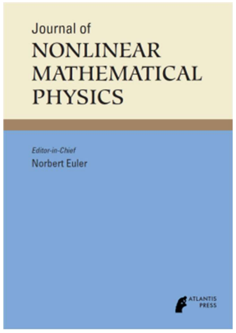

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Article

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AUTOMORPHISMS OF CURVES

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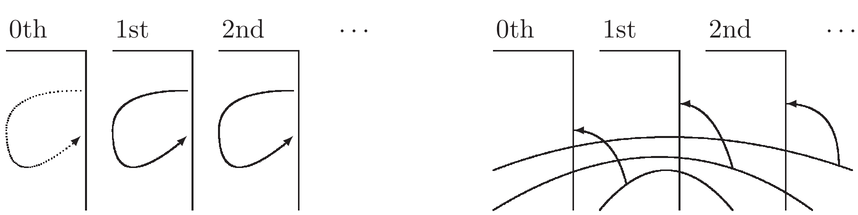
The article deals with local symmetries of the infinite-order jet space of C^∞-smooth curves in $\mathbb{R}^{m+1}$ ($m \geq 1$). Transformations under consideration are the most general possible: they need not preserve the distinction between dependent and independent variables and the order of derivatives may be arbitrarily changed. Unlike the common prolonged point and Lie's contact transformations, they destroy the finite-order jet spaces.

Keywords: Infinite order jet space; automorphisms of jet space; higher order symmetries of jets.

Mathematics Subject Classification 2000: 58A20, 58H99

Automorphisms of various structures belong to the most important subjects of mathematical theories. In the theory of (partial) differential equations, this area rests on the famous Lie's and Cartan's methods of infinitesimal transformations or moving frames. Such methods are as a rule applied in a certain finite-order jet space of a given order and it follows that all automorphisms which change the order of derivatives are omitted. In other words, the most general automorphisms of differential equations cannot be included if the common classical methods are mechanically applied.

We restrict ourselves to the extremely modest task, to the most general automorphisms of trivial (empty) systems of ordinary differential equations, that is, to the general automorphisms of the family of all C^∞-smooth curves in a finite-dimensional space. Our task is precisely formulated in Introduction below. For better clarity, it may be also described as follows. The common (prolonged) point or Lie's contact transformations preserve the finite-order jet spaces, see the left-hand figure. On the contrary, the right-hand scheme elucidates the new transformations. It should be noted that in comparison with rather narrow and well-aranged (pseudo-)group of classical point and Lie's contact transformations, there exists an immense amount of generalized automorphisms with peculiar properties and the overall group composition structure looks rather mysterious.



In a certain sense, our article can be related to moving frame method by Cartan. We intend to deal with the method of generalized (or: Lie–Bäcklund) infinitesimal symmetries in subsequent paper.

The article is self-contained. We restrict ourselves to the local theory on certain open subsets of generic points, i.e., some closed and nowhere dense subsets of exceptional points are omitted.

1. Introduction

1.1. *Transformations of smooth curves*

Our reasonings start in the space $\mathbb{R}^{m+1}$ $(m = 1, 2, \ldots)$ with coordinates x, w^i $(i = 1, \ldots, m)$ and we are interested in the family of C^∞-smooth curves

$$w^i = w^i(x) \quad (i = 1, \ldots, m) \tag{1.1}$$

where the definition domains (open intervals of $\mathbb{R}$) are not specified. (More precisely: we deal with germs of curves.) Our transformations will be locally defined by equations

$$\bar{x} = F(x, \ldots, w^j_s, \ldots), \quad \bar{w}^i = F^i(x, \ldots, w^j_s, \ldots) \quad (i = 1, \ldots, m) \tag{1.2}$$

where $w^j_s = d^s w^j / dx^s$ should be substituted. Here F, F^i are C^∞-smooth functions, each depending on a finite number of arguments

$$x, w^j_s \quad (j = 1, \ldots, m;\ s = 0, 1, \ldots). \tag{1.3}$$

By virtue of formulae (1.2), a given curve (1.1) in $\mathbb{R}^{m+1}$ is transformed into a curve

$$\bar{w}^i = \bar{w}^i(\bar{x}) \quad (i = 1, \ldots, m) \tag{1.4}$$

again lying in $\mathbb{R}^{m+1}$ and this is achieved as follows.

A given curve (1.1) is inserted into (1.2_1) with the result

$$\bar{x} = F\left(x, \ldots, \frac{d^s w^j}{dx^s}(x), \ldots\right) = \mathcal{F}(x). \tag{1.5}$$

Then, assuming

$$\mathcal{F}'(x) = DF\left(x, \ldots, \frac{d^s w^j}{dx^s}(x), \ldots\right) \neq 0 \quad \left(D = \frac{\partial}{\partial x} + \sum w^j_{s+1} \frac{\partial}{\partial w^j_s}\right), \tag{1.6}$$

Eq. (1.5) can be inverted as $x = \bar{\mathcal{F}}(\bar{x})$ by using the implicit function theorem and we obtain the desired functions

$$\bar{w}^i(\bar{x}) = F^i\left(\bar{\mathcal{F}}(\bar{x}), \ldots, \frac{d^s w^j}{dx^s}(\bar{\mathcal{F}}(\bar{x})), \ldots\right) \quad (i = 1, \ldots, m) \tag{1.7}$$

by using (1.2_2).

The obvious identities

$$\bar{\mathcal{F}}(\mathcal{F}(x)) = x, \quad \bar{\mathcal{F}}'(\bar{x})\mathcal{F}'(x) = 1, \quad \bar{\mathcal{F}}'(\bar{x}) DF\left(x, \ldots, \frac{d^s w^j}{dx^s}(x), \ldots\right) = 1 \tag{1.8}$$

appearing on this occasion will be soon referred to.

1.2. *The prolongation procedure*

Formulae

$$\bar{w}^i_r = F^i_r(x, \ldots, w^j_s, \ldots) \quad (i = 1, \ldots, m;\ r = 0, 1, \ldots)$$

for the transformed higher order derivatives $\bar{w}^i_r = d^r\bar{w}^i/d\bar{x}^r$ can be obtained as well. Assume that they are known for a certain r. (In particular $F^i_0 = F^i$ if $r = 0$ and we denote $\bar{w}^i_0 = \bar{w}^i$.) Then

$$\begin{aligned}\bar{w}^i_{r+1}(\bar{x}) &= \frac{d}{d\bar{x}}\bar{w}^i_r(\bar{x}) = \frac{d}{d\bar{x}}F^i_r\left(\bar{\mathcal{F}}(\bar{x}),\ldots,\frac{d^s w^j}{dx^s}(\bar{\mathcal{F}}(\bar{x})),\ldots\right)\\ &= DF^i_r(x,\ldots,w^j_s,\ldots)\bar{\mathcal{F}}'(\bar{x}) = \frac{DF^i_r(x,\ldots,w^j_s,\ldots)}{DF(x,\ldots,w^j_s,\ldots)}\end{aligned}$$

by using (1.8).

Altogether taken, we have the infinite system

$$\bar{x} = F(x,\ldots,w^j_s,\ldots),\quad \bar{w}^i_r = F^i_r(x,\ldots,w^j_s,\ldots)\quad (i = 1,\ldots,m;\ r = 0,1,\ldots) \tag{1.9}$$

subjected to the recurrence

$$F^i_{r+1} = \frac{DF^i_r}{DF}\quad \left(D = \frac{\partial}{\partial x} + \sum w^j_{r+1}\frac{\partial}{\partial w^j_r}\right) \tag{1.10}$$

where $DF \neq 0$ is supposed. At this place, functions $F \neq$ const. and $F^i = F^i_0$ can be quite arbitrarily chosen and then the total system (1.9) satisfying (1.10) is uniquely determined.

1.3. *Invertible transformations*

We will be interested in such equations (1.9) that can be *locally inverted* by appropriate C^∞-smooth formulae

$$x = \bar{F}(\bar{x},\ldots,\bar{w}^j_s,\ldots),\quad w^i_r = \bar{F}^i_r(\bar{x},\ldots,\bar{w}^j_s,\ldots) \tag{1.11}$$

$(i = 1,\ldots,m;\ r = 0,1,\ldots)$ analogous to formulae (1.9). If this is possible, we shall see later that the recurrence

$$\bar{F}^i_{r+1} = \frac{\bar{D}\bar{F}^i_r}{\bar{D}\bar{F}}\quad \left(\bar{D} = \frac{\partial}{\partial \bar{x}} + \sum \bar{w}^j_{r+1}\frac{\partial}{\partial \bar{w}^j_r}\right) \tag{1.12}$$

and the inequality $\bar{D}\bar{F} \neq 0$ are automatically satisfied. It follows that curves (1.4) are conversely transformed into curves (1.1).

1.4. *Definition*

We speak of *morphism* (1.9) if the recurrence (1.10) holds true and of *automorphism* (1.9) if moreover the inverse (1.11) exists.

Our task is to investigate the automorphisms, in particular an algorithm for explicit calculation of all automorphisms will be proposed as the concluding achievement of this article.

1.5. *Example*

We omit the common *point transformations*

$$\bar{x} = F(x,w^1,\ldots,w^m),\quad \bar{w}^i = F^i(x,w^1,\ldots,w^m)\quad (i = 1,\ldots,m)$$

(abbreviations $w^i = w^i_0, \bar{w}^i = \bar{w}^i_0$) which are (locally) invertible if and only if the Jacobi determinant is nonvanishing. Instead, we mention automorphisms of quite other kind.

Theorem 1. *Let* $f(\bar{x}, \bar{w}_0^1, \ldots, \bar{w}_0^m, x, w_0^1, \ldots, w_0^m)$ *be a function of* $2m+2$ *independent variables mentioned. Suppose that the system*

$$f = 0, \quad Df = 0, \ldots, D^m f = 0 \quad \left(D = \frac{\partial}{\partial x} + \sum w_{s+1}^j \frac{\partial}{\partial w_s^j} \right) \tag{1.13}$$

admits a certain solution

$$\bar{x} = F(x, w_0^1, \ldots, w_m^m), \quad \bar{w}_0^i = F_0^i(x, w_0^1, \ldots, w_m^m) \quad (i = 1, \ldots, m) \tag{1.14}$$

and moreover the system

$$f = 0, \quad \bar{D}f = 0, \ldots, \bar{D}^m f = 0 \quad \left(\bar{D} = \frac{\partial}{\partial \bar{x}} + \sum \bar{w}_{s+1}^j \frac{\partial}{\partial \bar{w}_s^j} \right) \tag{1.15}$$

admits a certain solution

$$x = \bar{F}(\bar{x}, \bar{w}_0^1, \ldots, \bar{w}_m^m), \quad w_0^i = \bar{F}_0^i(\bar{x}, \bar{w}_0^1, \ldots, \bar{w}_m^m) \quad (i = 1, \ldots, m) \tag{1.16}$$

by applying the implicit function theorem. If Eqs. (1.14) *and* (1.16) *are regarded as transformations of curves, they are inverse one to the other.* (*Alternatively*: *prolongations of Eqs.* (1.14) *and* (1.16) *provide mutually inverse automorphisms.*)

For the particular case $m = 1$, this is the classical Lie's contact transformation. We state a *tricky proof* only for the case $m = 2$, however, it may be easily carried over the case of general m.

Proof. Let us consider a curve (1.1) where $m = 2$. The transformed curve (1.4) is defined by Eq. (1.14) which are equivalent to identities (1.13), by definition. We shall see that identities (1.13) imply (1.15) and therefore imply (1.16). Altogether (1.14) implies (1.16). Quite analogously, Eqs. (1.16) clearly imply (1.14), so we indeed have an automorphism.

Passing to the proper proof, we begin with Eq. (1.13_1) which reads

$$f(\bar{x}, \bar{w}_0^1(\bar{x}), \bar{w}_0^2(\bar{x}), x, w_0^1(x), w_0^2(x)) = 0 \quad (\bar{x} = \mathcal{F}(x))$$

by using (1.5). Consequently

$$\frac{d}{dx} f(\cdots) = \bar{D}f(\cdots)\mathcal{F}'(x) + Df(\cdots) = 0$$

identically. Then (1.13_2) implies $\bar{D}f(\cdots) = 0$ identically. Analogously

$$\frac{d}{dx} Df(\cdots) = \bar{D}Df(\cdots)\mathcal{F}'(x) + D^2 f(\cdots) = 0$$

therefore $\bar{D}Df(\cdots) = D\bar{D}f(\cdots) = 0$ by using (1.13_3). In the same manner

$$\frac{d}{dx} \bar{D}f(\cdots) = \bar{D}^2 f(\cdots)\mathcal{F}'(x) + D\bar{D}f(\cdots) = 0$$

and therefore $\bar{D}^2 f(\cdots) = 0$ identically. The proof is done. □

Continuing with $m = 2$ and more explicit formulae in order to transparently illustrate the Theorem 1, then the function

$$f = \bar{w}_0^2 - w_0^2 + w_0^1 \bar{x} + \bar{w}_0^1 x$$

represents the simplest possible choice. The system

$$f = 0, \quad Df = -w_1^2 + w_1^1 \bar{x} + \bar{w}_0^1 = 0, \quad D^2 f = -w_2^2 + w_2^1 \bar{x} = 0$$

admits the solution

$$\bar{x} = \frac{w_2^2}{w_2^1}, \quad \bar{w}_0^1 = w_1^2 + w_1^1 \frac{w_2^2}{w_2^1}, \quad \bar{w}_0^2 = w_0^2 - (w_0^1 + w_1^1 x)\frac{w_2^2}{w_2^1} - w_1^2 x$$

and the system

$$f = 0, \quad \bar{D}f = \bar{w}_1^2 + w_0^1 + \bar{w}_1^1 x = 0, \quad \bar{D}^2 f = \bar{w}_2^2 + \bar{w}_2^1 x = 0$$

provides the inversion

$$x = -\frac{\bar{w}_2^2}{\bar{w}_2^1}, \quad w_0^1 = -\bar{w}_1^2 + \bar{w}_1^1 \frac{\bar{w}_2^2}{\bar{w}_2^1}, \quad w_0^2 = \bar{w}_0^2 - (\bar{w}_0^1 - \bar{w}_1^1 \bar{x})\frac{\bar{w}_2^2}{\bar{w}_2^1} - \bar{w}_1^2 \bar{x}$$

where $w_2^1 \neq 0, \bar{w}_2^1 \neq 0$ is supposed. More interesting is the choice

$$f = (\bar{x} - x)^2 + (\bar{w}_0^1 - w_0^1)^2 + (\bar{w}_0^2 - w_0^2)^2 - r^2 \quad (r = \text{const.} > 0).$$

Then the result can be expressed in nice geometrical terms. For this aim, let $\mathbb{P} : \{x, w_0^1(x), w_0^2(x)\} \subset \mathbb{R}^3$ denote the original curve. The transformed curve $\bar{\mathbb{P}}$ is determined by the equations with scalar products

$$(\bar{\mathbb{P}} - \mathbb{P})^2 = r^2, \quad (\bar{\mathbb{P}} - \mathbb{P})\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}x} = 0, \quad (\bar{\mathbb{P}} - \mathbb{P})\frac{\mathrm{d}^2\mathbb{P}}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}x}\right)^2 = 0.$$

Employing the Frenet formulae, two transformed curves

$$\bar{\mathbb{P}}_\pm : \{\bar{x}, \bar{w}_0^1(\bar{x}), \bar{w}_0^2(\bar{x})\} \subset \mathbb{R}^3, \quad \bar{\mathbb{P}}_\pm = \mathbb{P} + \frac{1}{\kappa}\mathbb{N} \pm \sqrt{r^2 - \frac{1}{\kappa^2}}\mathbb{B}$$

can be obtained if $r\kappa > 1$. Here $\kappa, \mathbb{N}, \mathbb{B}$ are the curvature, unit normal and unit binormal of curve $\mathbb{P}$. In a certain sense, we have obtained two notable curves "parallel at the distance r" to the original curve $\mathbb{P}$. The transformation is involutive if the $\pm$ branches are appropriately composed since the function f behaves symmetrically with respect to variables x, w_0^1, w_0^2 and $\bar{x}, \bar{w}_0^1, \bar{w}_0^2$.

2. The Technical Background

2.1. *General invertible mappings*

Passing to the general theory, let us introduce the infinite-dimensional space $\mathbf{M}(m)$ equipped with coordinates x, w_s^j $(j = 1, \ldots, m; s = 0, 1, \ldots)$, see also (1.3). We will study C^∞-smooth mappings $\mathbf{m}$ of open subsets of the space $\mathbf{M}(m)$ into $\mathbf{M}(m)$ given by formulae

$$\mathbf{m}^* x = F(x, \ldots, w_s^j, \ldots), \quad \mathbf{m}^* w_r^i = F_r^i(x, \ldots, w_s^j, \ldots) \tag{2.1}$$

$(i = 1, \ldots, m; r = 0, 1, \ldots)$ in terms of coordinates. We are interested in such mappings (2.1) that admit the inverse $\bar{\mathbf{m}}$ $(= \mathbf{m}^{-1})$ given by analogous formulae

$$\bar{\mathbf{m}}^* x = \bar{F}(x, \ldots, w_s^j, \ldots), \quad \bar{\mathbf{m}}^* w_r^i = \bar{F}_r^i(x, \ldots, w_s^j, \ldots) \tag{2.2}$$

as above. In the invertible case, both mappings $\mathbf{m}^*$ and $\bar{\mathbf{m}}^*$ provide a local automorphism of the algebra of differential forms on $\mathbf{M}(m)$, of course.

2.2. *On the jet structure*

We introduce the *contact module* Ω of all differential 1-forms

$$\omega = \sum a_r^i \omega_r^i \quad (\omega_r^i = \mathrm{d}w_r^i - w_{r+1}^i \mathrm{d}x, \text{ finite sum})$$

with arbitrary C^∞-smooth coefficients locally defined on $\mathbf{M}(m)$ and recall the vector field

$$D = \frac{\partial}{\partial x} + \sum w_{s+1}^j \frac{\partial}{\partial w_s^j} \quad \text{(infinite sum)}.$$

The obvious identities

$$\omega(D) = D\rfloor\omega = 0, \quad \mathcal{L}_{fD}\omega = f\mathcal{L}_D\omega, \quad \mathcal{L}_D\omega_r^i = \omega_{r+1}^i, \quad \mathcal{L}_D\Omega \subset \Omega,$$

$$\mathrm{d}\omega_r^i = \mathrm{d}x \wedge \omega_{r+1}^i, \quad \mathrm{d}f = Df\mathrm{d}x + \sum \frac{\partial f}{\partial w_s^j}\omega_s^j$$

with $\omega \in \Omega, f$ an arbitrary function and $\mathcal{L}_D = D\rfloor d + dD\rfloor$ the Lie derivative will be frequently employed. So we obtain the congruence

$$\mathbf{m}^*\omega_r^i = \mathrm{d}F_r^i - F_{r+1}^i\mathrm{d}F \cong (DF_r^i - F_{r+1}^i DF)\mathrm{d}x \pmod{\Omega}$$

for every mapping (2.1). It follows that the requirements

$$F_{r+1}^i = \frac{DF_r^i}{DF}, \quad \mathbf{m}^*\omega_r^i \in \Omega \quad (\text{fixed } i \text{ and } r) \tag{2.3}$$

are equivalent. Denoting by $\mathbf{m}^*\Omega$ the module of all forms $\sum a_r^i\mathbf{m}^*\omega_r^i$ (finite sum), it follows that the total recurrence (1.10) is equivalent to the inclusion

$$\mathbf{m}^*\Omega \subset \Omega \tag{2.4}$$

of modules. Continuing in this direction, assume

$$\mathbf{m}^*\omega_r^i = \sum a_{rs}^{ij}\omega_s^j \quad (\text{finite sum}) \tag{2.5}$$

in accordance with the inclusion (2.4). Then

$$\mathbf{m}^*\mathrm{d}\omega_r^i = \mathbf{m}^*(\mathrm{d}x \wedge \omega_{r+1}^i) = \mathrm{d}F \wedge \mathbf{m}^*\omega_{r+1}^i \cong DF\mathrm{d}x \wedge \mathbf{m}^*\omega_{r+1}^i,$$

$$\mathrm{d}\mathbf{m}^*\omega_r^i \cong \sum Da_{rs}^{ij}\mathrm{d}x \wedge \omega_s^j + \sum a_{rs}^{ij}\mathrm{d}x \wedge \omega_{s+1}^j \pmod{\Omega \wedge \Omega}$$

and so we have obtained the important recurrence

$$DF\mathbf{m}^*\omega_{r+1}^i = \sum Da_{rs}^{ij}\omega_s^j + \sum a_{rs}^{ij}\omega_{s+1}^j = \mathcal{L}_D \sum a_{rs}^{ij}\omega_s^j = \mathcal{L}_D\mathbf{m}^*\omega_r^i \tag{2.6}$$

for the contact forms.

2.3. *On the invertible case*

In accordance with Sec. 1.4, we speak of a *morphism* $\mathbf{m}$ if the inclusion (2.4) holds true and of an *automorphism* $\mathbf{m}$ if moreover the (both left- and right-) inverse $\bar{\mathbf{m}}$ exists.

Lemma 1. *The inverse* $\bar{\mathbf{m}}$ *of a morphism* $\mathbf{m}$ *again is a morphism.*

Proof. We wish to prove $\bar{\mathbf{m}}^*\Omega \subset \Omega$. So let $\omega \in \Omega$ and assume the congruence $\bar{\mathbf{m}}^*\omega \cong f\mathrm{d}x \pmod{\Omega}$. Then

$$\omega = \mathbf{m}^*\bar{\mathbf{m}}^*\omega \cong \mathbf{m}^*(f\mathrm{d}x) \pmod{\mathbf{m}^*\Omega} \text{ hence } \pmod{\Omega}$$

by using (2.4). So we have

$$\omega \cong \mathbf{m}^*(f\mathrm{d}x) = \mathbf{m}^*f \cdot \mathrm{d}F \cong \mathbf{m}^*f \cdot DF\mathrm{d}x \pmod{\Omega}.$$

It follows that $\mathbf{m}^*f = 0$ hence $f = 0$ identically and we are done. □

Consequence 1. *If* $\mathbf{m}$ *is automorphism then* $\mathbf{m}^*\Omega = \Omega$.

Proof. Clearly $\mathbf{m}^*\Omega \subset \Omega, \bar{\mathbf{m}}^*\Omega \subset \Omega$ but the latter inclusion reads $\Omega \subset \mathbf{m}^*\Omega$. □

2.4. *Digression*

For every mapping (2.1) with the inverse (2.2), the images $\mathbf{m}_*Z$ and $\bar{\mathbf{m}}_*Z$ of any vector field Z defined by

$$(\mathbf{m}_*Z)f = \bar{\mathbf{m}}^*(Z\mathbf{m}^*f), \quad (\bar{\mathbf{m}}_*Z)f = \mathbf{m}^*(Z\bar{\mathbf{m}}^*f)$$

make a good sense. In particular, if $\mathbf{m}$ is an automorphism, then both vector fields $\mathbf{m}_*D, \bar{\mathbf{m}}_*D$ are multiples of D. (Hint: they satisfy $\omega(\mathbf{m}_*D) = \omega(\bar{\mathbf{m}}_*D) = 0$ for all $\omega \in \Omega$.) Then the identities

$$(\bar{\mathbf{m}}_*D)\mathbf{m}^*x = \mathbf{m}^*Dx = 1, \quad (\bar{\mathbf{m}}_*D)x = \mathbf{m}^*D\bar{\mathbf{m}}^*x$$

with $\mathbf{m}^*x = F, \bar{\mathbf{m}}^*x = \bar{F}$ imply

$$\bar{\mathbf{m}}_*D = \frac{1}{DF}D, \quad \bar{\mathbf{m}}_*D = \mathbf{m}^*D\bar{F}\cdot D, \quad DF\cdot\mathbf{m}^*D\bar{F} = 1.$$

This provides a simple alternative proof of recurrence (2.6) on one line

$$\frac{1}{DF}\mathcal{L}_D(\mathbf{m}^*\omega_r^i) = \mathcal{L}_{\bar{\mathbf{m}}_*D}(\mathbf{m}^*\omega_r^i) = \mathbf{m}^*\mathcal{L}_D\omega_r^i = \mathbf{m}^*\omega_{r+1}^i,$$

alas, only for the invertible case.

2.5. *The main result*

Theorem 2 (Invertibility Theorem). *Let $\mathbf{m}$ be a morphism such that $\omega_0^i \in \mathbf{m}^*\Omega$ for all $i = 1,\ldots,m$. Then $\mathbf{m}$ is an automorphism.*

Alternatively saying, by virtue of recurrence (2.6), $\mathbf{m}$ *is automorphism if all forms ω_0^j $(j = 1,\ldots,m)$ can be expressed as finite linear combinations of forms*

$$\mathbf{m}^*\omega_r^i = \left(\frac{1}{DF}\mathcal{L}_D\right)^r\mathbf{m}^*\omega_0^i \quad (i = 1,\ldots,m\,; r = 0,1,\ldots)$$

or, equivalently, of more appropriate forms

$$\ell_r^i = \mathcal{L}_D^r\mathbf{m}^*\omega_0^i = \sum\binom{r}{k}D^{r-k}a_{0s}^{ij}\omega_{s+k}^j \quad (i = 1,\ldots,m;\ r = 0,1,\ldots)$$

not depending on the factor $1/DF$.

Note 1. A very close interrelation between forms $\mathbf{m}^*\omega_r^i$ and ℓ_r^i is expressed by the formulae

$$\ell_0^i = \mathbf{m}^*\omega_0^i, \quad \ell_1^i = DF\mathbf{m}^*\omega_1^i, \quad \ell_r^i = D^rF\mathbf{m}^*\omega_1^i + \cdots + (DF)^r\mathbf{m}^*\omega_r^i \quad (r \geq 2)$$

which follow from the trivial recurrence $\ell_{r+1}^i = \mathcal{L}_D\ell_r^i$ $(r \geq 0)$. Consequently both families of forms $\mathbf{m}^*\omega_r^i$ and ℓ_r^i can be taken for generators of module $\mathbf{m}^*\Omega$. The invertibility theorem can be expressed by saying that *a morphism* $\mathbf{m}$ *is invertible if and only if the forms* $\mathbf{m}^*\omega_r^i$ *(hence the forms ℓ_r^i) generate module* Ω, see the point (ι) below. *In this case, they provide even a basis of* Ω, see $(\iota\iota)$ below.

The proof is lengthy and consists of five steps which are of independent interests and will be also referred to later.

2.6. *Towards the proof*

We shall see in (ι) that the assumption in the Main Theorem implies the equality $\Omega = \mathbf{m}^*\Omega$, morever it follows from ($\iota\iota\iota$) that the inverse $\mathbf{m}^{-1}$ does exist (as a mere abstract mapping) and finally ($\iota\nu$) implies that the inverse is in fact a true morphism. This concludes the proof.

(ι) *A simple reasoning.* Assume

$$\omega_r^i = \sum a_s^j \mathbf{m}^*\omega_s^j \in \mathbf{m}^*\Omega \quad (\text{fixed } i \text{ and } r, \text{finite sum}).$$

Then

$$\omega_{r+1}^i = \mathcal{L}_D\omega_r^i = \sum Da_s^j \mathbf{m}^*\omega_s^j + \sum a_s^j DF\mathbf{m}^*\omega_{s+1}^j \in \mathbf{m}^*\Omega \tag{2.7}$$

by using recurrence (2.6). It follows that the primary assumption $\omega_0^i \in \mathbf{m}^*\Omega$ implies even the inclusion $\omega_r^i \in \mathbf{m}^*\Omega$ for all r, therefore $\Omega \subset \mathbf{m}^*\Omega$ and the equality $\mathbf{m}^*\Omega = \Omega$ is obvious. Let us reformulate this simple reasoning in terms of filtrations for future needs.

($\iota\iota$) *Some filtrations and gradations.* We introduce the submodules $\Omega_l \subset \Omega$ of all differential forms $\omega = \sum a_r^i\omega_r^i$ (sum over all i and $r \le l$) of order l at most. Then

$$\Omega_* : \cdots \subset \Omega_{-1} = 0 \subset \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega = \cup\Omega_l \tag{2.8}$$

is a filtration. Analogously, let $\bar{\Omega}_l \subset \Omega$ be the submodule of all differential forms $\omega = \sum a_r^i\mathbf{m}^*\omega_r^i$ (sum over all i and $r \le l$). Then

$$\bar{\Omega}_* : \cdots \subset \bar{\Omega}_{-1} = 0 \subset \bar{\Omega}_0 \subset \bar{\Omega}_1 \subset \cdots \subset \Omega = \cup\bar{\Omega}_l \tag{2.9}$$

again is a filtration (which follows from the equality $\Omega = \mathbf{m}^*\Omega$). In particular, we have (locally) $\Omega_0 \subset \bar{\Omega}_S$ if S is large enough and then

$$\Omega_l \subset \bar{\Omega}_{l+S} \quad (l = 0, 1, \ldots) \tag{2.10}$$

as follows from Eq. (2.7).

We also introduce the gradation $\mathcal{N}_l = \bar{\Omega}_l/\bar{\Omega}_{l-1}$. Then the mapping $\mathcal{L}_D : \bar{\Omega}_l \to \bar{\Omega}_{l+1}$ defined by Eq. (2.7) naturally induces the morphism $\mathcal{L}_D : \mathcal{N}_l \to \mathcal{N}_{l+1}$ of modules denoted by the same letter. Recurrence (2.6) implies that the morphism is surjective if $l \ge 0$ whence

$$\dim \mathcal{N}_0 \ge \dim \mathcal{N}_1 \ge \cdots \ge \dim \mathcal{N}_K = \dim \mathcal{N}_{K+1} = \cdots \tag{2.11}$$

is stationary if K is large enough.

($\iota\iota\iota$) *On the bijectivity of* $\mathbf{m}$. Bijectivity of $\mathbf{m}$ is clearly equivalent to the bijectivity of $\mathbf{m}^*$, even to the bijectivity of the restriction $\mathbf{m}^* : \Omega \to \Omega$. (Hint: Ω together with $\mathrm{d}x$ generate the module of all differential forms on $\mathbf{M}(m)$, use moreover $\mathbf{m}^*\mathrm{d}x \cong DF\mathrm{d}x \pmod{\Omega}$ where $DF \ne 0$ is supposed.) We have proved in (ι) that $\mathbf{m}^* : \Omega \to \Omega$ is surjective. *We shall prove that it is even bijective*, therefore the (abstract) inverse $\mathbf{m}^{-1}$ exists. As a by-product, *the linear independence of differential forms is preserved after applying* $\mathbf{m}^*$.

Passing to the proof, let us employ inclusion (2.10) which provides the inequality

$$(l+1)m = \dim \Omega_l \le \dim \bar{\Omega}_{L+l} = \dim \mathcal{N}_0 + \cdots + \dim \mathcal{N}_{L+l}.$$

If l is taken large enough, the inequality together with (2.11) imply $m = \dim\mathcal{N}_K = \dim\mathcal{N}_{K+1} = \cdots$. On the other hand, clearly $m \ge \dim\mathcal{N}_0 \ge \dim\mathcal{N}_K$ whence $\dim\mathcal{N}_l = m$ for all l.

Now we recall the surjection $\mathbf{m}^* : \Omega_l \to \bar{\Omega}_l$ and therefore the naturally induced surjection

$$\mathbf{m}^* : \Omega_l/\Omega_{l-1} \to \bar{\Omega}_l/\bar{\Omega}_{l-1} = \mathcal{N}_l \quad (l \geq 0) \tag{2.12}$$

of gradations. However,

$$\dim \Omega_l/\Omega_{l-1} = m = \dim \mathcal{N}_l \quad (l \geq 0), \tag{2.13}$$

therefore (2.12) *is even a bijective mapping* between gradations. We recall two filtrations (2.8, 2.9) of module Ω. By virtue of well-known principle of algebra, we conclude that $\mathbf{m}^* : \Omega \to \Omega$ is a bijective mapping (even between filtrations Ω_* and $\bar{\Omega}_*$).

($\iota\nu$) *On the inverse morphism*. In order to determine the inverse morphism $\bar{\mathbf{m}}$ $(= \mathbf{m}^{-1})$ in explicit terms, we may start with inclusion (2.10) which is expressed by certain identities

$$\mathrm{d}w^i_r - w^i_{r+1}\mathrm{d}x = \sum b^{ij}_{rs}\mathbf{m}^*\omega^j_s = \sum b^{ij}_{rs}(\mathrm{d}F^j_s - F^j_{s+1}\mathrm{d}F) \tag{2.14}$$

and with the congruence $\mathrm{d}F \cong DF\mathrm{d}x$ (mod $\mathbf{m}^*\Omega$) which gives

$$\mathrm{d}x = \frac{1}{DF}\mathrm{d}F + \sum b^j_s(\mathrm{d}F^j_s - F^j_{s+1}\mathrm{d}F). \tag{2.15}$$

Identities (2.14, 2.15) together imply that we (locally) deal with certain composed functions

$$x = \bar{F}(F, \ldots, F^j_s, \ldots), \quad w^i_r = \bar{F}^i_r(F, \ldots, F^j_s, \ldots) \tag{2.16}$$

which are uniquely determined since differentials $\mathrm{d}F, \ldots, \mathrm{d}F^i_s, \ldots$ are linearly independent. Comparing (2.16) with Eqs. (2.1) and (2.2), it follows that we have just obtained explicit formulae for the inverse morphism $\bar{\mathbf{m}}$.

(ν) *Note*. In fact only the existence of functions $\bar{F}, \bar{F}^i_0$ $(i = 1, \ldots, m)$ cause the main difficulties since the remaining functions $\bar{F}^i_r$ with $r \geq 0$ follow from the recurrence (1.10). In particular there exists a *common definition domain for all functions* $\bar{F}, \bar{F}^i_r$ $(i = 1, \ldots, m;\ r = 0, 1, \ldots)$.

Consequence 2. *A morphism* $\mathbf{m}$ *is automorphism if and only if* $\mathbf{m}^*\Omega = \Omega$.

3. Examples

3.1. *Three-dimensional space*

We shall more systematically deal with curves in $\mathbb{R}^3$, hence $m = 2$. The notation in the space $\mathbf{M}(2)$ will be simplified as follows. Coordinates and contact forms:

$$x, \quad y_r = w^1_r, \quad z_r = w^2_r, \quad \eta_r = \omega^1_r, \quad \zeta_r = \omega^2_r,$$

equations of morphisms:

$$\begin{gathered} \mathbf{m}^*x = F(x, \ldots, y_s, z_s, \ldots), \\ \mathbf{m}^*y_r = G_r(x, \ldots, y_s, z_s, \ldots), \quad \mathbf{m}^*z_r = H_r(x, \ldots, y_s, z_s, \ldots), \end{gathered} \tag{3.1}$$

and the inverse morphisms:

$$\begin{gathered} \bar{\mathbf{m}}^*x = \bar{F}(x, \ldots, y_s, z_s, \ldots), \\ \bar{\mathbf{m}}^*y_r = \bar{G}_r(x, \ldots, y_s, z_s, \ldots), \quad \bar{\mathbf{m}}^*z_r = \bar{H}_r(x, \ldots, y_s, z_s, \ldots). \end{gathered} \tag{3.2}$$

Formulae (2.5) will be needed only for the particular case $r = 0$ and we denote

$$\begin{aligned} \mathbf{m}^*\eta_0 &= a_0\eta_0 + b_0\zeta_0 + \cdots + a_S\eta_S + b_S\zeta_S, \\ \mathbf{m}^*\zeta_0 &= A_0\eta_0 + B_0\zeta_0 + \cdots + A_S\eta_S + B_S\zeta_S \end{aligned} \tag{3.3}$$

with appropriate $S \geq 0$, where either $a_S \neq 0$ or $b_S \neq 0$ may be supposed without any loss of generality. Then

$$\begin{aligned} \mathbf{m}^*\eta_r &= \cdots + \frac{1}{(DF)^r}(a_S\eta_{S+r} + b_S\zeta_{S+r}), \\ \mathbf{m}^*\zeta_r &= \cdots + \frac{1}{(DF)^r}(A_S\eta_{S+r} + B_S\zeta_{S+r}) \end{aligned} \tag{3.4}$$

by using recurrence (2.6) with the vector field

$$D = \frac{\partial}{\partial x} + \sum y_{r+1}\frac{\partial}{\partial y_r} + \sum z_{r+1}\frac{\partial}{\partial z_r}.$$

We state only the top order terms in (3.4), more precise formulae will be necessary case by case.

We wish to determine such morphisms (3.1) that the inverse (3.2) exists. The algorithm consists of two parts. In the easier *algebraic part*, requirements on the coefficients a_i, b_i, A_i, B_i $(i = 1, \ldots, m)$ in (3.3) ensuring the inclusion $\eta_0, \zeta_0 \in \mathbf{m}^*\Omega$ and therefore ensuring the invertibility are determined. In the subsequent *analytic part*, the method $(\iota\nu)$ of Sec. 2.6 is applied in order to determine functions F, G_0, H_0 in transformation formulae (3.1).

We shall thoroughly discuss the cases $S = 0$ and $S = 1$, but only particular results will be stated if $S \geq 2$.

3.2. *The zeroth-order case*

Assume $S = 0$ in Eqs. (3.3) and (3.4). This will provide the simplest invitation into the substance of our method.

(ι) *The algebra.* We have

$$\mathbf{m}^*\eta_r = \cdots + \frac{1}{(DF)^r}(a_0\eta_r + b_0\zeta_r), \quad \mathbf{m}^*\zeta_r = \cdots + \frac{1}{(DF)^r}(A_0\eta_r + B_0\zeta_r)$$

for all r, see formulae (3.4). In invertible case, η_0 and ζ_0 should be expressed in terms of all forms $\mathbf{m}^*\eta_r, \mathbf{m}^*\zeta_r$. This is obviously possible if and only if

$$\triangle = \det\begin{pmatrix} a_0 & b_0 \\ A_0 & B_0 \end{pmatrix} \neq 0$$

which is supposed from now on.

$(\iota\iota)$ *The analysis.* We have

$$\begin{aligned} \mathbf{m}^*\eta_0 &= \mathrm{d}G_0 - G_1\mathrm{d}F = a_0\eta_0 + b_0\zeta_0, \\ \mathbf{m}^*\zeta_0 &= \mathrm{d}H_0 - H_1\mathrm{d}F = A_0\eta_0 + B_0\zeta_0. \end{aligned} \tag{3.5}$$

In principle two subcases are (locally, on open subsets) possible. Either differentials $\mathrm{d}F, \mathrm{d}x, \mathrm{d}y_0, \mathrm{d}z_0$ are linearly independent (*subcase* $\mathcal{C}$) or not (*subcase* $\mathcal{P}$). In the first subcase $\mathcal{C}$, we may suppose

$$G_0 = \mathcal{G}(F, x, y_0, z_0), \quad H_0 = \mathcal{H}(F, x, y_0, z_0).$$

In the second subcase $\mathcal{P}$, clearly

$$F = F(x, y_0, z_0), \quad G_0 = G(x, y_0, z_0), \quad H_0 = H(x, y_0, z_0)$$

and we deal with the common *point transformation* which need not any comments.

$(\iota\iota\iota)$ *On the subcase* $\mathcal{C}$. Inserting

$$\mathrm{d}G_0 = \mathcal{G}_F\mathrm{d}F + D\mathcal{G}\mathrm{d}x + \mathcal{G}_{y_0}\eta_0 + \mathcal{G}_{z_0}\zeta_0$$

into (3.5_1), then identities $\mathcal{G}_F = F_1$, $D\mathcal{G} = 0$, $\mathcal{G}_{y_0} = a_0$, $\mathcal{G}_{z_0} = b_0$ easily follow. Analogously $\mathcal{H}_F = H_1$, $D\mathcal{H} = 0$, $\mathcal{H}_{y_0} = A_0$, $\mathcal{H}_{z_0} = B_0$ follows from (3.5_2). In particular we have two linear equations

$$D\mathcal{G} = \mathcal{G}_x + y_1\mathcal{G}_{y_0} + z_1\mathcal{G}_{z_0} = 0, \quad D\mathcal{H} = \mathcal{H}_x + y_1\mathcal{H}_{y_0} + z_1\mathcal{H}_{z_0} = 0$$

with nonvanishing determinant $\triangle$ whence

$$y_1 = \frac{1}{\triangle}(\mathcal{H}_x\mathcal{G}_{y_0} - \mathcal{G}_x\mathcal{H}_{y_0}), \quad z_1 = \frac{1}{\triangle}(\mathcal{H}_x\mathcal{G}_{z_0} - \mathcal{G}_x\mathcal{H}_{z_0}).$$

This is a contradictory system for the function F. *Subcase $\mathcal{C}$ cannot be realized*.

3.3. *The first-order case*

Assume $S = 1$ in Eqs. (3.3) and (3.4). The general method will apply with only little additional effort.

(ι) *The algebra.* We suppose

$$\begin{aligned} \mathbf{m}^*\eta_0 &= a_0\eta_0 + b_0\zeta_0 + a_1\eta_1 + b_1\zeta_1, \\ \mathbf{m}^*\zeta_0 &= A_0\eta_0 + B_0\zeta_0 + A_1\eta_1 + B_1\zeta_1 \end{aligned} \tag{3.6}$$

where $a_1 \neq 0$ or $b_1 \neq 0$ may be assumed for certainty. Then

$$\mathbf{m}^*\eta_r = \cdots + \frac{1}{(DF)^r}(a_1\eta_{r+1} + b_1\zeta_{r+1}),$$

$$\mathbf{m}^*\zeta_r = \cdots + \frac{1}{(DF)^r}(A_1\eta_{r+1} + B_1\zeta_{r+1})$$

as the top order terms are concerned. In contrast with the zeroth-order case, necessarily

$$\det\begin{pmatrix} a_1 & b_1 \\ A_1 & B_1 \end{pmatrix} = 0$$

in the invertible case. (Hint: otherwise both η_0 and ζ_0 cannot be calculated in terms of all forms $\mathbf{m}^*\eta_r, \mathbf{m}^*\zeta_r$.) So we assume $A_1 = ca_1,\ B_1 = cb_1$ from now on. Then

$$\mu_0 = \mathbf{m}^*\zeta_0 - c\cdot\mathbf{m}^*\eta_0 = c_0\eta_0 + d_0\zeta_0 \in \mathbf{m}^*\Omega \quad (c_0 = A_0 - ca_0, d_0 = B_0 - cb_0)$$

and (roughly saying) one of the forms η_0, ζ_0 is lying in $\mathbf{m}^*\Omega$ (modulo the other). Continuing, we introduce the form

$$\mu_1 = \mathcal{L}_D\mu_0 = Dc_0\eta_0 + Dd_0\zeta_0 + c_0\eta_1 + d_0\zeta_1 \in \mathbf{m}^*\Omega$$

and analogous arguments as above imply that necessarily $c_0 = ua_1, d_0 = ub_1$ in the invertible case. Assuming this, we introduce the form

$$\nu_0 = \mu_1 - u\mathbf{m}^*\eta_0 = (Dc_0 - ua_0)\eta_0 + (Dd_0 - ub_0)\zeta_0 \in \mathbf{m}^*\Omega.$$

If μ_0, ν_0 are linearly independent, both forms η_0 and ζ_0 can be expressed in terms of these forms and we have the invertible case.

The independence is equivalent to the inequality

$$\det\begin{pmatrix} c_0 & Dc_0 - ua_0 \\ d_0 & Dd_0 - ub_0 \end{pmatrix} = u^2\det\begin{pmatrix} a_1 & Da_1 - a_0 \\ b_1 & Db_1 - b_0 \end{pmatrix} \neq 0. \tag{3.7}$$

Summarizing the achievements, *we have the final formulae*

$$\begin{aligned} \mathbf{m}^*\eta_0 &= a_0\eta_0 + b_0\zeta_0 + a_1\eta_1 + b_1\zeta_1, \\ \mathbf{m}^*\zeta_0 - c\cdot\mathbf{m}^*\eta_0 &= u(a_1\eta_0 + b_1\zeta_0) \end{aligned} \tag{3.8}$$

with inequality (3.7) *in the invertible case.*

($\iota\iota$) *The analysis.* Inserting

$$\mathbf{m}^*\eta_0 = \mathrm{d}G_0 - G_1\mathrm{d}F, \quad \mathbf{m}^*\zeta_0 = \mathrm{d}H_0 - H_1\mathrm{d}F$$

into identity (3.8), we conclude that two subcases are possible. Either differentials $\mathrm{d}F, \mathrm{d}x, \mathrm{d}y_0, \mathrm{d}z_0, \mathrm{d}y_1, \mathrm{d}z_1$ are linearly independent (*subcase* $\mathcal{C}$) and then

$$G_0 = \mathcal{G}(F, x, y_0, z_0, y_1, z_1), \quad H_0 = \mathcal{H}(F, G_0, x, y_0, z_0), \tag{3.9}$$

or, they are dependent (*subcase* $\mathcal{P}$) and then we may assume

$$\begin{gathered} F = F(x, y_0, z_0, y_1, z_1), \\ G_0 = G_0(x, y_0, z_0, y_1, z_1), H_0 = \mathcal{H}(F, G_0, x, y_0, z_0) \end{gathered} \tag{3.10}$$

without loss of generality.

($\iota\iota\iota$) *On the subcase* $\mathcal{C}$. Inserting (3.9) into (3.8) we obtain the identities

$$\begin{gathered} \mathcal{G}_F = G_1, \quad D\mathcal{G} = 0, \quad \mathcal{G}_{y_0} = a_0, \quad \mathcal{G}_{z_0} = b_0, \quad \mathcal{G}_{y_1} = a_1, \quad \mathcal{G}_{z_1} = b_1, \\ \mathcal{H}_F + \mathcal{H}_G\mathcal{G}_F = H_1, \quad D\mathcal{H} = 0, \\ (\mathcal{H}_G - c)\mathcal{G}_{y_0} + \mathcal{H}_{y_0} = ua_1\,, \ldots, (\mathcal{H}_G - c)\mathcal{G}_{z_1} = 0 \end{gathered}$$

with abbreviation $G = G_0$ and some symmetrical formulae omitted. Since either $\mathcal{G}_{y_1} \neq 0$ or $\mathcal{G}_{z_1} \neq 0$, we have $\mathcal{H}_G = c$ and the identities simplify to

$$\mathcal{G}_F = G_1, \quad D\mathcal{G} = 0, \quad \mathcal{H}_F + \mathcal{H}_G\mathcal{G}_F = H_1, \quad D\mathcal{H} = 0 \tag{3.11}$$

with the compatibility requirement

$$\mathcal{H}_{y_0}\mathcal{G}_{z_1} = \mathcal{H}_{z_0}\mathcal{G}_{y_1}$$

which is satisfied as a consequence of (3.11_4). (Hint: direct verification.)

Summary $\mathcal{C}$. *"A not too special" function $\mathcal{H}$ can be arbitrarily chosen and then the automorphism* $\mathbf{m}$ *is determined. In more detail, let $\mathcal{H}(F, G_0, x, y_0, z_0)$ be such a function that the equation*

$$(D\mathcal{H})(F, G_0, x, y_0, z_0, y_1, z_1) = \frac{\partial \mathcal{H}}{\partial x} + y_1\frac{\partial \mathcal{H}}{\partial y_0} + z_1\frac{\partial \mathcal{H}}{\partial z_0} = 0$$

determines a function $G_0 = \mathcal{G}(F, x, y_0, z_0, y_1, z_1)$ by using the implicit function theorem. Analogously assume that the equation

$$(D\mathcal{G})(F, x, y_0, \ldots, z_2) = \frac{\partial \mathcal{G}}{\partial x} + y_1\frac{\partial \mathcal{G}}{\partial y_0} + \cdots + z_2\frac{\partial \mathcal{G}}{\partial z_1} = 0$$

determines a function $F = F(x, y_0, \ldots, z_2)$. The inequality (3.7) *is "in general" satisfied. The remaining equations* ($3.11_{1,3}$) *may be regarded as a prolongation formulae.*

One can check that inequality (3.7) is satisfied if

$$\operatorname{rank}\begin{pmatrix} \mathcal{H}_{y_0} & \mathcal{H}_{y_0y_0} & \mathcal{H}_{y_0z_0} \\ \mathcal{H}_{z_0} & \mathcal{H}_{z_0y_0} & \mathcal{H}_{z_0z_0} \end{pmatrix} = 2$$

by a routine verification.

($\iota\nu$) *A notice.* This result was already obtained in Sec. 1.5. Indeed, let us choose

$$f(\bar{x}, \bar{w}_0^1, \bar{w}_0^2, x, w_0^1, w_0^2) = \mathcal{H}(\bar{x}, \bar{w}_0^1, x, w_0^1, w_0^2) - \bar{w}_0^2$$

in terms of the original notation. The automorphism was defined by equations

$$f = \mathcal{H} - \bar{w}_0^2 = 0, \quad Df = D\mathcal{H} = 0, \quad D^2 f = D^2\mathcal{H} = 0$$

in Sec. 1.5. Our automorphism $\mathbf{m}$ is defined by

$$(\mathbf{m}^* z_0 =)\ \mathbf{m}^* w_0^2 = H_0 = \mathcal{H}, \quad D\mathcal{H} = 0, \quad D\mathcal{G} = 0.$$

The first and the second of the equations are clearly identical. The last equations $D^2\mathcal{H} = 0$ and $D\mathcal{G} = 0$ are equivalent. (Hint. We have the identity

$$\mathcal{K}(F, \mathcal{G}(F, x, w_0^1, w_0^2, w_1^1, w_1^2), x, w_0^1, w_0^2, w_1^1, w_1^2) = 0 \quad (\mathcal{K} = D\mathcal{H})$$

for the function $\mathcal{G}$. Here F is a mere parameter whence the equation

$$D\mathcal{K}(\cdots) + \mathcal{K}_\mathcal{G}(\cdots)D\mathcal{G} = 0$$

is identically satisfied. Since $\mathcal{K}_G = (D\mathcal{H})_G \neq 0$ is assumed, we are done.) Altogether taken, *our morphism* $\boldsymbol{m}$ *in the subcase* $\mathcal{C}$ *is identical with the automorphism of* Sec. 1.5.

(ν) *On the subcase* $\mathcal{P}$. Inserting (3.10) into (3.8) we obtain the identities

$$DG = G_1 DF, \quad G_{y_0} - G_1 F_{y_0} = a_0, \ldots, \quad G_{y_1} - G_1 F_{y_1} = a_1, \ldots \tag{3.12}$$

$$\mathcal{H}_F DF + \mathcal{H}_G DG + D\mathcal{H} = H_1 DF, \tag{3.13}$$

$$\mathcal{H}_F F_{y_0} + \mathcal{H}_G G_{y_0} + \mathcal{H}_{y_0} = H_1 F_{y_0} + ca_0 + ua_1, \ldots \tag{3.14}$$

$$\mathcal{H}_F F_{y_1} + \mathcal{H}_G G_{y_1} = H_1 F_{y_1} + ca_1, \ldots \tag{3.15}$$

with abbreviation $G = G_0$. Indications to the system (3.12)–(3.15) are as follows.

1. Identities (3.15) and (3.12) provide the linear equations

$$(\mathcal{H}_F - H_1 + cG_1)F_{y_1} + (\mathcal{H}_G - c)G_{y_1} = 0, \quad (\mathcal{H}_F - H_1 + cG_1)F_{z_1} + (\mathcal{H}_G - c)G_{z_1} = 0$$

and it follows that either of the requirements

$$\det\begin{pmatrix} F_{y_1} & G_{y_1} \\ F_{z_1} & G_{z_1} \end{pmatrix} = 0, \quad \mathcal{H}_F - H_1 + cG_1 = \mathcal{H}_G - c = 0 \tag{3.16}$$

must be satisfied.

2. Requirement (3.16_2) implies (3.16_1) and this may be proved as follows. Assuming (3.16_2) then (3.13, 3.14) simplify as

$$D\mathcal{H} = 0, \quad \mathcal{H}_{y_0} = u(G_{y_1} - G_1 F_{y_1}), \quad \mathcal{H}_{z_0} = u(G_{z_1} - G_1 F_{z_1})$$

(direct verification) whence

$$\det\begin{pmatrix} \mathcal{H}_{y_0} & G_{y_1} - G_1 F_{y_1} \\ \mathcal{H}_{z_0} & G_{z_1} - G_1 F_{z_1} \end{pmatrix} = \frac{1}{DF} \det\begin{pmatrix} \mathcal{H}_{y_0} & G_{y_1} DF - F_{y_1} DG \\ \mathcal{H}_{z_0} & G_{z_1} DF - F_{z_1} DG \end{pmatrix} = 0$$

by employing (3.12_1). The top order terms in DF, DG provide the identity

$$\det\begin{pmatrix} \mathcal{H}_{y_0} & z_2 \\ \mathcal{H}_{z_0} & -y_2 \end{pmatrix} \cdot \det\begin{pmatrix} F_{y_1} & G_{y_1} \\ F_{z_1} & G_{z_1} \end{pmatrix} = 0$$

and this implies (3.16_1).

3. Assuming (3.16_1), then either of the possibilities

$$F = F(x, y_0, z_0), \quad (G_0 =)G = \mathcal{G}(F, x, y_0, z_0) \tag{3.17}$$

must be taken into account.

4. Suppose (3.17_1). Then the composed functions (3.10) can be replaced with the more convenient

$$F = F(x, y_0, z_0), \quad (G_0 =)G = G(x, y_0, z_0, y_1, z_1), \quad H_0 = \mathcal{H}(G, x, y_0, z_0). \tag{3.18}$$

Assuming (3.18), identities ($3.12_{4,5}$) and ($3.15_{1,2}$) give

$$\mathcal{H}_G G_{y_1} = cG_{y_1}, \quad \mathcal{H}_G G_{z_1} = cG_{z_1}$$

whence $c = \mathcal{H}_G$. Employing moreover ($3.12_{2,3}$) and ($3.14_{1,2}$), we obtain

$$\mathcal{H}_{y_0} = (H_1 - \mathcal{H}_G G_1 - \mathcal{H}_F)F_{y_0} + uG_{y_1}, \quad \mathcal{H}_{z_0} = (H_1 - \mathcal{H}_G G_1 - \mathcal{H}_F)F_{z_0} + uG_{z_1}$$

where $H_1 - \mathcal{H}_G G_1 - \mathcal{H}_F = D\mathcal{H}/DF$ may be inserted by using ($3.12_1, 3.13$). Altogether we obtain the system

$$\mathcal{H}_{y_0} = \frac{D\mathcal{H}}{DF}F_{y_0} + uG_{y_1}, \quad \mathcal{H}_{z_0} = \frac{D\mathcal{H}}{DF}F_{z_0} + uG_{z_1}$$

which provides the differential equation

$$(\mathcal{H}_{y_0}DF - F_{y_0}D\mathcal{H})G_{z_1} = (\mathcal{H}_{z_0}DF - F_{z_0}D\mathcal{H})G_{y_1} \quad (G = G_0) \tag{3.19}$$

for the function G_0. In more detail, Eq. (3.19) reads

$$(A + Bz_1)G_{z_1} + (C + By_1)G_{y_1} = 0 \quad (G = G_0) \tag{3.20}$$

where

$$A = \mathcal{H}_{y_0}F_x - F_{y_0}\mathcal{H}_x, \quad B = \mathcal{H}_{y_0}F_{z_0} - F_{y_0}\mathcal{H}_{z_0}, \quad C = F_{z_0}\mathcal{H}_x - \mathcal{H}_{z_0}F_x.$$

5. Suppose (3.17_2). Then the simple composed functions

$$F = F(x, y_0, z_0, y_1, z_1), \quad G_0 = \mathcal{G}(F, x, y_0, z_0), \quad H = \mathcal{H}(F, x, y_0, z_0) \tag{3.21}$$

can be introduced instead of (3.10). Identities (3.12)–(3.15) cannot be mechanically applied since G_0 is a composed function here, however, the equations

$$\begin{aligned}
&(\mathcal{G}_F - G_1)DF + D\mathcal{G} = 0, && (\mathcal{H}_F - H_1)DF + D\mathcal{H} = 0,\\
&(\mathcal{G}_F - G_1)F_{y_0} + \mathcal{G}_{y_0} = a_0, && \mathcal{H}_{y_0} - c\mathcal{G}_{y_0} = u(\mathcal{G}_F - G_1)F_{y_1},\\
&(\mathcal{G}_F - G_1)F_{z_0} + \mathcal{G}_{z_0} = b_0, && \mathcal{H}_{z_0} - c\mathcal{G}_{z_0} = u(\mathcal{G}_F - G_1)F_{z_1},\\
&(\mathcal{G}_F - G_1)F_{y_1} = a_1, && \mathcal{H}_F - H_1 = c(\mathcal{G}_F - G_1),\\
&(\mathcal{G}_F - G_1)F_{z_1} = b_1 &&
\end{aligned} \tag{3.22}$$

can be obtained with a little effort. It follows that $D\mathcal{G} = cD\mathcal{H}$ and then the differential equation

$$(\mathcal{H}_{y_0}D\mathcal{G} - G_{y_0}D\mathcal{H})F_{z_1} = (\mathcal{H}_{z_0}D\mathcal{G} - G_{z_0}D\mathcal{H})F_{y_1} \tag{3.23}$$

quite analogous to (3.19) appears.

Summary $\mathcal{P}$. *Assuming* (3.18), *functions $F, \mathcal{H}$ can be in principle arbitrarily chosen and we have differential equation* (3.19) *for the remaining function G_0. Assuming* (3.20), *we may choose functions $\mathcal{G}, \mathcal{H}$ and then F satisfies differential equation* (3.23).

($\nu\iota$) *A few notices.* We will not discuss inequality (3.7) since this is a mere routine task. Differential equations (3.19) and (3.22) can be explicitly resolved. For instance, general solution G of Eq. (3.19) written in the alternative transcription (3.20) is given by either of the implicit equations

$$(A+Bz_1)U=(C+By_1)V \quad (\text{if } B\neq 0), \quad Cz_1-Ay_1=W \quad (\text{if } B=0)$$

where U,V,W may be *arbitrary functions* of variables G,x,y_0,z_0. Recall that A,B,C are *given functions* of the same variables. Even very simple solutions are rather instructive, for instance the solution

$$\mathbf{m}^*x=x, \quad \mathbf{m}^*y_0=y_0+\lambda z_1, \quad \mathbf{m}^*z_0=z_0 \quad (\lambda\in\mathbb{R})$$

depending on parameter λ with the obvious inverse

$$\bar{\mathbf{m}}^*x=x, \quad \bar{\mathbf{m}}^*y_0=y_0-\lambda z_1, \quad \bar{\mathbf{m}}^*z_0=z_0 \quad (\lambda\in\mathbb{R})$$

is a one-parameter group. On the other hand, analogous solution

$$\mathbf{m}^*x=x, \quad \mathbf{m}^*y_0=y_0+\lambda^2y_1+\lambda z_1, \quad \mathbf{m}^*z_0=\lambda y_0+z_0 \quad (\lambda\in\mathbb{R})$$

with the inverse

$$\bar{\mathbf{m}}^*x=x, \quad \bar{\mathbf{m}}^*y_0=y_0-\lambda z_1, \quad \bar{\mathbf{m}}^*z_0=z_0-\lambda y_0+\lambda^2z_1$$

does not provide a group and it seems that this automorphism with a fixed parameter λ cannot be included into any one-parameter group except the trivial case $\lambda=0$.

Equation (3.23) can be explicitly resolved, as well. The resulting automorphisms (especially the prolongations) are more involved. We state the simplest example

$$\bar{\mathbf{m}}^*x=x+\lambda\frac{y_1}{z_1}, \quad \bar{\mathbf{m}}^*y_0=y_0, \quad \bar{\mathbf{m}}^*z_0=z_0 \quad \lambda\in\mathbb{R}$$

with the inverse

$$\bar{\mathbf{m}}^*x=x-\lambda\frac{y_1}{z_1}, \quad \bar{\mathbf{m}}^*y_0=y_0, \quad \bar{\mathbf{m}}^*z_0=z_0.$$

This is a one-parameter group.

3.4. *The second-order case*

Assuming $S=2$, our approach applies but a thorough discussion is rather lengthy. So we will indicate only few particular results in order to point out some new aspects currently appearing if $S\geq 2$.

(ι) *The algebra.* We recall formulae (3.3) which read

$$\bar{\mathbf{m}}^*\eta_0=a_0\eta_0+b_0\zeta_0+\cdots+a_2\eta_2+b_2\zeta_2, \quad \bar{\mathbf{m}}^*\zeta_0=A_0\eta_0+B_0\zeta_0+\cdots+A_2\eta_2+B_2\zeta_2$$

where either $a_2\neq 0$ or $b_2\neq 0$ is supposed. Using (3.4), it follows easily that $A_2=ca_0$, $B_2=cb_0$ in the invertible case. So we may introduce the first-order form

$$\mu_1=\bar{\mathbf{m}}^*\zeta_0-c\,\bar{\mathbf{m}}^*\eta_0=c_0\eta_0+d_0\zeta_0+c_1\eta_1+d_1\zeta_1\in\mathbf{m}^*\Omega$$

where $c_i=A_i-ca_i$, $d_i=B_i-cb_i$ $(i=0,1)$. Let us omit the "*residual*" *subcase* where $c_1=d_1=0$ identically. Clearly

$$\mu_{r+1}=\mathcal{L}_D^r\mu_1=\cdots+c_1\eta_{r+1}+d_1\zeta_{r+1}\in\mathbf{m}^*\Omega \quad (r=0,1,\ldots).$$

In the invertible case, both forms η_0,ζ_0 can be expressed in terms of forms $\mathbf{m}^*\eta_r$ and $\mathbf{m}^*\zeta_r$ $(r=0,1,\ldots)$. Using (3.4_1) with $S=2$, it follows that necessarily

$$c_1=ua_1, \quad d_1=ub_1 \tag{3.24}$$

hence

$$\mu_2 - u\,\mathbf{m}^*\eta_0 = (Dc_0 - ua_0)\eta_0 + (Dd_0 - ub_0)\zeta_0 + (Dc_1 + c_0 - ua_1)\eta_1 + (Dd_1 + d_0 - ub_1)\zeta_1$$

again is a first-order form in $\mathbf{m}^*\Omega$. Since either $c_1 \neq 0$ or $d_1 \neq 0$, we conclude that

$$Dc_1 + c_0 - ua_1 = vc_1, \quad Dd_1 + d_0 - ub_1 = vd_1 \tag{3.25}$$

in the invertible case. We introduce the zeroth-order form

$$\nu_0 = \mu_2 - u\,\mathbf{m}^*\eta_0 - v\mu_1 = (Dc_0 - ua_0 - vc_0)\eta_0 + (Dd_0 - ub_0 - vd_0)\zeta_0 \in \mathbf{m}^*\Omega$$

and let us consider form

$$\nu_1 = \mathcal{L}_D\nu_0 = \cdots + (Dc_0 - ua_0 - vc_0)\eta_1 + (Dd_0 - ub_0 - vd_0)\zeta_1 \in \mathbf{m}^*\Omega.$$

It easily follows that

$$Dc_0 - ua_0 - vc_0 = wc_1, \quad Dd_0 - ub_0 - vd_0 = wd_1 \tag{3.26}$$

in the invertible case. We obtain the zeroth-order form $\pi_0 = \nu_1 - w\mu_1 \in \mathbf{m}^*\Omega$. Linear independence of forms ν_0, π_0 is expressed by

$$\det\begin{pmatrix} Dc_0 - ua_0 - vc_0 & D(Dc_0 - ua_0 - vc_0) - wc_0 \\ Dd_0 - ub_0 - vd_0 & D(Dd_0 - ub_0 - vd_0) - wd_0 \end{pmatrix} \neq 0 \tag{3.27}$$

and this ensures the invertibility of the morphism $\mathbf{m}$. Summarizing the achievements, *we have the final formulae*

$$\bar{\mathbf{m}}^*\eta_0 = a_0\eta_0 + b_0\zeta_0 + \cdots + a_2\eta_2 + b_2\zeta_2,$$

$$\bar{\mathbf{m}}^*\zeta_0 - c\,\bar{\mathbf{m}}^*\eta_0 = c_0\eta_0 + d_0\zeta_0 + c_1\eta_1 + d_1\zeta_1$$

with coefficients satisfying (3.24, …, 3.27) *in the invertible case.*

($\iota\iota$) *The analysis.* In this article, we mention only the *subcase* $\mathcal{C}$ where differentials $\mathrm{d}F, \mathrm{d}x, \mathrm{d}y_0, \ldots, \mathrm{d}z_2$ are linearly independent. Then

$$G = \mathcal{G}(F, x, y_0, z_0, y_1, z_1, y_2, z_2), \quad H = \mathcal{H}(F, G, x, y_0, z_0, y_1, z_1)$$

(abbreviation $G = G_0$) may be assumed and identities

$$\mathcal{G}_F = G_1, \quad D\mathcal{G} = 0, \quad \mathcal{G}_{y_0} = a_0, \ldots, \mathcal{G}_{z_2} = b_2 \tag{3.28}$$

$$\mathcal{H}_F = H_1 + \mathcal{H}_G G_1, \quad D\mathcal{H} = 0, \quad \mathcal{H}_{y_0} = c_0, \ldots, \mathcal{H}_{z_1} = d_1 \tag{3.29}$$

immediately follow. Identities $(3.28_1, 3.29_1)$ are a mere prolongation formulae. Assuming function $\mathcal{H}$ for known, identity (3.29_2) reads

$$\mathcal{H}_x(\cdot) + y_1\mathcal{H}_{y_0}(\cdot) + \cdots + z_2\mathcal{H}_{z_1}(\cdot) = 0, \quad (\cdot) = (F, G, x, y_0, \ldots, z_1) \tag{3.30}$$

and determines the composed function $G = \mathcal{G}$. (We suppose $D\mathcal{H}_G \neq 0$ here.) With this function $\mathcal{G}$, identity (3.28_2) reads

$$\mathcal{G}_x(\cdot\cdot) + y_1\mathcal{G}_{y_0}(\cdot\cdot) + \cdots + z_3\mathcal{G}_{z_2}(\cdot\cdot) = 0, \quad (\cdot\cdot) = (F, x, y_0, \ldots, z_2) \tag{3.31}$$

and determines the remaining function

$$\mathbf{m}^*x = F(x, y_0, z_0, \ldots, y_3, z_3)$$

by using the implicit function theorem. (We suppose $D\mathcal{G}_F \neq 0$.) So, altogether taken, *our task is to determine such function* $\mathcal{H}$ *that requirements* (3.24)–(3.27) *are satisfied if functions* $\mathcal{G}, F$ *are determined from equations* (3.30) *and* (3.31).

($\iota\iota\iota$) *Auxiliary calculations.* We will express derivatives of functions $\mathcal{G}, F$ in terms of function $\mathcal{H}$. Let us begin with the identity (3.30) that is,

$$(D\mathcal{H})(F, \mathcal{G}(F, x, y_0, \ldots, z_2), x, y_0, \ldots, z_2) = 0.$$

It follows that derivatives of function $\mathcal{G}$ satisfy

$$\mathcal{K}_F + \mathcal{K}_G \mathcal{G}_F = 0, \quad \mathcal{K}_G D\mathcal{G} + D\mathcal{K} = 0,$$

$$\mathcal{K}_G \mathcal{G}_{y_0} + \mathcal{K}_{y_0} = 0, \ldots, \mathcal{K}_G \mathcal{G}_{z_2} + \mathcal{K}_{z_2} = 0$$

where

$$\mathcal{K}_F = D(\mathcal{H}_F), \quad \mathcal{K}_G = D(\mathcal{H}_G), \quad D\mathcal{G} = 0,$$

$$\mathcal{K}_{y_0} = D(\mathcal{H}_{y_0}), \quad \mathcal{K}_{y_1} = D(\mathcal{H}_{y_1}) + \mathcal{H}_{y_0}, \quad \mathcal{K}_{y_2} = \mathcal{H}_{y_1}, \ldots, \mathcal{K}_{z_2} = \mathcal{H}_{z_1}$$

may be substituted. Analogously (3.31) reads

$$(D\mathcal{G})(F(x, y_0, \ldots, z_3), x, y_0, \ldots, z_3) = 0$$

and this implies the identities

$$D\mathcal{G}_F DF + D^2\mathcal{G} = 0, \quad D\mathcal{G}_F F_{y_0} + D(\mathcal{G}_{y_0}) = 0, \ldots, D\mathcal{G}_F F_{z_3} + \mathcal{G}_{z_2} = 0$$

determining derivatives of F.

($\iota\nu$) *Fundamental requirements.* Employing identities (3.28), (3.29) and ($\iota\iota\iota$), requirements (3.24)–(3.27) on coefficients $a_0, \ldots, b_2$ and $c_0, \ldots, d_1$ can be expressed only in terms of function $\mathcal{H}$. We state the final result.

Requirement (3.24) reads

$$\mathcal{H}_{y_1} + u\frac{\mathcal{H}_{y_1}}{\mathcal{H}_G} = 0, \quad \mathcal{H}_{z_1} + u\frac{\mathcal{H}_{z_1}}{\mathcal{H}_G} = 0$$

whence $u = -\mathcal{K}_G = -D\mathcal{H}_G$. Analogously (3.25) turns into the system

$$\left(\mathcal{H}_{y_1F} - \mathcal{H}_{y_1G}\frac{\mathcal{K}_F}{\mathcal{K}_G}\right) DF = v\mathcal{H}_{y_1}, \quad \left(\mathcal{H}_{z_1F} - \mathcal{H}_{z_1G}\frac{\mathcal{K}_F}{\mathcal{K}_G}\right) DF = v\mathcal{H}_{z_1}$$

for the unknown v with the compatibility condition

$$\det\begin{pmatrix} \mathcal{H}_{y_1F}\mathcal{K}_G - \mathcal{H}_{y_1G}\mathcal{K}_F & \mathcal{H}_{y_1} \\ \mathcal{H}_{z_1F}\mathcal{K}_G - \mathcal{H}_{z_1G}\mathcal{K}_F & \mathcal{H}_{z_1} \end{pmatrix} = 0 \tag{3.32}$$

and requirement (3.26) consists of two linear equations (not written here) for the function w with the compatibility condition

$$\det\begin{pmatrix} (\mathcal{H}_{y_0F}\mathcal{H}_{z_1} - \mathcal{H}_{z_1F}\mathcal{H}_{y_0})\mathcal{K}_G - (\mathcal{H}_{y_0G}\mathcal{H}_{z_1} - \mathcal{H}_{z_1G}\mathcal{H}_{y_0})\mathcal{K}_F & 1 \\ (\mathcal{H}_{z_0F}\mathcal{H}_{y_1} - \mathcal{H}_{y_1F}\mathcal{H}_{z_0})\mathcal{K}_G - (\mathcal{H}_{z_0G}\mathcal{H}_{y_1} - \mathcal{H}_{y_1G}\mathcal{H}_{z_0})\mathcal{K}_F & 1 \end{pmatrix} = 0. \tag{3.33}$$

We shall not discuss inequality (3.27) here. Altogether taken, we have derived two equations (3.32) and (3.33) for the function $\mathcal{H}$ and both are of the special kind

$$A\mathcal{K}_F = B\mathcal{K}_G \quad (\mathcal{K} = D\mathcal{H}) \tag{3.34}$$

where coefficients A, B are expressed in terms of derivatives of function $\mathcal{H}$.

(ν) *A particular final result.* First of all, Eq. (3.34) is satisfied if $A = B = 0$ identically. *Assume this for both requirements* (3.32) *and* (3.33). Then (3.32) provides the condition

$$\operatorname{rank}\begin{pmatrix} \mathcal{H}_{y_1} & \mathcal{H}_{y_1F} & \mathcal{H}_{y_1G} \\ \mathcal{H}_{z_1} & \mathcal{H}_{z_1F} & \mathcal{H}_{z_1G} \end{pmatrix} = 1$$

which is satisfied if $\mathcal{H}$ is a composed function

$$\mathcal{H} = h(F, G, x, y_0, z_0, k) \quad \text{where } k = k(x, y_0, z_0, y_1, z_1).$$

With this result, (3.33) provides the system

$$\left(\frac{h_{y_0}}{h_k}\right)_F k_{z_1} = \left(\frac{h_{z_0}}{h_k}\right)_F k_{y_1}, \quad \left(\frac{h_{y_0}}{h_k}\right)_G k_{z_1} = \left(\frac{h_{z_0}}{h_k}\right)_G k_{y_1} \tag{3.35}$$

for the functions h and k. The system is compatible if

$$\frac{h_{y_0}}{h_k} = l\left(x, y_0, z_0, k, \frac{h_{z_0}}{h_k}\right)$$

is a composed function. However, Eqs. (3.35) with given h should have a solution k independent of parameters F, G. This is satisfied if

$$l = a(x, y_0, z_0, k)\frac{h_{z_0}}{h_{y_0}} + b(x, y_0, z_0, k)$$

is a linear function of argument h_{z_0}/h_{y_0}.

Summary, particular case of $\mathcal{C}$. *Let $h = h(F, G, x, y_0, z_0, k)$ be a solution of the equation*

$$h_{y_0} = a(x, y_0, z_0, k)h_{z_0} + b(x, y_0, z_0, k)h_k.$$

Let moreover $k = k(x, y_0, z_0, y_1, z_1)$ be a solution of the equation

$$a(x, y_0, z_0, k)k_{z_1} = k_{y_1}$$

equivalent to the system (3.35). *Then the composed function*

$$\mathcal{H} = h(F, G, x, y_0, z_0, k(x, y_0, z_0, y_1, z_1))$$

together with implicit equations (3.30) *and* (3.31) *determine an automorphism. Here a, b may be in principle quite arbitrary functions and we suppose the "general case" where $\mathcal{H}_G \neq 0, \mathcal{G}_F \neq 0$ and inequality* (3.27) *are satisfied.*

Quite explicit illustrative examples can be easily found, however, they do not look very instructive. For instance, if a and b are supposed constant, then

$$h = (ay_0 + z_0)G + (by_0 + k)(F + x), \quad k = ay_1 + z_1$$

may be chosen here but we omit the resulting clumsy transformation formulae.

($\nu\iota$) *Concluding note.* Equation (3.34) is of independent interests. We have already discussed the case $A = B = 0$ identically. One can easily check that the cases $A = 0, B \neq 0$ or $A \neq 0, B = 0$ do not give any automorphism. (Hint: $\mathcal{K}_G = D\mathcal{H}_G = 0$ implies $\mathcal{H}_G = a(F)$, $\mathcal{H} = a(F)G + b(x, y_0, \ldots, z_1)$ and then the Eq. (3.29_2) expressed by $D\mathcal{H} = Db = 0$ does not determine the desired composed function $G = \mathcal{G}$.) The remaining case $AB \neq 0$ is rather interesting, however, we can state only brief indications here.

First of all, (3.34) is satisfied if

$$\mathcal{H}_F = h(F, G, x, y_0, z_0, g) \quad \text{where } g = \mathcal{H}_G \tag{3.36}$$

and moreover the identities

$$Ah_g = B, \quad Dh = h_x + y_1 h_{y_0} + z_1 h_{z_0} = 0 \tag{3.37}$$

hold true (direct verification). Employing this result, (3.37_2) should determine a certain function $g = \mathcal{H}_G$, then $f = \mathcal{H}_F$ is determined by (3.36). Assuming the compatibility $f_G = \mathcal{H}_{FG} = \mathcal{H}_{GF} = g_F$, the desired function

$$\mathcal{H} = \int f dF + g dG = \int \mathcal{H}_F dF + \mathcal{H}_G dG$$

follows by integration.

However, the compatibility causes the main difficulties. It is expressed by

$$h_{\mathcal{G}} Dh_G - h_G Dh_{\mathcal{G}} = Dh_F. \tag{3.38}$$

(Hint. Use the identities

$$f_G = h_G + h_{\mathcal{G}} g_G, \quad Dh_F + Dh_{\mathcal{G}} g_F = 0$$

following from (3.36), (3.37_2).) The equation (3.38) must be a consequence of (3.37_2). Together with (3.37_1), this provides the final system of equations (not written here) for the function h.

4. The Main Algorithm

4.1. *Preliminaries*

Let us return to the Invertibility Theorem of Sec. 2.5. *We propose a universal algorithm for the verification of the assumption of this theorem, namely of the inclusion* $\omega_0^i \in \mathbf{m}^*\Omega$. This is a purely algebraical problem.

For this aim, we recall the submodules $\Omega_l \subset \Omega$ $(l = 0, 1, \ldots)$ with generators ω_r^i $(i = 1, \ldots, m;$ $r = 0, \ldots, l)$ introduced in (ι) of Sec. 2.6. Then $\mathbf{m}$ is invertible if and only if $\Omega_0 \subset \mathbf{m}^*\Omega$. This inclusion will be investigated by using the alternative generators

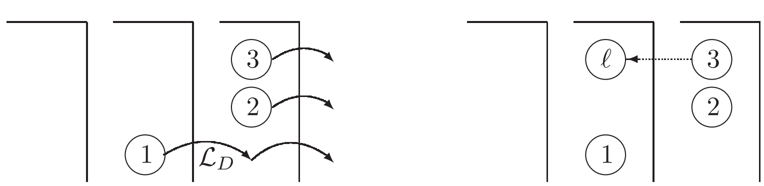
$$\ell_r^i = \mathcal{L}_D^r \mathbf{m}^* \omega_0^i = \sum \binom{r}{k} D^{r-k} a_{0s}^{ij} \omega_{s+k}^j \quad (i = 1, \ldots, m;\ r = 0, 1, \ldots)$$

of module $\mathbf{m}^*\Omega$, see Note 1 in Sec. 2.5. *The algorithm consists of four arrangements* A, B, C, D *which are successively and repeatedly applied. Assuming the inclusions*

$$\ell_0^i = \mathbf{m}^*\omega_0^i = \sum a_{0s}^{ij} \omega_s^j \in \Omega_S \quad (i = 1, \ldots, m, \text{ sum over } s \le S), \tag{4.1}$$

the algorithm will be "of the length mS*" at most.*

The idea of the algorithm can be elucidated as follows. Let the left-hand figure schematically describe the initial localization of forms $\ell_0^i \in \mathbf{m}^*\Omega$ in filtration Ω_*.



We repeatedly apply $\mathcal{L}_D$. Then the forms $\mathcal{L}_D^k \ell_0^i$ should cover all the module Ω including also the lower order elements. This implies that the top order summands of forms 1, 2, 3 cannot be linearly independent and permits to replace, e.g., the form 3 with a certain linear combination $\ell \in \mathbf{m}^*\Omega$ of lower order, see the right-hand figure. The procedure is repeated again and again. After a finite number of steps, we obtain even some forms of module $\mathbf{m}^*\Omega$ lying in Ω_0. *If the final forms are linearly independent, we have the invertible case, otherwise the invertibility fails.*

4.2. *The formal procedure*

Let us turn to thorough exposition. Assume (4.1).

A. *The sequence $\ell_0^1, \ldots, \ell_0^m$ is permuted to ensure the nondecreasing order of terms* (A mere technical measure which can be omitted in particular examples.) In more detail, we introduce the permutation $\ell^1(0), \ldots, \ell^m(0)$ of sequence $\ell_0^1, \ldots, \ell_0^m$ such that the order of $\ell^i(0)$ is K if the inequality $I(K-1) < i \le I(K)$ is satisfied where

$$-1 = I(-1) \le I(0) \le \cdots \le I(S) = m$$

is a certain sequence of integers. We shall occasionally denote

$$\ell^i(0)_K = \ell^i(0) = \sum A_{00}^{ij}\omega_0^j + \cdots + \sum A_{0K}^{ij}\omega_K^j \tag{4.2}$$

for better clarity of the order. (Coefficients A_{0s}^{ij} differ from a_{0s}^{ij} by a mere permutation with respect to i).

B. Module $\mathbf{m}^*\Omega(m)$ is generated by all forms $\ell_r^i = \mathcal{L}_D^r \ell_0^i$, hence by all forms $\mathcal{L}_D^r \ell^i(0)$. In particular, we mention generators

$$\mathcal{L}_D^{S-K+r}\ell^i(0)_K = \cdots + \sum A_{0K}^{ij}\omega_{S+r}^j \tag{4.3}$$

which are exactly of the order $S+r$. Assuming $S > 0$ (the case $S = 0$ is trivial), it follows that in the invertible case, *the determinant of the top-order coefficients necessarily vanishes*:

$$\det \begin{pmatrix} A_{0\ 0}^{i_0 1} & \cdots & A_{0\ 0}^{i_0 m} \\ A_{0\ 1}^{i_1 1} & \cdots & A_{0\ 1}^{i_1 m} \\ \cdots & & \\ A_{0\ S}^{i_S 1} & \cdots & A_{0\ S}^{i_S m} \end{pmatrix} = 0 \qquad \begin{array}{c} 0 \le i_0 \le I(0), \\ I(0) < i_1 \le I(1), \\ \cdots \\ I(S-1) < i_S \le I(S). \end{array}$$

(*Hint*. Otherwise forms (4.3) with $r = 0, 1, \ldots$ are linearly independent and of large order $S + r$.) It follows that all forms ω_s^i of lower order s, in particular forms ω_0^i, cannot be expressed as linear combination of forms $\mathcal{L}_D^r \ell^i(0)$ hence of forms $\mathcal{L}_D^r \ell_0^i = \ell_r^i$, see the left-hand figure. On the contrary, assuming the top order dependences, some lower order forms can be included, see the right-hand schema with the form ℓ defined by (4.5).) Therefore there exists *a nontrivial linear dependences*,

$$\sum c_{i_0} A_{0\ 0}^{i_0 j} + \sum c_{i_1} A_{0\ 1}^{i_1 j} + \cdots + \sum c_{i_S} A_{0\ S}^{i_S j} = 0 \quad (j = 1, \ldots, m) \tag{4.4}$$

among the rows.

C. *We choose a fixed identity* (4.4). Assuming $c_{i_{K+1}} = \cdots = c_{i_S} = 0$ but the summand $\sum c_{i_K} A_{0\ K}^{i_K j}$ nonvanishing, let moreover $c_I A_{0\ K}^{I\ j} \ne 0$ for a certain indice I (where $I(K-1) < I \le I(K)$) and appropriate j $(j = 1, \ldots, m)$. Clearly $K > 0$ in the invertible case. We introduce the form

$$\ell = \sum c_{i_0} \mathcal{L}_X^K \ell^{i_0}(0)_0 + \sum c_{i_1} \mathcal{L}_X^{K-1} \ell^{i_1}(0)_1 + \cdots + \sum c_{i_K} \ell^{i_K}(0)_K \tag{4.5}$$

which is *of the order less than* K by virtue of (4.4).

D. *The form* $\ell^I(0)_K = \ell^I(0)$ *of the order* K *is replaced by the lower order form* ℓ. We obtain the forms

$$\ell^1(0), \dots, \ell^{I-1}(0), \ell, \ell^{I+1}(0), \dots, \ell^m(0) \in \mathbf{m}^*\Omega \tag{4.6}$$

which overtake the role of the primary family $\ell_0^1, \dots, \ell_0^m$ in the step A above.

The procedure is repeated.

A. *Forms* (4.6) *are permuted to obtain sequence* $\ell^1(1), \dots, \ell^m(1)$ *of nondecreasing order* (only the localization of ℓ is changed). Analogously to (4.2) we denote

$$\ell^i(1)_K = \ell^i(1) = \sum A_{10}^{i\,j}\omega_0^j + \cdots + \sum A_{1K}^{i\,j}\omega_K^j$$

for better clarity, where the order of $\ell^i(1)_K$ is exactly K.

B. Module $\mathbf{m}^*\Omega$ is generated by the forms $\mathcal{L}_D^r\ell^i(1)$ and we determine linear dependence analogous to (4.4) among the top-order coefficients $A_{1K}^{i\,j}$.

C. A fixed dependence leads to a lower-order form in $\mathbf{m}^*\Omega$ analogous to the form (4.5).

D. Some of the forms $\ell^i(1)$ can be replaced with the relevant form ℓ and we obtain sequence $\ell^1(2), \dots, \ell^m(2)$ of nondecreasing order (the following step A), and so on.

One should remember: *in every particular step, the forms* $\ell^1(k), \dots, \ell^m(k)$ *are linearly independent in the invertible case. The algorithm does finish. Either the desired dependences in B do not exist for a certain step and we deal with the noninvertible case, or the orders of forms* $\ell^i(k)$ *cannot be already reduced* (*being of the order zero*). *In the latter case, the final forms*

$$\ell^1(k), \dots, \ell^m(k) \in \mathbf{m}^*\Omega \cap \Omega_0$$

either are linearly independent therefore generate Ω_0 *and we have the invertible case or, they are linearly dependent which implies the noninvertibility.* The number of reductions is clearly mS at most which is the "length of the algorithm".

4.3. *Concluding example*

The algorithm can be employed for explicit construction of invertible systems. For instance, let us deal with the particular case $S = 1$ and general $m \geq 1$, hence

$$\mathbf{m}^*\omega_0^i = \ell_0^i = \sum a_{00}^{ij}\omega_0^j + \sum a_{01}^{ij}\omega_1^j. \tag{4.7}$$

Assuming the dependences

$$c_1^1 a_{01}^{1j} + \cdots + c_m^1 a_{0m}^{1j} = 0 \quad (j = 1, \dots, m;\ c_1^1 \neq 0)$$

among top-order coefficients, we introduce the form

$$\ell^1 = \sum c_i^1 \ell_0^i = \sum A^{1j}\omega_0^j \quad (A^{1j} = \sum c_i^1 a_{00}^{ij}). \tag{4.8}$$

Then ℓ_0^1 is replaced by this ℓ^1 of the zeroth-order. Assuming the dependences

$$c_1^2 A^{1j} + c_2^2 a_{01}^{1j} + \cdots + c_m^2 a_{0m}^{1j} = 0 \quad (j = 1, \dots, m;\ c_2^2 \neq 0)$$

we introduce the form

$$\ell^2 = c_1^2 \mathcal{L}_D \ell^1 + \sum_{i>1} c_i^2 \ell_0^i = \sum A^{2j}\omega_0^j \quad \left(A^{2j} = c_1^2 D A^{1j} + \sum_{i>1} c_i^2 a_{00}^{ij}\right).$$

Then ℓ_0^2 is replaced by ℓ^2, and so on until we arrive to the resulting forms $\ell^1, \dots, \ell^m$ of zeroth order.

Let us summarize. Assuming the invertible case, *we have the zeroth-order forms* $\ell^1, \ldots, \ell^m$ *defined by* (4.8) *and the finite recurrence*

$$\ell^{k+1} = \sum_{i \le k} c_i^k \mathcal{L}_D \ell^i + \sum_{i > k} c_i^k \ell_0^i = \sum A^{k+1\,j} \omega_0^j \tag{4.9}$$

(*where* $c_k^k \neq 0$ *is supposed for* $k = 1, \ldots, m$) *which are linearly independent.*

The coefficients A^{kj}, c_i^k satisfy the identities

$$A^{kj} = \sum_{i<k} c_i^k A^{ij} + \sum_{i \ge k} c_i^k a_{00}^{ij}, \quad \sum_{i<k} c_i^k A^{ij} + \sum_{i \ge k} c_i^k a_{01}^{ij} = 0 \tag{4.10}$$

where $j = 1, \ldots, m$ and $c_k^k \neq 0$ $(k = 1, \ldots, m)$ is supposed. In the opposite direction, if we wish to write down an invertible system, then coefficients c_i^k (with $c_k^k \neq 0$) and a_{00}^{ij} may be arbitrarily chosen. Functions A^{kj} and the remaining coefficients a_{01}^{ij} can be determined and invertibility is ensured if $\det(A^{kj}) \neq 0$.

This is the *algebraic part* of the invertibility problem and we shall not deal with the *analytic part* since the general mechanisms of calculations were already illustrated in Secs. 3.2–4.1 above. In general, it is interesting to note that the subcase $\mathcal{C}$ with independent differentials $\mathrm{d}F, \mathrm{d}x, \mathrm{d}w_0^1, \ldots, \mathrm{d}w_1^m$ cannot be realized if $S = 1$ and $m > 2$ but we omit the proof here.

5. Comments

The true setting of our task rests upon the forgotten Cartan's idea to develop the theory of differential equations in the absolute sense, without any preferred hierarchy of variables and additional structures. Alas, the actual investigations are as a rule carried out in the rigid finite-order jet spaces, see [6, 10] and extensive literature therein. Then, due to the Lie–Bäcklund theorem [1, 5], only the prolonged point symmetries and Lie's contact transformations (for the case of one dependent variable) are admitted.

Some more general transformations adapted to special differential equations occasionally appear even in the classical literature. We recall the *Laplace substitutions* $\bar{u} = u_x + au$ in the theory of equations $u_{xy} + au_x + bu_y + cu = M$ [4] and *differential substitutions* $\bar{u} = g(x, u, u_x, \ldots, u_{x \ldots x})$ applied to evolutional equations [7–9, 11]. However, these are unsufficient and particular exceptions.

Our approach employs a somewhat unusual calculus in $\mathbb{R}^\infty$ treated in [2] and it can be applied even to the case of several independent variables [3]. We believe that it will be useful in the revised theory of symmetries and equivalences of differential equations and variational integrals.

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