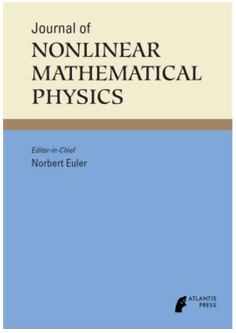

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CONSERVATION LAWS FOR HEATED LAMINAR RADIAL LIQUID AND FREE JETS

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The conserved quantities for the heated radial liquid jet and the heated radial free jet are established by using conservation laws. The flow in a heated radial jet is described by Prandtl's momentum boundary layer equation, the continuity equation and the energy equation. Viscous dissipation is neglected. The multiplier approach is used to derive the conservation laws for the system of three equations for the velocity components and the temperature and three conserved vectors are obtained. The conservation laws for the system of two partial differential equations for the stream function formulation are also computed by the multiplier approach and three conserved vectors are obtained. One of these is a non-local conserved vector for the system. The conserved quantities for the heated radial liquid jet and the heated radial free jet, emitted into a stationary fluid of uniform temperature θ_∞, are derived by integrating the conservation laws across the jet.

Keywords: Heated radial liquid jet; heated radial free jet; conservation laws; Prandtl's boundary layer equations; multiplier approach; Euler operator.

1. Introduction

The conserved quantities for laminar jets have been established either from physical arguments or by integrating Prandtl's momentum boundary layer equation across the jet and using the boundary conditions and the continuity equation. By this method, Schlichting [10, 11] established the conserved quantity for the two-dimensional free jet, Schwarz [12] for the radial free jet and Glauert [5] for the two-dimensional and radial wall jets. The conserved quantity for the two-dimensional liquid jet and radial liquid jet was obtained from a physical argument [14]. The conserved quantities for axisymmetric free and wall jets were derived by Goldstein [6] and Duck and Bodonyi [3]. Recently, Naz, Mason and Mahomed [7] presented a new method of driving the conserved quantities by utilizing conservation laws. The conserved quantities for liquid, free and wall jets for two-dimensional and radial flows were rederived using conservation laws.

The heated radial liquid jet was studied by Chaudhury [2]. The only conserved quantity obtained was total mass flux per radian. Schwarz [12] formulated the problem of the radial laminar heated free jet. Two conserved quantities were used to derive the similarity solution. One was the radial flux of momentum in the radial direction of the jet which was obtained by integrating the momentum equation across the jet. The second conserved quantity was the amount of heat per unit time ejected by the jet into the ambient fluid and was derived by integrating the energy equation across the jet.

In this paper, we derive the conserved quantities for radial laminar heated jet flows, emitting into a stationary fluid of temperature θ_∞, by using the conservation laws for the partial differential equations describing the jet flows. The flow in the heated radial liquid jet and the heated radial free jet is governed by Prandtl's momentum boundary layer equation, the continuity equation and the energy equation. The effects of viscous dissipation in the energy equation are neglected. We will use the multiplier approach [13, 9, 1, 8] to derive the conservation laws for the system of three partial differential equations for the velocity components and temperature as well as the system of two partial differential equations for the stream function and temperature. We derive the conserved quantity for the heated radial liquid jet, emitted into a stationary fluid of uniform temperature θ_∞, by integrating the conservation law across the jet. For the heated radial free jet, emitted into a stationary fluid of temperature θ_∞, two conserved quantities are derived with the help of two conservation laws.

2. The Laminar Heated Radial Jet

Cylindrical polar coordinates (x, λ, y) are used. The radial coordinate is x, the axis of symmetry is $x = 0$ and all quantities are independent of λ. The y-coordinate is along the axis of symmetry.

We will consider the laminar heated radial liquid jet and laminar heated radial free jet. The absolute temperature in the jet is denoted by $\theta(x, y)$.

The radial liquid jet is formed when a circular jet of liquid strikes a plane boundary normally and spreads over it [14]. The effects of heat transfer in a radial liquid jet were studied in [2]. The solid boundary of the jet is at $y = 0$. The x-axis is along the boundary and the y-axis is perpendicular to the boundary. There is no suction or blowing of fluid at the boundary. The surrounding fluid is a gas and the equation of the free surface is $y = \phi(x)$.

Schwarz [12] described the radial free jet. It is formed when fluid emerges from a pair of parallel circular plates into the surrounding fluid. In the free jet the surrounding fluid consists of the same fluid as the jet and is at rest far from the jet. The free jet is symmetrical about the plane $y = 0$.

The fluid in the liquid jet and free jet is viscous and incompressible and the flow is steady. We consider the case when viscous dissipation is negligible in the energy equation. The liquid jet and free jet are emitted into a fluid at rest at uniform temperature θ_∞. The equations for the heated radial jet flows are derived by assuming the boundary layer approximations to the equations of motion. The fluid velocity vanishes outside the jet for both jets. It follows from Euler's equation that the pressure gradient $\partial p/\partial x$ vanishes outside the jet. From the boundary layer approximations, pressure $p = p(x)$ and therefore dp/dx vanishes in the jet. The Reynolds number is assumed to be high enough so that the boundary layer approximations are applicable but low enough for the flow to be laminar. The equations governing the flow in a heated radial jet, in the absence of a pressure gradient, are Prandtl's momentum boundary layer equation, the continuity equation and the energy equation:

$$uu_x + vu_y = \nu u_{yy}, \tag{2.1}$$

$$(xu)_x + (xv)_y = 0, \tag{2.2}$$

$$uT_x + vT_y = \alpha T_{yy}, \tag{2.3}$$

where $u(x, y)$ and $v(x, y)$ are the velocity components in the x and y directions, ν is the kinematic viscosity of the fluid, $T = \theta - \theta_\infty$ is the temperature of the fluid measured relative to the uniform background temperature θ_∞, $\alpha = k/\rho c_p$ is the thermal diffusivity of the fluid, ρ is the density of the fluid, c_p is the heat capacity of the fluid at constant pressure and k is the thermal conductivity of the fluid.

Introduce a stream function $\psi(x,y)$ defined by

$$u = \frac{1}{x}\psi_y, \quad v = -\frac{1}{x}\psi_x. \tag{2.4}$$

Equation (2.2) is identically satisfied while (2.1) and (2.3) become

$$\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy} = 0 \tag{2.5}$$

and

$$\frac{1}{x}\psi_y T_x - \frac{1}{x}\psi_x T_y - \alpha T_{yy} = 0. \tag{2.6}$$

3. Conservation Laws for System for the Velocity Components and Temperature

In this section we will derive the conservation laws for the system (2.1)–(2.3) by the multiplier approach. Multipliers Λ_1, Λ_2 and Λ_3 for the system (2.1)–(2.3) have the property that

$$\begin{aligned}\Lambda_1(uu_x + vu_y - \nu u_{yy}) + \Lambda_2(u + xu_x + xv_y) + \Lambda_3(uT_x + vT_y - \alpha T_{yy})\\ = D_x T^1 + D_y T^2,\end{aligned} \tag{3.1}$$

for all functions $u(x,y), v(x,y)$ and $T(x,y)$. The total derivative operators D_x and D_y are defined by

$$\begin{aligned}D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + T_x\frac{\partial}{\partial T} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + T_{xx}\frac{\partial}{\partial T_x}\\ + u_{xy}\frac{\partial}{\partial u_y} + v_{xy}\frac{\partial}{\partial v_y} + T_{xy}\frac{\partial}{\partial T_y} + \cdots,\end{aligned} \tag{3.2}$$

$$\begin{aligned}D_y = \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + v_y\frac{\partial}{\partial v} + T_y\frac{\partial}{\partial T} + u_{yy}\frac{\partial}{\partial u_y} + v_{yy}\frac{\partial}{\partial v_y} + T_{yy}\frac{\partial}{\partial T_y}\\ + u_{yx}\frac{\partial}{\partial u_x} + v_{yx}\frac{\partial}{\partial v_x} + T_{yx}\frac{\partial}{\partial T_x} + \cdots.\end{aligned} \tag{3.3}$$

We will consider multipliers of the form $\Lambda_1 = \Lambda_1(x,y,u,v,T), \Lambda_2 = \Lambda_2(x,y,u,v,T)$ and $\Lambda_3 = \Lambda_3(x,y,u,v,T)$. The determining equations for the multipliers are

$$E_u[\Lambda_1(uu_x + vu_y - \nu u_{yy}) + \Lambda_2(u + xu_x + xv_y) + \Lambda_3(uT_x + vT_y - \alpha T_{yy})] = 0, \tag{3.4}$$

$$E_v[\Lambda_1(uu_x + vu_y - \nu u_{yy}) + \Lambda_2(u + xu_x + xv_y) + \Lambda_3(uT_x + vT_y - \alpha T_{yy})] = 0, \tag{3.5}$$

$$E_T[\Lambda_1(uu_x + vu_y - \nu u_{yy}) + \Lambda_2(u + xu_x + xv_y) + \Lambda_3(uT_x + vT_y - \alpha T_{yy})] = 0. \tag{3.6}$$

The standard Euler operators E_u, E_v and E_T are defined as follows:

$$E_u = \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + D_x^2\frac{\partial}{\partial u_{xx}} + D_xD_y\frac{\partial}{\partial u_{xy}} + D_y^2\frac{\partial}{\partial u_{yy}} - \cdots, \tag{3.7}$$

$$E_v = \frac{\partial}{\partial v} - D_x\frac{\partial}{\partial v_x} - D_y\frac{\partial}{\partial v_y} + D_x^2\frac{\partial}{\partial v_{xx}} + D_xD_y\frac{\partial}{\partial v_{xy}} + D_y^2\frac{\partial}{\partial v_{yy}} - \cdots, \tag{3.8}$$

$$E_T = \frac{\partial}{\partial T} - D_x\frac{\partial}{\partial T_x} - D_y\frac{\partial}{\partial T_y} + D_x^2\frac{\partial}{\partial T_{xx}} + D_xD_y\frac{\partial}{\partial T_{xy}} + D_y^2\frac{\partial}{\partial T_{yy}} - \cdots. \tag{3.9}$$

The expansion of (3.4)–(3.6) gives rise to the following equations:

$$\Lambda_{1u}(uu_x + vu_y - \nu u_{yy}) + \Lambda_{2u}(u + xu_x + xv_y) + \Lambda_{3u}(uT_x + vT_y - \alpha T_{yy}) + \Lambda_1 u_x + \Lambda_2 + \Lambda_3 T_x - D_x(\Lambda_1 u + x\Lambda_2) - D_y(\Lambda_1 v) - \nu D_y^2(\Lambda_1) = 0, \quad (3.10)$$

$$\Lambda_{1v}(uu_x + vu_y - \nu u_{yy}) + \Lambda_{2v}(u + xu_x + xv_y) + \Lambda_{3v}(uT_x + vT_y - \alpha T_{yy}) + \Lambda_1 u_y + \Lambda_3 T_y - D_y(x\Lambda_2) = 0, \quad (3.11)$$

and

$$\Lambda_{1T}(uu_x + vu_y - \nu u_{yy}) + \Lambda_{2T}(u + xu_x + xv_y) + \Lambda_{3T}(uT_x + vT_y - \alpha T_{yy}) - D_x(u\Lambda_3) - D_y(v\Lambda_3) - \alpha D_y^2(\Lambda_3) = 0. \quad (3.12)$$

Equations (3.10)–(3.12) are separated by equating the coefficients of the partial derivatives of $u(x, y), v(x, y)$ and $T(x, y)$. Now

$$D_y^2(\Lambda_1) = u_{yy}\Lambda_{1u} + v_{yy}\Lambda_{1v} + T_{yy}\Lambda_{1T} + \text{lower derivative terms} \quad (3.13)$$

and

$$D_y^2(\Lambda_3) = u_{yy}\Lambda_{3u} + v_{yy}\Lambda_{3v} + T_{yy}\Lambda_{3T} + \text{lower derivative terms.} \quad (3.14)$$

Thus equating to zero the coefficients of u_{yy}, v_{yy} and T_{yy} in (3.10), after expansion, gives

$$\Lambda_{1u} = 0, \quad \Lambda_{1v} = 0, \quad \nu\Lambda_{1T} + \alpha\Lambda_{3u} = 0 \quad (3.15)$$

and the coefficient of T_{yy} in (3.11) and (3.12) yields

$$\Lambda_{3v} = 0, \quad \Lambda_{3T} = 0. \quad (3.16)$$

Equation (3.11), after using (3.15) and (3.16), splits into following equations:

$$\Lambda_{2v} = 0, \quad \Lambda_1 - x\Lambda_{2u} = 0, \quad \Lambda_3 - x\Lambda_{2T} = 0, \quad \Lambda_{2y} = 0. \quad (3.17)$$

The solution of Eqs. (3.15), (3.16) and (3.17) is

$$\Lambda_1 = A(x), \quad \Lambda_2 = \frac{u}{x}A(x) + \frac{T}{x}B(x) + C(x), \quad \Lambda_3 = B(x). \quad (3.18)$$

Substitution of (3.18) in (3.10) and (3.12) yields

$$2u\left(A'(x) - \frac{A}{x}\right) + T\left(B'(x) - \frac{B}{x}\right) + xC'(x) = 0, \quad (3.19)$$

$$u\left(B'(x) - \frac{B}{x}\right) = 0. \quad (3.20)$$

Equations (3.19) and (3.20) result in

$$A(x) = c_2 x, \quad B(x) = c_3 x, \quad C(x) = c_1, \quad (3.21)$$

which together with (3.18) yields

$$\Lambda_1 = c_2 x, \quad \Lambda_2 = c_1 + c_2 u + c_3 T, \quad \Lambda_3 = c_3 x, \quad (3.22)$$

where c_1, c_2 and c_3 are constants.

Thus from (3.1) and (3.22),

$$c_2x(uu_x + vu_y - \nu u_{yy}) + (c_1 + c_2u + c_3T)(u + xu_x + xv_y) + c_3x(uT_x + vT_y - \alpha T_{yy})$$
$$= D_x[c_1xu + c_2xu^2 + c_3xuT] + D_y[c_1xv + c_2x(uv - \nu u_y) + c_3x(vT - \alpha T_y)], \qquad (3.23)$$

for arbitrary $u(x, y), v(x, y)$ and $T(x, y)$ and therefore when $u(x, y), v(x, y)$ and $T(x, y)$ are the solutions of the system (2.1)–(2.3), we obtain

$$D_x[c_1xu + c_2xu^2 + c_3xuT] + D_y[c_1xv + c_2x(uv - \nu u_y) + c_3x(vT - \alpha T_y)] = 0. \qquad (3.24)$$

Thus for the system (2.1)–(2.3) we obtain following three conserved vectors:

$$T^1 = xu, \qquad T^2 = xv, \qquad (3.25)$$

$$T^1 = xu^2, \qquad T^2 = x(uv - \nu u_y), \qquad (3.26)$$

$$T^1 = xuT, \qquad T^2 = x(vT - \alpha T_y). \qquad (3.27)$$

4. Conservation Laws for System for Stream Function and Temperature

Consider a multiplier of the form $\Lambda_1 = \Lambda_1(x, y, \psi, T)$ and $\Lambda_2 = \Lambda_2(x, y, \psi, T)$ for the system (2.5)–(2.6). The determining equations for the multipliers Λ_1 and Λ_2 are

$$E_\psi\left[\Lambda_1\left(\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy}\right) + \Lambda_2\left(\frac{1}{x}\psi_yT_x - \frac{1}{x}\psi_xT_y - \alpha T_{yy}\right)\right] = 0, \qquad (4.1)$$

$$E_T\left[\Lambda_1\left(\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy}\right) + \Lambda_2\left(\frac{1}{x}\psi_yT_x - \frac{1}{x}\psi_xT_y - \alpha T_{yy}\right)\right] = 0, \qquad (4.2)$$

where the standard Euler operator E_ψ is defined by

$$E_\psi = \frac{\partial}{\partial\psi} - D_x\frac{\partial}{\partial\psi_x} - D_y\frac{\partial}{\partial\psi_y} + D_x^2\frac{\partial}{\partial\psi_{xx}} + D_xD_y\frac{\partial}{\partial\psi_{xy}} + D_y^2\frac{\partial}{\partial\psi_{yy}} - \cdots, \qquad (4.3)$$

and E_T is defined in (3.9). The total derivative operators D_x and D_y are defined by

$$D_x = \frac{\partial}{\partial x} + \psi_x\frac{\partial}{\partial\psi} + T_x\frac{\partial}{\partial T} + \psi_{xx}\frac{\partial}{\partial\psi_x} + T_{xx}\frac{\partial}{\partial T_x} + \psi_{xy}\frac{\partial}{\partial\psi_y} + T_{xy}\frac{\partial}{\partial T_y} + \cdots, \qquad (4.4)$$

$$D_y = \frac{\partial}{\partial y} + \psi_y\frac{\partial}{\partial\psi} + T_y\frac{\partial}{\partial T} + \psi_{yy}\frac{\partial}{\partial\psi_y} + T_{yy}\frac{\partial}{\partial T_y} + \psi_{yx}\frac{\partial}{\partial\psi_x} + T_{yx}\frac{\partial}{\partial T_x} + \cdots. \qquad (4.5)$$

Expansion of (4.1) and (4.2) gives

$$\left(\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy}\right)\Lambda_{1\psi} + \left(\frac{1}{x}\psi_yT_x - \frac{1}{x}\psi_xT_y - \alpha T_{yy}\right)\Lambda_{2\psi}$$
$$+ D_x\left(\frac{1}{x}\Lambda_1\psi_{yy} + \frac{1}{x}\Lambda_2T_y\right) + D_y\left[\left(\frac{2}{x^2}\psi_y - \frac{1}{x}\psi_{xy}\right)\Lambda_1 - \frac{1}{x}\Lambda_2T_x\right]$$
$$+ D_xD_y\left(\frac{1}{x}\Lambda_1\psi_y\right) - D_y^2\left(\frac{1}{x}\Lambda_1\psi_x\right) + \nu D_y^3(\Lambda_1) = 0, \qquad (4.6)$$

and

$$\left(\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy}\right)\Lambda_{1T} + \left(\frac{1}{x}\psi_yT_x - \frac{1}{x}\psi_xT_y - \alpha T_{yy}\right)\Lambda_{2T}$$
$$- D_x\left(\frac{1}{x}\Lambda_2\psi_y\right) + D_y\left(\frac{1}{x}\Lambda_2\psi_x\right) - \alpha D_y^2(\Lambda_2) = 0. \qquad (4.7)$$

Now,

$$D_y^2(\Lambda_2) = \psi_{yy}\Lambda_{2\psi} + T_{yy}\Lambda_{2T} + \text{lower derivative terms} \tag{4.8}$$

and

$$D_y^3(\Lambda_1) = \psi_{yyy}\Lambda_{1\psi} + T_{yyy}\Lambda_{1T} + \text{lower derivative terms.} \tag{4.9}$$

The coefficient of T_{yyy}, after expansion of Eq. (4.6), yields

$$\Lambda_{1T} = 0. \tag{4.10}$$

The coefficients of ψ_{yy} and T_{yy} in the expansion of (4.7) give

$$\Lambda_{2\psi} = 0, \quad \Lambda_{2T} = 0. \tag{4.11}$$

From Eqs. (4.10) and (4.11), we conclude that

$$\Lambda_1 = A(x, y, \psi), \quad \Lambda_2 = B(x, y). \tag{4.12}$$

The substitution of (4.12) in (4.6) and (4.7) yields

$$\begin{aligned}
&\left(\frac{2}{x}A_x + 3\nu A_{\psi y}\right)\psi_{yy} - \frac{2}{x}A_y\psi_{xy} + 3\nu A_{\psi\psi}\psi_y\psi_{yy} - \frac{1}{x}A_{\psi y}\psi_x\psi_y + \nu A_{\psi\psi\psi}\psi_y^3 \\
&\quad + \left(\frac{1}{x}A_{\psi x} + 3\nu A_{\psi\psi y}\right)\psi_y^2 + \left(\frac{1}{x^2}A_y + \frac{1}{x}A_{xy} + 3\nu A_{\psi yy}\right)\psi_y - \frac{1}{x}\psi_x A_{yy} + \nu A_{yyy} \\
&\quad + \left(\frac{1}{x}B_x - \frac{1}{x^2}B\right)T_y - \frac{1}{x}T_x B_y = 0
\end{aligned} \tag{4.13}$$

and

$$\frac{1}{x}B_y\psi_x - \left(\frac{1}{x}B_x - \frac{1}{x^2}B\right)\psi_y - \alpha B_{yy} = 0. \tag{4.14}$$

Equations (4.13) and (4.14) finally yield

$$A = \Lambda_1 = c_4 + c_5\psi, \quad B = \Lambda_2 = c_6 x, \tag{4.15}$$

where c_4, c_5 and c_6 are constants.

Now

$$\begin{aligned}
&(c_4 + c_5\psi)\left(\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy}\right) + c_6 x\left(\frac{1}{x}\psi_y T_x - \frac{1}{x}\psi_x T_y - \alpha T_{yy}\right) \\
&\quad = D_x\left[c_4\left(\frac{1}{x}\psi_y^2\right) + c_5\left(\frac{1}{x}\psi\psi_y^2\right) + c_6 T\psi_y\right] + D_y\left[c_4\left(-\frac{1}{x}\psi_x\psi_y - \nu\psi_{yy}\right)\right. \\
&\qquad \left. + c_5\left(-\frac{1}{x}\psi\psi_x\psi_y + \frac{\nu}{2}\psi_y^2 - \nu\psi\psi_{yy}\right) + c_6(-\psi_x T - \alpha x T_y)\right],
\end{aligned} \tag{4.16}$$

holds for arbitrary functions $\psi(x, y)$ and $T(x, y)$. Therefore

$$\begin{aligned}
&D_x\left[c_4\left(\frac{1}{x}\psi_y^2\right) + c_5\left(\frac{1}{x}\psi\psi_y^2\right) + c_6 T\psi_y\right] + D_y\left[c_4\left(-\frac{1}{x}\psi_x\psi_y - \nu\psi_{yy}\right)\right. \\
&\qquad \left. + c_5\left(-\frac{1}{x}\psi\psi_x\psi_y + \frac{\nu}{2}\psi_y^2 - \nu\psi\psi_{yy}\right) + c_6(-\psi_x T - \alpha x T_y)\right] = 0
\end{aligned} \tag{4.17}$$

for all solutions $\psi(x,y)$ and $T(x,y)$ of the system (2.5)–(2.6). We therefore obtain three conserved vectors

$$T^1 = \frac{1}{x}\psi_y^2, \qquad T^2 = -\frac{1}{x}\psi_x\psi_y - \nu\psi_{yy}, \tag{4.18}$$

$$T^1 = T\psi_y, \qquad T^2 = -\psi_x T - \alpha x T_y, \tag{4.19}$$

$$T^1 = \frac{1}{x}\psi\psi_y^2, \quad T^2 = -\frac{1}{x}\psi\psi_x\psi_y + \frac{\nu}{2}\psi_y^2 - \nu\psi\psi_{yy}. \tag{4.20}$$

The conserved vectors (3.26) and (4.18) and the conserved vectors (3.27) and (4.19) are equivalent.

5. Conserved Quantities

In this section we will present a new method of deriving the conserved quantity for the heated radial liquid jet and for the heated radial free jet emitting into the same fluid which is at rest and at uniform temperature θ_∞. All the conserved vectors (T^1, T^2) derived here depend on $u(x,y), v(x,y), T(x,y)$ or $\psi(x,y)$ and satisfy

$$D_x T^1 + D_y T^2 = \frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y}. \tag{5.1}$$

For a conserved vector, $D_x T^1 + D_y T^2 = 0$, and therefore

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y} = 0. \tag{5.2}$$

The conserved quantities will be derived by integrating (5.2) for the corresponding conservation law across the jet and using the boundary conditions.

5.1. *Conserved quantity for the heated radial liquid jet*

For the heated radial liquid jet it will be assumed that there is no slip or suction or blowing at the lower boundary over which the liquid is flowing. The shear stress falls to zero at the free surface $y = \phi(x)$, since the viscosity of air is negligible. The heat transfer is zero along the free surface $y = \phi(x)$. Thus the boundary conditions for the heated radial liquid jet are:

$$y = 0: \quad u = 0, \quad v = 0, \tag{5.3}$$

$$y = \phi(x): \quad u_y = 0, \quad T_y = 0. \tag{5.4}$$

Now

$$v(x,\phi(x)) = \frac{D}{Dt}[\phi(x)] = u(x,\phi(x))\frac{d\phi(x)}{dx}, \tag{5.5}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u(x,y)\frac{\partial}{\partial x} + v(x,y)\frac{\partial}{\partial y} \tag{5.6}$$

is the material time derivative.

The conserved quantity for the heated radial liquid jet is obtained from the conserved vector (3.25) and is the same as for the radial liquid jet because viscous dissipation was neglected in (2.3). We briefly outline the derivation of the conserved quantity [7]. Integrate (5.2) with respect to y from $y = 0$ to $y = \phi(x)$ keeping x fixed during the integration. Then, for the conserved vector (3.25),

$$\int_0^{\phi(x)} \left[\frac{\partial(xu)}{\partial x} + \frac{\partial(xv)}{\partial y}\right] dy = 0. \tag{5.7}$$

Using (5.5) for $v(x, \phi(x))$, the boundary condition $v(x, 0) = 0$ and the formula for differentiation under an integral sign [4], we obtain from (5.7) that

$$x \int_0^{\phi(x)} u(x, y)dy = \text{constant, independent of } x. \tag{5.8}$$

Equation (5.8) states that the total volume flux is constant, independent of the radius. Thus

$$F = x \int_0^{\phi(x)} u(x, y)dy \tag{5.9}$$

is the conserved quantity for the heated radial liquid jet [2].

5.2. *Conserved quantities for the heated laminar radial free jet*

Two conserved quantities were used by Scwarz [12] to derive the similarity solution for the case of a heated jet emitting into a fluid at rest at uniform temperature θ_∞. At $y = \pm\infty, \theta = \theta_\infty$ and therefore $T = 0$ at $y = \pm\infty$.

The boundary conditions for the heated radial free jet are [12]

$$y = 0: \quad v = 0, \quad u_y = 0, \quad T_y = 0, \tag{5.10}$$

$$y = \pm\infty: \quad u = 0, \quad u_y = 0, \quad T = 0, \quad T_y = 0. \tag{5.11}$$

We will first use the conserved vectors (3.26) and (3.27) to derive two conserved quantities for the heated laminar radial free jet. Integrating (5.2), for the conserved vector (3.26), with respect to y from $y = -\infty$ to $y = \infty$ keeping x fixed during the integration, we obtain

$$\frac{d}{dx}\left[x \int_{-\infty}^{\infty} u^2 dy\right] + [xuv - \nu x u_y]_{-\infty}^{\infty} = 0. \tag{5.12}$$

Using boundary conditions (5.11) in (5.12) and the requirement that $v(x, \pm\infty)$ is finite, the second term vanishes and we obtain

$$x \int_{-\infty}^{\infty} u^2(x, y)dy = \text{constant, independent of } x. \tag{5.13}$$

Equation (5.13) multiplied by $2\pi\rho$ states that the total momentum flux in the radial direction is constant, independent of radius. Therefore

$$F = 2\pi x\rho \int_{-\infty}^{\infty} u^2(x, y)dy, \tag{5.14}$$

is one conserved quantity for the heated radial free jet [12].

The second conserved quantity is derived from the conserved vector (3.27). Integrating (5.2), for the conserved vector (3.27), with respect to y from $y = -\infty$ to $y = \infty$ keeping x fixed during the integration, we obtain

$$\frac{d}{dx}\left[x \int_{-\infty}^{\infty} uT dy\right] + x[vT - \alpha T_y]_{-\infty}^{\infty} = 0. \tag{5.15}$$

The second term in (5.15) vanishes due to boundary conditions (5.11) and we obtain

$$\frac{d}{dx}\left[x \int_{-\infty}^{\infty} u(x, y)T(x, y)dy\right] = 0, \tag{5.16}$$

which yields

$$x \int_{-\infty}^{\infty} u(x,y)T(x,y)dy = \text{constant, independent of } x. \tag{5.17}$$

Thus

$$G = 2\pi x \rho c_p \int_{-\infty}^{\infty} u(x,y)T(x,y)dy, \tag{5.18}$$

is the second conserved quantity for the heated laminar radial free jet and is the same as that derived by Schwarz [12].

The conserved quantities (5.14) and (5.18) can also be derived using the conserved vectors (4.18) and (4.19) for the system (2.5)–(2.6). We now briefly outline the derivation.

In terms of the stream function the boundary conditions (5.10) and (5.11) take the form

$$y = 0: \quad \psi_x = 0, \quad \psi_{yy} = 0, \quad T_y = 0, \tag{5.19}$$

$$y = \pm\infty: \quad \psi_y = 0, \quad \psi_{yy} = 0, \quad T = 0, \quad T_y = 0. \tag{5.20}$$

Integrate (5.2) with respect to y from $y = -\infty$ to $y = \infty$ with x kept fixed during the integration. For the conserved vector (4.18) we obtain

$$\frac{d}{dx}\left[\frac{1}{x}\int_{-\infty}^{\infty} \psi_y^2 dy\right] + \left[-\frac{1}{x}\psi_x\psi_y - \nu\psi_{yy}\right]_{-\infty}^{\infty} = 0. \tag{5.21}$$

Now $\psi_x(x,\pm\infty) = -xv(x,\pm\infty)$ is assumed to be finite. The second term in (5.21) is zero due to boundary conditions (5.20) and hence we conclude that

$$\frac{1}{x}\int_{-\infty}^{\infty} \psi_y^2(x,y)dy = \text{constant, independent of } x. \tag{5.22}$$

Equation (5.22) is equivalent to (5.13) and hence we obtain again the conserved quantity F given in (5.14).

Integrating (5.2), for the conserved vector (4.19), with respect to y from $y = -\infty$ to $y = \infty$ keeping x fixed during the integration, we obtain

$$\frac{d}{dx}\left[\int_{-\infty}^{\infty} T\psi_y dy\right] + [-\psi_x T - \alpha x T_y]_{-\infty}^{\infty} = 0. \tag{5.23}$$

Imposing the boundary conditions (5.20) on (5.23), we obtain

$$\int_{-\infty}^{\infty} T(x,y)\psi_y(x,y)dy = \text{constant, independent of } x, \tag{5.24}$$

which is equivalent to (5.17) and hence we obtain the conserved quantity (5.18).

The conserved vector (4.20) gives the conserved quantity for the radial wall jet [7]. But the heated radial wall jet cannot be formulated from the results we have derived because there is no second conservation law.

In Table 1 the boundary layer equations, conservation laws and conserved quantities for heated radial liquid and free jets are compared with those of radial liquid and free jets [7].

6. Conclusions

The conservation laws for the system of equations for the velocity components and the temperature as well as for the system of equations for the stream function and the temperature governing the flow in a heated radial laminar jet with negligible viscous dissipation were derived. The multiplier

Table 1. Comparison between heated radial jets and radial jets.

	Heated radial liquid and free jets	Radial liquid and free jets
Boundary layer equations	$uu_x + vu_y = \nu u_{yy}$ $uT_x + vT_y = \alpha T_{yy}$ $(xu)_x + (xv)_y = 0$	$uu_x + vu_y = \nu u_{yy}$ $(xu)_x + (xv)_y = 0$
Conserved vectors	$T^1 = xu$ $T^2 = xv$	$T^1 = xu$ $T^2 = xv$
	$T^1 = xu^2$ $T^2 = x(uv - \nu u_y)$	$T^1 = xu^2$ $T^2 = x(uv - \nu u_y)$
	$T^1 = xuT$ $T^2 = x(vT - \alpha T_y)$	
Stream function formulation	$\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2$ $-\frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy} = 0$ $\frac{1}{x}\psi_y T_x - \frac{1}{x}\psi_x T_y - \alpha T_{yy} = 0$	$\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy} = 0$
Conserved vectors	$T^1 = \frac{1}{x}\psi_y^2$ $T^2 = -\frac{1}{x}\psi_x\psi_y - \nu\psi_{yy}$	$T^1 = \frac{1}{x}\psi_y^2$ $T^2 = -\frac{1}{x}\psi_x\psi_y - \nu\psi_{yy}$
	$T^1 = \frac{1}{x}\psi\psi_y^2$ $T^2 = -\frac{1}{x}\psi\psi_x\psi_y + \frac{\nu}{2}\psi_y^2 - \nu\psi\psi_{yy}$	$T^1 = \frac{1}{x}\psi\psi_y^2$ $T^2 = -\frac{1}{x}\psi\psi_x\psi_y + \frac{\nu}{2}\psi_y^2 - \nu\psi\psi_{yy}$
	$T^1 = T\psi_y$ $T^2 = -\psi_x T - \alpha x T_y$	
Conserved quantities		
For liquid jet	$x\int_0^{\phi(x)} u(x,y)dy$	$x\int_0^{\phi(x)} u(x,y)dy$
For free jet	$2\pi x\rho\int_{-\infty}^{\infty} u^2(x,y)dy$ $2\pi x\rho c_p\int_{-\infty}^{\infty} u(x,y)T(x,y)dy$	$2\pi x\rho\int_{-\infty}^{\infty} u^2(x,y)dy$

approach applied to the system of three partial differential equations for the velocity components and the temperature gave three conserved vectors. For the system of two partial differential equations for the stream function and the temperature, the multiplier approach also gave three conserved vectors.

The conserved quantities for the heated radial liquid jet and for the radial free jet emitting into the same fluid at rest and at uniform temperature θ_∞ were established using the conserved vectors. One of the conserved vectors for the system of equations for the velocity components and temperature gave the conserved quantity for the heated radial liquid jet. The remaining two conserved vectors were used to derive two conserved quantities for the heated radial free jet. For the stream function formulation two conserved vectors were used to give an alternative derivation of the conserved quantities for the heated radial free jet. The third conserved vector gives the conserved quantity for the radial wall jet and could be used as one of the two conserved quantities for the heated radial wall jet if another nonlocal conserved vector can be found.

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