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HOW TO EXTEND ANY DYNAMICAL SYSTEM SO THAT IT BECOMES ISOCHRONOUS, ASYMPTOTICALLY ISOCRONOUS OR MULTI-PERIODIC

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We indicate how one can extend any dynamical system (namely, any system of nonlinearly coupled autonomous ordinary differential equations) so that the extended dynamical system thereby obtained is either *isochronous* or *asymptotically isochronous* or *multi-periodic*, namely its *generic* solutions are either *completely periodic* with a *fixed* period or tend *asymptotically*, in the remote future, to such *completely periodic* functions or are *multi-periodic* (or become *multi-periodic* only *asymptotically*, in the remote future). In all cases the scale of the periodicity can be arbitrarily assigned. Moreover, the solutions of the extended systems are generally well approximated by those of the original, unmodified, systems, up to a *constant* rescaling of the independent variable (time), as long as their evolution is considered over time intervals short with respect to the (arbitrarily assigned) periodicities characterizing the extended systems. Several examples are displayed. In some cases the *general solution* of these dynamical systems is also exhibited; in others, this is impossible inasmuch as the models being manufactured are extensions of dynamical systems displaying *chaotic* evolutions, such as, for instance, the well-known Lorenz model of 3 nonlinearly coupled ODEs.

Keywords: Dynamical systems; nonlinear ODEs; periodic systems; isochronous systems; asymptotically isochronous systems; multiperiodic dynamical systems.

1. Introduction

In this paper we review a simple technique allowing to extend any dynamical system — namely, any system of nonlinearly coupled autonomous Ordinary Differential Equations (ODEs) — so that the extended system thereby obtained is either *isochronous* or *asymptotically isochronous* or *multi-periodic*, namely its *generic* solution is either *completely periodic* (*periodic* in *all* its degrees of freedom) with a *fixed* period independent of the initial data, or tends *asymptotically*, in the remote future, to such *completely periodic* functions, or is *multi-periodic* (as defined below), or acquires

this property only *asymptotically*, in the remote future (this last case is merely outlined below). In all cases the scale of the periodicity can be arbitrarily assigned. Moreover, the solutions of the extended systems are generally well approximated by those of the original, unmodified, systems, up to a *constant* rescaling of the independent variable (time), as long as their evolution is considered over time intervals short with respect to the (arbitrarily assigned) periodicities characterizing the extended systems. We then exhibit simple yet nontrivial examples of such *isochronous* or *asymptotically isochronous* or *multi-periodic* systems composed of a few nonlinearly coupled *autonomous* ODEs. The basic idea behind the technique to manufacture such systems is not quite new [1–5], but its extension to *asymptotically isochronous*, and especially to *multi-periodic*, systems is new, and in any case the procedure reported below provides some new twists of it and allows to exhibit rather neat models which, to the best of our knowledge, are novel and remarkable. Many other analogous models can moreover be manufactured, once this simple technique is mastered; indeed we expect that it shall become a standard tool used by researchers interested in the mathematical modeling of phenomena displaying *isochronous* or *periodic* behaviors, and also by experimenters and practitioners involved in manufacturing devices exhibiting such phenomenologies.

In the following Sec. 2 we identify a class of *autonomous* dynamical systems displaying such behaviors — either *isochronous* or *asymptotically isochronous* or *multi-periodic* — and we then describe how they can be manufactured starting from an arbitrary *autonomous* system of N ODEs and extending it so that the *generic* solution of the extended system display such behaviors. That treatment details our methodology to manufacture such systems, and it also clarifies the similarities and differences among the approach introduced herein and previous treatments [1–5]. In Secs. 3, 4 respectively 5 we indicate how our treatment allows to manufacture *isochronous*, *asymptotically isochronous* respectively *multi-periodic* dynamical systems, and in Subsecs. 3.1, 4.1 respectively 5.1 we report several neat, and apparently nontrivial, examples of *isochronous*, *asymptotically isochronous* respectively *multi-periodic* dynamical systems obtained in this manner. The reader mainly interested to see quite simple instances of the dynamical systems yielded by our technique is advised to have an immediate look at these three subsections. In Sec. 6 we dwell on the similarities and differences among the behavior of the original systems and their extended — *isochronous* or *asymptotically isochronous* or *multi-periodic* — versions; this is particularly interesting when the original (unmodified) systems yield *chaotic* evolutions. A Sec. 7 entitled “Outlook” concludes the paper: in it we tersely outline various generalizations of our methodology.

2. A Class of Systems Displaying Various Periodicity Behaviors, and How to Manufacture Them

In this section we firstly define *isochronous*, *asymptotically isochronous* and *multi-periodic* dynamical systems, and we identify a special class of such systems. We then describe a technique to extend any *autonomous* dynamical system so that the extended dynamical system thereby obtained, while still *autonomous*, belongs to that class, hence has the property to be either *isochronous* or *asymptotically isochronous* or *multi-periodic*, according to the specific methodology employed to manufacture it.

2.1. *Isochronous, asymptotically isochronous, multi-periodic dynamical systems*

In this Subsec. 2.1 we firstly provide a definition of *isochronous*, *asymptotically isochronous* respectively *multi-periodic* dynamical systems, and we then identify a special class of such systems.

Notation. For simplicity all quantities below are *real*, unless otherwise stated. The *independent variable* is the time t , and differentiations with respect to this variable are denoted by superimposed dots. *Dependent variables* are denoted by Latin letters (towards the end of this alphabet, such as $x, y, X \dots$, often with a subscripted index); *arbitrary* (constant) *parameters* are denoted by Greek letters (generally towards the beginning of this alphabet, such as $\alpha, \beta, \gamma, \dots$, the exceptions being

Ω and ω); and *arbitrary* constants (appearing in displayed *general* solutions) are denoted by upper case Latin letters (towards the beginning of this alphabet, such as $A, B, C \dots$). Hence hereafter a dynamical system is a set of, generally nonlinear, ODEs, written, say, in either one of the following two ways:

$$\dot{\underline{z}} = \underline{h}(\underline{z}, t), \quad (1a)$$

or

$$\dot{z}_j = h_j(\underline{z}, t), \quad j = 1, \dots, J. \quad (1b)$$

Here and hereafter underlined quantities denote vectors, the dimensionality of which will be clear from the context (for instance, in this case \underline{z} is clearly a J -vector), and appended indices denote of course the components of the corresponding vector; J is an arbitrary *positive integer*. The J -vector $\underline{h}(\underline{z}, t)$ is assumed to be given; whenever it does not feature any *explicit* time-dependence, $\underline{h}(\underline{z}, t) \equiv \underline{h}(\underline{z})$, the dynamical system is called *autonomous*.

Definition 2.1. A vector $\underline{z}(t; \underline{\alpha})$, depending on the time t and on a set of parameters denoted by the vector $\underline{\alpha}$ (which need not have the same dimensionality as \underline{z}), is *isochronous* (with period T) if *all* its components are periodic with the *same* period T (independent of $\underline{\alpha}$), namely for all time

$$\underline{z}(t + T; \underline{\alpha}) = \underline{z}(t; \underline{\alpha}); \quad z_j(t + T; \underline{\alpha}) = z_j(t; \underline{\alpha}), \quad j = 1, \dots, J. \quad (2)$$

Here and hereafter whenever we mention a period such as T , we generally mean the *primitive* period; of course this formula, (2), remains valid if T is replaced by any *integer* multiple of T .

Definition 2.2. A time-dependent J -vector $\underline{z}(t; \underline{\alpha})$ is *asymptotically isochronous* if there exists an *isochronous* J -vector $\tilde{\underline{z}}(t; \underline{\alpha})$ such that

$$\lim_{t \rightarrow +\infty} \|\underline{z}(t; \underline{\alpha}) - \tilde{\underline{z}}(t; \underline{\alpha})\| = 0. \quad (3)$$

Here the symbol $\|\cdot\|$ denotes some convenient norm, which we shall not need to specify; for instance the standard norm $\|\underline{z}\| = \max_{j=1, \dots, J} |z_j|$ will generally do. Of course this definition does not identify uniquely the J -vector $\tilde{\underline{z}}(t; \underline{\alpha})$. Clearly *isochronous* J -vectors are as well *asymptotically isochronous*, but generally in the following when referring to the property of *asymptotic isochrony* we will have in mind — without necessarily specifying this explicitly — quantities that possess this property without possessing the stronger property of *isochrony*.

Definition 2.3. A J -vector $\underline{z}(t)$ is *multi-periodic* if each of its J components is a *multi-periodic* function. A practical definition, appropriate in the context of our treatment, of a *multi-periodic* function $f(t)$ of the time t , is that $f(t)$ be expressible as a function of a finite number of arguments $s_\ell = \sin[\Omega_\ell(t - t_\ell)]$, and is *continuous* in all these arguments in the set characterized by the restrictions $|s_\ell| \leq 1$. Such a *multi-periodic* function is clearly (a subcase of) an *almost periodic* function, whose precise definition requires that it be the uniform in t limit of linear superpositions of a *finite* number of *continuous periodic* functions $f_\ell(t)$, each of which is *periodic* with a different period $T_\ell = 2\pi/\Omega_\ell$. Let us recall — since this is useful for our purposes, see below — that an *almost periodic* function can as well be characterized by the property that, given any arbitrary *positive* quantity ε , $\varepsilon > 0$, there exist a “quasi-period” $T(\varepsilon)$ such that, for all time t ,

$$|f[t + T(\varepsilon)] - f(t)| < \varepsilon. \quad (4)$$

Note incidentally that the additional requirement that the “quasi-period” $T(\varepsilon)$ be a *rational* number (in whichever units) would entail no significant restriction.

In the case of a J -vector $\underline{z}(t; \underline{\alpha})$ depending on a set of parameters $\underline{\alpha}$ an additional issue is whether the periods T_ℓ , as defined above, do or do not depend on the parameters $\underline{\alpha}$, and also whether they

are different for the different components $z_j(t; \underline{\alpha})$ of the J -vector $\underline{z}(t; \underline{\alpha})$. In the case in which neither of these possible dependences is present — namely the periods T_ℓ , hence as well the overall quasi-period $T(\varepsilon)$, are the same for all the J components $z_j(t; \underline{\alpha})$ of the J -vector $\underline{z}(t; \underline{\alpha})$ and are moreover independent of the parameters $\underline{\alpha}$; as is indeed generally the case for the *multi-periodic* dynamical systems treated in this paper — one might call such vectors, and the corresponding dynamical systems (see below), *multi-isochronous*, since clearly if *all* these periods T_ℓ are *congruent* the *multi-periodicity* reduces to *isochrony*; but we prefer not to encumber the reader by introducing new terminology. And of course in the following, when referring to this property of *multi-periodicity*, we will generally have in mind — without necessarily specifying this explicitly — quantities that possess this property without possessing the stronger property of *isochrony*.

Definition 2.4. A dynamical system is called *isochronous*, *asymptotically isochronous* respectively *multi-periodic* if its *generic* solution is *isochronous*, *asymptotically isochronous* respectively *multi-periodic*. This definition leaves open the possible existence of a subset of *nongeneric* solutions not possessing the relevant property: they might be solutions that feature singularities at some specific times. Such a subset of *nongeneric* solutions — which need not exist — should in any case have *positive* codimension.

Note that the above definition of *asymptotically isochronous* system includes both the case of standard limit cycles — provided *all* trajectories of the system under consideration approach limit cycles *all* having the same period, which is of course the case if *all* trajectories are attracted to a *single* limit cycle: for instance the van der Pol oscillator is, in our terminology, *asymptotically isochronous* — as well as cases in which the *isochronous* limit orbits form a continuous set. In fact, the examples we shall treat *all* belong to the latter class. This indicates that our examples of *asymptotically isochronous* systems are, just as the *isochronous* systems themselves, *nongeneric*. This is, of course, part of the reason for their interest.

Lemma 2.5 (“Transitivity”). *If the J -vector $\underline{Z}(t)$ is globally defined, continuous and bounded — i.e., each of its J components $z_j(t)$ is a continuous and bounded function of time for all time — and if the scalar function $\tau(t)$ (is defined for all time and) is *isochronous*, *asymptotically isochronous* respectively *multi-periodic*, then the J -vector $\underline{z}(t) = \underline{Z}[\tau(t)]$ is *isochronous*, *asymptotically isochronous* respectively *multi-periodic*.*

The proof of this *Lemma* is sufficiently obvious that we leave its details to be filled in by the diligent reader — who shall take advantage, in the *multi-periodic* case, of the property associated with the inequality (4).

Lemma 2.6. *If the generic solution of the (autonomous) dynamical system*

$$\dot{\underline{Z}} = \underline{h}(\underline{Z}); \quad \dot{Z}_j = h_j(\underline{Z}), \quad j = 1, \dots, J \quad (5a)$$

*is, for all time, continuous and bounded, and the scalar function $\tau(t)$ (is defined for all time and) is *isochronous*, *asymptotically isochronous* respectively *multi-periodic*, then the (nonautonomous) dynamical system*

$$\dot{\underline{z}} = \dot{\tau}(t) \underline{h}(\underline{z}); \quad \dot{z}_j = \dot{\tau}(t) h_j(\underline{z}) \quad (5b)$$

*is *isochronous*, *asymptotically isochronous* respectively *multi-periodic*.*

The proof of this *Lemma* 2.6 is an immediate consequence of the preceding *Lemma* 2.5, via the observation that, if $\underline{Z}(t)$ is a solution of (5a), then

$$\underline{z}(t) = \underline{Z}[\tau(t)]; \quad z_j(t) = Z_j[\tau(t)], \quad j = 1, \dots, J$$

is a solution of (5b).

2.2. How to manufacture autonomous dynamical systems that are either isochronous, or asymptotically isochronous, or multi-periodic

As indicated above, our point of view is to start from an *autonomous*, but otherwise largely arbitrary, dynamical system, say

$$\dot{\underline{X}} = \underline{h}(\underline{X}); \quad \dot{X}_n = h_n(\underline{X}), \quad n = 1, \dots, N, \quad (6a)$$

and modify it so that the modified system is either *isochronous*, or *asymptotically isochronous* or *multi-periodic*. The clue of how to proceed is provided by the last result of the preceding Subsec. 2.1, Lemma 2.6. Hence our *first step* is to modify the system (6a) to read

$$\dot{\underline{x}} = \dot{\tau}(t) \underline{h}(\underline{x}); \quad \dot{x}_n = \dot{\tau}(t) h_n(\underline{x}), \quad n = 1, \dots, N, \quad (6b)$$

with the scalar function $\tau(t)$ either *periodic* with period T , or *asymptotically periodic* with the same period, or *multi-periodic*. It is then plain that the general solution of this system reads

$$\underline{x}(t) = \underline{X}[\tau(t)]; \quad x_n(t) = X_n[\tau(t)], \quad n = 1, \dots, N, \quad (6c)$$

where $\underline{X}(\tau)$ is the general solution of the system

$$\underline{X}' = \underline{h}(\underline{X}); \quad X'_n = h_n(\underline{X}), \quad n = 1, \dots, N, \quad (6d)$$

which coincides with (6a), except that here the appended prime indicates of course differentiation with respect to the variable τ of $\underline{X}(\tau)$. Hence whenever $\underline{h}(\underline{X})$ is such that the solution of this system of N nonlinear ODEs, (6d), exists globally — i.e., for all (real) values of the independent variable τ — then clearly (6c) imply that $\underline{x}(t)$ inherits from $\tau(t)$ the property to be either *isochronous* with period T , or *asymptotically isochronous* with the same period, or *multi-periodic*; hence the system (6b) also possesses the corresponding property.

However, the system (6b) is *not* autonomous. To eliminate this “defect”, we perform the *second step* of our treatment, replacing the system (6b) with the system

$$\dot{\underline{x}} = \varphi \underline{h}(\underline{x}); \quad \dot{x}_n = \varphi h_n(\underline{x}), \quad n = 1, \dots, N, \quad (7)$$

which is of course equivalent to (6b) provided the time-evolution of the scalar quantity φ is such that, by setting

$$\dot{\tau}(t) = \varphi, \quad (8a)$$

with (merely for convenience)

$$\tau(0) = 0, \quad (8b)$$

one obtains a scalar function $\tau(t)$ having the desired property: either *isochrony* or *asymptotic isochrony* or *multi-periodicity*.

There are now two options to obtain a quantity φ that qualifies for this purpose and makes the modified system (7) *autonomous*. The first option is to identify a *collective variable* $\varphi(\underline{x})$ that, as a consequence of the very evolution entailed by the dynamical system (7), has a time evolution, $\varphi(t) \equiv \varphi[\underline{x}(t)]$, such that, via (8), it defines a function $\tau(t)$ having the desired properties. This is, by and large, the approach followed in previous papers, where various ways to identify such a variable were analyzed in various contexts, mainly to manufacture *isochronous* systems [1–5]. A second — perhaps more obvious — possibility is to treat φ as an *additional* dependent variable, and to *extend* the system (7) by attaching to it a few additional ODEs involving φ and possibly other, *additional* dependent variables, so as to guarantee that the time evolution of φ yield, via (8), a function $\tau(t)$ having the desired properties. Specific instances of how to achieve this goal are detailed in the following Secs. 3–5, where it will also be shown how a *third step* of our treatment — consisting essentially in a change of the additional dependent variables, entangling them with the

original variables — is quite convenient in as much as it allows to manufacture rather large classes of extended dynamical systems having the required properties of *isochrony*, *asymptotic isochrony* or *multi-periodicity*.

It will moreover be clear — for instance from the very simple instances explicitly displayed in the following three Subsecs. 3.1, 4.1 and 5.1 — that in this manner it is possible to manufacture systems having a rather neat look, therefore likely to become useful tools in the context of mathematical modeling. The large freedom to manufacture such models should be quite clear from our treatment, see above and its developments presented below, where however we deliberately often opt for simplicity rather than generality. A terse outline of possible generalizations is reported in the last section.

3. Isochronous Systems

In this Sec. 3 we present a very simple way to implement the program outlined above, in order to extend a quite general (*autonomous*) dynamical system involving N ODEs such as (6a), into an *isochronous* system involving $N + 2$ ODEs — defined according to Definition 2.4 with Definition 2.1. A few examples are then presented in Subsec. 3.1.

We take as starting point the standard harmonic oscillator equations of motion:

$$\dot{f}_1 = \Omega f_2, \quad \dot{f}_2 = -\Omega f_1, \quad (9)$$

entailing of course

$$f_1(t) = A \cos(\Omega t) + B \sin(\Omega t), \quad (10a)$$

$$f_2(t) = -A \sin(\Omega t) + B \cos(\Omega t). \quad (10b)$$

Here and hereafter Ω is an arbitrary positive constant. These functions, $f_1(t)$ and $f_2(t)$, are of course *isochronous*, with period

$$T = \frac{2\pi}{\Omega}. \quad (11)$$

But let us, temporarily, forget the explicit expressions, (10), detailing their time-evolution, considering rather these functions f_1 and f_2 as *dependent variables* characterized by the simple ODEs (9).

Next let us consider the general system of ODEs in N variables

$$\dot{x}_n = f_n h_n(\underline{x}), \quad n = 1, \dots, N, \quad (12)$$

corresponding to (7) via the very simple assignment $\varphi = f_1$, which via (8) entails for $\tau(t)$ the expression

$$\tau(t) = \Omega^{-1} \{A \sin(\Omega t) + B[1 - \cos(\Omega t)]\} \quad (13a)$$

$$= \frac{\sqrt{A^2 + B^2}}{\Omega} \{ \sin[\Omega(t - t_0)] + \sin(\Omega t_0) \}, \quad (13b)$$

$$\tan(\Omega t_0) = \frac{B}{A}. \quad (13c)$$

The fact that this function is periodic with period T is of course plain. Hence the system of $N + 2$ ODEs constituted by (12) with (9) is generally *isochronous* with period T , indeed its general solution is provided by the formulas (6c) and (10).

Our *third step* is to introduce the two functions $y_1(t)$ and $y_2(t)$ via the definitions

$$f_1 = F_1(\underline{x})y_1, \quad f_2 = F_2(\underline{x})y_2, \quad (14)$$

where we reserve the privilege to assign the two functions $F_1(\underline{x})$ and $F_2(\underline{x})$ at our convenience.

The insertion of these definitions, (14), in (12) and (9) yields the following system of $N + 2$, generally nonlinear, ODEs for the $N + 2$ dependent variables $x_n(t)$, $y_1(t)$ and $y_2(t)$:

$$\dot{x}_n = y_1 F_1(\underline{x}) h_n(\underline{x}), \quad (15a)$$

$$\dot{y}_1 = \Omega y_2 \frac{F_2(\underline{x})}{F_1(\underline{x})} - y_1^2 \sum_{n=1}^N \left[\frac{\partial F_1(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (15b)$$

$$\dot{y}_2 = -y_1 \frac{F_1(\underline{x})}{F_2(\underline{x})} \left\{ \Omega + y_2 \sum_{n=1}^N \left[\frac{\partial F_2(\underline{x})}{\partial x_n} h_n(\underline{x}) \right] \right\}. \quad (15c)$$

Clearly the solution of the initial-value problem for this dynamical system is given by the formulas

$$\underline{x}(t) = \underline{X}[\tau(t)]; \quad x_n(t) = X_n[\tau(t)], \quad n = 1, \dots, N, \quad (16a)$$

$$y_1(t) = \frac{A \cos(\Omega t) + B \sin(\Omega t)}{F_1[\underline{x}(t)]}, \quad (16b)$$

$$y_2(t) = \frac{-A \sin(\Omega t) + B \cos(\Omega t)}{F_2[\underline{x}(t)]}. \quad (16c)$$

Here $\tau(t)$ is given by (13), while $\underline{X}(\tau)$ is the solution of the ODE (6d) with the initial condition $\underline{X}(0) = \underline{x}(0)$ (see (16a) and (8b) or (13)), and the two constants A and B , see (13), (16b) and (16c), are defined as follows:

$$A = F_1[\underline{x}(0)]y_1(0), \quad B = F_2[\underline{x}(0)]y_2(0). \quad (16d)$$

This solution shows clearly that, whenever $\underline{h}(\underline{X})$ is such that the solution of the system of N ODEs (6d) exists *globally* — i.e., for all (real) values of the dependent variable τ — then clearly $\underline{x}(t)$ is *isochronous* with period T . And this conclusion holds as well for the entire solution (16) of the system of $N + 2$ ODEs (15) — including also the two dependent variables $y_1(t)$ and $y_2(t)$ — provided the possibility that $y_1(t)$ or $y_2(t)$ blow-up due to the vanishing at some time of $F_1(t) \equiv F_1[\underline{x}(t)]$ or $F_2(t) \equiv F_2[\underline{x}(t)]$ can be excluded or disregarded. [The possibility that this be disregarded must be discussed on a case-by-case basis, as indicated by the example of the dynamical system described by the simple evolution ODE $\dot{y} = \Omega(1 + y^2)$ with initial condition $y(0) = 0$, yielding the solution $y(t) = \tan(\Omega t)$, which is of course periodic (for all time t) with period $T/2$, see (11). But if some phenomenological significance is associated with the dependent variable y , it might or it might not make good sense when the value of this quantity becomes infinite. In the former case, one can then conclude that this quantity displays a periodic evolution; in the latter this conclusion is not justified, in as much as the model loses its phenomenological significance at the time $T/4$, when $y(t)$ blows up].

Note that the conclusion about the *isochrony* of the system (15) remains valid even when the solution of the system (6d) cannot be obtained in explicit form, as is generally the case whenever the dynamical system (6a) yields a *chaotic* evolution (see Example 3.3 below). On the other hand whenever the solution of the system (6d) can be obtained in explicit form, the formulas (16) provide an explicit solution of the *isochronous* system (15) (see Examples 3.0, 3.1 and 3.2 below).

We leave as a simple exercise for the diligent reader to repeat the above treatment, but replacing the role played in (12) by f_1 with a linear combination of f_1 and f_2 , or even allowing these two functions to depend in a more complicated manner on both y_1 and y_2 than is entailed by (14); an outline of this, and other, more general approaches is provided in the last section. And of course the alert reader may now play and manufacture lots of *isochronous* systems, by inserting different choices of the $N + 2$ functions $h_n(\underline{x})$, $F_1(\underline{x})$, $F_2(\underline{x})$ in (15): we display some neat examples in the next subsection.

Before proceeding to report some examples let us also display the neater version of (15) corresponding to the special case with $F_1(\underline{x}) = F_2(\underline{x}) = F(\underline{x})$:

$$\dot{x}_n = y_1 F(\underline{x}) h_n(\underline{x}), \tag{17a}$$

$$\dot{y}_1 = \Omega y_2 - y_1^2 \sum_{n=1}^N \left[\frac{\partial F(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \tag{17b}$$

$$\dot{y}_2 = -y_1 \left\{ \Omega + y_2 \sum_{n=1}^N \left[\frac{\partial F(\underline{x})}{\partial x_n} h_n(\underline{x}) \right] \right\}. \tag{17c}$$

3.1. Examples of isochronous dynamical systems

The systems below (in this subsection) are *all*, for generic initial data, *isochronous* with period T , see (11). As it is clear from the above treatment, these examples are quite special cases of much more general *isochronous* systems.

Example 3.0. *2 coupled first order evolution ODEs:*

$$\dot{y}_1 = \Omega y_2 + \gamma y_1^2, \quad \dot{y}_2 = (-\Omega + \gamma y_2) y_1. \tag{18}$$

This system obtains from (17) with $N = 1$ via the extremely simple assignment

$$F(x) = x, \quad h(x) = -\gamma, \tag{19}$$

whereby the second and third equations (17) get decoupled from the first. It can also be reformulated as a single second-order ODE for the function $y = y_1$:

$$\ddot{y} + \Omega^2 y = \gamma(3\dot{y}y - \gamma y^3). \tag{20}$$

The *general solution* of this system, (18), reads

$$y_1(t) = \frac{\Omega \cos[\Omega(t - t_0)]}{\gamma\{C - \sin[\Omega(t - t_0)]\}}, \tag{21a}$$

$$y_2(t) = -\frac{\Omega \sin[\Omega(t - t_0)]}{\gamma\{C - \sin[\Omega(t - t_0)]\}}, \tag{21b}$$

with C and t_0 two arbitrary constants whose values are determined, in the context of the initial-value problem, by the initial data $y_1(0), y_2(0)$. The *isochronous* character (with period T , see (11)) of this solution is plain; but of course this solution blows up periodically unless $|C| > 1$. In view of its simplicity it is unlikely that this system is new.

Example 3.1. *3 coupled first-order evolution ODEs:*

$$\dot{x} = \gamma x^2 y_1, \tag{22a}$$

$$\dot{y}_1 = \Omega y_2 - \gamma x y_1^2, \tag{22b}$$

$$\dot{y}_2 = -(\Omega + \gamma x y_2) y_1. \tag{22c}$$

This system of 3 first-order ODEs can be reformulated as a system of 2 ODEs (one of first-order and one of second-order), for instance

$$\dot{x} = \gamma x^2 y_1, \tag{23a}$$

$$\ddot{y}_1 + \Omega^2 y_1 = -3\gamma x \dot{y}_1 y_1 - 2\gamma^2 x^2 y_1^3, \tag{23b}$$

or a single third-order ODE, for instance

$$\ddot{x} + \Omega^2 \dot{x} - 3\ddot{x}\dot{x}^{-1} + 2\dot{x}^3 x^{-2} = 0. \quad (24)$$

This system obtains from (17) by the quite simple assignment

$$N = 1, \quad F(x) = x, \quad h(x) = \gamma x. \quad (25)$$

With this assignment of $h(x)$ the ODE (6d) is trivially solvable. Hence the *general solution* of the system (22) can be explicitly exhibited:

$$x(t) = C \exp[\gamma\tau(t)], \quad (26a)$$

$$y_1(t) = C^{-1}[A \cos(\Omega t) + B \sin(\Omega t)] \exp[-\gamma\tau(t)], \quad (26b)$$

$$y_2(t) = C^{-1}[-A \sin(\Omega t) + B \cos(\Omega t)] \exp[-\gamma\tau(t)], \quad (26c)$$

with $\tau(t)$ defined by (13). The 3 constants A, B and C are of course determined, in the context of the *initial-value* problem, by the *initial data* $x(0), y_1(0), y_2(0)$. The *isochronous* character of this solution is plain.

Example 3.2. 4 coupled first-order evolution ODEs:

$$\dot{x}_1 = x_1 x_2 y_1, \quad (27a)$$

$$\dot{x}_2 = -\beta x_1^2 y_1, \quad (27b)$$

$$\dot{y}_1 = \Omega y_2 - x_2 y_1^2, \quad (27c)$$

$$\dot{y}_2 = -\Omega y_1 - x_2 y_1 y_2. \quad (27d)$$

This system of 4 coupled first-order ODEs can be replaced by a system of 2 second-order coupled ODEs, for instance

$$\ddot{x}_1 = \left(\frac{\dot{x}_1}{x_1} + \frac{\dot{y}_1}{y_1} \right) \dot{x}_1 - \beta x_1^3 y_1^2, \quad (28a)$$

$$\ddot{y}_1 + \Omega^2 y_1 = -3 \left(\frac{\dot{y}_1}{y_1} \right) \dot{x}_1 - \left(\frac{\dot{x}_1}{x_1} \right)^2 y_1 + \beta x_1^2 y_1^3. \quad (28b)$$

These equations arise from (17) with $N = 2$ and the simple assignment

$$F(\underline{x}) = x_1, \quad h_1(\underline{x}) = x_2, \quad h_2(\underline{x}) = -\beta x_1. \quad (29a)$$

With this assignment of $h_1(\underline{x})$ and $h_2(\underline{x})$ the linear system of 2 ODEs (6d) is trivially solvable. Thereby the *general solution* of this model, (27), is easily seen to read as follows:

$$x_1(t) = C \cos[\omega\tau(t)] + D \sin[\omega\tau(t)], \quad (30a)$$

$$x_2(t) = \omega \{-C \sin[\omega\tau(t)] + D \cos[\omega\tau(t)]\}, \quad (30b)$$

$$y_1(t) = \frac{A \cos(\Omega t) + B \sin(\Omega t)}{C \cos[\omega\tau(t)] + D \sin[\omega\tau(t)]}, \quad (30c)$$

$$y_2(t) = \frac{-A \sin(\Omega t) + B \cos(\Omega t)}{C \cos[\omega\tau(t)] + D \sin[\omega\tau(t)]}, \quad (30d)$$

with $\tau(t)$ defined again by (13). The 4 constants A, B, C and D are of course determined, in the context of the *initial-value* problem, by the *initial data* $x_1(0), x_2(0), y_1(0), y_2(0)$, while $\omega = \sqrt{\beta}$. Of course this constant ω is *real* (for definiteness, *positive*) only if β is *positive*; it is instead *imaginary* if β is *negative*, but this does not spoil the *real* character of this solution (with A, B, C *real* and D

imaginary), nor its *isochrony* with period T , see (11). On the other hand this solution is *nonsingular* for all time t only if the denominator in the last two formulas never vanishes, which, if ω is *real*, is only possible — for an appropriate range of values of the two constants C and D — provided the two constants A and B satisfy the restriction

$$2\sqrt{A^2 + B^2} < \frac{\pi\Omega}{\omega}, \quad (30e)$$

see (13).

In the following we will generally omit to discuss the possibility that the solution of the model under consideration experiences divergences, leaving this issue to be treated on a case-by-case basis by the diligent reader.

Example 3.3. *5 coupled first-order evolution ODEs:*

$$\dot{x}_1 = -\alpha x_1(x_1 - x_2)y_1, \quad (31a)$$

$$\dot{x}_2 = x_1(\beta x_1 - x_2 - x_1 x_3)y_1, \quad (31b)$$

$$\dot{x}_3 = x_1(x_1 x_2 - \gamma x_3)y_1, \quad (31c)$$

$$\dot{y}_1 = \Omega y_2 + \alpha(x_1 - x_2)y_1^2, \quad (31d)$$

$$\dot{y}_2 = -\Omega y_1 + \alpha(x_1 - x_2)y_1 y_2. \quad (31e)$$

The 2 first-order ODEs (31d) and (31e) can be replaced by the single second-order ODE

$$\ddot{y}_1 + \Omega^2 y_1 = 3\alpha(x_1 - x_2)\dot{y}_1 y_1 - \alpha[(1 + \alpha + \beta)x_1^2 - 2(1 + \alpha)x_1 x_2 + \alpha x_2^2 - x_1^2 y_1]y_1^3. \quad (32)$$

This system obtains from (17) with $N = 3$ via the assignment

$$F(\underline{x}) = x_1, \quad (33a)$$

$$h_1(\underline{x}) = -\alpha(x_1 - x_2), \quad (33b)$$

$$h_2(\underline{x}) = \beta x_1 - x_2 - x_1 x_3, \quad (33c)$$

$$h_3(\underline{x}) = x_1 x_2 - \gamma x_3, \quad (33d)$$

entailing that (6d) becomes the Lorenz system [6] (see (34) below), which is well-known to exhibit *chaotic* behavior for suitable values of the parameters α , β and γ .

For $\Omega = 0$ one can set $y_2 = 0$ and $y_1 = 1/x_1$: indeed then (31e) is trivially satisfied, (31d) coincides with (31a), while the 3 ODEs (31a), (31b) and (31c) coincide (up to notational changes) with the 3 ODEs of the Lorenz model [6, 7],

$$X'_1 = -\alpha(X_1 - X_2), \quad (34a)$$

$$X'_2 = \beta X_1 - X_2 - X_1 X_3, \quad (34b)$$

$$X'_3 = X_1 X_2 - \gamma X_3. \quad (34c)$$

Here the appended prime indicates of course differentiation with respect to the independent variable τ .

The solution of our *isochronous* model (31) can be exhibited, but only in terms of the solutions $X_n(\tau)$ of the Lorenz model (34) (with $X_n(0) = x_n(0)$, $n = 1, 2, 3$):

$$x_n(t) = X_n[\tau(t)], \quad n = 1, 2, 3, \quad (35a)$$

$$y_1(t) = \frac{A \cos(\Omega t) + B \sin(\Omega t)}{x_1(t)}, \quad (35b)$$

$$y_2(t) = \frac{\Omega \{-A \sin(\Omega t) + B \cos(\Omega t)\}}{x_1(t)}, \quad (35c)$$

again with $\tau(t)$ defined by (13). But let us emphasize that these formulas do *not* yield the solutions of the model (31) explicitly, because the solutions $X_n(\tau)$ of the Lorenz model (34) are generally unknown (except in very special cases), indeed the time evolution of the Lorenz model is generally *chaotic*; but since these solutions *exist*, the formulas (35) reveal, via (13), the *isochronous* character (with period T , see (11)) of the solutions of the model (31) apart from the possible appearance of singularities: these may arise either through the vanishing of the denominators in the last two formulas (35), as discussed above, see Example 3.2, or possibly due to singularities in the solutions of the Lorenz system (34) when its independent variable τ evolves backwards. This latter possibility is discussed in Sec. 6.

4. *Asymptotically Isochronous Systems*

In this Sec. 4 we show how to extend a quite general (*autonomous*) dynamical system involving N ODEs such as (6a), into an *asymptotically isochronous* system involving $N + 3$ ODEs — defined of course according to Definition 2.4 with Definition 2.2. Since the procedure is analogous to that described in Sec. 3 to obtain an extended system which is *isochronous*, the following treatment is terse. We display two alternative routes, both quite simple, yielding somewhat different results; the alert reader will have no difficulty to devise additional ones (see also Sec. 7). A few simple examples are then presented in Subsec. 4.1.

The first alternative we report uses as starting point the following 3 ODEs:

$$\dot{f}_1 = \Omega f_2, \quad (36a)$$

$$\dot{f}_2 = -\Omega f_3, \quad (36b)$$

$$\dot{f}_3 = -\eta f_3 - \Omega f_2 - \eta f_1, \quad (36c)$$

entailing of course

$$f_1(t) = A \cos(\Omega t) + B \sin(\Omega t) + C \exp(-\eta t), \quad (37a)$$

$$f_2(t) = -A \sin(\Omega t) + B \cos(\Omega t) - C(\eta/\Omega) \exp(-\eta t), \quad (37b)$$

$$f_3(t) = A \cos(\Omega t) + B \sin(\Omega t) - C(\eta/\Omega)^2 \exp(-\eta t). \quad (37c)$$

Here and hereafter η (and of course Ω) are two *positive* constants, entailing — see Definition 2.2 — that this solution is *asymptotically isochronous* with period T , see (11).

Let us now introduce the — clearly equally *asymptotically isochronous* — function

$$\tilde{\tau}(t) = \Omega^{-1} \{A \sin(\Omega t) + B[1 - \cos(\Omega t)]\} + C\eta^{-1}[1 - \exp(-\eta t)], \quad (38a)$$

such that

$$\dot{\tilde{\tau}} = f_1 \quad (38b)$$

and $\tilde{\tau}(0) = 0$. We then proceed analogously to what we did above, see the treatment starting with (12), where we of course replace now (14) with

$$f_m = F_m(\underline{x})y_m, \quad m = 1, 2, 3. \quad (39)$$

We thus get the following system of $N + 3$ coupled ODEs:

$$\dot{x}_n = y_1 F_1(\underline{x})h_n(\underline{x}), \quad (40a)$$

$$\dot{y}_1 = \Omega y_2 \frac{F_2(\underline{x})}{F_1(\underline{x})} - y_1^2 \sum_{n=1}^N \left[\frac{\partial F_1(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (40b)$$

$$\dot{y}_2 = -\Omega y_3 \frac{F_3(\underline{x})}{F_2(\underline{x})} - y_1 y_2 \frac{F_1(\underline{x})}{F_2(\underline{x})} \sum_{n=1}^N \left[\frac{\partial F_2(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (40c)$$

$$\dot{y}_3 = -\eta y_3 - \Omega y_2 \frac{F_2(\underline{x})}{F_3(\underline{x})} - y_1 \frac{F_1(\underline{x})}{F_3(\underline{x})} \left\{ \eta + y_3 \sum_{n=1}^N \left[\frac{\partial F_3(\underline{x})}{\partial x_n} h_n(\underline{x}) \right] \right\}. \quad (40d)$$

This derivation entails that this system is generally — for a large class of the $3 + N$ functions $F_1(\underline{x}), F_2(\underline{x}), F_3(\underline{x})$ and $h_n(\underline{x})$ — *asymptotically isochronous*, with its general solution reading as follows:

$$\underline{x}(t) = \underline{X}[\tilde{\tau}(t)]; \quad x_n(t) = X_n[\tilde{\tau}(t)], \quad (41a)$$

$$y_1(t) = \frac{A \cos(\Omega t) + B \sin(\Omega t) + C \exp(-\eta t)}{F_1[\underline{x}(t)]}, \quad (41b)$$

$$y_2(t) = \frac{-A \sin(\Omega t) + B \cos(\Omega t) - C(\eta/\Omega) \exp(-\eta t)}{F_2[\underline{x}(t)]}, \quad (41c)$$

$$y_3(t) = \frac{A \cos(\Omega t) + B \sin(\Omega t) - C(\eta/\Omega)^2 \exp(-\eta t)}{F_3[\underline{x}(t)]}, \quad (41d)$$

where of course $\underline{X}(\tau)$ is the general solution of (6d) and $\tilde{\tau}(t)$ is defined by (38a). And clearly this system is *explicitly solvable* whenever the system (6d) is itself *explicitly solvable* (see below Examples 4.1 and 4.2).

Again, let us also display the neater form taken by this *asymptotically isochronous* system, (40), in the special case with $F_1(\underline{x}) = F_2(\underline{x}) = F_3(\underline{x}) = F(\underline{x})$:

$$\dot{x}_n = y_1 F(\underline{x}) h_n(\underline{x}), \quad (42a)$$

$$\dot{y}_1 = \Omega y_2 - y_1^2 \sum_{n=1}^N \left[\frac{\partial F(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (42b)$$

$$\dot{y}_2 = -\Omega y_3 - y_1 y_2 \sum_{n=1}^N \left[\frac{\partial F(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (42c)$$

$$\dot{y}_3 = -\eta y_3 - \Omega y_2 - \eta y_1 - y_1 y_3 \sum_{n=1}^N \left[\frac{\partial F(\underline{x})}{\partial x_n} h_n(\underline{x}) \right]. \quad (42d)$$

The second alternative we report takes as starting point, in place of the system of 3 ODEs (36), the following 3 ODEs:

$$\dot{f}_1 = \Omega f_2, \quad (43a)$$

$$\dot{f}_2 = -\Omega f_1, \quad (43b)$$

$$\dot{f}_3 = \eta(1 - f_3^2). \quad (43c)$$

Note that the first 2 of these 3 ODEs coincide with the 2 (linear) ODEs (9), while the third is instead nonlinear (yet also trivially solvable). The general solution of this system can of course be explicitly displayed, for instance as follows:

$$f_1(t) = C \sin[\Omega(t - t_0)], \quad (44a)$$

$$f_2(t) = C \cos[\Omega(t - t_0)], \quad (44b)$$

$$f_3(t) = \tanh[\eta(t - t_1)], \quad (44c)$$

where C, t_0, t_1 are 3 arbitrary constants that can be fixed in terms of the initial data. We moreover introduce the function

$$\begin{aligned} \check{\tau}(t) &= \Omega^{-1}[f_1(t)f_3(t) - f_1(0)f_3(0)] \\ &= \frac{C}{\Omega}\{\sin[\Omega(t - t_0)] \tanh[\eta(t - t_1)] - \sin(\Omega t_0) \tanh(\eta t_1)\}, \end{aligned} \quad (45)$$

which is clearly *asymptotically isochronous* (and clearly such that $\check{\tau}(0) = 0$); and we notice that the first and third of the 3 ODEs (43) entail the formula

$$\dot{\check{\tau}} = f_2 f_3 + (\eta/\Omega) f_1 (1 - f_3^2). \quad (46)$$

Hence we conclude that, for a largely arbitrary assignment of the N functions $h_n(\underline{x})$, the *autonomous* system of N ODEs

$$\dot{x}_n = [f_2 f_3 + (\eta/\Omega) f_1 (1 - f_3^2)] h_n(\underline{x}), \quad n = 1, \dots, N, \quad (47)$$

is *asymptotically isochronous* provided the 3 functions f_1, f_2 and f_3 evolve according to the system (43). Then we replace the 3 dependent variables f_m with the 3 new dependent variables y_m via the assignment (39). And we thereby obtain the following, *asymptotically isochronous*, system of $N + 3$ ODEs satisfied by the $N + 3$ dependent variables x_n and y_m :

$$\dot{x}_n = [y_2 y_3 F_2(\underline{x}) F_3(\underline{x}) + (\eta/\Omega) y_1 F_1(\underline{x}) (1 - y_3^2 [F_3(\underline{x})]^2)] h_n(\underline{x}), \quad (48a)$$

$$\dot{y}_1 = \Omega y_2 \frac{F_2(\underline{x})}{F_1(\underline{x})} - y_1 \left[\frac{F_2(\underline{x}) F_3(\underline{x})}{F_1(\underline{x})} y_2 y_3 + (\eta/\Omega) y_1 (1 - y_3^2 [F_3(\underline{x})]^2) \right] \sum_{n=1}^N \left[\frac{\partial F_1(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (48b)$$

$$\dot{y}_2 = -\Omega y_1 \frac{F_1(\underline{x})}{F_2(\underline{x})} - y_2 \left[F_3(\underline{x}) y_2 y_3 + (\eta/\Omega) y_1 \frac{F_1(\underline{x})}{F_2(\underline{x})} (1 - y_3^2 [F_3(\underline{x})]^2) \right] \sum_{n=1}^N \left[\frac{\partial F_2(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (48c)$$

$$\dot{y}_3 = \eta \frac{1 - y_3^2 [F_3(\underline{x})]^2}{F_3(\underline{x})} - y_3 \left[y_2 y_3 F_2(\underline{x}) + (\eta/\Omega) y_1 \frac{F_1(\underline{x})}{F_3(\underline{x})} (1 - y_3^2 [F_3(\underline{x})]^2) \right] \sum_{n=1}^N \left[\frac{\partial F_3(\underline{x})}{\partial x_n} h_n(\underline{x}) \right]. \quad (48d)$$

This derivation entails that this system is generally — for a large class of the $3 + N$ functions $F_1(\underline{x}), F_2(\underline{x}), F_3(\underline{x})$ and $h_n(\underline{x})$ — *asymptotically isochronous*, with its general solution reading as follows:

$$\underline{x}(t) = \underline{X}[\check{\tau}(t)]; \quad x_n(t) = X_n[\check{\tau}(t)], \quad (49a)$$

$$y_1(t) = \frac{C \sin[\Omega(t - t_0)]}{F_1[\underline{x}(t)]}, \quad (49b)$$

$$y_2(t) = \frac{C \cos[\Omega(t - t_0)]}{F_2[\underline{x}(t)]}, \quad (49c)$$

$$y_3(t) = \frac{\tanh[\eta(t - t_1)]}{F_3[\underline{x}(t)]}, \quad (49d)$$

where of course $\underline{X}(\tau)$ is the general solution of (6d) and $\check{\tau}(t)$ is defined by (45). And clearly this system is *explicitly solvable* whenever the system (6d) is itself *explicitly solvable* (see below Examples 4.4 and 4.5).

Again, let us also display the neater form taken by this *asymptotically isochronous* system, (48), in the special case with $F_1(\underline{x}) = F_2(\underline{x}) = F_3(\underline{x}) = F(\underline{x})$:

$$\dot{x}_n = \{y_2 y_3 [F(\underline{x})]^2 + (\eta/\Omega) y_1 F(\underline{x}) (1 - y_3^2 [F(\underline{x})]^2)\} h_n(\underline{x}), \quad (50a)$$

$$\dot{y}_1 = \Omega y_2 - y_1 [F(\underline{x}) y_2 y_3 + (\eta/\Omega) y_1 (1 - y_3^2 [F(\underline{x})]^2)] \sum_{n=1}^N \left[\frac{\partial F(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (50b)$$

$$\dot{y}_2 = -\Omega y_1 - y_2 [F(\underline{x}) y_2 y_3 + (\eta/\Omega) y_1 (1 - y_3^2 [F(\underline{x})]^2)] \sum_{n=1}^N \left[\frac{\partial F(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (50c)$$

$$\dot{y}_3 = \eta \frac{1 - y_3^2 [F(\underline{x})]^2}{F(\underline{x})} - y_3 [y_2 y_3 F(\underline{x}) + (\eta/\Omega) y_1 (1 - y_3^2 [F(\underline{x})]^2)] \sum_{n=1}^N \left[\frac{\partial F(\underline{x})}{\partial x_n} h_n(\underline{x}) \right]. \quad (50d)$$

In the following Subsec. 4.1 we report a few simple examples of *asymptotically isochronous* systems, obtained via the technique described herein.

4.1. *Examples of asymptotically isochronous systems*

In this Subsec. 4.1 we exhibit several neat examples of *asymptotically isochronous* systems (and some limit cases which are neither *isochronous* nor *asymptotically isochronous*, yet sufficiently neat to deserve explicit mention): it is again clear from the above treatment that these examples are quite special cases of much more general *asymptotically isochronous* systems. Hereafter η is an arbitrary *positive* constant, $\eta > 0$ (but occasionally we will consider the limiting case with $\eta = 0$). The first half of these examples obtain from (40) (in fact, from the simpler system (42)), and the subsequent ones from (48) (in fact, from the simpler system (50)); indeed all of them via the very simple assignment

$$F_m(\underline{x}) = x_m, \quad m = 1, 2, 3. \quad (51)$$

And the assignments of the functions $h_n(\underline{x})$ are the same as for the analogous examples reported in Subsec. 3.1.

Example 4.1. *3 coupled first-order evolution ODEs:*

$$\dot{y}_1 = \Omega y_2 + \gamma y_1^2, \quad (52a)$$

$$\dot{y}_2 = -\Omega y_3 + \gamma y_1 y_2, \quad (52b)$$

$$\dot{y}_3 = -\eta y_3 - \Omega y_2 - \eta y_1 + \gamma y_1 y_3. \quad (52c)$$

The *general solution* of this system reads as follows:

$$y_1(t) = \frac{A\Omega \cos(\Omega t) + B\Omega \sin(\Omega t) + C\eta \exp(-\eta t)}{\gamma\{1 - A \sin(\Omega t) + B[1 - \cos(\Omega t)] + C[1 - \exp(-\eta t)]\}}, \quad (53a)$$

$$y_2(t) = \frac{-A\Omega \sin(\Omega t) + B\Omega \cos(\Omega t) - C\eta(\eta/\Omega) \exp(-\eta t)}{\gamma\{1 - A \sin(\Omega t) + B[1 - \cos(\Omega t)] + C[1 - \exp(-\eta t)]\}}, \quad (53b)$$

$$y_3(t) = \frac{A\Omega \cos(\Omega t) + B\Omega \sin(\Omega t) - C\eta(\eta/\Omega)^2 \exp(-\eta t)}{\gamma\{1 - A \sin(\Omega t) + B[1 - \cos(\Omega t)] + C[1 - \exp(-\eta t)]\}}. \quad (53c)$$

The 3 constants A, B , and C are of course determined, in the context of the initial-value problem, by the initial data $y_1(0), y_2(0), y_3(0)$. The *asymptotically isochronous* character of this solution is plain, and the restriction

$$|A| + |B| + |C| < 1 \quad (53d)$$

is clearly sufficient to exclude that, for $t > 0$, the 3 dependent variables $y_m(t)$ blow up. There is however a two-parameter subset of solutions which are *isochronous*, namely those with $C = 0$.

Example 4.1bis. 3 coupled first-order evolution ODEs:

$$\dot{y}_1 = \Omega y_2 + \gamma y_1^2, \quad (54a)$$

$$\dot{y}_2 = \Omega y_3 + \gamma y_1 y_2, \quad (54b)$$

$$\dot{y}_3 = -\Omega y_2 + \gamma y_1 y_3. \quad (54c)$$

The general solution of this system reads as follows:

$$y_1(t) = \frac{A\Omega \cos(\Omega t) + B\Omega \sin(\Omega t) + c}{\gamma\{1 - A \sin(\Omega t) + B[1 - \cos(\Omega t)] + ct\}}, \quad (55a)$$

$$y_2(t) = \frac{-A\Omega \sin(\Omega t) + B\Omega \cos(\Omega t)}{\gamma\{1 - A \sin(\Omega t) + B[1 - \cos(\Omega t)] + ct\}}, \quad (55b)$$

$$y_3(t) = \frac{A\Omega \cos(\Omega t) + B\Omega \sin(\Omega t)}{\gamma\{1 - A \sin(\Omega t) + B[1 - \cos(\Omega t)] + ct\}}. \quad (55c)$$

Here A, B, c are 3 arbitrary constants. The subset of solutions with $c = 0$ is clearly *isochronous*, while the solutions with $c \neq 0$ are neither *isochronous* nor *asymptotically isochronous*: they tend asymptotically to the equilibrium configuration $y_1 = y_2 = y_3 = 0$. This system is clearly the limiting case of Example 4.1, with $\eta = 0$ (and $C\eta = c$).

Example 4.2. 4 coupled first-order evolution ODEs:

$$\dot{x} = \gamma x^2 y_1, \quad (56a)$$

$$\dot{y}_1 = \Omega y_2 - \gamma x y_1^2, \quad (56b)$$

$$\dot{y}_2 = -\Omega y_3 - \gamma x y_1 y_2, \quad (56c)$$

$$\dot{y}_3 = -\eta y_3 - \Omega y_2 - (\eta + \gamma x y_3) y_1. \quad (56d)$$

Hereafter we assume γ to be a *nonvanishing* real constant. The general solution of this system reads as follows:

$$x(t) = D \exp[\gamma \tilde{\tau}(t)], \quad (57a)$$

$$y_1(t) = \frac{A \cos(\Omega t) + B \sin(\Omega t) + C \exp(-\eta t)}{D} \exp[-\gamma \tilde{\tau}(t)], \quad (57b)$$

$$y_2(t) = \left[-A \sin(\Omega t) + B \cos(\Omega t) - C \frac{\eta}{\Omega} \exp(-\eta t) \right] \frac{\exp[-\gamma \tilde{\tau}(t)]}{D}, \quad (57c)$$

$$y_3(t) = \left[A \cos(\Omega t) + B \sin(\Omega t) - C \frac{\eta^3}{\Omega^2} \exp(-\eta t) \right] \frac{\exp[-\gamma \tilde{\tau}(t)]}{D}, \quad (57d)$$

now with $\tilde{\tau}(t)$ defined by (38a). The 4 constants A, B, C and D are of course determined, in the context of the *initial-value* problem, by the *initial* data $x(0), y_1(0), y_2(0), y_3(0)$. It is again plain that these solutions are generally *asymptotically isochronous*, except for their subset with $C = 0$, which are *isochronous*.

Example 4.2bis. 4 coupled first-order evolution ODEs:

$$\dot{x} = \gamma x^2 y_1, \quad (58a)$$

$$\dot{y}_1 = \Omega y_2 - \gamma x y_1^2, \quad (58b)$$

$$\dot{y}_2 = -\Omega y_3 - \gamma x y_1 y_2, \quad (58c)$$

$$\dot{y}_3 = -\Omega y_2 - \gamma x y_3 y_1. \quad (58d)$$

This system is the limit case of Example 4.1 with $\eta = 0$. Accordingly, its *general solution* is given by the *general solution* of Example 4.1 with $\eta = 0$ and $\tilde{\tau}(t)$ replaced by its limiting expression with $\eta = 0$,

$$\tilde{\tau}_0(t) = \Omega^{-1}\{A \sin(\Omega t) + B[1 - \cos(\Omega t)]\} + Ct, \quad (59)$$

see (38a). The subset of these solutions with $C = 0$ are *isochronous*, but those with $C \neq 0$ spiral away to infinity as $t \rightarrow \infty$: more specifically, the dependent variable $x(t)$ does so if $\gamma C > 0$ while the other 3 dependent variables $y_m(t)$ spiral to zero, and vice versa if $\gamma C < 0$.

Example 4.3. *6 coupled first-order evolution ODEs.* The first 3 ODEs of this model coincide with the first 3 of the 5 ODEs (31), and the last 3 read as follows:

$$\dot{y}_1 = \Omega y_2 + \alpha(x_1 - x_2)y_1^2, \quad (60a)$$

$$\dot{y}_2 = -\Omega y_3 + \alpha x_1(x_1 - x_2)y_1 y_2, \quad (60b)$$

$$\dot{y}_3 = -\eta y_3 - \Omega y_2 - [\eta - \alpha x_1(x_1 - x_2)y_3]y_1. \quad (60c)$$

Note that also the first one of these 3 ODEs coincide with the fourth one of the 5 ODEs (31).

As in the case of Example 3.3 (see Subsec. 3.1), the solutions of this model can be formally written in terms of the solutions $X_n(t)$ of the Lorenz model (34) (with $X_n(0) = x_n(0)$, $n = 1, 2, 3$), as follows:

$$x_n(t) = X_n[\tilde{\tau}(t)], \quad (61a)$$

$$y_1(t) = \frac{A \cos(\Omega t) + B \sin(\Omega t) + C \exp(-\eta t)}{x_1(t)}, \quad (61b)$$

$$y_2(t) = \frac{-A \sin(\Omega t) + B \cos(\Omega t) - C(\eta/\Omega) \exp(-\eta t)}{x_1(t)}, \quad (61c)$$

$$y_3(t) = \frac{A \cos(\Omega t) + B \sin(\Omega t) - C(\eta/\Omega)^2 \exp(-\eta t)}{x_1(t)}, \quad (61d)$$

with $\tilde{\tau}(t)$ defined by (38a). And (as in the preceding Example 4.2) the considerations made above at the end of Example 3.3 (see Subsec. 3.1) are again applicable, up to obvious adjustments.

Example 4.4. *4 coupled first-order evolution ODEs:*

$$\dot{x} = -\gamma x[y_2 y_3 x + (\eta/\Omega)y_1(1 - y_3^2 x^2)], \quad (62a)$$

$$\dot{y}_1 = \Omega y_2 + \gamma y_1[x y_2 y_3 + (\eta/\Omega)y_1(1 - y_3^2 x^2)], \quad (62b)$$

$$\dot{y}_2 = -\Omega y_1 + \gamma y_2[x y_2 y_3 + (\eta/\Omega)y_1(1 - y_3^2 x^2)], \quad (62c)$$

$$\dot{y}_3 = \eta \frac{1 - y_3^2 x^2}{x} + \gamma y_3[x y_2 y_3 + (\eta/\Omega)y_1(1 - y_3^2 x^2)]. \quad (62d)$$

The *general solution* of this system reads as follows:

$$x(t) = A - B \sin[\Omega(t - t_0)] \tanh[\eta(t - t_1)], \quad (63a)$$

$$y_1(t) = \frac{B \sin[\Omega(t - t_0)]}{A - B \sin[\Omega(t - t_0)] \tanh[\eta(t - t_1)]}, \quad (63b)$$

$$y_2(t) = \frac{B \cos[\Omega(t - t_0)]}{A - B \sin[\Omega(t - t_0)] \tanh[\eta(t - t_1)]}, \quad (63c)$$

$$y_3(t) = \frac{\tanh[\eta(t - t_1)]}{A - B \sin[\Omega(t - t_0)] \tanh[\eta(t - t_1)]}. \quad (63d)$$

Here of course A, B, t_0 and t_1 are 4 *arbitrary* constants which can be adjusted to fit the initial data $x(0)$ and $y_m(0), m = 1, 2, 3$. The *asymptotically isochronous* character of this solution is plain; a sufficient condition to exclude that this solution blow up is provided by the restriction $|A| > |B|$. The two subsets of solutions obtained by setting $t_1 = \pm\infty$ hence replacing the tanh functions with ∓ 1 are of course *isochronous*.

As a side observation (also useful for later reference) the diligent reader may check that the 3 quantities

$$Q_1 = x^2 (y_1^2 + y_2^2), \quad (64a)$$

$$Q_2 = x (1 + xy_1y_3), \quad (64b)$$

$$Q_3 = \frac{\operatorname{arctanh}(xy_3)}{\eta} - \frac{\operatorname{arctan}(y_1/y_2)}{\Omega}, \quad (64c)$$

provide 3, functionally independent, conserved quantities for this dynamical system, (62).

Example 4.5. 4 coupled first-order evolution ODEs:

$$\dot{x} = \gamma x^2 [xy_2y_3 + (\eta/\Omega)y_1(1 - x^2y_3^2)], \quad (65a)$$

$$\dot{y}_1 = \Omega y_2 - \gamma xy_1 [xy_2y_3 + (\eta/\Omega)y_1(1 - x^2y_3^2)], \quad (65b)$$

$$\dot{y}_2 = -\Omega y_1 - \gamma xy_2 [xy_2y_3 + (\eta/\Omega)y_1(1 - x^2y_3^2)], \quad (65c)$$

$$\dot{y}_3 = \eta(x^{-1} - xy_3^2) - \gamma xy_3 [xy_2y_3 + (\eta/\Omega)y_1(1 - x^2y_3^2)]. \quad (65d)$$

The *general solution* of this system reads as follows:

$$x(t) = A \exp[\gamma \check{\tau}(t)], \quad (66a)$$

$$y_1(t) = C \sin[\Omega(t - t_0)] \exp[-\gamma \check{\tau}(t)], \quad (66b)$$

$$y_2(t) = C \cos[\Omega(t - t_0)] \exp[-\gamma \check{\tau}(t)], \quad (66c)$$

$$y_3(t) = \tanh[\eta(t - t_1)] \exp[-\gamma \check{\tau}(t)], \quad (66d)$$

with $\check{\tau}(t)$ defined by (45). Here of course A, C, t_0 and t_1 are 4 arbitrary constants which can be adjusted to fit the initial data $x(0)$ and $y_m(0), m = 1, 2, 3$.

Example 4.6. 6 coupled first-order evolution ODEs:

$$\dot{x}_1 = -\alpha(x_1 - x_2)x_1[x_1y_2y_3 + (\eta/\Omega)y_1(1 - x_1^2y_3^2)], \quad (67a)$$

$$\dot{x}_2 = (\beta x_1 - x_2 - x_1x_3)x_1[x_1y_2y_3 + (\eta/\Omega)y_1(1 - x_1^2y_3^2)], \quad (67b)$$

$$\dot{x}_3 = (x_1x_2 - \gamma x_3)x_1[x_1y_2y_3 + (\eta/\Omega)y_1(1 - x_1^2y_3^2)], \quad (67c)$$

$$\dot{y}_1 = \Omega y_2 + \alpha(x_1 - x_2)y_1[x_1y_2y_3 + (\eta/\Omega)y_1(1 - x_1^2y_3^2)], \quad (67d)$$

$$\dot{y}_2 = -\Omega y_1 + \alpha(x_1 - x_2)y_2[x_1y_2y_3 + (\eta/\Omega)y_1(1 - x_1^2y_3^2)], \quad (67e)$$

$$\dot{y}_3 = \eta(x_1^{-1} - x_1y_3^2) + \alpha(x_1 - x_2)y_3[x_1y_2y_3 + \eta y_1(1 - x_1^2y_3^2)]. \quad (67f)$$

As in the cases of Example 3.3 (see Subsec. 3.1) and of Example 4.4, the solutions of this model can be formally written in terms of the solutions of the Lorenz model (34) (with $X_n(0) = x_n(0), n = 1, 2, 3$), as follows:

$$x_n(t) = X_n[\check{\tau}(t)], \tag{68a}$$

$$y_1(t) = \frac{C \sin[\Omega(t - t_0)]}{x_1(t)}, \tag{68b}$$

$$y_2(t) = \frac{C \cos[\Omega(t - t_0)]}{x_1(t)}, \tag{68c}$$

$$y_3(t) = \frac{\tanh[\eta(t - t_1)]}{x_1(t)}, \tag{68d}$$

with $\check{\tau}(t)$ defined by (45) and the 3, *a priori* arbitrary, constants C, t_0 and t_1 determined by the *initial* data $y_m(0), m = 1, 2, 3$, as well as $x_1(0)$. The two subsets of solutions corresponding to $t_1 = \pm\infty$ hence to the replacement of the function \tanh with ∓ 1 in the last of these formulas and in the definition (45) of $\check{\tau}(t)$ are of course *isochronous*.

5. Multi-Periodic Systems

In this Sec. 5 we show how to extend a quite general (*autonomous*) dynamical system involving N ODEs such as (6a), into a *multi-periodic*, as well *autonomous*, system involving $N + 4$ ODEs — defined of course according to Definition 2.4 with Definition 2.3. Since the procedure is analogous to that described above in Secs. 3 and 4 to obtain *isochronous* and *asymptotically isochronous* systems, the following treatment is quite terse. We report only one procedure, and display in Subsec. 5.1 only two simple examples; the alert reader will have no difficulty to devise other analogous procedures (possibly also using the hints provided in Sec. 7) as well as additional examples.

We take as starting point the standard equations of motion of two harmonic oscillators with different frequencies — which shall have to be *noncongruent* in order that the extended dynamical system we will manufacture be indeed *multi-periodic* and *not* isochronous.

So we start from the following 4 linear ODEs:

$$\dot{f}_1 = \Omega f_2, \quad \dot{f}_2 = -\Omega f_1, \tag{68e}$$

$$\dot{f}_3 = \lambda\Omega f_4, \quad \dot{f}_4 = -\lambda\Omega f_3, \tag{68f}$$

entailing of course

$$f_1(t) = A \cos(\Omega t) + B \sin(\Omega t), \tag{69a}$$

$$f_2(t) = -A \sin(\Omega t) + B \cos(\Omega t), \tag{69b}$$

$$f_3(t) = C \cos(\lambda\Omega t) + D \sin(\lambda\Omega t), \tag{69c}$$

$$f_4(t) = -C \sin(\lambda\Omega t) + D \cos(\lambda\Omega t). \tag{69d}$$

Here, as usual, Ω is an arbitrary positive constant (dimensionally, an inverse time), and λ is a positive *irrational* number. The two functions $f_1(t)$ and $f_2(t)$ are of course periodic with period T , see (11), while the two functions $f_3(t)$ and $f_4(t)$ are as well periodic but with the *noncongruent* period T/λ .

We then introduce the *multi-periodic* function $\hat{\tau}(t)$ via the very simple assignment

$$\hat{\tau}(t) = \frac{A \sin(\Omega t) + B[1 - \cos(\Omega t)]}{\Omega} + \left(\frac{\mu}{\lambda}\right) \frac{C \sin(\lambda\Omega t) + D[1 - \cos(\lambda\Omega t)]}{\Omega}, \tag{70a}$$

entailing $\hat{\tau}(0) = 0$ and

$$\dot{\hat{\tau}}(t) = f_1(t) + \mu f_3(t). \tag{71}$$

Here and hereafter μ is an arbitrary (nonvanishing) number.

Next we set

$$\dot{\underline{x}} = (f_1 + \mu f_3)\underline{h}(\underline{x}); \quad \dot{x}_n = (f_1 + \mu f_3)h_n(\underline{x}), \quad n = 1, \dots, N, \quad (72)$$

so that the time evolution of \underline{x} is *multi-periodic*; and we moreover set

$$f_m = F_m(\underline{x})y_m, \quad m = 1, 2, 3, 4, \quad (73)$$

thereby getting, via the insertion of these formulas in (72) and (69), the following system of $N + 4$ ODEs:

$$\dot{x}_n = [F_1(\underline{x})y_1 + \mu F_3(\underline{x})y_3]h_n(\underline{x}), \quad (74a)$$

$$\dot{y}_1 = \Omega \frac{F_2(\underline{x})}{F_1(\underline{x})}y_2 - \left[y_1 + \mu \frac{F_3(\underline{x})}{F_1(\underline{x})}y_3 \right] y_1 \sum_{n=1}^N \left[\frac{\partial F_1(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (74b)$$

$$\dot{y}_2 = -\Omega \frac{F_1(\underline{x})}{F_2(\underline{x})}y_1 - \frac{F_1(\underline{x})y_1 + \mu F_3(\underline{x})y_3}{F_2(\underline{x})}y_2 \sum_{n=1}^N \left[\frac{\partial F_2(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (74c)$$

$$\dot{y}_3 = \lambda \Omega \frac{F_4(\underline{x})}{F_3(\underline{x})}y_4 - \left[\frac{F_1(\underline{x})}{F_3(\underline{x})}y_1 + \mu y_3 \right] y_3 \sum_{n=1}^N \left[\frac{\partial F_3(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (74d)$$

$$\dot{y}_4 = -\lambda \Omega \frac{F_3(\underline{x})}{F_4(\underline{x})}y_3 - \frac{F_1(\underline{x})y_1 + \mu F_3(\underline{x})y_3}{F_4(\underline{x})}y_4 \sum_{n=1}^N \left[\frac{\partial F_4(\underline{x})}{\partial x_n} h_n(\underline{x}) \right]. \quad (74e)$$

Clearly the general solution of this system reads as follows:

$$x_n(t) = X_n[\hat{\tau}(t)], \quad n = 1, 2, 3, \quad (75a)$$

$$y_m(t) = \frac{f_m(t)}{F_m(\underline{x})}, \quad m = 1, 2, 3, 4, \quad (75b)$$

where $X_n(\tau)$ is the general solution of (6d), $\hat{\tau}(t)$ is defined by (70a) and the 4 functions $f_m(t)$ are given by (69). The *multi-periodic* character of this general solution is plain — for a largely arbitrary assignment of the $N + 4$ functions $h_n(\underline{x})$, $n = 1, \dots, N$ and $F_m(\underline{x})$, $m = 1, 2, 3, 4$.

We also report the somewhat neater form of the *multi-periodic* dynamical system (74) in the special case when the 4 functions $F_m(\underline{x})$ coincide, $F_m(\underline{x}) = F(\underline{x})$:

$$\dot{x}_n = F(\underline{x})(y_1 + \mu y_3)h_n(\underline{x}), \quad (76a)$$

$$\dot{y}_1 = \Omega y_2 - (y_1 + \mu y_3)y_1 \sum_{n=1}^N \left[\frac{\partial F(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (76b)$$

$$\dot{y}_2 = -\Omega y_1 - (y_1 + \mu y_3)y_2 \sum_{n=1}^N \left[\frac{\partial F(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (76c)$$

$$\dot{y}_3 = \lambda \Omega y_4 - (y_1 + \mu y_3)y_3 \sum_{n=1}^N \left[\frac{\partial F(\underline{x})}{\partial x_n} h_n(\underline{x}) \right], \quad (76d)$$

$$\dot{y}_4 = -\lambda \Omega y_3 - (y_1 + \mu y_3)y_4 \sum_{n=1}^N \left[\frac{\partial F(\underline{x})}{\partial x_n} h_n(\underline{x}) \right]. \quad (76e)$$

Two simple specific examples are presented below: the alert reader will have no difficulty in devising many others.

5.1. Examples of multi-periodic dynamical systems

In this Subsec. 5.1 we present, without any comment, two simple examples of *autonomous* dynamical systems yielding *multi-periodic* evolutions.

Example 5.1. 5 coupled first-order evolution ODEs:

$$\dot{x} = \gamma x^2 (y_1 + \mu y_3), \quad (77a)$$

$$\dot{y}_1 = \Omega y_2 - \gamma x (y_1 + \mu y_3) y_1, \quad (77b)$$

$$\dot{y}_2 = -\Omega y_1 - \gamma x (y_1 + \mu y_3) y_2, \quad (77c)$$

$$\dot{y}_3 = \lambda \Omega y_4 - \gamma x (y_1 + \mu y_3) y_3, \quad (77d)$$

$$\dot{y}_4 = -\lambda \Omega y_3 - \gamma x (y_1 + \mu y_3) y_4. \quad (77e)$$

This system corresponds to (76) with $N = 1$, $F(x) = x$, $h(x) = \gamma x$. Its *general* solution reads

$$x(t) = x(0) \exp[\gamma \hat{\tau}(t)], \quad (78a)$$

$$y_m(t) = \frac{f_m(t)}{x(0)} \exp[-\gamma \hat{\tau}(t)], \quad m = 1, 2, 3, 4, \quad (78b)$$

with $\hat{\tau}(t)$ defined by (70a) and the 4 functions $f_m(t)$ given by (69). The *multi-periodic* character of this model is plain. Note that, for $\mu = 0$, the subcase of this model characterized by $y_3 = y_4 = 0$ reduces to Example 3.1.

Example 5.2. 7 coupled first-order evolution ODEs:

$$\dot{x}_1 = -\alpha x_1 (x_1 - x_2) (y_1 + \mu y_3), \quad (79a)$$

$$\dot{x}_2 = x_1 (\beta x_1 - x_2 - x_1 x_3) (y_1 + \mu y_3), \quad (79b)$$

$$\dot{x}_3 = x_1 (x_1 x_2 - \gamma x_3) (y_1 + \mu y_3), \quad (79c)$$

$$\dot{y}_1 = \Omega y_2 + \alpha (x_1 - x_2) (y_1 + \mu y_3) y_1, \quad (79d)$$

$$\dot{y}_2 = -\Omega y_1 + \alpha (x_1 - x_2) (y_1 + \mu y_3) y_2, \quad (79e)$$

$$\dot{y}_3 = \Omega y_4 + \alpha (x_1 - x_2) (y_1 + \mu y_3) y_3, \quad (79f)$$

$$\dot{y}_4 = -\Omega y_3 + \alpha (x_1 - x_2) (y_1 + \mu y_3) y_4. \quad (79g)$$

This model obtains from (76) via the assignment (33), and its *general* solution is given by the formulas (75) with $X_n(\tau)$, $n = 1, 2, 3$, being the general solution of the Lorenz model (34), and of course $\hat{\tau}(t)$ again defined by (70a) and the 4 functions $f_m(t)$ again given by (69). Note that, for $\mu = 0$, the subcase of this model characterized by $y_3 = y_4 = 0$ reduces to Example 3.3.

6. Comparisons

In this section we compare the behavior of the *generic* solutions of the extended — *isochronous* or *asymptotically isochronous* or *multi-periodic* — models introduced above with those of their original counterparts — with particular attention to the case when the corresponding original system features a *chaotic* behavior. This question — that we consider quite interesting hence presumably worthy of future elaborations — shall however be treated tersely here since it has already been discussed, in the context of Hamiltonian systems, in previous papers [2, 9, 10].

A first issue concerns the fact that the original system does not move in the same space as its extension: indeed the latter has 2, 3 or 4 degrees of freedom more than the former, depending whether one is considering the *isochronous*, *asymptotically isochronous* or *multi-periodic* extension

(as introduced above: see (15), or (40) and (48), or (74)). However in the extended systems we constructed in Sec. 2 there are two kinds of dependent variables: the variables x_n (with $n = 1, \dots, N$) which essentially coincide with the *original* variables (although their time evolution is of course different in the extended case), and the *additional* variables y_m (with $m = 1, 2$ or $m = 1, 2, 3$ or $m = 1, 2, 3, 4$). We may therefore *project* the extended dynamics on the x_n variables. The result can then be straightforwardly compared with the original dynamics.

To illustrate this comparison in the particularly interesting case in which the original, unmodified dynamical system evolves *chaotically* we focus here on the specific such examples reported in preceding sections, but we trust the reader to appreciate the general relevance of this discussion. Let us refer, to begin with, to the *isochronous* dynamical system (31) treated as Example 3.3 in Subsec. 3.1. The results obtained there (see (35a) with (13)) entail that the behavior of this system can be described as follows. The initial data $x_n(0)$, $n = 1, 2, 3$, define, via the simple rule

$$X_n(0) = x_n(0), \quad n = 1, 2, 3 \quad (80)$$

(entailed by the property $\tau(0) = 0$, see (13)), initial data, hence a trajectory, for the Lorenz system (34). The trajectory of the corresponding solution of the *isochronous* dynamical system (31) coincides then with a piece of this trajectory of the Lorenz system (34), but traveled back and forth in time — *isochronously*, with period T — starting from the same initial data. At this point one should mention that, while the solution of the Lorenz model (for the interesting set of its 3 defining constants α, β, γ , which is the case we always refer to) always exists (as a continuous function) for *positive* time, it need not remain singularity-free for all *negative* time: this is therefore a potential source of singularity that might spoil the *isochronous* character of some solutions of the model (31). In the following discussion we confine our consideration to initial data — which certainly exist — that exclude this from happening.

To continue our qualitative discussion of the comparative behavior of the solutions of the *isochronous* system (31) and the Lorenz model (34) we now point out that the definition (13) of $\tau(t)$ entails that, for t in the neighborhood of a *generic* time \bar{t} ,

$$\tau(t) = U + Vt + O\left(\frac{t - \bar{t}}{T}\right), \quad (81a)$$

with the constants U and V defined as follows:

$$U \equiv U(\bar{t}) = \tau(\bar{t}) - V\bar{t}, \quad (81b)$$

$$V \equiv V(\bar{t}) = A \cos(\Omega\bar{t}) + B \sin(\Omega\bar{t}), \quad (81c)$$

and the *isochrony* period T defined by (11). Hence the behavior of (31) and (34) only differ — over time intervals much shorter than T , which, for the sake of this discussion, is now assumed to be itself much larger than all the characteristic times of the Lorenz model (34) — by a *constant* time rescaling and shift. Of course while, when the time rescaling constant V is *positive*, this behavior of (31) can be considered to resemble that of the Lorenz model (34), when the rescaling constant V is *negative* it resembles instead the *time-reversed* behavior of the Lorenz model; and each of these two behaviors of the *isochronous* system (31) alternate within each period T , so that the trajectories of this model, within each period T , shall alternatively approach and get away from the strange attractor featured by the Lorenz model (34), this phenomenology being of course only apparent if the period T is much larger than all the characteristic times of the Lorenz model. The phenomenon worth emphasizing is that — under such circumstances, with T large with respect to the characteristic times of the Lorenz system — this *isochronous* system will behave for quite some time quite similarly to a typically *chaotic* system (as indeed the Lorenz model is known to be); in spite of the overall regularity of its dynamical behavior, as entailed by its *isochrony*. Note that

this situation can always be realized, since T can be chosen arbitrarily when manufacturing the *isochronous* system (31).

The peculiarity of this finding has already been highlighted in Refs. [9, 10] — albeit in the different context of the classical many-body problem characterized by original equations of motion that are invariant under time-reversal: which is not the case here. Moreover in that case both the original problem and its modified, *isochronous* version are Hamiltonian, so that the relevant discussion of this phenomenology — as it were, forcing an *isochronous* evolution over an initially *chaotic* behavior — was naturally framed in the context of the properties of *integrability* and *superintegrability*, and in that context it was shown that any *isochronous* system is *integrable* indeed *maximally superintegrable* (in fact it corresponds to a special instance of such systems: *all* confined solutions of *maximally superintegrable* systems are *completely periodic*, but not necessarily *all* with the *same* period, as is instead the case for *isochronous* systems) [9, 10]. In the present, more general context of (not necessarily Hamiltonian) dynamical systems the issue is whether the system under consideration does or does not have the maximal number of (functionally independent and globally defined) constants of motion, namely $N - 1$ if N is the number of degrees of freedom of the system. We refer for short to such systems as being characterized by a “maximally conserved” dynamics. Note, however, that in the general (non Hamiltonian) case a maximally conserved dynamics need not be periodic. Indeed one can show, along the same lines as the proof in [10] (but averaging the trajectory over the entire time evolution), that all *asymptotically isochronous* systems of type (48) yield a maximally conserved dynamics. This was explicitly pointed out at the end of Example 4.4 (see (64)). Note however that the fact that *isochronous* and *asymptotically isochronous* systems yield a maximally conserved dynamics does not necessarily entail that their dynamics is “simple” (even in the *isochronous* case!): the constants of motion may be very complicated functions that cannot be exhibited explicitly, and obtaining the actual orbit requires moreover inversions that generally entail additional complications. On the other hand, this complication cannot, for our *isochronous* systems, be so large as to qualify the motion as *chaotic*: this follows from the observation that their trajectories coincide with just a *finite* piece of the corresponding trajectory of the original system, traveled periodically over and over, back and forth.

An analogous discussion can be made in connection with the *asymptotically isochronous* dynamical systems treated as Examples 4.3 and 4.6 in Subsec. 4.1. It is indeed clear that the definition (38a) of $\tilde{\tau}(t)$ entails, in the neighborhood of a generic time \bar{t} ,

$$\tilde{\tau}(t) = \tilde{U} + \tilde{V}t + O\left(\frac{t - \bar{t}}{T}, \frac{t - \bar{t}}{\bar{T}}\right), \tag{82a}$$

with the constants \tilde{U} and \tilde{V} defined as follows:

$$\tilde{U} \equiv \tilde{U}(\bar{t}) = \tilde{\tau}(\bar{t}) - \tilde{V}\bar{t}, \tag{82b}$$

$$\tilde{V} \equiv \tilde{V}(\bar{t}) = A \cos(\Omega\bar{t}) + B \sin(\Omega\bar{t}) + C \exp(-\eta\bar{t}), \tag{82c}$$

and likewise the definition (45) of $\check{\tau}(t)$ entails

$$\check{\tau}(t) = \check{U} + \check{V}t + O\left(\frac{t - \bar{t}}{T}, \frac{t - \bar{t}}{\bar{T}}\right), \tag{83a}$$

with the constants \check{U} and \check{V} defined as follows:

$$\check{U} \equiv \check{U}(\bar{t}) = \check{\tau}(\bar{t}) - \check{V}\bar{t}, \tag{83b}$$

$$\check{V} \equiv \check{V}(\bar{t}) = C \left\{ \cos[\Omega(\bar{t} - t_0)] \tanh[\eta(\bar{t} - t_1)] + \frac{\eta \sin[\Omega(\bar{t} - t_0)]}{\Omega \cosh^2[\eta(\bar{t} - t_1)]} \right\}, \tag{83c}$$

and T defined again by (11) while $\bar{T} = 1/\eta$.

This clearly entails also in these cases the remarkable phenomenon discussed above: the behavior of these systems, around a generic time, may resemble over long periods of time that of a *chaotic* system (the Lorenz system (34)), yet their overall behavior becomes eventually quite regular.

Note moreover that the trajectories of these systems straddle again only a finite piece of a corresponding trajectory of the Lorenz system (34), when t goes from zero to infinity. This is implied by (61a) with (38a) and by (68a) with (45); a trajectory being again traveled back and forth over time, albeit now not quite periodically.

This fact might suggest that also these modified systems are characterized by a *maximally conserved* dynamics. But the following remark shows that this argument is not really relevant for the system of Example 4.3. Indeed the conserved quantities, being time-independent, should exist for *all* time ($-\infty < t < +\infty$), and clearly when the time t spans this *infinite* interval, the corresponding span of $\tilde{\tau}(t)$, see (38a), is *semi-infinite* rather than being *finite*. This counter-argument, however, does not apply to the model of Example 4.6, because when the time t spans the entire interval from $-\infty$ to $+\infty$, the quantity $\tilde{\tau}(t)$ only spans the interval from $-C$ to $+C$, see (45), which is always *finite* (although it might be made arbitrarily large). Indeed in this case the maximal number of conserved quantities can be manufactured following the same treatment as detailed in [10], except that the formula defining the mean \bar{f} , modulated by the *a priori* arbitrary function $f(\underline{z})$, over the (periodic) orbit of the *isochronous* system, reading

$$\bar{f} = \frac{1}{T} \int_0^T dt f[\underline{z}(t)], \quad (84a)$$

where the vector \underline{z} includes all the degrees of freedom of the system and T is the *isochrony* period, should be replaced by the following formula,

$$\bar{f} = \lim_{\bar{T} \rightarrow \infty} \left\{ \frac{1}{2\bar{T}} \int_{-\bar{T}}^{\bar{T}} dt f[\underline{z}(t)] \right\}, \quad (84b)$$

which makes sense also for nonperiodic trajectories but yields the same result as the previous one, (84a), in the case of periodic trajectories with period T . Of course the construction of the constants of motion is only given by this prescription in the phase space region in which the trajectories of the system do not run into singularities, either due to the backward time-evolution in the Lorenz model or to a vanishing of the coordinate $x_1(t)$, see (68).

The extension of this discussion to include the case of *multi-periodic* systems is left as a task for the diligent reader, who is then advised to focus on the Example 5.2 and to note that in that case — just as in the case of the Example 4.6 — the definition (84b) is adequate provided the trajectory of the system does not run into singularities.

Let us end this section by re-emphasizing that, while for simplicity our presentation has been focused on specific examples, the essence of our findings is easily seen to have an easily identifiable general validity — both regarding the comparison of the time evolution of the extended systems with those of the original, unmodified systems, and as well the maximally conserved character of the time evolution of the extended systems (which is the case in an open, hence fully-dimensional, phase space region of the extended models of Examples 4.6 and 5.2, but not in the case of the extended model of Example 4.3).

7. Outlook

In this paper we have introduced a somewhat novel technique to extend an *autonomous* dynamical system so that the extended *autonomous* system thereby obtained is either *isochronous* or *asymptotically isochronous* or *multi-periodic*, and it moreover generally entails, over times much shorter than the (arbitrarily assigned) period or periods associated with the extended system, a dynamical

evolution which differs from the original one only by a *constant rescaling* of the independent variable (*time*). As already emphasized above, the examples exhibited in this paper are merely special instances of those that can be manufactured via this methodology: the results reported herein open therefore an ample vista of further investigations, both in the direction of *applicable* models, as well as of models evoking a fundamental physical interest such as the classical many-body problem.

To give a glimpse of such possible developments let us now outline three generalizations, and a variation, of the methods developed in Sec. 2 and more specifically in Secs. 3–5. For simplicity we limit here our consideration to the case of *isochronous* systems; the alert reader will have no difficulty to extend these considerations to the case of *asymptotically isochronous* and *multi-periodic* systems, or for that matter to devise additional generalizations of our approach, rather obviously suggested by the previous treatment, as well as by its generalizations outlined below.

Again, for simplicity and clarity, the three possible generalizations, and the variation, are outlined below by focussing on representative examples.

(i) Consider the dynamical system of 4 ODEs in the 4 dependent variables $\bar{x}, \bar{y}_1, \bar{y}_2, z$ defined as follows:

$$\dot{\bar{x}} = \gamma \bar{x}^2 \bar{y}_1 F_0(z) F_1(z) - \bar{x} F_0'(z) \bar{G}(\bar{x}, \bar{y}_1, \bar{y}_2, z), \tag{85a}$$

$$\dot{\bar{y}}_1 = \Omega \frac{F_2(z)}{F_1(z)} \bar{y}_2 - \gamma \bar{x} \bar{y}_1^2 F_0(z) F_1(z) - \frac{F_1'(z)}{F_1(z)} \bar{y}_1 \bar{G}(\bar{x}, \bar{y}_1, \bar{y}_2, z), \tag{85b}$$

$$\dot{\bar{y}}_2 = -\Omega \frac{F_1(z)}{F_2(z)} \bar{y}_1 - \gamma \bar{x} \bar{y}_1 \bar{y}_2 F_0(z) F_1(z) - \frac{F_2'(z)}{F_2(z)} \bar{y}_2 \bar{G}(\bar{x}, \bar{y}_1, \bar{y}_2, z), \tag{85c}$$

$$\dot{z} = \bar{G}(\bar{x}, \bar{y}_1, \bar{y}_2, z), \tag{85d}$$

where

$$\begin{aligned} \bar{G}(\bar{x}, \bar{y}_1, \bar{y}_2, z) = & \delta z \{ \Omega [\beta_1 \bar{y}_2 F_2(z) - \beta_2 \bar{y}_1 F_1(z)] + \alpha \gamma \bar{x}^2 \bar{y}_1 F_0^2(z) F_1(z) - \gamma \bar{x} \bar{y}_1 F_0(z) F_1(z) \\ & \times [\beta_1 \bar{y}_1 F_1(z) + \beta_2 \bar{y}_2 F_2(z)] \}. \end{aligned} \tag{85e}$$

Here $\alpha, \beta_1, \beta_2, \gamma, \delta$ and Ω are 6 arbitrary constants (for definiteness, $\Omega > 0$), and $F_0(z), F_1(z), F_2(z)$ are 3 arbitrary functions.

The starting point to arrive at this system, (85), is the system (22), complemented by the assignments

$$\bar{x} = x F_0(z); \quad \bar{y}_m = y_m F_m(z), \quad m = 1, 2, \tag{85f}$$

and the ODE

$$\dot{z} = \{ \alpha \gamma x^2 y_1 + \beta_1 [\Omega y_2 - \gamma x y_1^2] - \beta_2 [\Omega y_1 + \gamma x y_1^2] \} \delta z, \tag{86a}$$

which, via (22) and (86c), can clearly be re-written as follows:

$$\dot{z} = \dot{\tau} \delta z, \tag{86b}$$

where

$$\dot{\tau}(t) = \alpha x(t) + \beta_1 y_1(t) + \beta_2 y_2(t) - [\alpha x(0) + \beta_1 y_1(0) + \beta_2 y_2(0)]. \tag{86c}$$

And this clearly entails that the general solution of this system, (85), is provided by the following formulae:

$$\bar{x}(t) = \frac{x(t)}{F_0[z(t)]}, \tag{87a}$$

$$\bar{y}_m(t) = \frac{y_m(t)}{F_m[z(t)]}, \quad m = 1, 2, \tag{87b}$$

$$z(t) = D \exp[\delta \tau(t)], \tag{87c}$$

with the 3 functions $x(t)$, $y_1(t)$, $y_2(t)$ defined by (26) with (13). The 4 constants A , B , C and D contained in these formulae (see (26) with (13) and (87c)) are arbitrary; in the context of the initial-value problem, they are determined in terms of the 4 initial values $\bar{x}(0)$, $\bar{y}_1(t)$, $\bar{y}_2(t)$, $z(0)$ (note that the second line in the right-hand side of the formula (86c) defining $\bar{\tau}(t)$ has been introduced merely for convenience, to cause $\bar{\tau}(0)$ to vanish; it could of course be omitted, since its presence amounts merely to a rescaling of the *a priori* arbitrary constant D , see (87c)). The *isochronous* character of this general solution is plain. The diligent reader will verify that these formulae, (87), provide indeed the general solution of the system (85), and shall thereby note the remarkable nature of this solution, as evidenced for instance by the nested character of the formula (87c) providing the explicit expression of the dependent variable $z(t)$. It shall thereby be clear how the construction of this system can be interpreted as the first step of an iterative procedure that might be continued.

(ii) Consider the $N + 2$ nonlinearly coupled ODEs satisfied by the $N + 2$ dependent variables x_n , y_1 and y_2 , where n runs from 1 to N :

$$\dot{x}_n = [bG_1(\underline{x}, y_1, y_2) + cG_2(\underline{x}, y_1, y_2)]h_n(\underline{x}), \quad (88a)$$

$$\dot{y}_1 = [F_1(\underline{x})]^{-1} \left\{ G_1(\underline{x}, y_1, y_2) - y_1[bG_1(\underline{x}, y_1, y_2) + cG_2(\underline{x}, y_1, y_2)] \sum_{n=1}^N \frac{\partial F_1(\underline{x})}{\partial x_n} h_n(\underline{x}) \right\}, \quad (88b)$$

$$\dot{y}_2 = [F_2(\underline{x})]^{-1} \left\{ G_2(\underline{x}, y_1, y_2) - y_2[bG_1(\underline{x}, y_1, y_2) + cG_2(\underline{x}, y_1, y_2)] \sum_{n=1}^N \frac{\partial F_2(\underline{x})}{\partial x_n} h_n(\underline{x}) \right\}, \quad (88c)$$

where b and c are two arbitrary constants, \underline{x} denotes of course the N -vector having the N dependent variables x_n as its N components, the N functions $h_n(\underline{x})$ depend arbitrarily on the N components x_n of this N -vector, and the two functions $G_m(\underline{x}, y_1, y_2)$, $m = 1, 2$, of the $N + 2$ dependent variables x_n , y_1 , y_2 are explicitly defined as follows,

$$G_1(\underline{x}, y_1, y_2) = -\alpha y_1^2 [F_1(\underline{x})]^2 + \gamma y_2^2 [F_2(\underline{x})]^2 - \frac{\alpha \Omega^2}{4}, \quad (89a)$$

$$G_2(\underline{x}, y_1, y_2) = y_2 F_2(\underline{x}) [-2\alpha y_1 F_1(\underline{x}) + \delta y_2 F_2(\underline{x})], \quad (89b)$$

with α , γ and δ three arbitrary constants and the two functions $F_1(\underline{x})$, $F_2(\underline{x})$ of the N dependent variables x_n assigned arbitrarily.

The starting point to prove that the dynamical system (88) is *isochronous* with period T , see (11), is the fact that the system of two ODEs

$$\dot{f}_1 = -\alpha \Omega^2 / 4 - \alpha f_1^2 + \gamma f_2^2, \quad \dot{f}_2 = (-2\alpha f_1 + \delta f_2) f_2, \quad (90)$$

is *isochronous* with period T , see (11), as implied by Proposition 4 of [5] (up to trivial notational changes). This entails that the function

$$\tau(t) = b f_1(t) + c f_2(t) \quad (91a)$$

is equally *isochronous*, while clearly

$$\dot{\tau} = b[-\alpha \Omega^2 / 4 - \alpha f_1^2 + \gamma f_2^2] + c[(-2\alpha f_1 + \delta f_2) f_2]. \quad (91b)$$

Hence if one couples the system (90) to the system of N ODEs

$$\dot{x}_n = \dot{\tau} h_n(\underline{x}), \quad n = 1, \dots, N, \quad (92)$$

with $\dot{\tau}$ defined by the preceding formula, one obtains an *isochronous* system of $N + 2$ ODEs. And it is then a matter of trivial algebra to see that this system corresponds to the dynamical system (88) via the assignment (14).

(iii) As third example of generalization we report a more general version of the results of Secs. 2 and 3, amounting to the assertion that the following system of $N + 2$ ODEs is *isochronous* with period T , see (11), where n runs from 1 to N and m is defined modulo 2:

$$\dot{x}_n = (c_1 F_1 + c_2 F_2) h_n(\underline{x}), \tag{93a}$$

$$\dot{y}_m = (-1)^m G^{-1} \left\{ -\frac{\Omega}{2} \frac{\partial(F_1^2 + F_2^2)}{\partial y_{m+1}} + (c_1 F_1 + c_2 F_2) \sum_{n=1}^N [G_{nm} h_n(\underline{x})] \right\}. \tag{93b}$$

Here the N functions $h_n(\underline{x})$ of the N dependent variables x_n are arbitrary. The two functions $F_m \equiv F_m(\underline{x}, y_1, y_2)$, $m = 1, 2$ of the $N + 2$ dependent variables x_n, y_1, y_2 must be invertible with respect to y_m for all \underline{x} , but can otherwise be chosen arbitrarily. The $1+2N$ functions $G \equiv G(\underline{x}, y_1, y_2)$, $G_{nm} \equiv G_{nm}(\underline{x}, y_1, y_2)$ are defined as follows in terms of the two functions $F_m(\underline{x}, y_1, y_2)$ (which must of course be assigned so that $G(\underline{x}, y_1, y_2)$ does not vanish):

$$G = \frac{\partial F_1}{\partial y_1} \frac{\partial F_2}{\partial y_2} - \frac{\partial F_2}{\partial y_1} \frac{\partial F_1}{\partial y_2}, \tag{94a}$$

$$G_{nm} = \frac{\partial F_1}{\partial x_n} \frac{\partial F_2}{\partial y_{m+1}} - \frac{\partial F_2}{\partial x_n} \frac{\partial F_1}{\partial y_{m+1}}. \tag{94b}$$

The *isochrony* of the *general solution* of this system of $N + 2$ ODEs is implied by the formulas

$$x_n(t) = X_n[\tau(t)], \tag{95a}$$

$$F_m[\underline{x}(t), y_1(t), y_2(t)] = f_m(t), \tag{95b}$$

$$\dot{\tau}(t) = c_1 f_1(t) + c_2 f_2(t). \tag{95c}$$

Here the functions $X_n(\tau)$ are the solutions of the system (6d) (indeed (95a) coincides with (6c)), while the two functions $f_m(t)$ are explicitly defined by (10), implying that

$$\tau(t) = (c_1 A + c_2 B) \frac{\sin(\Omega t)}{\Omega} + (c_1 B - c_2 A) \frac{1 - \cos(\Omega t)}{\Omega}, \tag{95d}$$

where we have chosen the integration constant so that $\tau(0) = 0$ implying $X_n(0) = x_n(0)$. These formulae, (95), with A and B two arbitrary constants (to be fixed by the initial data), provide, in somewhat implicit form, the *general solution* of the system (93), in terms of the *general solution* $X_n(\tau)$ of the system (6d), thereby demonstrating the *isochronous* character of these solutions. And we trust that the clue provided by the formulas written above is sufficient to indicate how they have been derived, via a treatment quite analogous, if somewhat more general, than that detailed in Secs. 2 and 3.

Clearly the *isochronous* system (93) — thanks to the large arbitrariness in the assignment of the two constants c_m and especially of the two functions $F_m(\underline{x}, y_1, y_2)$ of the $N + 2$ dependent variables x_n and y_m , as well as the arbitrariness in the assignments of the N functions $h_n(\underline{x})$ of the N dependent variables x_n — provides a wider scope for applications than the system (40), to which, as the diligent reader will verify, it reduces for the special choice $c_1 = 1, c_2 = 0, F_m(\underline{x}, y_1, y_2) = F_m(\underline{x}) y_m, m = 1, 2$.

The alert reader who will repeat this more general derivation will also notice that the *isochronous* character of the system (93) might be preserved even in the more general case in which the functions h_n are allowed to depend also on the two dependent variables y_m , although in that case the connection of the solution of (93) with the solutions $X_n(\tau)$ of the system (6d) would have to be replaced by a different, somewhat more complicated, relation than (95a).

Finally, a variation: let us indicate a way (out of many possible ones) to extend a largely arbitrary dynamical system involving N dependent variables (such as (6d)) so that the extended (*isochronous!*)

dynamical system thereby obtained features only $N + 1$ dependent variables (rather than $N + 2$ as it is the case for (40) or its more general version (93)). Such a system reads as follows

$$\dot{x}_n = \Omega\{\alpha^2 - [F(\underline{x}, y) - \beta]\}^{1/2} h_n(\underline{x}), \quad (96a)$$

$$\dot{y} = \Omega\{\alpha^2 - [F(\underline{x}, y) - \beta]\}^{1/2} \left[\frac{\partial F(\underline{x}, y)}{\partial y} \right]^{-1} \cdot \left\{ 1 - \sum_{n=1}^N \left[\frac{\partial F(\underline{x}, y)}{\partial x_n} h_n(\underline{x}) \right] \right\}, \quad (96b)$$

where n runs from 1 to N . Here the N functions $h_n(\underline{x})$ of the N dependent variables x_n are arbitrary. The function $F(\underline{x}, y)$ of the $N + 1$ dependent variables x_n and y must be invertible with respect to y but is otherwise largely arbitrary. The two constants α and β are also arbitrary, and the determination of the square root appearing in the right-hand side of these ODEs must be assigned so as to guarantee the smoothness of the time-dependence of the dependent variables $x_n(t)$ and $y(t)$, namely continuity of their time-derivatives — in addition to continuity of the functions themselves; of course this also entails some conditions on the functions $h_n(\underline{x})$ and $F(\underline{x}, y)$.

The *general solution* of this system can be written in terms of the *general solution* $X_n(\tau)$ of the corresponding system (6d) in the following form, which is somewhat implicit with respect to the dependent variable $y(t)$:

$$x_n(t) = X_n(\beta + \alpha \sin[\Omega(t - t_0)]), \quad (97a)$$

$$F[\underline{x}(t), y(t)] = \beta + \alpha \sin[\Omega(t - t_0)]. \quad (97b)$$

The *isochronous* character of this solution is clear, and its validity could be easily verified by direct substitution, but let us outline for completeness how this *isochronous* system, (96), has been manufactured. The starting point is the single evolution ODE

$$\dot{f} = \Omega[\alpha^2 - (f - \beta)^2]^{1/2}, \quad (98a)$$

where α and β are two arbitrary constants and the determination of the square root must again be chosen so as to guarantee that the dependent variable, $f \equiv f(t)$, depend smoothly on time, i.e. it should not only be a continuous functions of time, but its time derivative should as well be a continuous function of time. Hence the general solution of this ODE reads

$$f(t) = \beta + \alpha \sin[\Omega(t - t_0)], \quad (98b)$$

with t_0 an arbitrary constant the (real or imaginary) value of which can be adjusted to fit the initial datum $f(0)$. The *isochronous* character of $f(t)$ is of course plain.

The subsequent procedure to manufacture an *isochronous* dynamical system is to write firstly

$$\dot{x}_n = \Omega[\alpha^2 - (f - \beta)^2]^{1/2} h_n(\underline{x}), \quad n = 1, \dots, N, \quad (99)$$

and to then introduce a new dependent variable $y \equiv y(t)$ by setting

$$f = F(\underline{x}, y), \quad (100)$$

where $F(\underline{x}, y)$ is a function of the $N + 1$ variables x_n and y that we reserve the privilege to assign later (at our convenience). It is then easily seen that one obtains in this manner just the dynamical system (96), and it is moreover plain that this derivation justifies the assertions made above about this system.

We end this paper, in which we indicated how to manufacture *autonomous* dynamical systems which are either *isochronous*, or *asymptotically isochronous* or *multi-periodic*, by mentioning the possibility to also manufacture *asymptotically multi-periodic* systems, whose solutions approach *multi-periodic* functions only *asymptotically*, at very large time. We feel that a precise definition of such systems can be left as an easy task — presumably interesting and possibly relevant for applications — for the reader who has internalized the main ideas, and the techniques, detailed in this

paper: who will easily devise procedures to manufacture such systems, and obtain explicit examples of such systems — both explicitly solvable ones and others, arrived at by extending *chaotic* systems hence generally not featuring explicit solutions.

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