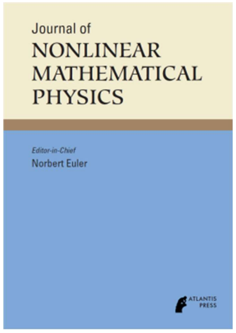

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INVARIANT LINEARIZATION CRITERIA FOR SYSTEMS OF CUBICALLY NONLINEAR SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Invariant linearization criteria for square systems of second-order quadratically nonlinear ordinary differential equations (ODEs) that can be represented as geodesic equations are extended to square systems of ODEs cubically nonlinear in the first derivatives. It is shown that there are two branches for the linearization problem via point transformations for an arbitrary system of second-order ODEs and its reduction to the simplest system. One is when the system is at most cubic in the first derivatives. One obtains the equivalent of the Lie conditions for such systems. We explicitly solve this branch of the linearization problem by point transformations in the case of a square system of two second-order ODEs. Necessary and sufficient conditions for linearization to the simplest system by means of point transformations are given in terms of coefficient functions of the system of two second-order ODEs cubically nonlinear in the first derivatives. A consequence of our geometric approach of projection is a rederivation of Lie's linearization conditions for a single second-order ODE and sheds light on more recent results for them. In particular we show here how one can construct point transformations for reduction to the simplest linear equation by going to the higher space and just utilizing the coefficients of the original ODE. We also obtain invariant criteria for the reduction of a linear square system to the simplest system. Moreover these results contain the quadratic case as a special case. Examples are given to illustrate our results.

Keywords: Invariant criteria; second-order systems; linearization; geometric approach; Lie algebras.

1. Introduction

A linearization problem involves the study of families of equations that are reducible via admissible transformations, which can be point, contact or more general, to linear equations. Lie [12] presented linearizability criteria, obtaining both algebraic and practical criteria, for a single second-order

ODE to be transformable to a linear equation via invertible changes of both the independent and dependent variables.

Lie [12] proved that necessary and sufficient conditions for a second-order ODE, $y'' = E(x, y, y')$, to be linearizable by means of invertible point transformations are that the ODE be at most cubic in the first derivative, viz.

$$y'' + E_3(x,y)y'^3 + E_2(x,y)y'^2 + E_1(x,y)y' + E_0(x,y) = 0 \tag{1.1}$$

and the coefficients E_0 to E_3 satisfy the over-determined system

$$\begin{aligned} b_x &= -\frac{1}{3}E_{1y} + \frac{2}{3}E_{2x} + be - E_0E_3, \\ b_y &= E_{3x} - b^2 + bE_2 - E_1E_3 + eE_3, \\ e_x &= E_{0y} + e^2 - eE_1 - bE_0 + E_0E_2, \\ e_y &= \frac{2}{3}E_{1y} - \frac{1}{3}E_{2x} - be + E_0E_3, \end{aligned} \tag{1.2}$$

where b and e are auxiliary variables and the suffices x and y here and hereafter refer to partial derivatives. Since the classic work of Lie there has been continuing interest in this topic. We, inter alia, rederive the Lie conditions (1.2) geometrically by projections.

Tressé [21] also studied the linearization problem for scalar second-order ODEs. He deduced two relative invariants of the equivalence group of point transformations, the vanishing of both of which gives necessary and sufficient conditions for linearization of Eq. (1.1). These conditions are equivalent to the Lie conditions (1.2) (see Mahomed and Leach [15]) and can be given as the compatibility of (1.2) as

$$\begin{aligned} &3(E_1E_3)_x - E_{1yy} + 2E_{2xy} - 3(E_0E_3)_y + E_2E_{1y} - 2E_2E_{2x} - 3E_{3xx} - 3E_3E_{0y} = 0, \\ &3(E_0E_3)_x + 2E_{1xy} - 3E_{0yy} - E_{2xx} - E_1E_{2x} + 2E_1E_{1y} - 3(E_0E_2)_y + 3E_0E_{3x} = 0. \end{aligned} \tag{1.3}$$

Note that under the interchange of E_3 by $-E_0$, E_2 by $-E_1$ and x by y, these conditions imply each other. Equations (1.3) provide practical criteria for linearization of Eq. (1.1) by point transformations. These conditions were also derived by the Cartan equivalence method (see Grissom *et al.* [9]) as well as recently using a geometric argument in Ibragimov and Magri [10]. The reader is also referred to the review of various approaches in Mahomed [13]. Linearization via point and other than point transformations is of great interest and has been investigated in several works (see, e.g. [2, 4, 8, 11, 16, 18–20, 23]).

The algebraic criteria of linearization of systems of second-order ODEs by means of point transformations have been considered in Wafo and Mahomed [23]. Practical criteria for quadratic nonlinear systems of second-order ODEs have been researched recently as well (see Mahomed and Qadir [17]). In this paper our intention is to extend these results to cubically nonlinear in the first derivatives square systems of second-order ODEs using geometric methods developed earlier (see Feroze *et al.* [5]). As a by-product of our approach we rederive the Lie conditions (1.2). Moreover we present practical criteria in terms of coefficients for cubically nonlinear in the first derivatives systems of second-order ODEs to be linearizable by point transformations. As a consequence we provide practical criteria for the class of linear second-order system of two ODEs to be reducible to the simplest system. Our results subsume the linearization criteria for the quadratic case.

The outline of this paper is as follows. In the next section we present mathematical preliminaries. In Sec. 3 we give an alternative method for obtaining the Lie conditions (1.2) as well as an alternate method for the construction of linearizing transformations for scalar second-order ODEs. Then in Sec. 4 we derive practical criteria for linearization for a system of two second-order ODEs cubically nonlinear in the first derivatives. Herein we state the relevant result for the reduction of linear square systems to the simplest system. Our theorem also contains the quadratically nonlinear equations as a corollary. In the next section we provide examples that amply illustrate our results. Finally in Sec. 6 we present a brief summary and conclusion.

2. Preliminaries

We firstly present some preliminaries. The system of geodesic equations is

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0, \quad i, j, k = 1, \ldots, n, \tag{2.1}$$

where the dot refers to total differentiation with respect to the parameter s and Γ^i_{jk} are the Christoffel symbols, which depend upon the x^i and are given in terms of the metric tensor as

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}). \tag{2.2}$$

The Christoffel symbols are symmetric in the lower pair of indices and have $n^2(n+1)/2$ elements. The Riemann curvature tensor is

$$R^i_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk}, \tag{2.3}$$

which is skew-symmetric in the lower last two indices and satisfies

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0. \tag{2.4}$$

A necessary and sufficient condition for a system of n second-order quadratically nonlinear ODEs for n dependent variables of the form (2.1) to be linearizable by point transformation and admit $sl(n+2, \mathbb{R})$ symmetry algebra is that the Riemann tensor vanishes [1, 16], i.e.

$$R^i_{jkl} = 0. \tag{2.5}$$

Practical criteria and the construction of point transformations are given in [17]. In particular for a system of two geodesic equations, (2.1), one has the linearization conditions (admittance of the $sl(4, \mathbb{R})$ symmetry algebra) on the coefficients given by

$$\begin{aligned} a_y - b_x + be - cd &= 0, \\ b_y - c_x + (ac - b^2) + (bf - ce) &= 0, \\ d_y - e_x - (ae - bd) - (df - e^2) &= 0, \\ (b + f)_x &= (a + e)_y, \end{aligned} \tag{2.6}$$

where the Christoffel symbols are

$$\Gamma^1_{11} = -a, \quad \Gamma^1_{12} = -b, \quad \Gamma^1_{22} = -c, \quad \Gamma^2_{11} = -d, \quad \Gamma^2_{12} = -e, \quad \Gamma^2_{22} = -f. \tag{2.7}$$

Now Eq. (2.2) together with (2.7) on setting $g_{11} = p$, $g_{12} = q = g_{21}$ and $g_{22} = r$ yield

$$\begin{aligned} p_x &= -2(ap + dq), \\ q_x &= -bp - (a + e)q - dr, \\ r_x &= -2(bq + er), \\ p_y &= -2(bp + eq), \\ q_y &= -cp - (b + f)q - er, \\ r_y &= -2(cq + fr). \end{aligned} \tag{2.8}$$

The construction of the linearization point transformations is as follows (see [17]). One invokes

$$g_{ab}(\mathbf{x}) = \frac{\partial u^i}{\partial x^a}\frac{\partial u^j}{\partial x^b} g_{ij}(\mathbf{u}), \tag{2.9}$$

where $\mathbf{x} = (x^1, \ldots, x^n)$, $\mathbf{u} = (u^1, \ldots, u^n)$ with the requirement that $g_{ij}(\mathbf{u})$ be the identity matrix. For the case of two variables we need to solve the equations

$$u_x^2 + v_x^2 = p, \quad u_x u_y + v_x v_y = q, \quad u_y^2 + v_y^2 = r, \tag{2.10}$$

for which we have set $(x^1, x^2) = (x, y)$, $(u^1, u^2) = (u, v)$, $g_{11} = p$, $g_{12} = q = g_{21}$ and $g_{22} = r$ in (2.9).

Following Aminova and Aminov [1] we project the system down by one dimension and write the geodesic equations (2.1) as

$$x^{a\prime\prime} + A_{bc}x^{a\prime}x^{b\prime}x^{c\prime} + B^a_{bc}x^{b\prime}x^{c\prime} + C^a_b x^{b\prime} + D^a = 0, \quad a = 2, \ldots, n, \tag{2.11}$$

where the prime now denotes differentiation with respect to the parameter x^1 (in [1] x^n is used as the parameter) and the coefficients in terms of the Γ^a_{bc}s are

$$A_{bc} = -\Gamma^1_{bc}, \quad B^a_{bc} = \Gamma^a_{bc} - 2\delta^a_{(c}\Gamma^1_{b)1}, \quad C^a_b = 2\Gamma^a_{1b} - \delta^a_b\Gamma^1_{11}, \quad D^a = \Gamma^a_{11}, \qquad a, b, c = 2, \ldots, n, \tag{2.12}$$

where we have used the notation $T_{(ab)} = (T_{ab} + T_{ba})/2$. It is straightforward to deduce (2.11) and (2.12). Indeed insert

$$\dot{x}^a = \frac{dx^a}{dx^1}\dot{x}^1, \quad a = 2, \ldots, n,$$

and its derivatives

$$\ddot{x}^a = \frac{d^2x^a}{dx^{1^2}}\dot{x}^{1^2} + \frac{dx^a}{dx^1}\ddot{x}^1, \quad a = 2, \ldots, n,$$

into system (2.1). These directly yield (2.11) and (2.12) after cancellation of $\dot{x}^{1^2}$. Note that in projecting down the Christoffel symbols there is degeneracy which results from the reduction of the range of the indices so that Γ^1_{11} and Γ^1_{b1} appear in the same combinations in C^a_b and B^a_{bc}, respectively. Consequently the set of coefficients $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ have n fewer elements than the coefficients Γ^i_{jk}.

3. Rederivation of the Lie Conditions

We invoke Eqs. (2.11) and (2.12) for $n = 2$. We also use (2.7) in identifying the Γ^i_{jk}s with the coefficients a to f of the system of two geodesic equations which projects to (2.11). Thus we have (setting $(x^1, x^2) = (x, y)$)

$$y'' + E_3(x, y)y'^3 + E_2(x, y)y'^2 + E_1(x, y)y' + E_0(x, y) = 0, \tag{3.1}$$

where

$$\begin{aligned} E_3 &= A_{22} = -\Gamma^1_{22} = c, \\ E_2 &= B^2_{22} = \Gamma^2_{22} - 2\Gamma^1_{12} = -f + 2b, \\ E_1 &= C^2_2 = 2\Gamma^2_{12} - \Gamma^1_{11} = -2e + a, \\ E_0 &= D^2 = \Gamma^2_{11} = -d. \end{aligned} \tag{3.2}$$

To rederive the Lie conditions (1.2) we use the system of two geodesic equations (2.1) from which Eq. (3.1) arises projectively. Hence we utilize the conditions (2.6) which are conditions for a flat space. This requires that the coefficients a to f be in terms of the E_is. From (3.2) we have

$$\begin{aligned} a &= E_1 + 2e, \\ c &= E_3, \\ d &= -E_0, \\ f &= 2b - E_2, \end{aligned} \tag{3.3}$$

where we have chosen b and e as yet arbitrary. These are constrained by the relations (2.6). We substitute (3.3) into (2.6). Equations (2.6) then yield

$$\begin{aligned} E_{1y} + 2e_y - b_x + be + E_3E_0 &= 0, \\ b_y - E_{3x} + E_1E_3 + eE_3 + b^2 - bE_2 &= 0, \\ E_{0y} + e_x + eE_1 + e^2 - bE_0 + E_0E_2 &= 0, \\ 3b_x - 3e_y - E_{2x} - E_{1y} &= 0. \end{aligned} \tag{3.4}$$

The first and last equations of (3.4) are easily seen to be equivalent to

$$\begin{aligned} b_x &= -\frac{1}{3}E_{1y} + \frac{2}{3}E_{2x} - be - E_0E_3, \\ e_y &= \frac{1}{3}E_{2x} - \frac{2}{3}E_{1y} - be - E_0E_3. \end{aligned} \tag{3.5}$$

When one replaces e by $-e$ the second and third equations of (3.4) as well as Eq. (3.5) are precisely the Lie conditions (1.2). Hence we have provided an alternative derivation of the Lie conditions (1.2) by viewing Eq. (3.1) in one higher space dimension and looking at the flat space requirement there. If we had projected the system of two geodesic equations to a single ODE of the form (3.1) by using x^2 instead of x^1, then by interchanging E_3 by $-E_0$, E_2 by $-E_1$ and $x^1 = x$ by $x^2 = y$, the coefficients (3.2) imply the coefficients of the projected equation with independent variable x^2. We state the following theorem.

Theorem 1. *A necessary and sufficient condition that the scalar second-order ODE* (3.1) *has* $sl(3,\mathbb{R})$ *symmetry algebra is that there is a corresponding system of two geodesic equations of the form* (2.1), *from which it is projected, that admits the* $sl(4,\mathbb{R})$ *symmetry algebra.*

Furthermore one can construct linearizing point transformations for (3.1) that satisfy (1.3) by resorting to the corresponding system of two geodesic equations from which (3.1) arises by projection. This is done by using the relations (2.10). This approach also results in the determination of at least one metric as a bonus. This method uses the coefficients of the equation which is linearizable and a transformation is then constructed via the relations (2.10). We consider two examples to illustrate this.

1. When one uses (3.3), the simple nonlinear equation

$$y'' + y'^3 - y' = 0 \tag{3.6}$$

has corresponding a to f values,

$$a = -1 + 2e, \quad c = 1, \quad d = 0, \quad f = 2b.$$

The substitution of these values into Eq. (2.6) results in an overdetermined system of first-order equations for b and e which has solution

$$b = \frac{\alpha'(y)}{\alpha(y) + \exp x}, \quad e = \frac{\exp x}{\alpha(y) + \exp x},$$

where α satisfies the ODE $\alpha'' - \alpha = 0$. The simplest solutions thus arise for $\alpha = 0$ and these correspond to $b = 0$ and $e = 1$. With these values inserted in a to f we obtain from (2.8) particular solutions for p, q and r given by

$$p = r = \exp(2y - 2x), \quad q = -\exp(2y - 2x).$$

When one invokes (2.10), a linearizing point transformation to the simplest second-order ODE is

$$u = \frac{1}{\sqrt{2}} \exp(-x + y), \quad v = \frac{1}{\sqrt{2}} \exp(-x - y),$$

where u is the new independent variable.

2. When one uses (3.3), the familiar nonlinear ODE (see, e.g. [14])

$$y'' + 3yy' + y^3 = 0$$

has

$$a = 3y + 2e, \quad c = 0, \quad d = -y^3, \quad f = 2b.$$

These and the choices $b = 1/y$ and $e = -y$ satisfy (2.6). A particular solution of (2.8) is then

$$p = 1 + x^2 - 2xy^{-1} + y^{-2}, \quad q = (1 + x^2)y^{-2} - xy^{-3}, \quad r = y^{-4}(1 + x^2).$$

After solution of (1.1) a point transformation that linearizes the original ODE to the simplest second-order equation then is

$$u = x - \frac{1}{y}, \quad v = \frac{1}{2}x^2 - \frac{x}{y},$$

where u is taken as the new independent variable. This transformation was previously obtained in [14] by mapping generators to canonical forms. We have presented another way to find such transformations.

4. Linearization Conditions for Square Systems

Encouraged by the success in obtaining the Lie conditions (1.2) by projection and then going back to the geodesic equations, we pursue similar conditions and practical criteria for linearization for a system of two second-order ODEs in a similar manner. Consequently, we study (2.11) for linearization via point transformations by resorting to a system of three geodesic equations (2.1). Before we do so we need firstly to understand what is meant by linearization for systems of ODEs. A system of two second-order linear ODEs can possess 5, 6, 7, 8 or 15 point symmetries (see [7, 22]). The maximal symmetry algebra, $sl(4, \mathbb{R})$, is reached for the simplest system. Here we consider practical linearization criteria in terms of the coefficients for a system of two cubically nonlinear second-order ODEs of the form (2.11) having the symmetry algebra $sl(4, \mathbb{R})$. The quadratically nonlinear case was treated in [17]. Also algebraic criteria for systems of second-order ODEs have been found in [23].

We once again invoke Eqs. (2.11) and (2.12) but now for $n = 3$. We therefore have

$$\begin{aligned} &x^{2''} + A_{22}(x^{2'})^3 + 2A_{23}(x^{2'})^2 x^{3'} + A_{33}x^{2'}(x^{3'})^2 + B^2_{22}(x^{2'})^2 + 2B^2_{23}x^{2'}x^{3'} \\ &\qquad + B^2_{33}(x^{3'})^2 + C^2_2 x^{2'} + C^2_3 x^{3'} + D^2 = 0, \\ &x^{3''} + A_{22}(x^{2'})^2 x^{3'} + 2A_{23}x^{2'}(x^{3'})^2 + A_{33}(x^{3'})^3 + B^3_{22}(x^{2'})^2 + 2B^3_{23}x^{2'}x^{3'} \\ &\qquad + B^3_{33}(x^{3'})^2 + C^3_2 x^{2'} + C^3_3 x^{3'} + D^3 = 0 \end{aligned} \tag{4.1}$$

with coefficients

$$A_{bc} = -\Gamma^1_{bc}, \quad B^a_{bc} = \Gamma^a_{bc} - 2\delta^a_{(c}\Gamma^1_{b)1}, \quad C^a_b = 2\Gamma^a_{1b} - \delta^a_b\Gamma^1_{11}, \quad D^a = \Gamma^a_{11}, \qquad a, b, c = 2, 3. \tag{4.2}$$

There is arbitrariness of two Cristoffel symbols for given coefficients of a 2×2 (quadratic) system of second-order ODEs. For the 3×3 (quadratic) system there is arbitrariness of three coefficients.

These correspond to the "auxiliary variables" in the Lie linearizability criteria. As such we can write the other Christoffel symbols in terms of the coefficients and any three Cristoffel symbols that we choose. We select Γ^1_{12}, Γ^2_{12} and Γ^3_{33} as arbitrary. We solve for the 15 Γ^a_{bc}s of (4.2) in terms of the 15 coefficients A_{bc}, B^a_{bc}, C^a_b, D^a as well as Γ^1_{12}, Γ^2_{12} and Γ^3_{33}. We only write down the Γ^a_{bc}s in which the arbitrary elements appear. They are

$$\begin{aligned}
\Gamma^1_{11} &= 2\Gamma^2_{12} - C^2_2,\\
\Gamma^1_{13} &= \frac{1}{2}(\Gamma^3_{33} - B^3_{33}),\\
\Gamma^2_{22} &= 2\Gamma^1_{12} + B^2_{22},\\
\Gamma^2_{23} &= \frac{1}{2}(\Gamma^3_{33} + 2B^2_{23} - B^3_{33}),\\
\Gamma^3_{13} &= \Gamma^2_{12} + \frac{1}{2}C^3_3 - \frac{1}{2}C^2_2,\\
\Gamma^3_{23} &= \Gamma^1_{12} + B^3_{23}.
\end{aligned} \tag{4.3}$$

The others can be read from Eq. (4.2).

The flat space requirements for the corresponding system of three geodesic equations (2.1) are now imposed by means of the vanishing of the Riemann tensor, viz. (2.5). They are (let $(x^1, x^2, x^3) = (x, y, z)$)

$$\begin{aligned}
(\Gamma^i_{j2})_x - (\Gamma^i_{j1})_y + \Gamma^i_{m1}\Gamma^m_{j2} - \Gamma^i_{m2}\Gamma^m_{j1} = 0,\\
(\Gamma^i_{j3})_x - (\Gamma^i_{j1})_z + \Gamma^i_{m1}\Gamma^m_{j3} - \Gamma^i_{m3}\Gamma^m_{j1} = 0,\\
(\Gamma^i_{j3})_y - (\Gamma^i_{j2})_z + \Gamma^i_{m2}\Gamma^m_{j3} - \Gamma^i_{m3}\Gamma^m_{j2} = 0,
\end{aligned} \tag{4.4}$$

which provide 27 conditions. Only 24 of them are linearly independent due to the identity (2.4). To see these equations in terms of the coefficients of the 3×3 system (4.1) we make the implicit summation over repeated indices explicit and use the above relations (4.3) to replace the Christoffel symbols which still leaves us with the "auxiliary variables". The details are given in the Appendix. A general algorithm for an $n \times n$ system along with a computer code to implement it is given in reference [6].

These are the 24 conditions, (6.1) to (6.3), given in the Appendix that arise from the vanishing of the Riemann tensor as given in (4.4). They are the integrability conditions of Lie-type for the Γ^i_{jk}. We find that there are 7 equations in (6.1) to (6.3) which are independent of the Γ^i_{jk}. The other 17 contain first-order partial derivatives of the Γ^i_{jk}. Of these, $\Gamma^2_{12,y}$ and $\Gamma^2_{12,z}$ appear once each, $\Gamma^3_{33,x}$ occurs three times and the rest twice each. Therefore, apart from the 7 conditions which are independent of the Γ^i_{jk} and given solely in terms of the coefficients of the system, there arise a further 8 conditions on the coefficients upon equating the respective Γ^i_{jk}. Hence we have 15 conditions or constraint equations on the coefficients. Now the Γ^i_{jk} which appear once each do not result in linearly independent equations as can easily be checked by equating them with the corresponding Γ^i_{jk} that were discarded. The resultant two equations that occur in this manner are linearly dependent. Thus the $\Gamma^2_{12,y}$ and $\Gamma^2_{12,z}$ are spurious. It is thus opportune to state the following theorem.

Theorem 2. *A necessary and sufficient condition for the system of two cubically nonlinear ODEs*

$$\begin{aligned}
y'' + A_{22}y'^3 + 2A_{23}y'^2z' + A_{33}y'z'^2 + B^2_{22}y'^2 + 2B^2_{23}y'z' + B^2_{33}z'^2 + C^2_2y' + C^2_3z' + D^2 = 0,\\
z'' + A_{22}y'^2z' + 2A_{23}y'z'^2 + A_{33}z'^3 + B^3_{22}y'^2 + 2B^3_{23}y'z' + B^3_{33}z'^2 + C^3_2y' + C^3_3z' + D^3 = 0,
\end{aligned} \tag{4.5}$$

where the prime denotes differentiation with respect to the independent variable x and the coefficients are in general functions of x, y, z, to be linearizable via point transformations to the simplest system of two second-order ODEs is that its coefficients satisfy the following fifteen conditions on the coefficients functions of (4.5), *viz.*

$$
\begin{aligned}
&\frac{1}{2}C^3_{2x} - D^3_y + \frac{1}{4}C^3_3C^3_2 + \frac{1}{4}C^2_2C^3_2 - D^2B^3_{22} - D^3B^3_{23} = 0,\\
&B^3_{22x} - \frac{1}{2}C^3_{2y} - A_{22}D^3 + \frac{1}{2}C^3_2B^2_{22} + \frac{1}{2}C^3_3B^3_{22} - \frac{1}{2}C^2_2B^3_{22} - \frac{1}{2}B^3_{23}C^3_2 = 0,\\
&B^3_{23x} - \frac{1}{3}B^2_{22x} + \frac{1}{6}C^2_{2y} - \frac{4}{3}D^3A_{23} - \frac{2}{3}B^3_{22}C^2_3 + \frac{2}{3}B^2_{23}C^3_2 - \frac{1}{2}C^3_{3y} = 0,\\
&\frac{1}{2}C^2_{3x} - D^2_z + \frac{1}{4}C^2_3C^3_3 + \frac{1}{4}C^2_3C^2_2 - B^2_{23}D^2 - B^2_{33}D^3 = 0,\\
&B^2_{33x} - \frac{1}{2}C^2_{3z} - D^2A_{33} + \frac{1}{2}C^2_3B^3_{33} - \frac{1}{2}B^2_{23}C^2_3 - \frac{1}{2}B^2_{33}C^3_3 + \frac{1}{2}B^2_{33}C^2_2 = 0,\\
&-A_{23y} + A_{22z} - A_{22}B^2_{23} - A_{23}B^3_{23} + A_{23}B^2_{22} + A_{33}B^3_{22} = 0,\\
&-A_{33y} + A_{23z} - A_{22}B^2_{33} - A_{23}B^3_{33} + A_{23}B^2_{23} + A_{33}B^3_{23} = 0,\\
&-A_{23x} + \frac{5}{6}A_{23}C^2_2 + \frac{1}{3}A_{33}C^3_2 - \frac{1}{3}B^3_{23z} + B^2_{33}B^3_{22} + \frac{1}{6}C^3_3A_{23} - B^2_{23}B^3_{23}\\
&\quad - \frac{2}{3}B^2_{23y} + \frac{1}{3}B^3_{33y} + \frac{2}{3}B^2_{22z} - \frac{1}{3}C^2_3A_{22} = 0,\\
&-A_{33x} + \frac{1}{2}C^2_2A_{33} + \frac{1}{2}A_{33}C^3_3 - B^2_{33y} + B^2_{23z} - B^2_{22}B^2_{33} + B^2_{23}B^2_{23} - B^2_{23}B^3_{33} + B^2_{33}B^3_{23} = 0,\\
&-\frac{2}{3}B^2_{22x} + \frac{1}{3}C^2_{2y} - \frac{1}{2}C^3_2B^3_{33} + D^2A_{22} - \frac{2}{3}D^3A_{23} - \frac{1}{3}C^2_3B^3_{22} + \frac{5}{6}B^2_{23}C^3_2\\
&\quad + B^3_{23x} - \frac{1}{2}C^3_{2z} + \frac{1}{2}C^3_3B^3_{23} - \frac{1}{2}C^2_2B^3_{23} = 0,\\
&-A_{22x} + \frac{1}{2}C^2_2A_{22} - B^3_{22}B^3_{33} + B^3_{23y} - B^3_{22z} + B^3_{22}B^2_{23} + B^3_{23}B^3_{23} + \frac{1}{2}C^3_3A_{22} - B^3_{23}B^2_{22} = 0,\\
&D^2_y + B^2_{22}D^2 + D^3B^2_{23} - D^3B^3_{33} + \frac{1}{2}C^3_{3x} - \frac{1}{2}C^2_{2x} - D^3_z + \frac{1}{4}C^3_3C^3_3 - \frac{1}{4}C^2_2C^2_2 - B^3_{23}D^2 = 0,\\
&-2A_{23x} + \frac{4}{3}B^3_{33y} + \frac{1}{3}A_{23}C^2_2 + \frac{5}{3}A_{23}C^3_3 + \frac{2}{3}C^2_3A_{22} - \frac{4}{3}B^3_{23z} - \frac{2}{3}C^3_2A_{33}\\
&\quad + 2B^3_{22}B^2_{33} - 2B^3_{23}B^2_{23} - \frac{2}{3}B^2_{23y} + \frac{2}{3}B^2_{22z} = 0,\\
&B^2_{23x} + \frac{1}{2}C^2_{3y} - 2D^2A_{23} + \frac{1}{2}C^2_3B^3_{23} + \frac{1}{2}C^2_3B^2_{22} + \frac{1}{2}C^3_3B^2_{23}\\
&\quad - \frac{1}{2}B^2_{23}C^2_2 - B^2_{33}C^3_2 - C^2_{2z} - D^3A_{33} = 0,\\
&-B^2_{23x} + B^3_{33x} + C^2_{3y} - C^2_3B^3_{23} + C^2_3B^2_{22} + B^2_{23}C^3_3 - B^2_{23}C^2_2 - \frac{1}{2}C^3_{3z} - \frac{1}{2}C^2_{2z} - 2D^3A_{33} = 0.
\end{aligned}
\tag{4.6}
$$

Proof. The proof follows from the preceding discussions. For, if the system of two equations (4.5) are linearizable by point transformation to the simplest system, then its coefficients can be written in terms of Γ^i_{jk} as in Eq. (4.2) which in turn gives rise to the integrability conditions of Lie-type on the Γ^i_{jk} and hence (4.6). Conversely, if the coefficients of the system of Eq. (4.5) satisfy the fifteen constraint conditions on the coefficients given by the relations (4.6) which is a consequence of the conditions of Lie-type (6.1) to (6.3), then the coefficients of the system (4.5) can be written in terms of the Γ^i_{jk} and the corresponding geodesic equations in three-space are linearizable as well as the projected equations (4.5).

Corollary 1. *The system of two quadratically nonlinear ODEs*

$$\begin{aligned} y'' + B^2_{22}y'^2 + 2B^2_{23}y'z' + B^2_{33}z'^2 = 0,\\ z'' + B^3_{22}y'^2 + 2B^3_{23}y'z' + B^3_{33}z'^2 = 0, \end{aligned} \tag{4.7}$$

where the B^a_{bc}s are functions of y and z and the dot denotes total derivative with respect to x, is linearizable by point transformations to the simplest system of two equations if and only if the B^a_{bc}s satisfy the four conditions on the coefficients given by

$$\begin{aligned} -B^3_{22}B^3_{33} + B^3_{23y} - B^3_{22z} + B^3_{22}B^2_{23} + B^3_{23}B^3_{23} - B^3_{23}B^2_{22} = 0,\\ \frac{4}{3}B^3_{33y} - \frac{4}{3}B^3_{23z} + 2B^3_{22}B^2_{33} - 2B^3_{23}B^2_{23} - \frac{2}{3}B^2_{23y} + \frac{2}{3}B^2_{22z} = 0,\\ -\frac{1}{3}B^3_{23z} + B^2_{33}B^3_{22} - B^2_{23}B^3_{23} - \frac{2}{3}B^2_{23y} + \frac{1}{3}B^3_{33y} + \frac{2}{3}B^2_{22z} = 0,\\ -B^2_{33y} + B^2_{23z} - B^2_{22}B^2_{33} + B^2_{23}B^2_{23} - B^2_{23}B^3_{33} + B^2_{33}B^3_{23} = 0. \end{aligned} \tag{4.8}$$

Remark. If one sets $B^2_{22} = -a, B^2_{23} = -b, B^2_{33} = -c, B^3_{22} = -d, B^3_{23} = -e$ and $B^3_{33} = -f$, one gets precisely the conditions (2.6). Hence Theorem 2 naturally contains the linearizability criteria for the quadratic case.

Corollary 2. *The system of two ODEs linear in the first derivatives*

$$\begin{aligned} y'' + C^2_2y' + C^2_3z' + D^2 = 0,\\ z'' + C^3_2y' + C^3_3z' + D^3 = 0, \end{aligned} \tag{4.9}$$

where the prime refers to differentiation with respect to x and the C^a_bs are independent of y and z, is linearizable by point transformations to the simplest system of two equations if and only if the C^a_bs and D^as satisfy the three conditions on the coefficients, viz.

$$\begin{aligned} \frac{1}{2}C^3_{2x} + \frac{1}{4}C^3_3C^3_2 + \frac{1}{4}C^2_2C^3_2 = D^3_y,\\ \frac{1}{2}C^2_{3x} + \frac{1}{4}C^2_3C^3_3 + \frac{1}{4}C^2_3C^2_2 = D^2_z,\\ \frac{1}{2}C^3_{3x} - \frac{1}{2}C^2_{2x} + \frac{1}{4}C^3_3C^3_3 - \frac{1}{4}C^2_2C^2_2 = D^3_z - D^2_y. \end{aligned} \tag{4.10}$$

We have provided practical criteria, necessary and sufficient conditions, for equations of the form (4.5) to be linearizable via point transformations to the simplest system. The question naturally arises if there are more general equations than (4.5) that can be linearizable to the simplest system. Indeed there are more general systems of two second-order ODEs which can be linearized.

The most general system of $n-1$ second-order ODEs linearizable is given by

$$J^i_jx^{j''} + G^i_{kj}x^{k'}x^{j''} + \Delta^i_{jkl}x^{j'}x^{k'}x^{l'} + \Lambda^i_{jk}x^{j'}x^{k'} + \Omega^i_jx^{j'} + E^i = 0, \quad i = 2, \dots, n, \tag{4.11}$$

where the prime refers to total differentiation with respect to x^1, the coefficient functions are dependent upon $x^1, \dots, x^n$ and are given by

$$\begin{aligned} J^i_j &= X^1_{,1}X^i_{,j} - X^1_{,j}X^i_{,1}, \\ G^i_{kj} &= X^1_{,k}X^i_{,j} - X^1_{,j}X^i_{,k}, \\ \Delta^i_{jkl} &= X^1_{,l}X^i_{,jk} - X^1_{,jk}X^i_{,l}, \\ \Lambda^i_{jl} &= 2X^1_{,l}X^i_{,1j} - 2X^1_{,1j}X^i_{,l} + X^1_{,1}X^i_{,jl} - X^i_{,1}X^1_{,jl}, \\ \Omega^i_j &= 2X^1_{,1}X^i_{,1j} - 2X^1_{,1j}X^i_{,1} + X^1_{,j}X^i_{,11} - X^1_{,11}X^i_{,j}, \\ E^i &= X^1_{,1}X^i_{,11} - X^1_{,11}X^i_{,1}, \quad i,j,k,l = 2,\dots,n, \end{aligned} \tag{4.12}$$

in which

$$X^1 = X^1(x^1,\dots,x^n), \quad X^i = X^i(x^1,\dots,x^n), \quad i = 2,\dots,n, \tag{4.13}$$

are invertible transformations. It is certainly not difficult to obtain (4.11). This is done by the substitution of (4.13) into the free particle system

$$X^{i\prime\prime} = 0, \quad i = 2,\dots,n; \quad {}' = \frac{d}{dX^1}. \tag{4.14}$$

This after routine calculations yields (4.11) with the coefficients satisfying (4.12). Equation (4.11) is the most general system of $n-1$ equations point transformable to the simplest system (4.14). Equation (4.11) has $n(n-1)(n^2+6n-1)/6$ coefficients.

Equation (4.11) can be written in normal form in terms of at most cubic first-order derivatives as

$$x^{i\prime\prime} + A^i_{jkl}x^{j\prime}x^{k\prime}x^{l\prime} + B^i_{jk}x^{j\prime}x^{k\prime} + C^i_j x^{j\prime} + D^i = 0, \quad i,j,k,l = 2,\dots,n, \tag{4.15}$$

provided

$$\begin{aligned} \Delta^i_{klm} &= J^i_j A^j_{klm} + G^i_{mj}B^j_{kl}, \\ \Lambda^i_{kl} &= J^i_j B^j_{kl} + G^i_{lj}C^j_k, \\ \Omega^i_k &= J^i_j C^j_k + G^i_{kj}D^j, \\ E^i &= J^i_j D^j, \\ G^i_{pj}A^j_{klm} &= 0. \end{aligned} \tag{4.16}$$

Relations (4.16) can be obtained by solving for the second derivative in terms of the first-order derivatives and inserting these into Eq. (4.11). The last equation of (4.16) tells us that not all the A^i_{jklm} coefficients are independent. As a matter of fact, if we replace these by A_{kl} in (4.15), it turns out that this relation in (4.16) is now identically satisfied. What transpires is that the quartic term disappears automatically due to G^i_{pj} being skew symmetric in the lower indices and $x^{p\prime}x^{j\prime}$ appearing symmetrically. One also needs then to adjust the relation (4.16a) in the latter case by

$$\Delta^i_{klm} = J^i_k A_{lm} + G^i_{mj}B^j_{kl}. \tag{4.17}$$

The remaining equations of (4.16) are the same.

There are two branches of the linearization problem by point transformations for a system of $n-1$ second-order ODEs. One is the general form (4.11) owing to the arbitrariness of the coefficients Δ^i_{jkl}. The other is the form, (2.11), in which the cubic coefficients are fewer in number. In the case of two second-order ODEs, Eq. (4.5), we have obtained explicit linearization criteria as encapsulated in Theorem 2 and in terms of their corollaries.

In the general equation (4.11) there are $(n-1)n(n^2+6n-1)/6$ coefficients while for (2.11) there are $(n-1)n(n+2)/2$ independent coefficients. It would be of interest to find practical criteria for the reduction of Eq. (4.11) to the simplest system via point transformations for $n = 3$. Of course it is of great interest to do this for the general system, (4.11), for $n \geq 4$.

If one has a system of the form (4.11) with known coefficients which is reducible to the free particle system (4.14) by point transformations, then one can utilize (4.12) to construct a linearizing point transformation. Also we can obtain linearizing point transformations for system (2.11) if it is linearizable to the simplest system (4.14) by invoking (4.12) together with (4.16).

In particular one can find linearizing point transformations for the system (4.5) in a similar manner by solving the system (4.16).

Instead of using the system (4.12) in order to construct a linearizing point transformation there are other ways as noted earlier. One is to go to the higher space, once one has the coefficients at hand, and use (2.9) for which $g_{ij}(\mathbf{u})$ must be the identity matrix in which we may set u^1 to be the independent variable. Yet a third approach is that of mapping symmetry generators of the linearizable system, if known, to the free particle generators.

5. Examples

We present examples to illustrate our results. We have y and z as the dependent variables. Also the prime below denotes differentiation with respect to x. Moreover we have included one example that does not satisfy our linearization criteria, but belongs to the more general class (4.11) which is linearizable.

1. Consider the anisotropic oscillator system

$$\begin{aligned} y'' + \omega_1(x)y &= 0, \\ z'' + \omega_2(x)z &= 0. \end{aligned} \tag{5.1}$$

The coefficients of system (5.1) satisfy the conditions (4.10) provided $\omega_1 = \omega_2$. Hence in order for the system (5.1) to be reducible to the free particle system one must have isotropy.

2. The simple linear system

$$\begin{aligned} y'' + z &= 0, \\ z'' + z &= 0, \end{aligned} \tag{5.2}$$

does not satisfy conditions (4.10). Thus this system is not transformable pointwise to the free particle system. This system does not have a first-order Lagrangian formulation as well [3].

3. For the quadratic system,

$$\begin{aligned} y'' - y' + y'^2 &= 0, \\ z'' - z' + z'^2 &= 0, \end{aligned} \tag{5.3}$$

all conditions (4.8) are satisfied. Therefore the system (5.3) is reducible to the simplest system. A point transformation that does the job is

$$u = \exp x, \quad v = \exp y, \quad w = \exp z, \tag{5.4}$$

where u is the independent variable. This can be constructed by going to the higher space as we have illustrated for scalar ODEs in Sec. 3.

4. Consider the cubically nonlinear system

$$\begin{aligned} y'' + \frac{1}{x}y' + y'^2 + \left(\frac{x}{y} + \frac{x}{y^2}\right) y'^3 &= 0, \\ z'' + \frac{1}{x}z' + z'^2 + 2y'z' + \left(\frac{x}{y} + \frac{x}{y^2}\right) y'^2 z' &= 0. \end{aligned} \tag{5.5}$$

For the system (5.5) all conditions (4.6) hold. A linearizing point transformation to the simplest system is

$$u = \ln xy, \quad v = \exp y, \quad w = \exp(y + z), \tag{5.6}$$

in which u is the independent variable.

5. Finally the system

$$\begin{aligned} 4yz^2y'^2 + 4y^2zy'z' + 2xz^2y'^3 + 8xyzy'^2z' + 2xy^2y'z'^2 + 2xy^2zy'z'' &= y^2z^2y'' + 2xy^2zz'y'', \\ y'' + xzy'y'' + xyz'y'' - xz^2y'^2y'' - xyzy'z'y'' &= y'(zy' + yz')(zy' + yz' + 2xy'z' + xyz''), \end{aligned} \tag{5.7}$$

is not of the form given in Theorem 2. It is of the form given in (4.11) and is linearizable by means of the point transformation

$$u = x\exp(yz), \quad v = xy^2z^2, \quad w = y, \tag{5.8}$$

where u is the independent variable.

6. Concluding Remarks

Aminova and Aminov [1] had provided a procedure of projecting down one dimension from a system of n geodesic equations to $n-1$ nonlinear in the first derivatives ODEs. Separately we had provided [17] linearizability criteria for a square quadratically nonlinear system. These were used together to derive linearizability criteria for a single cubically nonlinear equation by projecting down from a system of two quadratically nonlinear equations. This provided an alternate method to prove Lie's general result for linearizability of a single nonlinear equation. It led naturally to an extension of the linearization criteria via point transformations from a scalar second-order ODE as obtained by Lie [12] to a system of two cubically nonlinear ODEs of the form (4.5). These provided necessary and sufficient conditions for reduction to the simplest system and hence the symmetry algebra $sl(4,\mathbb{R})$ for equations of the form (4.5). Moreover Theorem 2 provides criteria for the reduction of linear systems of two equations to the free particle system.

Lie had demonstrated [12] that only cubically nonlinear scalar equations of order two are linearizable in general. It could have been hoped that the projection procedure would provide the complete solution of the linearizability problem for the system of two nonlinear ODEs. That hope is doomed from the start as there are five classes of systems of two cubically nonlinear equations that are linearizable by point transformations which have different symmetry algebras. Moreover the maximum symmetry algebra class of such systems of two equations is one branch of the linearization problem via point transformations as the general class is represented by (4.11). Why do we get a unique class in the former case and five in the latter? Furthermore how many distinct classes should there be for a system of n cubically nonlinear ODEs?

We start by noting that the projection procedure and linearizability can be equally well adopted for an arbitrary system of n quadratically nonlinear second-order ODEs reduced to $n-1$ cubically nonlinear second-order ODEs. There are two branches for the linearization problem for systems admitting the maximal algebra for $n \geq 3$. There are enormous computational complications that arise. One would need an algebraic computational code to deal with larger systems. A code has indeed been prepared to construct the metric coefficients given the Christoffel symbols [6]. That can be extended to treat the linearization of larger systems. Indeed one can obtain the equivalent of the Lie conditions for systems of $n > 3$ equations. Note that the number of scalar equations given by the linearizability requirement (2.5) is n^4. However the number of linearly independent components of the Riemann tensor is $N(n) = n^2(n^2-1)/12$. To see the enormous reduction it is worth giving the first four values of n^4 and $N(n)$: $2^4 = 16, N(2) = 1$; $3^4 = 81, N(3) = 6$; $4^4 = 256, N(4) = 20$; $5^4 = 3125, N(5) = 50$. As pointed out in [6], not every system of second-order (quadratic) ODEs can be related to a metric. There is a further set of compatibility conditions required. For the 4×4 (quadratic) system there are four compatibility conditions that arise from the four auxiliary variables. For an $n \times n$ (quadratic) system there are n auxiliary variables which lead to n compatibility conditions. Thus for an $n \times n$ (quadratic) system the total number of conditions that correspond to (4.4) is $n^2(n^2-1)/12 + n$. Therefore for $n = 5$ we obtain 55 equations. Now observe that in projecting down from the system of n dependent variables to $n-1$ variables the Christoffel symbols are reduced from $n^2(n+1)/2$ by n to give $(n-1)n(n+2)/2$ independent coefficients. Since we now have $n-1$ equations, each with its own cubic function, there are $(n-1)n/2$ cubic coefficients for the reduced system. If the number of coefficients left over after losing n equals the number of coefficients of the reduced system, we can determine one set of coefficients in terms of the other. The two expressions are obviously equal for $n = 2$ and the former is greater than the latter for $n > 2$. The coefficients of the cubic system can be determined uniquely in terms of the quadratic system for $n = 2$, i.e. for a scalar cubically nonlinear system. For larger systems there are infinitely many ways to write the former in terms of the latter. Hence there is a unique solution to the linearizability problem only for the scalar cubically nonlinear equation and many solutions for systems of cubically nonlinear systems!

The second question remains and has in fact been compounded. It is known that there are five and not infinitely many distinct classes. Why? The point is that all distinct ways of writing the cubic system coefficients in terms of the quadratic system coefficients do not give independent criteria as there are transformations permissible from one definition to another. The point is to determine those that are distinct. Another way of looking at what we have done is to note that we have asked that the original system correspond to a system of geodesic equations in flat space. Then the projection gives the reduced system, which must also be of geodesics in an $(n-1)$-dimensional flat space. Even if the original geodesics were curved, the projected geodesics could correspond to straight lines. For example, if the original space was a sphere and one projects along the plane containing the geodesic to a plane perpendicular to it, the resulting projected curve would be a straight line.

The minimal dimension of the symmetry algebra for a system of n second-order ODEs to be linearizable by point transformation is $2n+1$. The maximum dimension of the symmetry algebra is $(n+1)(n+3)$ which corresponds to $sl(n+2, \mathbb{R})$. The other submaximal symmetry algebras besides that of dimension $2n+1$ range from $2n+2$ to $(n+2)^2/2$ for n even and $[(n+2)^2+1]/2$ for n odd. Thus for $n = 2$ we have the minimum dimension to be 5 and other submaximal algebra dimensions are 6, 7 and 8. The maximum dimension for $n = 2$ is 15. For $n = 3$ the minimum dimension is 7 and the next to maximum is 13. The maximum is 24. Thus for this case there are 8 classes. Generally, for $n = 2m$, the number of classes is $2m^2 + 3$ and for $n = 2m - 1$ it is $2m^2 - 2m + 4$.

It would be important to find ways of providing the linearizability criteria for the cases of the other symmetry algebras.

Appendix

We take $j = 1$ in the third set of (4.4) as the three dependent equations and discard them. The invocation of the first set of nine equations of (4.4) gives

$$
\begin{aligned}
&\frac{1}{2}C^3_{2x} - D^3_y + \frac{1}{4}C^3_3C^3_2 + \frac{1}{4}C^2_2C^3_2 - D^2B^3_{22} - D^3B^3_{23} = 0,\\
&B^3_{22x} - \frac{1}{2}C^3_{2y} - A_{22}D^3 + \frac{1}{2}C^3_2B^2_{22} + \frac{1}{2}C^3_3B^3_{22} - \frac{1}{2}C^2_2B^3_{22} - \frac{1}{2}B^3_{23}C^3_2 = 0,\\
&\Gamma^1_{12,y} = -A_{22x} - A_{22}\Gamma^2_{12} + C^2_2A_{22} + \Gamma^1_{12}B^2_{22} + \Gamma^1_{12}\Gamma^1_{12} + \frac{1}{2}B^3_{22}\Gamma^3_{33} - \frac{1}{2}B^3_{22}B^3_{33} + \frac{1}{2}C^3_2A_{23},\\
&\Gamma^2_{12,x} = D^2_y + D^2\Gamma^1_{12} + \Gamma^2_{12}\Gamma^2_{12} - \frac{1}{4}C^2_3C^3_2 - \Gamma^2_{12}C^2_2 + B^2_{22}D^2 + D^3B^2_{23} - \frac{1}{2}D^3B^3_{33} + \frac{1}{2}D^3\Gamma^3_{33},\\
&\Gamma^2_{12,y} = -\frac{1}{3}B^2_{22x} + \frac{2}{3}C^2_{2y} + \Gamma^1_{12}\Gamma^2_{12} + \frac{1}{4}C^3_2\Gamma^3_{33} - \frac{1}{4}C^3_2B^3_{33}\\
&\qquad + D^2A_{22} + \frac{2}{3}D^3A_{23} - \frac{1}{6}C^2_3B^3_{22} + \frac{1}{6}B^2_{23}C^3_2,\\
&\Gamma^1_{12,x} = -\frac{2}{3}B^2_{22x} + \frac{1}{3}C^2_{2y} + \Gamma^1_{12}\Gamma^2_{12} + \frac{1}{4}C^3_2\Gamma^3_{33} - \frac{1}{4}C^3_2B^3_{33} + D^2A_{22}\\
&\qquad + \frac{1}{3}D^3A_{23} - \frac{1}{3}C^2_3B^3_{22} + \frac{1}{3}B^2_{23}C^3_2\\
&B^3_{23x} - \frac{1}{3}B^2_{22x} + \frac{1}{6}C^2_{2y} - \frac{4}{3}D^3A_{23} - \frac{2}{3}B^3_{22}C^2_3 + \frac{2}{3}B^2_{23}C^3_2 - \frac{1}{2}C^3_{3y} = 0,\\
&\Gamma^3_{33,y} = -2A_{23x} + B^3_{33y} - 2\Gamma^2_{12}A_{23} + A_{23}C^2_2 + 2\Gamma^1_{12}B^2_{23} + \Gamma^1_{12}\Gamma^3_{33} - \Gamma^1_{12}B^3_{33}\\
&\qquad + \Gamma^3_{33}B^3_{23} - B^3_{33}B^3_{23} + A_{22}C^2_3 + A_{23}C^3_3\\
&\Gamma^3_{33,x} = -2B^2_{23x} + B^3_{33x} + C^2_{3y} + 2D^2A_{23} - C^2_3B^3_{23} + C^2_3\Gamma^1_{12} - \Gamma^2_{12}B^3_{33} + C^2_3B^2_{22} + B^2_{23}C^3_3\\
&\qquad - B^2_{23}C^2_2 - \frac{1}{2}C^3_3B^3_{33} + \frac{1}{2}B^3_{33}C^2_2 + \frac{1}{2}C^3_3\Gamma^3_{33} - \frac{1}{2}C^2_2\Gamma^3_{33} + \Gamma^3_{33}\Gamma^2_{12}.
\end{aligned}
\tag{6.1}
$$

The second set of nine equations of (4.4) yields

$$
\begin{aligned}
&\Gamma^1_{12,x} = -B^3_{23x} + \frac{1}{2}C^3_{2z} + D^3A_{23} - \frac{1}{2}C^3_2B^2_{23} + \frac{1}{4}C^3_2B^3_{33} + \frac{1}{4}C^3_2\Gamma^3_{33}\\
&\qquad - \frac{1}{2}C^3_3B^3_{23} + \frac{1}{2}C^2_2B^3_{23} + \Gamma^2_{12}\Gamma^1_{12},\\
&\Gamma^1_{12,z} = -A_{23x} - A_{23}\Gamma^2_{12} + A_{23}C^2_2 + \Gamma^1_{12}B^2_{23} + \frac{1}{2}\Gamma^3_{33}B^3_{23}\\
&\qquad \frac{1}{2}\Gamma^3_{33}\Gamma^1_{12} - \frac{1}{2}B^3_{33}B^3_{23} - \frac{1}{2}B^3_{33}\Gamma^1_{12} + \frac{1}{2}A_{33}C^3_2,\\
&\Gamma^2_{12,x} = -\frac{1}{2}C^3_{3x} + \frac{1}{2}C^2_{2x} + D^3_z + \frac{1}{2}\Gamma^3_{33}D^3 + \frac{1}{2}D^3B^3_{33} - \frac{1}{4}C^3_2C^2_3 - \frac{1}{4}C^3_3C^3_3\\
&\qquad + \frac{1}{4}C^2_2C^2_2 + \Gamma^2_{12}\Gamma^2_{12} - C^2_2\Gamma^2_{12} + B^2_{23}D^2 + \Gamma^1_{12}D^2,
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^2_{12,z} &= -B^2_{23x} + 2A_{23}D^2 - \frac{1}{2}C^2_3 B^3_{23} + \frac{1}{2}B^2_{33}C^3_2 + C^2_{2z} + \frac{1}{2}C^2_3\Gamma^1_{12} + \frac{1}{4}C^3_3\Gamma^3_{33} \\
&\quad - \frac{1}{4}C^2_2\Gamma^3_{33} + \frac{1}{2}\Gamma^3_{33}\Gamma^2_{12} - \frac{1}{4}B^3_{33}C^3_3 + \frac{1}{4}C^2_2 B^3_{33} - \frac{1}{2}B^3_{33}\Gamma^2_{12} + A_{33}D^3, \\
&\quad \frac{1}{2}C^2_{3x} - D^2_z + \frac{1}{4}C^2_3 C^3_3 + \frac{1}{4}C^2_3 C^2_2 - B^2_{23}D^2 - B^2_{33}D^3 = 0, \\
&B^2_{33x} - \frac{1}{2}C^2_{3z} - D^2 A_{33} + \frac{1}{2}C^2_3 B^3_{33} - \frac{1}{2}B^2_{23}C^2_3 - \frac{1}{2}B^2_{33}C^3_3 + \frac{1}{2}B^2_{33}C^2_2 = 0, \\
\Gamma^3_{33,x} &= B^3_{33x} - 4B^2_{23x} + 6A_{23}D^2 - 2C^2_3 B^3_{23} + 2B^2_{33}C^3_2 + 2C^2_{2z} + C^2_3\Gamma^1_{12} + \frac{1}{2}C^3_3\Gamma^3_{33} \\
&\quad - \frac{1}{2}C^2_2\Gamma^3_{33} + \Gamma^3_{33}\Gamma^2_{12} - \frac{1}{2}B^3_{33}C^3_3 + \frac{1}{2}C^2_2 B^3_{33} - B^3_{33}\Gamma^2_{12} + 2A_{33}D^3, \\
\Gamma^3_{33,x} &= \frac{1}{2}C^3_{3z} + \frac{1}{2}C^2_{2z} - B^2_{23x} + 2A_{23}D^2 + C^2_3\Gamma^1_{12} + \frac{1}{2}C^3_3\Gamma^3_{33} - \frac{1}{2}C^2_2\Gamma^3_{33} \\
&\quad + \Gamma^2_{12}\Gamma^3_{33} - \frac{1}{2}C^3_3 B^3_{33} + \frac{1}{2}C^2_2 B^3_{33} - B^3_{33}\Gamma^2_{12} + 2A_{33}D^3, \\
\Gamma^3_{33,z} &= -2A_{33x} + B^3_{33z} - 2A_{33}\Gamma^2_{12} + C^2_2 A_{33} + 2\Gamma^1_{12}B^2_{33} + \frac{1}{2}\Gamma^3_{33}\Gamma^3_{33} - \frac{1}{2}B^3_{33}B^3_{33} \\
&\quad + A_{23}C^2_3 + A_{33}C^3_3.
\end{aligned} \tag{6.2}
$$

The last set of the nine equations of (4.4) result in six independent conditions

$$
\begin{aligned}
\Gamma^1_{12,y} &= -B^3_{23y} + B^3_{22z} + \frac{1}{2}C^3_2 A_{23} - B^3_{22}B^2_{23} + \frac{1}{2}B^3_{22}B^3_{33} + \frac{1}{2}B^3_{22}\Gamma^3_{33} \\
&\quad - B^3_{23}B^3_{23} + \Gamma^1_{12}\Gamma^1_{12} - \frac{1}{2}C^3_3 A_{22} + \frac{1}{2}C^2_2 A_{22} - A_{22}\Gamma^2_{12} + B^3_{23}B^2_{22} + B^2_{22}\Gamma^1_{12}, \\
\Gamma^1_{12,z} &= \frac{1}{3}B^3_{23z} + \frac{1}{6}C^3_2 A_{33} - B^2_{33}B^3_{22} - \frac{1}{6}C^3_3 A_{23} + \frac{1}{6}C^2_2 A_{23} - A_{23}\Gamma^2_{12} + B^2_{23}B^3_{23} - \frac{1}{2}B^3_{23}B^3_{33} \\
&\quad + \frac{1}{2}B^3_{23}\Gamma^3_{33} + B^2_{23}\Gamma_{12} - \frac{1}{2}B^3_{33}\Gamma^1_{12} + \frac{1}{2}\Gamma^1_{12}\Gamma^3_{33} + \frac{2}{3}B^2_{23y} - \frac{1}{3}B^3_{33y} - \frac{2}{3}B^2_{22z} + \frac{1}{3}C^2_3 A_{22}, \\
\Gamma^3_{33,y} &= \frac{4}{3}B^3_{23z} + \frac{2}{3}C^3_2 A_{33} - 2B^2_{33}B^3_{22} - \frac{2}{3}C^3_3 A_{23} + \frac{2}{3}C^2_2 A_{23} - 2A_{23}\Gamma^2_{12} + 2B^3_{23}B^2_{23} \\
&\quad - B^3_{23}B^3_{33} + B^3_{23}\Gamma^3_{33} + 2B^2_{23}\Gamma^1_{12} - B^3_{33}\Gamma^1_{12} + \Gamma^1_{12}\Gamma^3_{33} + \frac{2}{3}B^2_{23y} - \frac{1}{3}B^3_{33y} - \frac{2}{3}B^2_{22z} + \frac{1}{3}C^2_3 A_{22}, \\
\Gamma^3_{33,z} &= 2B^2_{33y} - 2B^2_{23z} + B^3_{33z} - 2A_{33}\Gamma^2_{12} + 2B^2_{22}B^3_{33} + 2B^2_{33}\Gamma^1_{12} + \frac{1}{2}\Gamma^3_{33}\Gamma^3_{33} \\
&\quad + C^2_3 A_{23} - 2B^2_{23}B^2_{23} + 2B^2_{23}B^3_{33} - \frac{1}{2}B^3_{33}B^3_{33} - 2B^2_{33}B^3_{23}, \\
&\quad - A_{23y} + A_{22z} - A_{22}B^2_{23} - A_{23}B^3_{23} + A_{23}B^2_{22} + A_{33}B^3_{22} = 0, \\
&\quad - A_{33y} + A_{23z} - A_{22}B^2_{33} - A_{23}B^3_{33} + A_{23}B^2_{23} + A_{33}B^3_{23} = 0,
\end{aligned} \tag{6.3}
$$

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