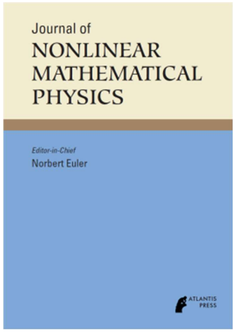

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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To cite this article: Yifang Kang, Zhiqi Chen (2009) Novikov Superalgebras in Low Dimensions, Journal of Nonlinear Mathematical Physics 16:3, 251–257, DOI: https://doi.org/10.1142/S1402925109000212

To link to this article: https://doi.org/10.1142/S1402925109000212

Published online: 04 January 2021

Article

Journal of Nonlinear Mathematical Physics, Vol. 16, No. 3 (2009) 251–257

NOVIKOV SUPERALGEBRAS IN LOW DIMENSIONS

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Received 12 May 2008
Accepted 26 February 2009

Novikov superalgebras are related to the quadratic conformal superalgebras which correspond to the Hamiltonian pairs and play fundamental role in the completely integrable systems. In this note, we divide Novikov superalgebras into two types: N and S. Then we show that the Novikov superalgebras of dimension up to 3 are of type N.

Keywords: Novikov superalgebra; Novikov algebra; admissible algebra; type N; type S.

1. Introduction

A Novikov *super*algebra A is a $\mathbb{Z}_2$-graded vector space $A = A_0 + A_1$ with a bilinear product $(u, v) \mapsto u \circ v$ for any $u \in A_i, v \in A_j$, $w \in A$ satisfying

$$(u \circ v) \circ w - u \circ (v \circ w) = (-1)^{ij}((v \circ u) \circ w - v \circ (u \circ w)), \tag{1.1}$$

$$(w \circ u) \circ v = (-1)^{ij}(w \circ v) \circ u. \tag{1.2}$$

The even part of a given Novikov superalgebra is what is said to be a Novikov algebra introduced in connection with the Poisson brackets of hydrodynamic type [4] and Hamiltonian operators in the formal variational calculus [7–9, 20, 21].

The supercommutator

$$[u, v] = u \circ v - (-1)^{ij} v \circ u, \quad \text{for any } u \in A_i, v \in A_j \tag{1.3}$$

makes any Novikov superalgebra A a Lie superalgebra denoted $S\mathrm{Lie}(A)$ in what follows. The passage from a Novikov algebra A to a Lie algebra denoted $\mathrm{Lie}(A)$ is analogous.

The notion of Novikov superalgebra was introduced in [15], as a particular case of Lie-super-admissible algebra (Gerstenhaber called them $\mathbb{Z}_2$-graded pre-Lie algebras [10]). Novikov superalgebras are also related to the quadratic conformal superalgebras [16].

For the notion of *conformal superalgebra*, see [11] (we do not touch priority questions here). Conformal superalgebras are related to the linear Hamiltonian operators in the Gel'fand–Dikii–Dorfman

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theory ([5–9]) and play an important role in quantum field theory [11] and vertex operator superalgebra theory [11, 17]. Recall that a conformal superalgebra $\Re = \Re_0 \oplus \Re_1$ (we denote it $(\Re, \partial, Y^+(\cdot, z))$) is a $\mathbb{Z}_2$-graded $\mathbb{C}[\partial]$-module with a $\mathbb{Z}_2$-graded linear map $Y^+(\cdot, z) : \Re \to \hom(\Re, \Re[z^{-1}]z^{-1})$ for any $u \in \Re_i, v \in \Re_j$ satisfying

$$Y^+(\partial u, z) = \frac{dY^+(u, z)}{dz}; \tag{1.4}$$

$$Y^+(u, z)v = (-1)^{ij} \mathrm{Re}\, s_x \frac{e^{x\partial} Y^+(v, -x)u}{z - x}; \tag{1.5}$$

$$Y^+(u, z_1)Y^+(v, z_2) = (-1)^{ij} Y^+(v, z_2)Y^+(u, z_1) + \mathrm{Re}\, s_x \frac{Y^+(Y^+(u, z_1 - x)v, x)}{z_2 - x}, \tag{1.6}$$

where $\mathrm{Re}\, s_z(z^n) = \delta^n_{-1}$ for $n \in \mathbb{Z}$. The even part $\Re_0$ is a *conformal algebra*. The definition of conformal superalgebra in the above generating-function form is equivalent to the definition in [11], where the author used a component formula with $Y^+(u, z) = \sum_{n=0}^{\infty} \frac{u_{(n)} z^{-n-1}}{n!}$.

A *quadratic conformal superalgebra* is a conformal superalgebra which is a free $\mathbb{C}[\partial]$-module over a $\mathbb{Z}_2$-graded subspace V, i.e.,

$$\Re = \mathbb{C}[\partial]V (\cong \mathbb{C}[\partial] \otimes_{\mathbb{C}} V), \tag{1.7}$$

such that the equation

$$Y^+(u, z)v = (w_1 + \partial w_2)z^{-1} + w_3 z^{-2} \quad \text{for any } u, v \in V \tag{1.8}$$

holds, where $w_1, w_2, w_3 \in V$. A quadratic conformal superalgebra corresponds to a Hamiltonian pair in [7], which plays a fundamental role in the theory of completely integrable systems. A super Gel'fand–Dorfman algebra is a $\mathbb{Z}_2$-graded vector space $A = A_0 + A_1$ with two operations $[\cdot, \cdot]$ and $\circ$ such that $(A, [\cdot, \cdot])$ is a Lie superalgebra, $(A, \circ)$ is a Novikov superalgebra, and the relation

$$[w \circ u, v] - (-1)^{ij}[w \circ v, u] + [w, u] \circ v - (-1)^{ij}[w, v] \circ u - w \circ [u, v] = 0 \tag{1.9}$$

holds for any $u \in A_i, v \in A_j, w \in A$. It was shown in [16] that there is a one-to-one correspondence between quadratic conformal superalgebras and super Gel'fand–Dorfman algebras. It was also pointed out in [16] that $(A, [\cdot, \cdot], \circ)$ is a super Gel'fand–Dorfman algebra for any Novikov superalgebra $(A, \circ)$ and the supercommutator $[\cdot, \cdot]$ relative $\circ$.

In this paper, we divide Novikov superalgebras into two types: N and S. We show that the Novikov superalgebras of dimension ≤ 3 are of type N. The full structure theory is yet to be developed; we hope that our answer will help.

The paper is organized as follows. In Sec. 2, we divide Novikov superalgebras into two types: N and S. The Novikov superalgebras of type N are both Lie-super-admissible algebras and Lie-admissible algebras. Let $A = A_0 + A_1$ be a Novikov superalgebra. Then the two algebraic structures coincide if $A_1 A_1 = 0$. We also show that $A_1 A_1 \neq 0$ if A is of type S. In Sec. 3, we show that the Novikov superalgebras of dimension ≤ 3 are of type N.

Throughout this paper we assume that the algebras are finite-dimensional over $\mathbb{C}$. Obvious proofs are omitted.

2. Novikov Superalgebras: Types *N* and *S*

First, we give some examples of Novikov superalgebras.

Example 2.1. Let A be an associative supercommutative superalgebra and D a left A-module. Then $\overline{A} = D \oplus A$ is a Novikov superalgebra if the product is defined by

$$(d_1 + a_1) \circ (d_2 + a_2) = (-1)^{ij} a_2 d_1 + a_1 a_2 \quad \text{for any } d_1 + a_1 \in \overline{A}_i, d_2 + a_2 \in \overline{A}_j.$$

Example 2.2. Let A be an associative supercommutative superalgebra. Let d be its even derivation and $c \in A_0$. Then the product defined by

$$u \circ v = ud(v) + cuv \quad \text{for any } u, v \in A$$

determines the structure of a Novikov superalgebra on the space A.

The former two examples are from [16], but we have omitted the details inessential for the purposes of our article.

Example 2.3. Let $A = A_0 + A_1$ be a $\mathbb{Z}_2$-graded vector space with $\dim A_0 = \dim A_1 = 1$. Let $\{e\}$ be a basis of A_0 and $\{v\}$ a basis of A_1. Define a multiplication on A by setting $vv = e$ (we give only nonzero products). Then A is a Novikov superalgebra and the Lie superalgebra $S\mathrm{Lie}(A)$ is defined by $[v, v] = 2e$. At the same time, A is a Novikov algebra under the above product and the Lie algebra $\mathrm{Lie}(A)$ is abelian.

Example 2.4. Let $A = A_0 + A_1$ be a $\mathbb{Z}_2$-graded vector space with $\dim A_0 = 1$ and $\dim A_1 = 2$. Let $\{e\}$ be a basis of A_0 and $\{u, v\}$ be a basis of A_1 such that the nonzero products are given by

$$uv = -vu = e.$$

Then A is a Novikov superalgebra and the Lie superalgebra $S\mathrm{Lie}(A)$ satisfies

$$[x, y] = 0 \quad \text{for any } x, y \in A,$$

which can be regarded as an abelian Lie algebra. At the same time, A is a Novikov algebra under the above product, but the Lie algebra $\mathrm{Lie}(A)$ is not abelian since $[u, v] = 2e$.

Example 2.5. Let $A = A_0 + A_1$ be a Novikov superalgebra satisfying $A_1A_1 = 0$. Then the Lie superalgebra $S\mathrm{Lie}(A)$ satisfies

$$\begin{aligned} [u, v] &= 0, \\ [x, y] &= -[y, x], \\ [w, [x, y]] &= [[w, x], y] - [[w, y], x], \\ [w, [u, v]] &= 0, \end{aligned}$$

where $u, v \in A_1, w \in A$ and either x or y belongs to A_0. Under the above product $S\mathrm{Lie}(A)$ can also be regarded as a Lie algebra with the parity forgotten. We also have

$$(wu)v = 0, \quad (uv)w - u(vw) = 0$$

for any $u, v \in A_1, w \in A$. The definitions imply that A is a Novikov algebra.

Definition 2.6. Let $A = A_0 + A_1$ be a Novikov superalgebra with multiplication $(u, v) \mapsto uv$. If A is also a Novikov algebra with respect to the same product and superstructure forgotten, then A is called a Novikov superalgebra of type N, otherwise A is said to be of type S.

By Definition 2.6, Novikov superalgebras in Examples 2.3–2.5 are of type N. For Novikov algebras, see [1–3, 12–14, 18, 19, 22].

Proposition 2.7. *Let $A = A_0 + A_1$ be a Novikov superalgebra. If $A_1 = 0$ or $A_0 = 0$, then $A_1A_1 = 0$ and A is of type N. In particular, 1-dimensional Novikov superalgebras are of type N.*

Proposition 2.8. *Let $A = A_0 + A_1$ be a Novikov superalgebra of type S. Then $A_1A_1 \neq 0$.*

Proposition 2.9. *Let $A = A_0 + A_1$ be a Novikov superalgebra with $\dim A_1 = 1$. Then A is of type N.*

Remark 2.10. Let $A = A_0 + A_1$ be a Novikov superalgebra. Then A is a Lie-super-admissible algebra. If A is of type N, then A is both a Lie-super-admissible algebra and a Lie-admissible algebra. By Example 2.5, these two structures coincide if $A_1A_1 = 0$. If A is of type N satisfying $A_1A_1 \neq 0$, then these two algebraic structures on A are different.

3. The Novikov Superalgebras of dim ≤ 3 are of Type N

Proposition 3.1. *Any 2-dimensional Novikov superalgebra is of type N.*

Proposition 3.2. *Let $A = A_0 + A_1$ be a 3-dimensional Novikov superalgebra of type S. Then* $\dim A_1 = 2$.

Proof. Assume that $A = A_0 + A_1$ is a 3-dimensional Novikov superalgebra of type S. By Propositions 2.7 and 2.9, we know that $\dim A_1 = 2$. □

The following discussion is based on the modules over Novikov algebras. The notion of a module over a Novikov algebra was introduced in [12], but a more explicit definition was given in [18]. A *module M over a Novikov algebra A* is a vector space endowed with two linear maps $L_M, R_M : A \to \mathrm{End}_{\mathbb{F}}(M)$ for any $x, y \in A$ satisfying

$$L_M(xy) = R_M(y)L_M(x), \tag{3.1}$$

$$R_M(xy) - R_M(y)R_M(x) = [L_M(x), R_M(y)], \tag{3.2}$$

$$R_M(x)R_M(y) = R_M(y)R_M(x), \tag{3.3}$$

$$[L_M(x), L_M(y)] = L_M([x, y]). \tag{3.4}$$

Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of M. Let α be a linear transformation on M with the matrix $(a_{ij})_{i,j=1}^n$ in the basis $\{e_1, e_2, \ldots, e_n\}$.

If A is 1-dimensional with a basis $\{e\}$, then Eqs. (3.3) and (3.4) are satisfied. Let M be a module with a basis $\{v_1, v_2\}$. Let L and R be the matrices of $L_M(e)$ and $R_M(e)$, respectively. If $ee = 0$, then by Eqs. (3.1) and (3.2) we have

$$RL = 0, \quad R^2 = -LR. \tag{3.5}$$

If $ee = e$, then by Eqs. (3.1) and (3.2) we have

$$RL = L, \quad R^2 = R + L - LR. \tag{3.6}$$

Claim 3.3. If $ee = 0$, then $R^2 = 0$.

Proof. Assume that $\det R \neq 0$. Then $L = 0$ and $R^2 = 0$. It is a contradiction. Hence $\det R = 0$. If $R^2 \neq 0$, then $R = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ for some $a \neq 0$. It follows that $L = \begin{pmatrix} 0 & 0 \\ b & c \end{pmatrix}$. Then $R^2 = \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} = -LR = -\begin{pmatrix} 0 & 0 \\ ac & 0 \end{pmatrix}$. Hence $a = 0$. So we have $R^2 = 0$. □

Claim 3.4. If $ee = e$ and $LR \neq RL$, then $L = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ for some $a \neq 0$.

Proof. If $LR \neq RL$, then $\det L = 0$. It follows that $L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ or $L = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ for some $a \neq 0$. For the former case, $R = \begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix}$. Then $R^2 = R$. Hence $LR = L = RL$. It is a contradiction. That is, $L = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ for some $a \neq 0$. □

Proposition 3.5. *In the above notations the classification of two-dimensional modules over one-dimensional Novikov algebras is given in the following table*:

Type	A	L	R	Type	A	L	R
$T1$	$ee=0$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$T2$	$ee=0$	$\begin{pmatrix}1&0\\0&a\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$
$T3$	$ee=0$	$\begin{pmatrix}0&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$T4$	$ee=0$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$
$T5$	$ee=0$	$\begin{pmatrix}0&0\\a&0\end{pmatrix}$	$\begin{pmatrix}0&0\\1&0\end{pmatrix}$	$T6$	$ee=0$	$\begin{pmatrix}0&0\\a&1\end{pmatrix}$	$\begin{pmatrix}0&0\\1&0\end{pmatrix}$
$T7$	$ee=e$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$T8$	$ee=e$	$\begin{pmatrix}a&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$
$T9$	$ee=e$	$\begin{pmatrix}a&0\\0&a\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$T10$	$ee=e$	$\begin{pmatrix}a&0\\1&a\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$
$T11$	$ee=e$	$\begin{pmatrix}a_1&0\\0&a_2\end{pmatrix}$ $a_1\neq a_2$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$T12$	$ee=e$	$\begin{pmatrix}-1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&b\\0&1\end{pmatrix}$ $b\neq 0$

Proof. Assume that A is a 1-dimensional Novikov algebra and M is a 2-dimensional A-module with a basis $\{v_1,v_2\}$ and the two linear maps L_M and R_M. Then there exists a basis $\{e\}$ of A such that $ee=0$ or $ee=e$. Let L and R be the matrices of $L_M(e)$ and $R_M(e)$, respectively.

(I) $ee=0$. By Claim 3.3, $R^2=0$. Then there exists another basis of M (also denoted by $\{v_1,v_2\}$) such that $R=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ or $R=\begin{pmatrix}0&0\\0&0\end{pmatrix}$. If $R=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, then there exists another basis of M (also denoted by $\{v_1,v_2\}$) such that $L=\begin{pmatrix}a_1&0\\0&a_2\end{pmatrix}$ or $L=\begin{pmatrix}a&0\\1&a\end{pmatrix}$. If $L=\begin{pmatrix}a&0\\1&a\end{pmatrix}$, replacing e by $\frac{e}{a}$ and v_2 by $\frac{v_2}{a}$ for $a\neq 0$, we can take $a=1$. If $R=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, then by the first part of Eq. (3.5), we have $L=\begin{pmatrix}0&0\\a&b\end{pmatrix}$. Replacing e by $\frac{e}{b}$ and v_2 by $\frac{av_2}{b}$ for $b\neq 0$, we can set $b=1$.

(II) $ee=e$. If $RL\neq LR$, then $L=\begin{pmatrix}a&0\\0&0\end{pmatrix}$ for some $a\neq 0$ by Claim 3.4. Furthermore $R=\begin{pmatrix}1&b\\0&c\end{pmatrix}$ for some $b\neq 0$. By the second part of Eq. (3.6), we have $a=-c=-1$. If $LR=RL$, then $R^2=R$ by Eq. (3.6). Hence there exists another basis of M (also denoted by $\{v_1,v_2\}$) such that $R=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, or $R=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, or $R=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. By the first part of Eq. (3.6), if $R=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, then $L=RL=\begin{pmatrix}0&0\\0&0\end{pmatrix}$. Similarly, if $R=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, then $L=\begin{pmatrix}a&b\\0&0\end{pmatrix}$. So $b=0$ by $LR=RL$. If $R=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, then there exists another basis of M (also denoted by $\{v_1,v_2\}$) such that $R=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $L=\begin{pmatrix}a_1&0\\0&a_2\end{pmatrix}$, where $a_1\neq a_2$, or $L=\begin{pmatrix}a&0\\0&a\end{pmatrix}$, or $L=\begin{pmatrix}a&0\\1&a\end{pmatrix}$. □

Proposition 3.6. *Let $A=A_0+A_1$ be a 3-dimensional Novikov superalgebra satisfying $A_1A_1\neq 0$. If $\dim A_1=2$, then $A_0A_0=A_0A_1=A_1A_0=0$.*

Proof. By assumption, A_1 is a 2-dimensional A_0-module. For the type $T2$ in Proposition 3.5,

$$(v_1v_1)v_1=0\Longrightarrow v_1v_1=0,$$

$$(v_2v_1)v_1=0\Longrightarrow v_2v_1=0,$$

$$a\neq 0,(v_1v_2)v_2=0\Longrightarrow v_1v_2=0.$$

$$a=0,(ev_1)v_2=-(ev_2)v_1\Longrightarrow v_1v_2=0,$$

$$a\neq 0,(v_2v_2)v_2=0\Longrightarrow v_2v_2=0,$$

$$a=0,(v_2v_2)v_1=-(v_2v_1)v_2=0\Longrightarrow v_2v_2=0.$$

That is, $A_1A_1 = 0$. We similarly show that $A_1A_1 = 0$ for the other cases, except $T1$. Since $A_1A_1 \neq 0$, we see that $A_0A_0 = A_0A_1 = A_1A_0 = 0$. □

Proposition 3.7. *Any* 3*-dimensional Novikov superalgebra is of type* N*.*

Proof. Assume that $A = A_0 + A_1$ is a 3-dimensional Novikov superalgebra of type S. By Propositions 2.8 and 3.2, $A_1A_1 \neq 0$ and $\dim A_1 = 2$. Then $A_0A_0 = A_0A_1 = A_1A_0 = 0$ by Proposition 3.6. It follows that A is of type N, which is a contradiction. □

Theorem 3.8. *The Novikov superalgebras of dimension up to* 3 *are of type* N*.*

Proof. The theorem follows from Propositions 2.7, 3.1 and 3.7. □

Remark 3.9. Since the Novikov superalgebras of type N are essentially Novikov algebras, there is a method to give the classification of the Novikov superalgebras of type N based on the classification of Novikov algebras. That is, we look for a grading for any Novikov algebra. But then we need to do it case-by-case. Moreover, the classification in higher dimensions is also an open problem, in particular, the classification of 4-dimensional Novikov algebras has not been finished yet. Then this method does not work for the Novikov algebras of dimension > 3.

Acknowledgments

We are grateful of the referees and the editors for their valuable comments and suggestions. Also we would like to acknowledge Professor Bai C M for his helpful suggestions.

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