists a potential function $K = K(x,y,w)$ such that $K_x = AG/(A^2+1)$ and $K_y = G/(A^2+1)$. These equations yields $K_x = AK_y$ whose general solution is $K(x,y,w) = P(u,w)$, where $P$ is an arbitrary function and $u = u(x,y)$ is a solution of the equation

$$u_x = A(x,y)u_y. \tag{3.14}$$

From these relations we obtain

$$G = u_y(A^2+1)S, \tag{3.15}$$

where $S(u,w)$ denotes the partial derivative $P_u(u,w)$. Thus, the solution of (3.10) is given by (3.15), where $u$ satisfies (3.14).

The substitution of (3.15) into (3.11) implies

$$q_1(S_{uw}S - S_uS_w + S_{uu}) + q_2S_u + q_3S = 0, \tag{3.16}$$

where the coefficients $q_i$ are expressed in terms of $A$ and $u_y$ as follows

$$\begin{aligned}
q_1 &= u_y^3(A^2+1)^3,\\
q_2 &= u_y(A^2+1)[3u_{yy}(A^2+1)^2 + u_y(5A_x + 3A^3A_y + 3A^2A_x + AA_y)],\\
q_3 &= u_{yyy}(A^2+1)^3 + u_{yy}(A^2+1)[(3A^2+5)A_x + (3A^2+1)AA_y]\\
&\quad + u_y[2A(A^2+1)A_{xx} + (A^2+1)(A^2+3)A_{xy} + A(A^2+1)^2A_{yy}\\
&\quad - 2(A^2-3)A_x^2 + 2A(A^2-3)A_xA_y + (A^4-1)A_y^2].
\end{aligned}$$

In particular, the method of separation of variables applied to (3.16) implies solutions of the form $S(u,w) = p(u)\mu(w)$, where $p = p(u)$ satisfies the equation

$$q_1p_{uu} + q_2p_u + q_3p = 0 \tag{3.17}$$

and $\mu = \mu(w)$ is an arbitrary function of its argument.

To summarize, Eq. (3.11) has been reduced to (3.16), where $A = A(x,y)$ satisfies (3.12) and $u = u(x,y)$ is a solution of (3.14).

### 3.2. *The infinitesimal* $\xi = A(x,y)$

In the following, we show that (3.12) can be reduced to a Monge–Ampère equation.

**Case A1.** For $A \equiv 0$, we obtain the trivial solution to (3.12).

**Case A2.** If $A = k$, with $k \neq 0$, then the constant solution to (3.12) is found.

**Case A3.** Assume $A$ is a nonconstant function. Then (3.12) can be rewritten as

$$\begin{aligned}
&B(B^2+1)B_{xx} + (B^2-1)(B^2+1)B_{xy} - B(B^2+1)B_{yy}\\
&\quad -(3B^2-1)B_x^2 - 2B(B^2-3)B_xB_y + (3B^2-1)B_y^2 = 0,
\end{aligned}$$

where $B = (A+1)/(A-1)$. The conservation form of the above PDE is

$$\left(\frac{B(B_x+BB_y)}{(B^2+1)^2}\right)_x - \left(\frac{B_x+BB_y}{(B^2+1)^2}\right)_y = 0. \tag{3.18}$$

We distinguish the following two cases:

**Case A3.1.** If $B = B(x, y)$ is a solution of the following Monge equation

$$B_x + BB_y = 0, \tag{3.19}$$

then (3.18) holds. Since the general solution of (3.19) is given implicitly by $y - xB = \nu(B)$, where $\nu$ is an arbitrary function, the corresponding solution of (3.12) is

$$y - x\frac{A+1}{A-1} = \nu\left(\frac{A+1}{A-1}\right). \tag{3.20}$$

**Case A3.2.** $B = B(x, y)$ does not satisfy (3.19). It follows from (3.18) that there exists a potential function $T = T(x, y)$ such that

$$\begin{cases} T_x = \dfrac{B_x + BB_y}{(B^2+1)^2}, \\ T_y = \dfrac{B(B_x + BB_y)}{(B^2+1)^2}. \end{cases} \tag{3.21}$$

Since $T_x$ cannot be identically zero, the above equations yield $B = T_y/T_x$ and, by substituting it into the first equation of (3.21) we have

$$T_xT_yT_{xx} - (T_x^2 - T_y^2)T_{xy} - T_xT_yT_{yy} + (T_x^2 + T_y^2)^2 = 0.$$

By using the following Legendre transformation [30, p. 353], $\mathcal{H}(a, b) + T(x, y) = xa + yb$, where $T_x = a$, $x = \mathcal{H}_a$, $T_y = b$, and $y = \mathcal{H}_b$, the above PDE turns into the following Monge–Ampère equation

$$\mathcal{H}_{aa}\mathcal{H}_{bb} - \mathcal{H}_{ab}^2 - \frac{ab}{(a^2+b^2)^2}\mathcal{H}_{aa} + \frac{a^2-b^2}{(a^2+b^2)^2}\mathcal{H}_{ab} + \frac{ab}{(a^2+b^2)^2}\mathcal{H}_{bb} = 0.$$

Furthermore, this can be reduced to the Monge–Ampère equation

$$V_{aa}V_{bb} - V_{ab}^2 = -\frac{1}{(a^2+b^2)^2}, \tag{3.22}$$

where

$$V(a, b) = \mathcal{H}(a, b) - \frac{1}{2}\arctan\left(\frac{a}{b}\right). \tag{3.23}$$

Special solutions to particular Monge–Ampère PDEs have been extensively analyzed in [27]. The PDE (3.22) may be included in Case 17 [27, p. 458] or Case 20 [27, p. 460].

### 3.3. *The determining equations of the extended nonclassical symmetries*

We distinguish the following four cases:

**Case 1.** $A \equiv 0$ and $G \equiv 0$.

**Proposition 2.** *If* $A \equiv 0$ *and* $G \equiv 0$, *the infinitesimal generator* (3.2) *becomes* $\mathcal{U} = \partial_y$, *which implies the invariance of* (1.5) *with respect to translations in the* $Y$*-space.*

This extended nonclassical symmetry is an extended classical symmetry that can be obtained from (3.13) for $\mu = 0$, $k_2 = 1$, and $k_i = 0$, where $i = 1, 3, 4$.

**Case 2.** Suppose $A \equiv 0$ and $G \not\equiv 0$. The PDE (3.10) takes the form $G_x = 0$ and it follows that $G = H(y, w)$. After substituting it into (3.9) and (3.11), we get $F = -H_w$ and, respectively,

$$H_{yy} + HH_{yw} - H_yH_w = 0, \tag{3.24}$$

where $H$ is an arbitrary function. Notice that $H \not\equiv 0$ in the above equation. The following cases may occur:

**Case 2.1.** Assume $H = \mu(w)$, where $\mu$ is an arbitrary function. Indeed, this is a particular solution to (3.24).

**Proposition 3.** *If* $A \equiv 0$ *and* $G = \mu(w)$, *where* $\mu$ *is an arbitrary function, the infinitesimal generator* (3.2) *turns into*

$$\mathcal{U} = \partial_y + \mu(w)\partial_w - \mu'(w)E\partial_E. \tag{3.25}$$

The above extended nonclassical symmetry is, in fact, an extended classical symmetry and corresponds to the case $k_2 = 1$, and $k_i = 0$, where $i = 1, 3, 4$ in (3.13).

**Case 2.2.** For $H_y \not\equiv 0$, the PDE (3.24) can be written in the following conservation form

$$\left(\frac{1}{H_y}\right)_y + \left(\frac{H}{H_y}\right)_w = 0.$$

After introducing the potential function $g = g(y, w)$, we get $g_y = H/H_y$, and $g_w = -1/H_y$. Clearly, $g_w \not\equiv 0$. After eliminating $H_y$ in these PDEs, we obtain $H = -g_y/g_w$. Next, substituting it into the second equation, the following PDE results

$$g_y(g_y - y)_w - g_w(g_y - y)_y = 0. \tag{3.26}$$

The following two cases occur:

**Case 2.2.1.** $g_y = y$. It results $g(y, w) = y^2/2 + h(w)$, where $h$ is an arbitrary nonconstant function (otherwise, $g_w \equiv 0$). Since $H = -g_y/g_w$, we get $H = y\mu(w)$, where $\mu = -1/h'$.

**Proposition 4.** *If* $A \equiv 0$ *and* $G = y\mu(w)$, *where* $\mu$ *is an arbitrary function, then the infinitesimal generator* (3.2) *becomes*

$$\mathcal{U} = \partial_y + y\mu(w)\partial_w - y\mu'(w)E\partial_E. \tag{3.27}$$

The extended nonclassical symmetry generated by (3.27) is a new symmetry reduction for (1.5) which cannot be obtained from (3.13).

**Case 2.2.2.** $g_y - y \not\equiv 0$. Since Eq. (3.26) is the Jacobian of the functions $g$ and $g_y - y$, there exists a function $\alpha$ such that

$$g_y = y + \alpha(g). \tag{3.28}$$

In the above relation, $w$ is viewed as a parameter. Equation (3.28) can be written as the following Abel ODE of second kind $(y + \alpha(g))dy/dg = 1$, for $y = y(g)$. The canonical substitutions $z = \alpha(g)$ and $v = y + \alpha(g)$ reduce the above ODE to its canonical form

$$vv' - v = \beta(z), \tag{3.29}$$

where $v = v(z)$ and $\beta = 1/(\alpha' \circ \alpha^{-1})$. A collection of the known cases of solvable Abel ODEs of the form (3.37) is presented in [26, pp. 107–120] and new results can be found, for example, in [10]. Each of these ODEs corresponds to an extended nonclassical symmetry related to (1.5), and, therefore, to new symmetry reductions.

**Proposition 5.** *If* $A \equiv 0$ *and* $G = -g_y/g_w$, *where* $g$ *is a solution of* (3.28), *then the infinitesimal generator* (3.2) *turns into*

$$\mathcal{U} = \partial_y - g_y/g_w\partial_w + E(g_y/g_w)_w\partial_E. \tag{3.30}$$

Since $g$ satisfies (3.26), the above vector field generates new symmetry reductions related to (1.5) that are not extended classical symmetries.

**Case 3.** $A \not\equiv 0$ and $G \equiv 0$. By (3.9), we obtain $F = 2(A_x - AA_y)/(A^2+1)$.

**Proposition 6.** *If $A \not\equiv 0$ is a solution of Eq.* (3.12) *and $G \equiv 0$, then the infinitesimal generator* (3.2) *becomes*

$$\mathcal{U} = A(x,y)\partial_x + \partial_y + \frac{2(A_x - AA_y)}{A^2+1}E\partial_E. \tag{3.31}$$

**Case 3.1.** $A = k$, where $k \neq 0$.

**Proposition 7.** *If $A = k$, where $k \neq 0$ is a constant, the infinitesimal generator* (3.31) *rewrites as $\mathcal{U} = k\partial_x + \partial_y$.*

Replacing $\mu = 0$, $k_2 = 1$, and $k_i = 0$ with $i = 1, 3, 4$ in (3.13), we obtain that the symmetry reduction generated by the above vector field is an extended classical symmetry.

**Case 3.2.** If $A$ is a nonconstant function, then we distinguish two subcases:

**Case 3.2.1.** $A$ is given implicitly by (3.20).

**Case 3.2.2.** The equation for $A$ is reduced to the Monge–Ampère equation (3.22).

In the above two cases, the vector field (3.31) generates new symmetry reductions for (1.5) except the case when $A(x,y) = (k_1 - k_3y + k_4x)/(k_2 + k_3x + k_4y)$.

**Case 4.** $A \not\equiv 0$ and $G \not\equiv 0$.

**Case 4.1.** Assume $A = k$, where $k \neq 0$. Without loss of generality, we consider the particular solution $u(x,y) = kx + y$ of Eq. (3.14). Since $u_y = 1$, the relation (3.15) yields $G = (k^2+1)S$, where $S$ is a nontrivial solution of Eq. (3.16) which rewrites as

$$S_{uu} + SS_{uw} - S_uS_w = 0. \tag{3.32}$$

The following cases may occur:

**Case 4.1.1.** Suppose $S = \mu(w)$, where $\mu$ is an arbitrary function. In this case, (3.32) is satisfied and (3.9) implies $F = -\mu'(w)$.

**Proposition 8.** *If $A = k$ with $k \neq 0$ and $G = (k^2+1)\mu(w)$, where $\mu$ is an arbitrary function, then the infinitesimal generator* (3.2) *becomes*

$$\mathcal{U} = k\partial_x + \partial_y + (k^2+1)\mu(w)\partial_w - (k^2+1)\mu'(w)E\partial_E. \tag{3.33}$$

The above nonclassical operator generates an extended classical symmetry that can be obtained from (3.13) for $k_1 = k/(k^2+1)$, $k_2 = 1/(k^2+1)$, $k_3 = 0$, and $k_4 = 0$.

**Case 4.1.2.** Suppose $S_u \not\equiv 0$. Notice that (3.32) can be written in the conservation form

$$\left(\frac{1}{S_u}\right)_u + \left(\frac{S}{S_u}\right)_w = 0,$$

and, hence, there exists a potential function $Q = Q(u,w)$ such that $Q_u = S/S_u$ and $Q_w = -1/S_u$. Eliminating $S$ in the above system, we obtain $S = -Q_u/Q_w$, where

$$Q_u(Q_u - u)_w - Q_w(Q_u - u)_u = 0. \tag{3.34}$$

We distinguish the following two cases:

**Case 4.1.2a.** $Q_u = u$. In this case, $Q(u,w) = u^2/2 + p(w)$, where $p$ is an arbitrary nonconstant function (otherwise $Q_w \equiv 0$). With the aid of $S = -Q_u/Q_w$, we get $S(u,w) = u\mu(w)$, where $\mu = -1/p'$ and $u(x,y) = kx + y$.

**Proposition 9.** *For $A(x,y) = k$ $(k \neq 0)$ and $G(x,y,w) = (k^2+1)(kx+y)\mu(w)$, where $\mu$ is an arbitrary nonconstant function, the infinitesimal generator* (3.2) *rewrites as follows*

$$\mathcal{U} = k\partial_x + \partial_y + (k^2+1)(kx+y)\mu(w)\partial_w - (k^2+1)(kx+y)\mu'(w)E\partial_E. \tag{3.35}$$

The vector field (3.35) generates a new class of symmetry reductions for (1.5) which are not extended classical symmetries.

**Case 4.1.2b.** Assume $Q_u - u \not\equiv 0$. Since (3.34) is the Jacobian of $Q$ and $Q_u - u$, there exists a function $\gamma$ such that

$$Q_u = u + \gamma(Q). \tag{3.36}$$

In the above ODE, $w$ is viewed as a parameter. Similar to Case 2.2.2, Eq. (3.36) can be reduced to $(u + \gamma(Q))du/dQ = 1$, which is an Abel ODE of second kind for $u = u(Q)$. Moreover, after using the substitutions $s = \gamma(Q)$ and $V = u + \gamma(Q)$, the above ODE can be reduced to the canonical form

$$VV' - V = \theta(s), \tag{3.37}$$

where $\theta = 1/(\gamma' \circ \gamma^{-1})$. This is an Abel equation of second kind for the unknown function $V = V(s)$. For each solution of (3.37), we obtain a nonclassical symmetry for (1.5). The solvable Abel ODEs of the form (3.37) that are known so far are listed in [26, pp. 107–120]. More recent results can be found, for instance, in [10].

**Proposition 10.** *Assume $A(x,y) = k$ with $k \neq 0$ and $G(x,y,w) = -(k^2+1)Q_u/Q_w$, where $u(x,y) = kx + y$, and $Q = Q(u,w)$ satisfies* (3.34). *Then the infinitesimal generator* (3.2) *becomes*

$$\mathcal{U} = k\partial_x + \partial_y - (k^2+1)Q_u/Q_w\partial_w + (k^2+1)E(Q_u/Q_w)_w\partial_E. \tag{3.38}$$

Since $Q$ satisfies (3.34), the above vector field generates new symmetry reductions related to (1.5) that are not extended classical symmetries.

**Case 4.2.** Suppose $A$ is a nonconstant function and $G \not\equiv 0$.

**Proposition 11.** *If $A$ is a nonconstant function satisfying Eq.* (3.12) *and $G = u_y(A^2+1)S$, where $u$ satisfies* (3.14) *and $S$ is a nonzero solution of* (3.16), *the infinitesimal generator* (3.2) *is written as*

$$\mathcal{U} = A\partial_x + \partial_y + u_y(A^2+1)S\partial_w + E\left(-u_y(A^2+1)S_w + \frac{2(A_x - AA_y)}{A^2+1}\right)\partial_E. \tag{3.39}$$

Two cases may occur:

**Case 4.2.1.** $A$ is given implicitly by (3.20).

**Case 4.2.2.** The equation for $A$ is reduced to the Monge–Ampère equation (3.22).

In the above two cases, the vector (3.39) generates new symmetry reductions that cannot be obtained from (3.13) except when $A(x,y) = (k_1 - k_3y + k_4x)/(k_2 + k_3x + k_4y)$.

Thus, we have found new symmetry reductions related to (1.5) and have shown that these are given by (3.27), (3.30), (3.35), (3.38), and, respectively, (3.31) and (3.39) except when $A(x,y) = (k_1 - k_3y + k_4x)/(k_2 + k_3x + k_4y)$.

## 4. Conclusion

In this paper we point out another systematic way of finding classes of symmetry reductions related to parameter identification problems of the form (1.1). Similar to the study in [3], we emphasize that the geometrical significance of the nonlinearity occurring between the data and parameter in (1.1) can be reflected by the group analysis tool. Seeking domains (or classes of data) for which the dimension of the problem can be reduced is not an easy task. Therefore, in this paper, we discuss the extended nonclassical symmetries related to (1.5). Briefly, to determine these group transformations, the data is included in the parameter space — a method that has been used before but not in connection with symmetry reductions for parameter identification problems — and then the nonclassical method is applied to the new nonlinear PDE.

For each given data $w$, Eq. (1.5) is a linear PDE in the unknown parameter $E$. The common approach for solving first-order PDEs is the well-known method of characteristics which consists of finding special curves called *characteristic curves* along which the PDE turns into an ODE system. Sometimes, but by no means always, the characteristic curves can be found explicitly and the reduced ODE can be solved, in which case the method of characteristics yields an analytic solution for the PDE. On the other hand, the `Maple pdsolve` subroutine is a powerful tool for finding analytical solutions to PDEs. This PDE solver is part of the `PDEtools` package (authors Cheb-Terrab and Von Bülow) which has been first incorporated in `Maple` in 1997 (see [11] for more details). The `pdsolve` routine currently recognizes a certain number of PDE families that can be solved by using standard methods. For instance, for first-order PDEs, it includes the standard method of characteristics. It should be pointed out that `pdsolve` fails or its output is almost unreadable for some classes of linear or quasilinear PDEs for which the characteristic equations cannot be solved explicitly. Surprisingly, the method of extended nonclassical symmetries can be successfully applied to handle classes of PDEs for which `Maple`'s `pdsolve` output is indecipherable, e.g., when the data given by (4.5). However, this comparison is not completely fair as `pdsolve` searches for the general solution while our method provides only a particular solution. There are other interesting cases of data $w$ in which `pdsolve`'s output is null (e.g., for $w(x, y) = x^3 + y^3 - 3xy$) but this case will be addressed in future work.

Let us summarize the methods of characteristics and of extended nonclassical symmetries in the case of Eq. (1.5). Since a normal vector to the surface $z = E(x, y)$ is $(E_x, E_y, -1)$, the PDE (1.5) is geometrically equivalent to the statement that the vector field $(w_x, w_y, -E\Delta w - 1)$ is tangent to the surface $z = E(x, y)$ at every point. Thus, the graph $z = E(x, y)$ of the solution $E$ is a union of integral curves of the vector field $(w_x, w_y, -E\Delta w - 1)$ (these integral curves are the characteristics of the PDE (1.5)). In this context, the method of extended nonclassical symmetries lies in solving the (quasilinear or linear) PDE system (3.4) and a reduced linear ODE (which is obtained by substituting the solutions of (3.4) into the original PDE). Thus, instead of solving the characteristic equations related to (1.5), we look for the characteristics of the PDEs (3.4) which might be easier to solve than the original PDE. In other words, we seek the infinitesimal generator (3.2) whose projection on the $(x, y)$-plane coincides with the projections of the vector fields $(\xi, \eta, \phi)$ and $(\xi, \eta, \psi)$ on the same plane. The latter two vector fields are tangent to the surfaces $z = w(x, y)$ and $z = E(x, y)$, respectively. Consequently, the surfaces generated by the data $w$ and the parameter $E$ are unions of integral curves of the vector fields $(\xi, \eta, \phi)$ and $(\xi, \eta, \psi)$, respectively.

The extended nonclassical symmetries associated with the PDE (1.5) yield classes of data and suitable domains for which the dimension of the problem can be reduced. Notice that these symmetry reductions generate new families of data that cannot be obtained by using the extended classical symmetry approach. Since the homogeneous Dirichlet boundary condition (1.3) is imposed, the data $w$ must satisfy (3.4) and the boundary $\partial\Omega$ must be "compatible" with the symmetry reduction as well. For instance, data of the form $w(x, y) = -y^2 + W(x)$ is invariant with respect

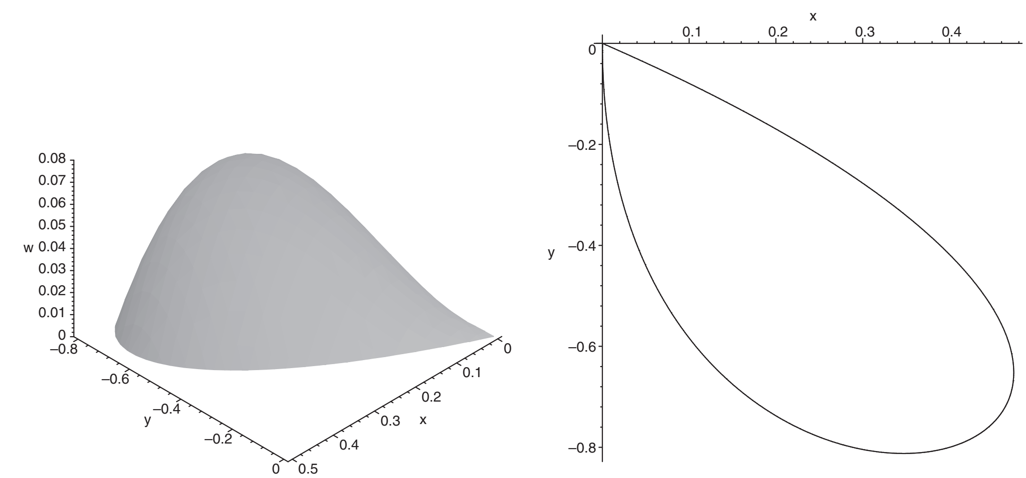


Fig. 1. The graph of the data $w(x,y) = (1/16)(-x^3 + 8y^3 + 6x^2y - 12xy^2 - 30x^2 - 40xy)$ and the boundary $\partial\Omega = \{(x,y)|w(x,y) = 0\}$.

to the nonclassical infinitesimal generator $\mathcal{U} = \partial_y - 2y\partial_w$, which is obtained from (3.27) by setting $\mu(w) = -2$. Next, we obtain that $E(x,y) = \Theta(x)$ and $w(x,y) + y^2 = W(x)$ are constant along the lines $x = \text{const}$. In this case, the PDE (1.5) is reduced to $W'(x)\Theta'(x) + (W''(x) - 2)\Theta(x) = -1$ if the problem is studied on a "compatible" domain whose boundary is, for instance, a circle, an ellipse, a generalized Lamé curve $x^{2p} + y^2 = 1$ $(p > 2)$, a Granville's egg curve $y^2x^2 = (x-b)(1-x)$, where $b \neq 0, 1$, or an elliptical curve — in particular, a Newton's egg curve $y^2 = (x^2-1)(x-a)$, where $a \neq \pm 1$. In all these situations, the parameter cannot be determined at the points $(x_0, y)$ for which $W'(x_0) = 0$. Despite this fact, additional information about the parameter on the corresponding domain can be obtained. Observe that the Maple ODE-solver `dsolve/numeric` cannot be used when solving certain initial value problems associated with specific reduced ODEs. For instance, for $w(x,y) = -y^2 + x^4(x^2-1)$ we obtain the reduced ODE $(6x^5 - 4x^3)\Theta' + (30x^4 - 12x^2 - 2)\Theta = -1$. If this is augmented with the initial condition $\Theta(0) = 1$, then it cannot be solved numerically by applying `dsolve/numeric`. This is due to the fact that zero is a singularity of the reduced ODE and, therefore, the existence and uniqueness theorem for first order initial value problems cannot be applied. However, the analytical solution of the reduced ODE can be found by using the `Maple` ODE-solver `dsolve`.

Next we will analyze a case of data for which the `Maple` `pdsolve` output is indecipherable. Similar results might be obtained when the data is described by high order polynomial functions. Consider the infinitesimal generator (3.35) with $\mu(w) = -1$. According to (3.4), we have

$$w(x,y) = \frac{1-k^4}{2k^2}x^2 - \frac{1+k^2}{k}xy + W(z) \quad \text{and} \quad E(x,y) = \Theta(z), \quad \text{where } z = y - \frac{x}{k}, \tag{4.1}$$

and after substituting them into (1.5), we obtain the following reduced ODE

$$(W'(z) + z)\,\Theta'(z) + \left(W''(z) + 1 - k^2\right)\Theta(z) = -\frac{k^2}{k^2+1} \tag{4.2}$$

whose solution is given by

$$\Theta(z) = C\exp(-U(z)) - \frac{k^2\exp(-U(z))}{1+k^2}\int_{z_0}^{z}\frac{\exp(U(s))}{s+W'(s)}ds, \quad U(z) = \int_{z_0}^{z}\frac{1-k^2+W''(s)}{s+W'(s)}ds \tag{4.3}$$

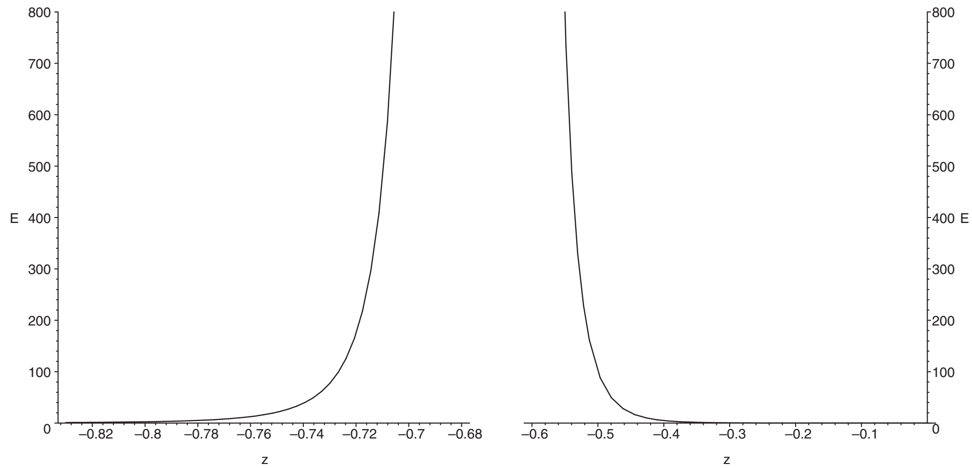


Fig. 2. The graph of the parameter $E(x, y) = \Theta(z)$ with $\Theta(1/2) = 100$ and $\Theta(5/7) = 300$.

where $C$ is a constant. For instance, for $k = 2$ and $W(z) = (1/2)z^3$, the PDE (1.5) reduces to $(5/8)(2z + 3z^2)\Theta'(z) + (15/4)(z - 1)\Theta(z) = -1$. The general solution of this ODE is

$$E = \Theta(z) = \frac{1}{15(2 + 3z)^5}[128 + 1152z + 5184z^2 + Cz^3 - 1944z^4 - 5184z^3 \ln(-z)], \tag{4.4}$$

where $z(x, y) = y - (1/2)x$ for $z$ not equal to $-2/3$ or 0. Notice that $E$ cannot be determined at the singular points 0 and $-2/3$ of the reduced ODE. For $\Theta(1/2) = 100$ and $\Theta(5/7) = 300$, the graph of the parameter is shown in Fig. 2. Consequently, for the data

$$w(x, y) = \frac{1}{16}\left(-x^3 + 8y^3 + 6x^2y - 12xy^2 - 30x^2 - 40xy\right), \tag{4.5}$$

we obtain the exact solution of the PDE (1.5). This is to be contrasted with `pdsolve`'s output, which is indecipherable. The parameter $E(x, y)$ and the expression $w(x, y) + (15/8)x^2 + (5/2)xy$ (depending on the data) are constant along the curves $z(x, y) = c$ and, hence, the PDE (1.5) is reduced to an ODE along these curves. Moreover, while the characteristic equations of this linear PDE are given by a nonlinear ODE system, the characteristic equations of the PDEs in (3.4) are linear and the reduced ODE (4.2) is linear as well.

To conclude, given a data function $w$, one should check its invariance in (3.4), where $\xi$, $\phi$, and $\psi$ are discussed in Sec. 3. Next, from the second PDE in (3.4), the parameter should be obtained in terms of the invariants of the symmetry reduction. At the end, substituting $E$ and $w$ into (1.5), the dimension of the model should be reduced by one. It would be interesting for further research to explore the applications of these results to the standard regularization methods used to investigate ill-posed problems of type (1.1).

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