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COHOMOLOGY OF THE LIE SUPERALGEBRA OF CONTACT VECTOR FIELDS ON $\mathbb{K}^{1|1}$ AND DEFORMATIONS OF THE SUPERSPACE OF SYMBOLS

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Following Feigin and Fuchs, we compute the first cohomology of the Lie superalgebra $\mathcal{K}(1)$ of contact vector fields on the $(1,1)$ -dimensional real or complex superspace with coefficients in the superspace of linear differential operators acting on the superspaces of weighted densities. We also compute the same, but $\mathfrak{osp}(1|2)$ -relative, cohomology. We explicitly give 1-cocycles spanning these cohomology. We classify generic formal $\mathfrak{osp}(1|2)$ -trivial deformations of the $\mathcal{K}(1)$ -module structure on the superspaces of symbols of differential operators. We prove that any generic formal $\mathfrak{osp}(1|2)$ -trivial deformation of this $\mathcal{K}(1)$ -module is equivalent to a polynomial one of degree ≤ 4 . This work is the simplest superization of a result by Bouarroudj [On $\mathfrak{sl}(2)$ -relative cohomology of the Lie algebra of vector fields and differential operators, *J. Nonlinear Math. Phys.* No. 1 (2007) 112–127]. Further superizations correspond to $\mathfrak{osp}(N|2)$ -relative cohomology of the Lie superalgebras of contact vector fields on $1|N$ -dimensional superspace.

Keywords: Superconformal algebra; cohomology; deformations; differential operators; orthosymplectic superalgebra; contact geometry; tensor densities.

1. Introduction

For motivations, see Bouarroudj's paper [7] of which this work is the most natural superization, other possibilities being cohomology of polynomial versions of various infinite dimensional “stringy” Lie superalgebras (for their list, see [21]). This list contains several infinite series and several exceptional superalgebras, but to consider cohomology relative a “middle” subsuperalgebra similar, in a sense, to $\mathfrak{sl}(2)$ is only possible when such a subsuperalgebra exists which only happens in a few cases. Here we consider the simplest of such cases.

Let $\mathbf{vect}(1)$ be the Lie algebra of polynomial vector fields on $\mathbb{K} := \mathbb{R}$ or \mathbb{C} . Consider the 1-parameter deformation of the $\mathbf{vect}(1)$ -action on $\mathbb{K}[x]$:

$$L_X^\lambda \frac{d}{dx}(f) = Xf' + \lambda X'f,$$

where $X, f \in \mathbb{K}[x]$ and $X' := \frac{dX}{dx}$. This deformation shows that on the level of Lie algebras (and similarly below, for Lie superalgebras) it is natural to choose \mathbb{C} as the ground field.

Denote by \mathcal{F}_λ the $\mathbf{vect}(1)$ -module structure on $\mathbb{K}[x]$ defined by L^λ for a fixed λ . Geometrically, $\mathcal{F}_\lambda = \{f dx^\lambda \mid f \in \mathbb{K}[x]\}$ is the space of polynomial weighted densities of weight $\lambda \in \mathbb{C}$. The space \mathcal{F}_λ coincides with the space of vector fields, functions and differential 1-forms for $\lambda = -1, 0$ and 1 , respectively.

Denote by $D_{\nu,\mu} := \text{Hom}_{\text{diff}}(\mathcal{F}_\nu, \mathcal{F}_\mu)$ the $\mathbf{vect}(1)$ -module of linear differential operators with the natural $\mathbf{vect}(1)$ -action denoted $L_X^{\nu,\mu}(A)$. Each module $D_{\nu,\mu}$ has a natural filtration by the order of differential operators; the graded module $\mathcal{S}_{\nu,\mu} := \text{gr} D_{\nu,\mu}$ is called the *space of symbols*. The quotient-module $D_{\nu,\mu}^k / D_{\nu,\mu}^{k-1}$ is isomorphic to the module of weighted densities $\mathcal{F}_{\mu-\nu-k}$; the isomorphism is provided by the principal symbol map σ_{pr} defined by:

$$A = \sum_{i=0}^k a_i(x) \left(\frac{\partial}{\partial x} \right)^i \mapsto \sigma_{\text{pr}}(A) = a_k(x)(dx)^{\mu-\nu-k},$$

(see, e.g., [16]). Therefore, as a $\mathbf{vect}(1)$ -module, the space $\mathcal{S}_{\nu,\mu}$ depends only on the difference $\beta = \mu - \nu$, so that $\mathcal{S}_{\nu,\mu}$ can be written as \mathcal{S}_β , and we have

$$\mathcal{S}_\beta = \bigoplus_{k=0}^{\infty} \mathcal{F}_{\beta-k}$$

as $\mathbf{vect}(1)$ -modules. The space of symbols of order $\leq n$ is

$$\mathcal{S}_\beta^n := \bigoplus_{k=0}^n \mathcal{F}_{\beta-k}.$$

In the last two decades, deformations of various types of structures have assumed an ever increasing role in mathematics and physics. For each such deformation problem a goal is to determine if all related deformation obstructions vanish and many beautiful techniques were developed to determine when this is so. Deformations of Lie algebras with base and versal deformations were already considered by Fialowski in 1986 [12]. In 1988, Fialowski [13] further introduced deformations whose base is a complete local algebra (the algebra is said to be *local* if it has a unique maximal ideal). Also, in [13], the notion of miniversal (or formal versal) deformation was introduced in general, and it was proved that under some cohomology restrictions, a versal deformation exists. Later Fialowski and Fuchs, using this framework, gave a construction for a versal deformation. Formal deformations of the $\mathbf{vect}(1)$ -module \mathcal{S}_β^n were studied in [1, 5]. Moreover, the formal deformations that become trivial once the action is restricted to $\mathfrak{sl}(2)$ were completely described in [6].

According to Nijenhuis-Richardson the space $H^1(\mathfrak{g}; \text{End}(V))$ classifies the infinitesimal deformations of a \mathfrak{g} -module V and the obstructions to integrability of a given infinitesimal deformation of V are elements of $H^2(\mathfrak{g}; \text{End}(V))$. More generally, if \mathfrak{h} is a subalgebra of \mathfrak{g} , then the \mathfrak{h} -relative cohomology $H^1(\mathfrak{g}, \mathfrak{h}; \text{End}(V))$ measures the infinitesimal deformations

that become trivial once the action is restricted to \mathfrak{h} (\mathfrak{h} -trivial deformations), while the obstructions to extension of any \mathfrak{h} -trivial infinitesimal deformation to a formal one are related to $H^2(\mathfrak{g}, \mathfrak{h}; \text{End}(V))$. Similarly, in the infinite dimensional setting, the infinitesimal deformations of the $\mathbf{vect}(1)$ -module \mathcal{S}_β^n are classified, from a certain point of view, by the space

$$H_{\text{diff}}^1(\mathbf{vect}(1); D) = \bigoplus_{0 \leq i, j \leq n} H_{\text{diff}}^1(\mathbf{vect}(1); D_{\beta-j, \beta-i}), \quad (1.1)$$

where $D := D(n, \beta)$ is the $\mathbf{vect}(1)$ -module of differential operators in \mathcal{S}_β^n and where H_{diff}^i denotes the differential cohomology; that is, only cochains given by differential operators are considered. The $\mathfrak{sl}(2)$ -trivial infinitesimal deformations are classified by the space

$$H_{\text{diff}}^1(\mathbf{vect}(1), \mathfrak{sl}(2); D) = \bigoplus_{0 \leq i, j \leq n} H_{\text{diff}}^1(\mathbf{vect}(1), \mathfrak{sl}(2); D_{\beta-j, \beta-i}). \quad (1.2)$$

Feigin and Fuchs computed $H_{\text{diff}}^1(\mathbf{vect}(1); D_{\lambda, \lambda'})$, see [11]. They showed that nonzero cohomology $H_{\text{diff}}^1(\mathbf{vect}(1); D_{\lambda, \lambda'})$ only appear for particular values of weights that we call *resonant* which satisfy $\lambda' - \lambda \in \mathbb{N}$. Therefore, in formulas (1.1) and (1.2), the summations are only over i and j such that $i \leq j$. Observe that, **whatever the ground field \mathbb{K} , the resonant values belong to \mathbb{R} .**

Bouarroudj and Ovsienko [9] computed $H_{\text{diff}}^1(\mathbf{vect}(1), \mathfrak{sl}(2); D_{\lambda, \lambda'})$, and Bouarroudj [8] solved a multi-dimensional version of the same problem on manifolds.

In this paper we study the simplest super analog of the problem solved in [11, 9, 8] namely, we consider the superspace $\mathbb{K}^{1|1}$ equipped with the contact structure determined by a 1-form α , and the Lie superalgebra $\mathcal{K}(1)$ of contact polynomial vector fields on $\mathbb{K}^{1|1}$. We introduce the $\mathcal{K}(1)$ -module \mathfrak{F}_λ of λ -densities on $\mathbb{K}^{1|1}$ and the $\mathcal{K}(1)$ -module of linear differential operators, $\mathfrak{D}_{\nu, \mu} := \text{Hom}_{\text{diff}}(\mathfrak{F}_\nu, \mathfrak{F}_\mu)$, which are super analogues of the spaces \mathcal{F}_λ and $D_{\nu, \mu}$, respectively. The Lie superalgebra $\mathfrak{osp}(1|2)$, a super analogue of $\mathfrak{sl}(2)$, can be realized as a subalgebra of $\mathcal{K}(1)$. We compute $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \lambda'})$ and $H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \lambda'})$ and we show that, as in the classical setting, nonzero cohomology $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \lambda'})$ only appear for resonant values of weights which satisfy $\lambda' - \lambda \in \frac{1}{2}\mathbb{N}$. So, the super analogue of the space \mathcal{S}_β^n is naturally the superspace (see [16]):

$$\mathfrak{S}_\beta^n = \bigoplus_{k=0}^{2n} \mathfrak{F}_{\beta - \frac{k}{2}}, \quad \text{where } n \in \frac{1}{2}\mathbb{N}.$$

We use the result to study formal deformations of the $\mathcal{K}(1)$ -module structure on \mathfrak{S}_β^n . Denote by $\mathfrak{D} := \mathfrak{D}(n, \beta)$ the $\mathcal{K}(1)$ -module of linear differential operators in \mathfrak{S}_β^n . The infinitesimal deformations of the $\mathcal{K}(1)$ -module structure on \mathfrak{S}_β^n are classified by the space

$$H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}) = \bigoplus_{0 \leq i \leq j \leq 2n} H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\beta - \frac{j}{2}, \beta - \frac{i}{2}}).$$

The $\mathfrak{osp}(1|2)$ -trivial infinitesimal deformations are classified by the space

$$H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}) = \bigoplus_{0 \leq i \leq j \leq 2n} H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\beta - \frac{j}{2}, \beta - \frac{i}{2}}).$$

Here, we study only the generic formal $\mathfrak{osp}(1|2)$ -trivial deformations of the action of $\mathcal{K}(1)$ on the space \mathfrak{S}_β^n . In order to study the integrability of a given $\mathfrak{osp}(1|2)$ -trivial infinitesimal deformation, we need the description of $\mathfrak{osp}(1|2)$ -invariant bilinear differential operators $\mathfrak{F}_\tau \otimes \mathfrak{F}_\lambda \rightarrow \mathfrak{F}_\mu$.

2. Definitions and Notations

2.1. The Lie superalgebra of contact vector fields on $\mathbb{K}^{1|n}$

Let $\mathbb{K}^{1|n}$ be the superspace with coordinates $(x, \theta_1, \dots, \theta_n)$, where the θ_i are odd indeterminates equipped with the standard contact structure given by the following 1-form:

$$\alpha_n = dx + \sum_{i=1}^n \theta_i d\theta_i.$$

On $\mathbb{K}[x, \theta] := \mathbb{K}[x, \theta_1, \dots, \theta_n]$, we consider the contact bracket

$$\{F, G\} = FG' - F'G - \frac{1}{2}(-1)^{p(F)} \sum_{i=1}^n \bar{\eta}_i(F) \cdot \bar{\eta}_i(G), \quad (2.1)$$

where $\bar{\eta}_i = \frac{\partial}{\partial \theta_i} - \theta_i \frac{\partial}{\partial x}$ and $p(F)$ is the parity of F .

Let $\text{Vect}_{\text{Pol}}(\mathbb{K}^{1|n})$ be the superspace of polynomial vector fields on $\mathbb{K}^{1|n}$:

$$\text{Vect}_{\text{Pol}}(\mathbb{K}^{1|n}) = \left\{ F_0 \partial_x + \sum_{i=1}^n F_i \partial_i \left| F_i \in \mathbb{K}[x, \theta] \text{ for all } i \right. \right\},$$

where $\partial_i = \frac{\partial}{\partial \theta_i}$ and $\partial_x = \frac{\partial}{\partial x}$, and consider the superspace $\mathcal{K}(n)$ of contact polynomial vector fields on $\mathbb{K}^{1|n}$. That is, $\mathcal{K}(n)$ is the superspace of vector fields on $\mathbb{K}^{1|n}$ preserving the distribution singled out by the 1-form α_n :

$$\mathcal{K}(n) = \{X \in \text{Vect}_{\text{Pol}}(\mathbb{K}^{1|n}) \mid \text{there exists } F \in \mathbb{K}[x, \theta] \text{ such that } L_X(\alpha_n) = F\alpha_n\}.$$

The Lie superalgebra $\mathcal{K}(n)$ is spanned by the fields of the form:

$$X_F = F\partial_x - \frac{1}{2} \sum_{i=1}^n (-1)^{p(F)} \bar{\eta}_i(F) \bar{\eta}_i, \quad \text{where } F \in \mathbb{K}[x, \theta].$$

In particular, we have $\mathcal{K}(0) = \mathfrak{vect}(1)$. Observe that $L_{X_F}(\alpha_n) = X_1(F)\alpha_n$. The bracket in $\mathcal{K}(n)$ can be written as: $[X_F, X_G] = X_{\{F, G\}}$.

2.2. The subalgebra $\mathfrak{osp}(1|2)$

In $\mathcal{K}(1)$, there is a subalgebra $\mathfrak{osp}(1|2)$ of projective transformations

$$\mathfrak{osp}(1|2) = \text{Span}(X_1, X_\theta, X_x, X_{x\theta}, X_{x^2}); \quad (\mathfrak{osp}(1|2))_{\bar{0}} = \text{Span}(X_1, X_x, X_{x^2}) \cong \mathfrak{sl}(2).$$

2.3. The space of polynomial weighted densities on $\mathbb{K}^{1|1}$

From now on, $n = 1$ and we will denote α_1 and $\bar{\eta}_1$ respectively by α and $\bar{\eta}$. We have analogous definition of weighted densities in super setting (see [2]) with dx replaced by α . The elements of these spaces are indeed (weighted) densities since all spaces of generalized tensor fields have just one parameter relative $\mathcal{K}(1)$ — the value of X_x on the lowest weight vector (the one annihilated by X_θ). From this point of view the volume element (roughly speaking, “ $dx \frac{\partial}{\partial \theta}$ ”) is indistinguishable from $\alpha^{\frac{1}{2}}$.

Consider the 1-parameter action of $\mathcal{K}(1)$ on $\mathbb{K}[x, \theta]$ given by the rule:

$$\mathfrak{L}_{X_F}^\lambda = X_F + \lambda F', \quad (2.2)$$

where $F' = \partial_x F$, or, in components:

$$\mathfrak{L}_{X_F}^\lambda(G) = L_{a\partial_x}^\lambda(g_0) + \frac{1}{2}bg_1 + \left(L_{a\partial_x}^{\lambda+\frac{1}{2}}(g_1) + \lambda g_0 b' + \frac{1}{2}g_0' b \right) \theta, \quad (2.3)$$

where $F = a + b\theta$, $G = g_0 + g_1\theta \in \mathbb{K}[x, \theta]$. We denote this $\mathcal{K}(1)$ -module by \mathfrak{F}_λ , the space of all polynomial weighted densities on $\mathbb{K}^{1|1}$ of weight λ :

$$\mathfrak{F}_\lambda = \{f(x, \theta)\alpha^\lambda \mid f(x, \theta) \in \mathbb{K}[x, \theta]\}. \quad (2.4)$$

Obviously:

- (1) The adjoint $\mathcal{K}(1)$ -module, is isomorphic to \mathfrak{F}_{-1} .
- (2) As a $\mathbf{vect}(1)$ -module, $\mathfrak{F}_\lambda \simeq \mathcal{F}_\lambda \oplus \Pi(\mathcal{F}_{\lambda+\frac{1}{2}})$.

Any differential operator A on $\mathbb{K}^{1|1}$ can be viewed as a linear mapping $F\alpha^\lambda \mapsto (AF)\alpha^\mu$ from \mathfrak{F}_λ to \mathfrak{F}_μ , thus the space of differential operators becomes a $\mathcal{K}(1)$ -module denoted $\mathfrak{D}_{\lambda,\mu}$ for the natural action:

$$\mathfrak{L}_{X_F}^{\lambda,\mu}(A) = \mathfrak{L}_{X_F}^\mu \circ A - (-1)^{p(A)p(F)} A \circ \mathfrak{L}_{X_F}^\lambda. \quad (2.5)$$

Proposition 2.1. *As a $\mathbf{vect}(1)$ -module, we have*

$$(\mathfrak{D}_{\lambda,\mu})_{\bar{0}} \simeq \mathfrak{D}_{\lambda,\mu} \oplus \mathfrak{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}} \quad \text{and} \quad (\mathfrak{D}_{\lambda,\mu})_{\bar{1}} \simeq \Pi(\mathfrak{D}_{\lambda+\frac{1}{2},\mu} \oplus \mathfrak{D}_{\lambda,\mu+\frac{1}{2}}).$$

Proof. It is clear that the map

$$\begin{aligned} \varphi_\lambda : \mathfrak{F}_\lambda &\rightarrow \mathcal{F}_\lambda \oplus \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \\ F\alpha^\lambda &\mapsto ((1 - \theta\partial_\theta)(F)(dx)^\lambda, \Pi(\partial_\theta(F)(dx)^{\lambda+\frac{1}{2}})) \end{aligned}$$

is $\mathbf{vect}(1)$ -isomorphism, see formulae (2.3). So, we deduce a $\mathbf{vect}(1)$ -isomorphism:

$$\begin{aligned} \Phi_{\lambda,\mu} : \mathfrak{D}_{\lambda,\mu} &\rightarrow \mathfrak{D}_{\lambda,\mu} \oplus \mathfrak{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}} \oplus \Pi(\mathfrak{D}_{\lambda,\mu+\frac{1}{2}}) \oplus \Pi(\mathfrak{D}_{\lambda+\frac{1}{2},\mu}) \\ A &\mapsto \varphi_\mu \circ A \circ \varphi_\lambda^{-1}. \end{aligned} \quad (2.6)$$

Here, we identify the $\mathfrak{vect}(1)$ -modules via the following isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \Pi(\mathcal{F}_{\mu+\frac{1}{2}})) &\rightarrow \Pi(D_{\lambda, \mu+\frac{1}{2}}), & A &\mapsto \Pi(\Pi \circ A), \\ \mathrm{Hom}_{\mathrm{diff}}(\Pi(\mathcal{F}_{\lambda+\frac{1}{2}}), \mathcal{F}_\mu) &\rightarrow \Pi(D_{\lambda+\frac{1}{2}, \mu}), & A &\mapsto \Pi(A \circ \Pi), \\ \mathrm{Hom}_{\mathrm{diff}}(\Pi(\mathcal{F}_{\lambda+\frac{1}{2}}), \Pi(\mathcal{F}_{\mu+\frac{1}{2}})) &\rightarrow D_{\lambda+\frac{1}{2}, \mu+\frac{1}{2}}, & A &\mapsto \Pi \circ A \circ \Pi. \end{aligned}$$

Note that the change of parity map Π commutes with the $\mathfrak{vect}(1)$ -action. \square

Consider a family of $\mathfrak{vect}(1)$ -modules on the space $D_{(\lambda_1, \dots, \lambda_m); \mu}$ of linear differential operators: $A : \mathcal{F}_{\lambda_1} \otimes \dots \otimes \mathcal{F}_{\lambda_m} \rightarrow \mathcal{F}_\mu$. The Lie algebra $\mathfrak{vect}(1)$ naturally acts on $D_{(\lambda_1, \dots, \lambda_m); \mu}$ (by the Leibniz rule). We similarly consider a family of $\mathcal{K}(1)$ -modules on the space $\mathfrak{D}_{(\lambda_1, \dots, \lambda_m); \mu}$ of linear differential operators: $A : \mathfrak{F}_{\lambda_1} \otimes \dots \otimes \mathfrak{F}_{\lambda_m} \rightarrow \mathfrak{F}_\mu$.

3. $\mathfrak{sl}(2)$ - and $\mathfrak{osp}(1|2)$ -invariant Bilinear Differential Operators

Proposition 3.1 (Gordon, [19]). *There exist $\mathfrak{sl}(2)$ -invariant bilinear differential operators, called transvectants,*

$$J_k^{\tau, \lambda} : \mathcal{F}_\tau \otimes \mathcal{F}_\lambda \rightarrow \mathcal{F}_{\tau+\lambda+k}, \quad (\varphi dx^\tau, \phi dx^\lambda) \mapsto J_k^{\tau, \lambda}(\varphi, \phi) dx^{\tau+\lambda+k}$$

given by

$$J_k^{\tau, \lambda}(\varphi, \phi) = \sum_{0 \leq i \leq k, i+j=k} c_{i,j} \varphi^{(i)} \phi^{(j)},$$

where $k \in \mathbb{N}$ and the coefficients $c_{i,j}$ are characterized as follows:

- (i) If $\tau, \lambda \notin \{0, -\frac{1}{2}, -1, \dots, -\frac{k-1}{2}\}$, then $c_{i,j} = \Gamma_{i,j,k-1}^{\tau, \lambda}$, see (3.2).
- (ii) If τ or $\lambda \in \{0, -\frac{1}{2}, -1, \dots, -\frac{k-1}{2}\}$, the coefficients $c_{i,j}$ satisfy the recurrence relation

$$(i+1)(i+2\tau)c_{i+1,j} + (j+1)(j+2\lambda)c_{i,j+1} = 0. \quad (3.1)$$

Moreover, the space of solutions of the system (3.1) is two-dimensional if $2\lambda = -s$ and $2\tau = -t$ with $t > k - s - 2$, and one-dimensional otherwise.

Gieres and Theisen [18] listed the $\mathfrak{osp}(1|2)$ -invariant bilinear differential operators, from $\mathfrak{F}_\tau \otimes \mathfrak{F}_\lambda$ to \mathfrak{F}_μ , called *supertransvectants*. Gargoubi and Ovsienko [17] gave an interpretation of these operators. In [18], the supertransvectants are expressed in terms of supercovariant derivative. Here, the supertransvectants appear in the context of the $\mathfrak{osp}(1|2)$ -relative cohomology. More precisely, we need to describe the $\mathfrak{osp}(1|2)$ -invariant linear differential operators from $\mathcal{K}(1)$ to $\mathfrak{D}_{\lambda, \lambda+k-1}$ vanishing on $\mathfrak{osp}(1|2)$. Thus, using the Gordan's transvectants and the isomorphism (2.6), we give, in the following theorem, another description and other explicit formulas.

Theorem 3.1. (i) *There are only the following $\mathfrak{osp}(1|2)$ -invariant bilinear differential operators acting in the spaces \mathfrak{F}_λ :*

$$\begin{aligned}\mathfrak{J}_k^{\tau,\lambda} : \mathfrak{F}_\tau \otimes \mathfrak{F}_\lambda &\rightarrow \mathfrak{F}_{\tau+\lambda+k} \\ (F\alpha^\tau, G\alpha^\lambda) &\mapsto \mathfrak{J}_k^{\tau,\lambda}(F, G)\alpha^{\tau+\lambda+k},\end{aligned}$$

where $k \in \frac{1}{2}\mathbb{N}$. The operators $\mathfrak{J}_k^{\tau,\lambda}$ labeled by semi-integer k are odd; they are given by

$$\mathfrak{J}_k^{\tau,\lambda}(F, G) = \sum_{i+j=[k]} \Gamma_{i,j,k}^{\tau,\lambda} ((-1)^{p(F)}(2\tau + [k] - j)F^{(i)}\bar{\eta}(G^{(j)}) - (2\lambda + [k] - i)\bar{\eta}(F^{(i)})G^{(j)}).$$

The operators $\mathfrak{J}_k^{\tau,\lambda}$, where $k \in \mathbb{N}$, are even; set $\mathfrak{J}_0^{\tau,\lambda}(F, G) = FG$ and

$$\mathfrak{J}_k^{\tau,\lambda}(F, G) = \sum_{i+j=k-1} (-1)^{p(F)} \Gamma_{i,j,k-1}^{\tau,\lambda} \bar{\eta}(F^{(i)})\bar{\eta}(G^{(j)}) - \sum_{i+j=k} \Gamma_{i,j,k-1}^{\tau,\lambda} F^{(i)}G^{(j)},$$

where $\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$ and $[k]$ denotes the integer part of k , $k > 0$, and

$$\Gamma_{i,j,k}^{\tau,\lambda} = (-1)^j \binom{2\tau + [k]}{j} \binom{2\lambda + [k]}{i}. \quad (3.2)$$

(ii) If $\tau, \lambda \notin \{0, -\frac{1}{2}, -1, \dots, -\frac{[k]}{2}\}$, then $\mathfrak{J}_k^{\tau,\lambda}$ is the unique (up to a scalar factor) bilinear $\mathfrak{osp}(1|2)$ -invariant bilinear differential operator $\mathfrak{F}_\tau \otimes \mathfrak{F}_\lambda \rightarrow \mathfrak{F}_{\tau+\lambda+k}$.

(iii) For $k \in \frac{1}{2}(\mathbb{N} + 5)$, the space of $\mathfrak{osp}(1|2)$ -invariant linear differential operators from $\mathcal{K}(1)$ to $\mathfrak{D}_{\lambda, \lambda+k-1}$ vanishing on $\mathfrak{osp}(1|2)$ is one dimensional.

Proof. (i) Let $\mathcal{T} : \mathfrak{F}_\tau \otimes \mathfrak{F}_\lambda \rightarrow \mathfrak{F}_\mu$ be an $\mathfrak{osp}(1|2)$ -invariant differential operator. Using the fact that, as $\mathfrak{vect}(1)$ -modules,

$$\mathfrak{F}_\tau \otimes \mathfrak{F}_\lambda \simeq \mathcal{F}_\tau \otimes \mathcal{F}_\lambda \oplus \Pi(\mathcal{F}_{\tau+\frac{1}{2}} \otimes \mathcal{F}_{\lambda+\frac{1}{2}}) \oplus \mathcal{F}_\tau \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \oplus \Pi(\mathcal{F}_{\tau+\frac{1}{2}}) \otimes \mathcal{F}_\lambda \quad (3.3)$$

and

$$\mathfrak{F}_\mu \simeq \mathcal{F}_\mu \oplus \Pi(\mathcal{F}_{\mu+\frac{1}{2}}),$$

we can deduce that the restriction of \mathcal{T} to each component of the right-hand side of (3.3) is a transvectant. So, the parameters τ, λ and μ must satisfy

$$\mu = \lambda + \tau + k, \quad \text{where } k \in \frac{1}{2}\mathbb{N}.$$

The corresponding operators will be denoted $\mathfrak{J}_k^{\tau,\lambda}$. Obviously, if k is integer, then the operator $\mathfrak{J}_k^{\tau,\lambda}$ is even and its restriction to each component of the right-hand side of (3.3) coincides

(up to a scalar factor) with the respective transvectants:

$$\begin{cases} J_k^{\tau, \lambda} & : \mathcal{F}_\tau \otimes \mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu, \\ J_{k-1}^{\tau+\frac{1}{2}, \lambda+\frac{1}{2}} & : \Pi(\mathcal{F}_{\tau+\frac{1}{2}}) \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \rightarrow \mathcal{F}_\mu, \\ J_k^{\tau, \lambda+\frac{1}{2}} & : \mathcal{F}_\tau \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \rightarrow \Pi(\mathcal{F}_{\mu+\frac{1}{2}}), \\ J_k^{\tau+\frac{1}{2}, \lambda} & : \Pi(\mathcal{F}_{\tau+\frac{1}{2}}) \otimes \mathcal{F}_\lambda \rightarrow \Pi(\mathcal{F}_{\mu+\frac{1}{2}}). \end{cases} \quad (3.4)$$

If k is semi-integer, then the operator $\mathfrak{J}_k^{\tau, \lambda}$ is odd and its restriction to each component of the right-hand side of (3.3) coincides (up to a scalar factor) with the respective transvectants:

$$\begin{cases} J_{[k]+1}^{\tau, \lambda} & : \mathcal{F}_\tau \otimes \mathcal{F}_\lambda \rightarrow \Pi(\mathcal{F}_{\mu+\frac{1}{2}}), \\ J_{[k]}^{\tau+\frac{1}{2}, \lambda+\frac{1}{2}} & : \Pi(\mathcal{F}_{\tau+\frac{1}{2}}) \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \rightarrow \Pi(\mathcal{F}_{\mu+\frac{1}{2}}), \\ J_{[k]}^{\tau, \lambda+\frac{1}{2}} & : \mathcal{F}_\tau \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \rightarrow \mathcal{F}_\mu, \\ J_{[k]}^{\tau+\frac{1}{2}, \lambda} & : \Pi(\mathcal{F}_{\tau+\frac{1}{2}}) \otimes \mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu. \end{cases} \quad (3.5)$$

More precisely, let $F\alpha^\tau \otimes G\alpha^\lambda \in \mathfrak{F}_\tau \otimes \mathfrak{F}_\lambda$, where $F = f_0 + \theta f_1$ and $G = g_0 + \theta g_1$, with $f_0, f_1, g_0, g_1 \in \mathbb{K}[x]$. Then if k is integer, we have

$$\mathfrak{J}_k^{\tau, \lambda}(\varphi, \psi) = [a_1 J_k^{\tau, \lambda}(f_0, g_0) + a_2 J_{k-1}^{\tau+\frac{1}{2}, \lambda+\frac{1}{2}}(f_1, g_1) + \theta(a_3 J_k^{\tau, \lambda+\frac{1}{2}}(f_0, g_1) + a_4 J_k^{\tau+\frac{1}{2}, \lambda}(f_1, g_0))] \alpha^\mu \quad (3.6)$$

and if k is semi-integer, we have

$$\begin{aligned} \mathfrak{J}_k^{\tau, \lambda}(\varphi, \psi) &= [b_1 J_{[k]}^{\tau, \lambda+\frac{1}{2}}(f_0, g_1) + b_2 J_{[k]}^{\tau+\frac{1}{2}, \lambda}(f_1, g_0) \\ &\quad + \theta(b_3 J_{[k]+1}^{\tau, \lambda}(f_0, g_0) + b_4 J_{[k]}^{\tau+\frac{1}{2}, \lambda+\frac{1}{2}}(f_1, g_1))] \alpha^\mu, \end{aligned} \quad (3.7)$$

where the a_i and b_i are constants. The invariance of $\mathfrak{J}_k^{\tau, \lambda}$ with respect to X_θ and $X_{x\theta}$ reads:

$$\mathfrak{L}_{X_\theta}^\mu \circ \mathfrak{J}_k^{\tau, \lambda} - (-1)^{2k} \mathfrak{J}_k^{\tau, \lambda} \circ \mathfrak{L}_{X_\theta}^{(\tau, \lambda)} = \mathfrak{L}_{X_{x\theta}}^\mu \circ \mathfrak{J}_k^{\tau, \lambda} - (-1)^{2k} \mathfrak{J}_k^{\tau, \lambda} \circ \mathfrak{L}_{X_{x\theta}}^{(\tau, \lambda)} = 0. \quad (3.8)$$

The formula (3.8) allows us to determine the coefficients a_i and b_i . More precisely, the invariance property with respect to X_θ and $X_{x\theta}$ yields

$$\begin{aligned} a_1 = a_2 = a_3 = a_4, \quad b_2 &= -\frac{2\lambda + k - 1}{2\tau + k - 1} b_1, \\ b_3 &= \frac{k}{2\tau + k - 1} b_1 \quad \text{and} \quad b_4 = -\left(1 + \frac{2\lambda}{2\tau + k - 1}\right) b_1. \end{aligned}$$

(ii) The uniqueness of supertransvectants follows from the uniqueness of transvectants.

(iii) In the non-super case, according to formulae (3.1), if $2\tau = -1$ and $k \geq 2$, the space of $\mathfrak{sl}(2)$ -invariant bilinear differential operators $\mathcal{F}_\tau \otimes \mathcal{F}_\lambda \rightarrow \mathcal{F}_{\tau+\lambda+k}$ is two-dimensional if and only if $2\lambda = -s$, where $s \in \{k-1, k-2\}$. This space is spanned by $J_k^{-\frac{1}{2}, -\frac{s}{2}}$ and $I_k^{-\frac{1}{2}, -\frac{s}{2}}$,

where

$$I_k^{-\frac{1}{2}, -\frac{s}{2}}(\varphi, \phi) = \begin{cases} \varphi\phi^{(k)} & \text{if } s = k - 1 \\ \varphi\phi^{(k)} + k\varphi'\phi^{(k-1)} & \text{if } s = k - 2 \end{cases}$$

and

$$J_k^{-\frac{1}{2}, -\frac{s}{2}}(\varphi, \phi) = \sum_{i+j=k, i \geq k-s+1} c_{i,j} \varphi^{(i)} \phi^{(j)},$$

where the coefficients $c_{i,j}$ satisfy (3.1). We see that only the operators $J_k^{-\frac{1}{2}, -\frac{s}{2}}$ vanish on the space of affine functions, i.e., of the form $\varphi(x) = ax + b$.

If $k \geq 3$, the space of $\mathfrak{sl}(2)$ -invariant bilinear differential operators $\mathcal{F}_{-1} \otimes \mathcal{F}_\lambda \rightarrow \mathcal{F}_{\lambda+k-1}$ is two-dimensional if and only if $2\lambda = -s$, where $s \in \{k-1, k-2, k-3\}$. This space is spanned by $J_k^{-1, -\frac{s}{2}}$ and $I_k^{-1, -\frac{s}{2}}$, where

$$I_k^{-1, -\frac{s}{2}}(\varphi, \phi) = \begin{cases} \varphi\phi^{(k)} & \text{if } s = k - 1 \\ \varphi\phi^{(k)} + \frac{k}{2}\varphi'\phi^{(k-1)} & \text{if } s = k - 2 \\ \varphi\phi^{(k)} + k\varphi'\phi^{(k-1)} + \frac{k(k-1)}{2}\varphi''\phi^{(k-2)} & \text{if } s = k - 3 \end{cases}$$

and where

$$J_k^{-1, -\frac{s}{2}}(\varphi, \phi) = \sum_{i+j=k, i \geq 3} c_{i,j} \varphi^{(i)} \phi^{(j)}.$$

We see that the operator $I_k^{-1, -\frac{s}{2}}$ does not vanish on $\mathfrak{sl}(2)$, but the operator $J_k^{-1, -\frac{s}{2}}$ vanishes on $\mathfrak{sl}(2)$.

Now, if $\tau = -1, -\frac{1}{2}$ and $2\lambda \notin \{1-k, 2-k, 3-k\}$ with $k \geq 3$, the space of $\mathfrak{sl}(2)$ -invariant bilinear differential operators $\mathcal{F}_\tau \otimes \mathcal{F}_\lambda \rightarrow \mathcal{F}_{\tau+\lambda+k}$ is one-dimensional. But, in this case, we see that the coefficients $c_{i,j}$ satisfying (3.1) are such that $c_{i,j} = 0$ if $i \leq 2$ for $\tau = -1$ and $c_{i,j} = 0$ if $i \leq 1$ for $\tau = -\frac{1}{2}$.

Thus, in the super setting, if $2k \geq 5$, according to Eqs. (3.6) and (3.7), we see that the space of $\mathfrak{osp}(1|2)$ -invariant linear differential operator from $\mathcal{K}(1)$ to $\mathfrak{D}_{\lambda, \lambda+k-1}$ vanishing on $\mathfrak{osp}(1|2)$ is one-dimensional. \square

4. Cohomology

Let us first recall some fundamental concepts from cohomology theory (see, e.g., [15]). Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra acting on a superspace $V = V_0 \oplus V_1$ and let \mathfrak{h} be a subalgebra of \mathfrak{g} . (If \mathfrak{h} is omitted it assumed to be $\{0\}$.) The space of \mathfrak{h} -relative n -cochains of \mathfrak{g} with values in V is the \mathfrak{g} -module

$$C^n(\mathfrak{g}, \mathfrak{h}; V) := \text{Hom}_{\mathfrak{h}}(\Lambda^n(\mathfrak{g}/\mathfrak{h}); V).$$

The coboundary operator $\delta_n : C^n(\mathfrak{g}, \mathfrak{h}; V) \rightarrow C^{n+1}(\mathfrak{g}, \mathfrak{h}; V)$ is a \mathfrak{g} -map satisfying $\delta_n \circ \delta_{n-1} = 0$. The kernel of δ_n , denoted $Z^n(\mathfrak{g}, \mathfrak{h}; V)$, is the space of \mathfrak{h} -relative n -cocycles, among them, the

elements in the range of δ_{n-1} are called \mathfrak{h} -relative n -coboundaries. We denote $B^n(\mathfrak{g}, \mathfrak{h}; V)$ the space of n -coboundaries.

By definition, the n^{th} \mathfrak{h} -relative cohomology space is the quotient space

$$H^n(\mathfrak{g}, \mathfrak{h}; V) = Z^n(\mathfrak{g}, \mathfrak{h}; V) / B^n(\mathfrak{g}, \mathfrak{h}; V).$$

We will only need the formula of δ_n (which will be simply denoted δ) in degrees 0 and 1: for $v \in C^0(\mathfrak{g}, \mathfrak{h}; V) = V^{\mathfrak{h}}$, $\delta v(g) := (-1)^{p(g)p(v)} g \cdot v$, where

$$V^{\mathfrak{h}} = \{v \in V \mid h \cdot v = 0 \text{ for all } h \in \mathfrak{h}\},$$

and for $\Upsilon \in C^1(\mathfrak{g}, \mathfrak{h}; V)$,

$$\delta(\Upsilon)(g, h) := (-1)^{p(g)p(\Upsilon)} g \cdot \Upsilon(h) - (-1)^{p(h)(p(g)+p(\Upsilon))} h \cdot \Upsilon(g) - \Upsilon([g, h]) \quad \text{for any } g, h \in \mathfrak{g}.$$

According to the \mathbb{Z}_2 -grading (parity) of \mathfrak{g} , for any $\Upsilon \in Z^1(\mathfrak{g}, V)$, we have

$$\Upsilon = \Upsilon' + \Upsilon'' \in Z^1(\mathfrak{g}_0; V) \oplus \text{Hom}(\mathfrak{g}_1, V)$$

subject to the following three equations:

$$\Upsilon'([g_1, g_2]) - g_1 \cdot \Upsilon'(g_2) + g_2 \cdot \Upsilon'(g_1) = 0 \quad \text{for any } g_1, g_2 \in \mathfrak{g}_0, \quad (4.1)$$

$$\Upsilon''([g, h]) - g \cdot \Upsilon''(h) + (-1)^{p(\Upsilon)} h \cdot \Upsilon'(g) = 0 \quad \text{for any } g \in \mathfrak{g}_0, h \in \mathfrak{g}_1, \quad (4.2)$$

$$\Upsilon'([h_1, h_2]) - (-1)^{p(\Upsilon)} (h_1 \cdot \Upsilon''(h_2) + h_2 \cdot \Upsilon''(h_1)) = 0 \quad \text{for any } h_1, h_2 \in \mathfrak{g}_1. \quad (4.3)$$

Formulas (4.1)–(4.3) show that $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu})$ and $H_{\text{diff}}^1(\mathbf{vect}(1); D_{\lambda, \mu})$ are closely related. Similarly, $H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$ is related to $H_{\text{diff}}^1(\mathbf{vect}(1), \mathfrak{sl}(2); D_{\lambda, \mu})$. Therefore, for comparison and to build upon, we recall the description of $H_{\text{diff}}^1(\mathbf{vect}(1); D_{\lambda, \mu})$. Note that $H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$ is also computed by Conley, see [10].

4.1. Relationship between $H_{\text{diff}}^1(\mathbf{vect}(1); D_{\lambda, \mu})$ and $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu})$

Feigin and Fuchs [11] calculated $H_{\text{diff}}^1(\mathbf{vect}(1); D_{\lambda, \mu})$. The result is as follows

$$H_{\text{diff}}^1(\mathbf{vect}(1); D_{\lambda, \mu}) \simeq \begin{cases} \mathbb{K} & \text{if } \mu - \lambda = 0, 2, 3, 4 \text{ for all } \lambda, \\ \mathbb{K}^2 & \text{if } \lambda = 0 \text{ and } \mu = 1, \\ \mathbb{K} & \text{if } \lambda = 0 \text{ or } \lambda = -4 \text{ and } \mu - \lambda = 5, \\ \mathbb{K} & \text{if } \lambda = -\frac{5 \pm \sqrt{19}}{2} \text{ and } \mu - \lambda = 6, \\ 0 & \text{otherwise.} \end{cases} \quad (4.4)$$

For $X \frac{d}{dx} \in \mathbf{vect}(1)$ and $f dx^\lambda \in \mathcal{F}_\lambda$, we write

$$C_{\lambda, \lambda+k} \left(X \frac{d}{dx} \right) (f dx^\lambda) = C_{\lambda, \lambda+k}(X, f) dx^{\lambda+k}.$$

The spaces $H_{\text{diff}}^1(\mathbf{vect}(1), D_{\lambda, \lambda+k})$ are generated by the cohomology classes of the following 1-cocycles:

$$C_{\lambda, \lambda}(X, f) = X' f$$

$$C_{0,1}(X, f) = X'' f$$

$$\begin{aligned}
\tilde{C}_{0,1}(X, f) &= (X'f)' \\
C_{\lambda, \lambda+2}(X, f) &= X^{(3)}f + 2X''f' \\
C_{\lambda, \lambda+3}(X, f) &= X^{(3)}f' + X''f'' \\
C_{\lambda, \lambda+4}(X, f) &= -\lambda X^{(5)}f + X^{(4)}f' - 6X^{(3)}f'' - 4X''f^{(3)} \\
C_{0,5}(X, f) &= 2X^{(5)}f' - 5X^{(4)}f'' + 10X^{(3)}f^{(3)} + 5X''f^{(4)} \\
C_{-4,1}(X, f) &= 12X^{(6)}f + 22X^{(5)}f' + 5X^{(4)}f'' - 10X^{(3)}f^{(3)} - 5X''f^{(4)} \\
C_{a_i, a_i+6}(X, f) &= \alpha_i X^{(7)}f - \beta_i X^{(6)}f' - \gamma_i X^{(5)}f'' - 5X^{(4)}f^{(3)} + 5X^{(3)}f^{(4)} + 2X''f^{(5)},
\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}
a_1 &= -\frac{5 + \sqrt{19}}{2}, & \alpha_1 &= -\frac{22 + 5\sqrt{19}}{4}, & \beta_1 &= \frac{31 + 7\sqrt{19}}{2}, & \gamma_1 &= \frac{25 + 7\sqrt{19}}{2} \\
a_2 &= -\frac{5 - \sqrt{19}}{2}, & \alpha_2 &= -\frac{22 - 5\sqrt{19}}{4}, & \beta_2 &= \frac{31 - 7\sqrt{19}}{2}, & \gamma_2 &= \frac{25 - 7\sqrt{19}}{2}.
\end{aligned}$$

Now, let us study the relationship between any 1-cocycle of $\mathcal{K}(1)$ and its restriction to the subalgebra $\mathbf{vect}(1)$. More precisely, we study the relationship between $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu})$ and $H_{\text{diff}}^1(\mathbf{vect}(1); D_{\lambda, \mu})$. According to Proposition 2.1, we see that $H_{\text{diff}}^1(\mathbf{vect}(1); \mathfrak{D}_{\lambda, \mu})$ can be deduced from the spaces $H_{\text{diff}}^1(\mathbf{vect}(1); D_{\lambda, \mu})$:

$$\begin{aligned}
H_{\text{diff}}^1(\mathbf{vect}(1); \mathfrak{D}_{\lambda, \mu}) &\simeq H_{\text{diff}}^1(\mathbf{vect}(1); D_{\lambda, \mu}) \oplus H_{\text{diff}}^1(\mathbf{vect}(1); D_{\lambda + \frac{1}{2}, \mu + \frac{1}{2}}) \\
&\oplus H_{\text{diff}}^1(\mathbf{vect}(1); \Pi(D_{\lambda, \mu + \frac{1}{2}})) \oplus H_{\text{diff}}^1(\mathbf{vect}(1); \Pi(D_{\lambda + \frac{1}{2}, \mu})). \tag{4.6}
\end{aligned}$$

Moreover, the following lemma shows the close relationship between the cohomology spaces $H^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu})$ and $H^1(\mathbf{vect}(1); D_{\lambda, \mu})$.

Lemma 4.1. *The 1-cocycle Υ of $\mathcal{K}(1)$ is a coboundary if and only if its restriction Υ' to $\mathbf{vect}(1)$ is a coboundary.*

Proof. It is easy to see that if Υ is a coboundary of $\mathcal{K}(1)$, then Υ' is a coboundary of $\mathbf{vect}(1)$. Now, assume that Υ' is a coboundary of $\mathbf{vect}(1)$, that is, there exist $A \in \mathfrak{D}_{\lambda, \mu}$ such that Υ' is defined by

$$\Upsilon'(X_f) = \mathfrak{L}_{X_f}^{\lambda, \mu} A \quad \text{for all } f \in \mathbb{K}[x].$$

By replacing Υ by $\Upsilon - \delta A$, we can suppose that $\Upsilon' = 0$. But, in this case, the map Υ must satisfy the following equations

$$\mathfrak{L}_{X_g}^{\lambda, \mu} \Upsilon(X_{h\theta}) - \Upsilon([X_g, X_{h\theta}]) = 0 \quad \text{for all } g, h \in \mathbb{K}[x], \tag{4.7}$$

$$\mathfrak{L}_{X_{h_1\theta}}^{\lambda, \mu} \Upsilon(X_{h_2\theta}) + \mathfrak{L}_{X_{h_2\theta}}^{\lambda, \mu} \Upsilon(X_{h_1\theta}) = 0 \quad \text{for all } h_1, h_2 \in \mathbb{K}[x]. \tag{4.8}$$

Equation (4.7) expresses the $\mathbf{vect}(1)$ -invariance of the map $\Upsilon : \Pi(\mathcal{F}_{-\frac{1}{2}}) \times \mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu$. Therefore, if Υ is an even 1-cocycle, then, according to Proposition 2.1, we can easily deduce the expression of Υ from the work of P. Grozman [20]. More precisely, Υ has, *a priori*, the

following form:

$$\Upsilon(X_{h\theta})(F\alpha^\lambda) = \begin{cases} (a_1 h f \theta) \alpha^{\lambda-1} & \text{if } \mu = \lambda - 1 \\ \left(a_2 h g + a_3 \left(\frac{1}{2} h f' + \lambda h' f \right) \theta \right) \alpha^\lambda & \text{if } \mu = \lambda \\ a_4 \left(\frac{1}{2} h g' + \lambda h' g \right) \alpha^{\lambda+1} & \text{if } \mu = \lambda + 1, \lambda \neq 0, -\frac{1}{2} \\ a_5 \left(\frac{1}{2} h g'' + h' g' \right) \alpha^{\frac{3}{2}} & \text{if } (\lambda, \mu) = \left(-\frac{1}{2}, \frac{3}{2} \right) \\ \left(a_6 (h g' + h' g) + a_7 \left(\frac{1}{2} h f'' + h' f' \right) \theta \right) \alpha & \text{if } (\lambda, \mu) = (0, 1) \\ a_8 (h g'' - h'' g) \alpha & \text{if } (\lambda, \mu) = (-1, 1) \\ (a_9 h g' + a_{10} (h f'' - h'' f) \theta) \alpha^{\frac{1}{2}} & \text{if } (\lambda, \mu) = \left(-\frac{1}{2}, \frac{1}{2} \right) \\ 0 & \text{otherwise,} \end{cases}$$

where $a_i \in \mathbb{K}$, $f, g \in \mathbb{K}[x]$ and $F = f + g\theta$. But, the map Υ must satisfy Eq. (4.8), so we obtain $a_1 = a_4 = a_5 = a_8 = 0$, $a_3 = -2a_2$, $a_7 = -2a_6$ and $a_{10} = -a_9$. More precisely, up to a scalar factor, Υ is given by:

$$\Upsilon = \begin{cases} \delta((1 - \theta \partial_\theta) \partial_x) & \text{if } (\lambda, \mu) = (0, 1), \\ \delta(\theta \partial_\theta \partial_x) & \text{if } (\lambda, \mu) = \left(-\frac{1}{2}, \frac{1}{2} \right), \\ \delta(\theta \partial_\theta) & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, if Υ is an odd 1-cocycle, then, Υ has, *a priori*, the following form (see [20]):

$$\Upsilon(X_{h\theta})(F\alpha^\lambda) = \begin{cases} (b_1 h f + b_2 h g) \alpha^{\lambda-\frac{1}{2}} & \text{if } \mu = \lambda - \frac{1}{2} \\ \left(b_3 \left(\frac{1}{2} h f' + \lambda h' f \right) + b_4 \left(\frac{1}{2} h g' + \left(\lambda + \frac{1}{2} \right) h' g \right) \theta \right) \alpha^{\lambda+\frac{1}{2}} & \text{if } \mu = \lambda + \frac{1}{2} \\ b_5 \left(\frac{1}{2} h f'' + h'' f \right) \alpha^{\frac{3}{2}} & \text{if } (\lambda, \mu) = \left(0, \frac{3}{2} \right) \\ \left(b_6 (h f'' - h'' f) + b_7 \left(\frac{1}{2} h g'' + h' g' \right) \theta \right) \alpha & \text{if } (\lambda, \mu) = \left(-\frac{1}{2}, 1 \right) \\ b_8 (h g'' - h'' g) \theta \alpha^{\frac{1}{2}} & \text{if } (\lambda, \mu) = \left(-1, \frac{1}{2} \right) \\ 0 & \text{otherwise,} \end{cases}$$

where $b_i \in \mathbb{K}$. But, the map Υ must satisfy Eq. (4.8), so we obtain $b_5 = b_8 = 0, b_1 = b_2, b_3 = b_4$ and $b_7 = 2b_6$. More precisely, up to a scalar factor, Υ is given by:

$$\Upsilon = \begin{cases} \delta(\partial_\theta) & \text{if } \mu = \lambda + \frac{1}{2}, \\ \delta(\theta) & \text{if } \mu = \lambda - \frac{1}{2}, \\ \delta(\partial_\theta \partial_x) & \text{if } (\lambda, \mu) = \left(-\frac{1}{2}, 1\right), \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof. \square

The following lemma gives the general form of any 1-cocycle of $\mathcal{K}(1)$.

Lemma 4.2. *Let $\Upsilon \in Z_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu})$. Up to a coboundary, the map Υ has the following general form*

$$\Upsilon(X_F) = \sum_{m,k} (a_{m,k} + \theta b_{m,k}) \bar{\eta}^k(F) \bar{\eta}^m, \quad (4.9)$$

where the coefficients $a_{m,k}$ and $b_{m,k}$ are constants.

Proof. Since $-\bar{\eta}^2 = \partial_x$, the operator Υ has the form (4.9), where, *a priori*, the coefficients $a_{m,k}$ and $b_{m,k}$ are functions (see [16]), but we will prove that, up to a coboundary, Υ is invariant with respect the vector field $X_1 = \partial_x$. The 1-cocycle condition reads:

$$\mathfrak{L}_{X_1}^{\lambda, \mu}(\Upsilon(X_F)) - (-1)^{p(F)p(\Upsilon)} \mathfrak{L}_{X_F}^{\lambda, \mu}(\Upsilon(X_1)) - \Upsilon([X_1, X_F]) = 0. \quad (4.10)$$

But, from (4.5), up to a coboundary, we have $\Upsilon(X_1) = 0$, and therefore Eq. (4.10) becomes

$$\mathfrak{L}_{X_1}^{\lambda, \mu}(\Upsilon(X_F)) - \Upsilon([X_1, X_F]) = 0$$

which is nothing but the invariance property of Υ with respect the vector field X_1 . \square

Lemma 4.3. *Any 1-cocycle $\Upsilon \in Z_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu})$ vanishing on $\mathfrak{osp}(1|2)$ is $\mathfrak{osp}(1|2)$ -invariant.*

Proof. The 1-cocycle relation of Υ reads:

$$(-1)^{p(F)p(\Upsilon)} \mathfrak{L}_{X_F}^{\lambda, \mu} \Upsilon(X_G) - (-1)^{p(G)(p(F)+p(\Upsilon))} \mathfrak{L}_{X_G}^{\lambda, \mu} \Upsilon(X_F) - \Upsilon([X_F, X_G]) = 0, \quad (4.11)$$

where $X_F, X_G \in \mathcal{K}(1)$. Thus, if $\Upsilon(X_F) = 0$ for all $X_F \in \mathfrak{osp}(1|2)$, Eq. (4.11) becomes

$$(-1)^{p(F)p(\Upsilon)} \mathfrak{L}_{X_F}^{\lambda, \mu} \Upsilon(X_G) - \Upsilon([X_F, X_G]) = 0 \quad (4.12)$$

expressing the $\mathfrak{osp}(1|2)$ -invariance of Υ . \square

Lemma 4.4 ([4] Lemma 3.3). *Up to a coboundary, any 1-cocycle $\Upsilon \in Z_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu})$ vanishing on $\mathfrak{sl}(2)$ is $\mathfrak{osp}(1|2)$ -invariant. That is, if $\Upsilon(X_1) = \Upsilon(X_x) = \Upsilon(X_{x^2}) = 0$, then the restriction of Υ to $\mathfrak{osp}(1|2)$ is trivial.*

Proof. Recall that, as $\mathfrak{sl}(2)$ -module, the subalgebra $\mathfrak{osp}(1|2)$ is isomorphic to $\mathfrak{sl}(2) \oplus \mathfrak{a}$, where $\mathfrak{a} = \text{Span}(X_\theta, X_{x\theta})$. Consider a linear operator $A : \mathfrak{a} \rightarrow D_{\lambda, \mu}$. By a straightforward computation, we show that if A is $\mathfrak{sl}(2)$ -invariant, then $\mu = \lambda - \frac{1}{2} + k$, where $k \in \mathbb{N}$ and the corresponding operator A_k has the following expression

$$A_k(X_{h\theta})(f dx^\lambda) = a_k \left(h f^{(k)} + k(2\lambda + k - 1) h' f^{(k-1)} \right) dx^{\lambda - \frac{1}{2} + k}, \quad (4.13)$$

where

$$k(k-1)(2\lambda + k - 1)(2\lambda + k - 2)a_k = 0.$$

Now, consider $\Upsilon \in Z_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu})$ such that $\Upsilon(X_1) = \Upsilon(X_x) = \Upsilon(X_{x^2}) = 0$. The 1-cocycle relations give, for all h, h_1, h_2 polynomial with degree 0 or 1 and g polynomial with degree 0, 1 or 2, the following equations

$$\mathfrak{L}_{X_g}^{\lambda, \mu} \Upsilon(X_{h\theta}) - \Upsilon([X_g, X_{h\theta}]) = 0, \quad (4.14)$$

$$\mathfrak{L}_{X_{h_1\theta}}^{\lambda, \mu} \Upsilon(X_{h_2\theta}) + \mathfrak{L}_{X_{h_2\theta}}^{\lambda, \mu} \Upsilon(X_{h_1\theta}) = 0. \quad (4.15)$$

(1) If Υ is an even 1-cocycle, then, according to Proposition 2.1, its restriction to \mathfrak{a} is decomposed into two maps: $\mathfrak{a} \rightarrow \Pi(D_{\lambda + \frac{1}{2}, \mu})$ and $\mathfrak{a} \rightarrow \Pi(D_{\lambda, \mu + \frac{1}{2}})$. Equation (4.14) tell us that these maps are $\mathfrak{sl}(2)$ -invariant. Therefore, their expressions are given by (4.13). So, we must have $\mu = \lambda + k = (\lambda + \frac{1}{2}) - \frac{1}{2} + k$ (and then $\mu + \frac{1}{2} = \lambda - \frac{1}{2} + k + 1$). More precisely, using Eq. (4.15), we get (up to a scalar factor):

$$\Upsilon|_{\mathfrak{osp}(1|2)} = \begin{cases} 0 & \text{if } k(k-1)(2\lambda + k)(2\lambda + k - 1) \neq 0 \\ & \text{or } k = 1 \text{ and } \lambda \notin \left\{0, -\frac{1}{2}\right\}, \\ \delta(\theta \partial_\theta \partial_x^k) & \text{if } (\lambda, \mu) = \left(\frac{-k}{2}, \frac{k}{2}\right), \\ \delta(\partial_x^k - \theta \partial_\theta \partial_x^k) & \text{if } (\lambda, \mu) = \left(\frac{1-k}{2}, \frac{1+k}{2}\right) \text{ or } \lambda = \mu. \end{cases}$$

(2) Similarly, if Υ is an odd 1-cocycle, we get:

$$\Upsilon|_{\mathfrak{osp}(1|2)} = \begin{cases} 0 & \text{if } k(k-1)(2\lambda + k - 1) \neq 0, \\ \delta(\theta) & \text{if } \mu = \lambda - \frac{1}{2}, \\ \delta(\partial_\theta) & \text{if } \mu = \lambda + \frac{1}{2}, \\ \delta(\theta \partial_x^k) & \text{if } (\lambda, \mu) = \left(\frac{1-k}{2}, \frac{k}{2}\right). \end{cases} \quad \square$$

Now, we can compute $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu})$ and the $\mathfrak{osp}(1|2)$ -relative cohomology $H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$. Let Υ be any 1-cocycle over $\mathcal{K}(1)$. According to Proposition 2.1, we have

$$\Upsilon|_{\mathfrak{vect}(1)} \in H_{\text{diff}}^1(\mathfrak{vect}(1); D_{\lambda, \mu}) \oplus H_{\text{diff}}^1(\mathfrak{vect}(1); D_{\lambda + \frac{1}{2}, \mu + \frac{1}{2}}) \quad \text{if } \Upsilon \text{ is even}$$

and

$$\Upsilon_{|\mathbf{vect}(1)} \in H_{\text{diff}}^1(\mathbf{vect}(1); \Pi(D_{\lambda, \mu + \frac{1}{2}})) \oplus H_{\text{diff}}^1(\mathbf{vect}(1); \Pi(D_{\lambda + \frac{1}{2}, \mu})) \quad \text{if } \Upsilon \text{ is odd.}$$

We know that nonzero cohomology $H_{\text{diff}}^1(\mathbf{vect}(1); D_{\lambda, \lambda'})$ only appear if $\lambda' - \lambda \in \mathbb{N}$. Thus, according to Lemma 4.1, the following statements hold:

- (i) If $\mu - \lambda \notin \frac{1}{2}(\mathbb{N} - 1)$, then $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu}) = 0$.
- (ii) If $\mu - \lambda$ is integer, then $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu})$ is spanned only by the cohomology classes of even cocycles.
- (iii) If $\mu - \lambda$ is semi-integer, then $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu})$ is spanned only by the cohomology classes of odd cocycles.

4.2. The space $H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$

The main result of this subsection is the following:

Theorem 4.1. $\dim H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu}) = 1$ if

$$\begin{aligned} \mu - \lambda &= \frac{3}{2} \quad \text{and} \quad \lambda \neq -\frac{1}{2}, \\ \mu - \lambda &= 2 \quad \text{for all } \lambda, \\ \mu - \lambda &= \frac{5}{2} \quad \text{and} \quad \lambda \neq -1, \\ \mu - \lambda &= 3 \quad \text{and} \quad \lambda \in \left\{0, -\frac{5}{2}\right\}, \\ \mu - \lambda &= 4 \quad \text{and} \quad \lambda = \frac{-7 \pm \sqrt{33}}{4}. \end{aligned}$$

Otherwise, $H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu}) = 0$.

The corresponding spaces $H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \lambda + \frac{k}{2}})$ are spanned by the cohomology classes of $\Upsilon_{\lambda, \lambda + \frac{k}{2}} = \mathfrak{J}_{\frac{k}{2}+1}^{-1, \lambda}$, where $k \in \{3, 4, 5, 6, 8\}$.

Proof. Note that, by Lemma 4.1, the $\mathfrak{osp}(1|2)$ -relative cocycles are related to its homologous in the classical setting, and by Lemma 4.3, they are supertransvectants. Bouarroudj and Ovsienko [9] showed that

$$H_{\text{diff}}^1(\mathbf{vect}(1), \mathfrak{sl}(2); D_{\lambda, \lambda+k}) \simeq \begin{cases} \mathbb{K} & \text{if } \begin{cases} k=2 & \text{and } \lambda \neq -\frac{1}{2}, \\ k=3 & \text{and } \lambda \neq -1, \\ k=4 & \text{and } \lambda \neq -\frac{3}{2}, \\ k=5 & \text{and } \lambda = 0, -4, \\ k=6 & \text{and } \lambda = -\frac{5 \pm \sqrt{19}}{2}, \end{cases} \\ 0 & \text{otherwise.} \end{cases} \quad (4.16)$$

These spaces are generated by the cohomology classes of the following non-trivial $\mathfrak{sl}(2)$ -relative 1-cocycles, $A_{\lambda, \lambda+k}$:

$$A_{\lambda, \lambda+2}(X, f) = X^{(3)}f, \quad \lambda \neq -\frac{1}{2},$$

$$A_{\lambda, \lambda+3}(X, f) = X^{(3)}f' - \frac{\lambda}{2}X^{(4)}f, \quad \lambda \neq -1,$$

$$A_{\lambda, \lambda+4}(X, f) = X^{(3)}f'' - \frac{2\lambda+1}{2}X^{(4)}f' + \frac{\lambda(2\lambda+1)}{10}X^{(5)}f, \quad \lambda \neq -\frac{3}{2},$$

$$A_{0,5}(X, f) = -3X^{(5)}f' + 15X^{(4)}f'' - 10X^{(3)}f^{(3)},$$

$$A_{-4,1}(X, f) = 28X^{(6)}f + 63X^{(5)}f' + 45X^{(4)}f'' + 10X^{(3)}f^{(3)},$$

$$A_{a_i, a_i+6}(X, f) = \alpha_i X^{(7)}f - 14\beta_i X^{(6)}f' - 126\gamma_i X^{(5)}f'' - 210\tau_i X^{(4)}f^{(3)} + 210X^{(3)}f^4$$

where $\tau_1 = -2 + \sqrt{19}$ and $\tau_2 = -2 - \sqrt{19}$. The a_i , α_i , β_i and γ_i are those given in (4.5).

So, we see first that if $2(\mu - \lambda) \notin \{3, \dots, 13\}$, then by Lemma 4.1, the corresponding cohomology $H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$ vanish. Indeed, let Υ be any element of $Z_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$. Then by (4.16) and (4.6), up to a coboundary, the restriction of Υ to $\mathfrak{vect}(1)$ vanishes, so $\Upsilon = 0$ by Lemma 4.1. By the same arguments, if $2(\mu - \lambda) > 9$, generically, the corresponding cohomology vanish.

For $2(\mu - \lambda) \in \{3, \dots, 13\}$, we study the supertransvectant $\mathfrak{J}_{\mu-\lambda+1}^{-1, \lambda}$. If it is a non-trivial 1-cocycle, then the corresponding cohomology space is one-dimensional, otherwise it is zero. To study any supertransvectant $\mathfrak{J}_{\mu-\lambda+1}^{-1, \lambda}$ satisfying $\delta(\mathfrak{J}_{\mu-\lambda+1}^{-1, \lambda}) = 0$, we consider the two components of its restriction to $\mathfrak{vect}(1)$ which we compare with $A_{\lambda, \mu}$ and $A_{\lambda+\frac{1}{2}, \mu+\frac{1}{2}}$ or $A_{\lambda+\frac{1}{2}, \mu}$ and $A_{\lambda, \mu+\frac{1}{2}}$ depending on whether $\lambda - \mu$ is integer or semi-integer. For instance, we show that $\mathfrak{J}_{\frac{5}{2}}^{-1, \lambda}$ is a 1-cocycle. Moreover, it is non-trivial for $\lambda \neq -\frac{1}{2}$ since, for $g, f \in \mathbb{K}[x]$, we have $\mathfrak{J}_{\frac{5}{2}}^{-1, \lambda}(X_g)(f) = -\theta A_{\lambda, \lambda+2}(g, f)$. More precisely, we get the following non-trivial 1-cocycles:

$$\left\{ \begin{array}{l} \Upsilon_{\lambda, \lambda+\frac{3}{2}}(X_G)(F\alpha^\lambda) = \overline{\eta}(G'')F\alpha^{\lambda+\frac{3}{2}} \quad \text{for } \lambda \neq -\frac{1}{2}, \\ \Upsilon_{\lambda, \lambda+\frac{5}{2}}(X_G)(F\alpha^\lambda) = (2\lambda\overline{\eta}(G^{(3)})F - 3\overline{\eta}(G'')F' - (-1)^{p(G)}G^{(3)}\overline{\eta}(F))\alpha^{\lambda+\frac{5}{2}} \quad \text{for } \lambda \neq -1, \\ \Upsilon_{\lambda, \lambda+2}(X_G)(F\alpha^\lambda) = \left(\frac{2}{3}\lambda G^{(3)}F - (-1)^{p(G)}\overline{\eta}(G'')\overline{\eta}(F) \right) \alpha^{\lambda+2} \quad \text{for all } \lambda, \\ \Upsilon_{\lambda, \lambda+3}(X_G)(F\alpha^\lambda) = \left((-1)^{p(G)}\overline{\eta}(G'')\overline{\eta}(F') - \frac{2\lambda+1}{3}((-1)^{p(G)}\overline{\eta}(G^{(3)})\overline{\eta}(F) + G^{(3)}F') \right. \\ \quad \left. + \frac{\lambda(2\lambda+1)}{6}G^{(4)}F \right) \alpha^{\lambda+3} \quad \text{for } \lambda = 0, -\frac{5}{2}, \\ \Upsilon_{\lambda, \lambda+4}(X_G)(F\alpha^\lambda) = \left((-1)^{p(G)}\overline{\eta}(G'')\overline{\eta}(F'') - \frac{2(\lambda+1)}{3}(2(-1)^{p(G)}\overline{\eta}(G^{(3)})\overline{\eta}(F') + G^{(3)}F'') \right. \\ \quad \left. + \frac{(\lambda+1)(2\lambda+1)}{6}((-1)^{p(G)}\overline{\eta}(G^{(4)})\overline{\eta}(F) + 2G^{(4)}F') \right. \\ \quad \left. - \frac{\lambda(\lambda+1)(2\lambda+1)}{15}G^{(5)}F \right) \alpha^{\lambda+4} \quad \text{for } \lambda = \frac{-7 \pm \sqrt{33}}{4}. \end{array} \right.$$

□

4.3. The space $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu})$ **Theorem 4.2.** $\dim H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu}) = 1$ if

$$\begin{aligned}
\mu - \lambda &= 0 && \text{for all } \lambda, \\
\mu - \lambda &= \frac{3}{2} && \text{for all } \lambda, \\
\mu - \lambda &= 2 && \text{for all } \lambda, \\
\mu - \lambda &= \frac{5}{2} && \text{for all } \lambda, \\
\mu - \lambda &= 3 && \text{and } \lambda \in \left\{0, -\frac{5}{2}\right\}, \\
\mu - \lambda &= 4 && \text{and } \lambda = \frac{-7 \pm \sqrt{33}}{4}.
\end{aligned}$$

$\dim H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{0, \frac{1}{2}}) = 2$. Otherwise, $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu}) = 0$.

The spaces $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu})$ are spanned by the cohomology classes of the 1-cocycles $\Upsilon_{\lambda, \mu}$ given in Theorem 4.1 and by the cohomology classes of the following 1-cocycles:

$$\begin{cases}
\Upsilon_{\lambda, \lambda}(X_G)(F\alpha^\lambda) = G'F\alpha^\lambda, \\
\Upsilon_{0, \frac{1}{2}}(X_G)(F) = \bar{\eta}(G')F\alpha^{\frac{1}{2}}, \\
\tilde{\Upsilon}_{0, \frac{1}{2}}(X_G)(F) = \bar{\eta}(G'F)\alpha^{\frac{1}{2}}, \\
\Upsilon_{-\frac{1}{2}, 1}(X_G)(F\alpha^{-\frac{1}{2}}) = (\bar{\eta}(G'')F + \bar{\eta}(G')F' + (-1)^{p(G)}G''\bar{\eta}(F))\alpha, \\
\Upsilon_{-1, \frac{3}{2}}(X_G)(F\alpha^{-1}) = ((-1)^{p(G)}(G'''\bar{\eta}(F) + 2G''\bar{\eta}(F')) + 2\bar{\eta}(G'')F' + \bar{\eta}(G')F'')\alpha^{\frac{3}{2}}.
\end{cases}$$

Proof. First, we recall the structure of the space $H_{\text{diff}}^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$ computed in [6]:

$$H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}) \simeq \begin{cases} \mathbb{K} & \text{if } \lambda = \mu, \\ \mathbb{K}^2 & \text{if } \lambda = \frac{1-k}{2}, \mu = \frac{k}{2}, k \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases} \quad (4.17)$$

Note that $H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu}) \subset H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{D}_{\lambda, \mu})$. Moreover, if $\mu \neq \lambda$, then by (4.17) and Lemma 4.4 we can see that $H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu}) = H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \mu})$, except for

$$(\lambda, \mu) \in \left\{ \left(0, \frac{1}{2}\right), \left(-\frac{1}{2}, 1\right), \left(-1, \frac{3}{2}\right), \left(-\frac{3}{2}, 2\right), \left(-2, \frac{5}{2}\right) \right\}. \quad (4.18)$$

Indeed, let Υ be any non-trivial element of $Z_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{D}_{\lambda, \mu})$ where $\mu \neq \lambda$. If $(\lambda, \mu) \neq (\frac{1-k}{2}, \frac{k}{2})$ where $k \in \mathbb{N} \setminus \{0\}$ then, by (4.17), we can see that $\Upsilon|_{\mathfrak{osp}(1|2)}$ is trivial, therefore, we deduce by using Lemma 4.3 that the 1-cocycle Υ defines a non-trivial cohomology class in $H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$. If $(\lambda, \mu) = (\frac{1-k}{2}, \frac{k}{2})$ where $k \in \mathbb{N} \setminus \{0\}$ then, by (4.4), we can see that, up to a coboundary, generically the 1-cocycle Υ vanishes on $\mathfrak{vect}(1)$ and then we conclude by using Lemma 4.4 since $\mathfrak{sl}(2) \subset \mathfrak{vect}(1)$.

Thus, we need to study only the case $\mu = \lambda$ together with the singular cases (4.18). According to Proposition 2.1, if $\mu - \lambda$ is integer, then

$$H_{\text{diff}}^1(\mathbf{vect}(1); \mathfrak{D}_{\lambda, \mu}) \simeq H_{\text{diff}}^1(\mathbf{vect}(1); D_{\lambda, \mu}) \oplus H_{\text{diff}}^1(\mathbf{vect}(1); D_{\lambda + \frac{1}{2}, \mu + \frac{1}{2}}),$$

and if $\mu - \lambda$ is semi-integer, then

$$H_{\text{diff}}^1(\mathbf{vect}(1); \mathfrak{D}_{\lambda, \mu}) \simeq H_{\text{diff}}^1(\mathbf{vect}(1); \Pi(D_{\lambda + \frac{1}{2}, \mu})) \oplus H_{\text{diff}}^1(\mathbf{vect}(1); \Pi(D_{\lambda, \mu + \frac{1}{2}})).$$

Thus, we deduce $H_{\text{diff}}^1(\mathbf{vect}(1); \mathfrak{D}_{\lambda, \mu})$ from (4.4).

Now, let Υ be a 1-cocycle from $\mathcal{K}(1)$ to $\mathfrak{D}_{\lambda, \lambda}$, that is, Υ is even. The map $\Upsilon|_{\mathbf{vect}(1)}$ is a 1-cocycle of $\mathbf{vect}(1)$. So, up to a coboundary, we have (here $\alpha, \beta \in \mathbb{K}$)

$$\Phi_{\lambda, \lambda} \circ \Upsilon|_{\mathbf{vect}(1)} = \alpha C_{\lambda, \lambda} + \beta C_{\lambda + \frac{1}{2}, \lambda + \frac{1}{2}}. \quad (4.19)$$

By Lemma 4.1, the 1-cocycle Υ is non-trivial if and only if $(\alpha, \beta) \neq (0, 0)$. By Lemma 4.2, we can write

$$\Upsilon(X_{h\theta}) = \sum_{m, k} b_{m, k} h^{(k)} \theta \partial_x^m + \sum_{m, k} \tilde{b}_{m, k} h^{(k)} \partial_\theta \partial_x^m,$$

where the coefficients $b_{m, k}$ and $\tilde{b}_{m, k}$ are constants. Moreover, the map Υ must satisfy the following equations

$$\begin{cases} \Upsilon([X_g, X_{h\theta}]) = \mathfrak{L}_{X_g}^{\lambda, \lambda} \Upsilon(X_{h\theta}) - \mathfrak{L}_{X_{h\theta}}^{\lambda, \lambda} \Upsilon(X_g), \\ \Upsilon([X_{h_1\theta}, X_{h_2\theta}]) = \mathfrak{L}_{X_{h_1\theta}}^{\lambda, \lambda} \Upsilon(X_{h_2\theta}) + \mathfrak{L}_{X_{h_2\theta}}^{\lambda, \lambda} \Upsilon(X_{h_1\theta}). \end{cases} \quad (4.20)$$

We solve Eqs. (4.19) and (4.20) for $\alpha, \beta, b_{k, m}, \tilde{b}_{m, k}$. We prove that $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{\lambda, \lambda})$ is spanned by the non-trivial cocycle $\Upsilon_{\lambda, \lambda}$ corresponding to the cocycle

$$\Phi_{\lambda, \lambda}^{-1} \circ (C_{\lambda, \lambda} + C_{\lambda + \frac{1}{2}, \lambda + \frac{1}{2}})$$

via its restriction to $\mathbf{vect}(1)$, see (4.5).

For the singular cases (4.18), by the same arguments as above, we get:

- (i) $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{0, \frac{1}{2}})$ is spanned by the non-trivial cocycles $\Upsilon_{0, \frac{1}{2}}$ and $\tilde{\Upsilon}_{0, \frac{1}{2}}$ corresponding, respectively, to the cocycles $\Phi_{0, \frac{1}{2}}^{-1} \circ \Pi \circ (-C_{0, 1})$ and $\Phi_{0, \frac{1}{2}}^{-1} \circ \Pi \circ (C_{\frac{1}{2}, \frac{1}{2}} - \tilde{C}_{0, 1})$, via their restrictions to $\mathbf{vect}(1)$.
- (ii) $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{-\frac{1}{2}, 1})$ is spanned by the non-trivial cocycle $\Upsilon_{-\frac{1}{2}, 1}$ corresponding to the cocycle $\Phi_{-\frac{1}{2}, 1}^{-1} \circ \Pi \circ (C_{0, 1} - C_{-\frac{1}{2}, \frac{3}{2}})$ via its restriction to $\mathbf{vect}(1)$.
- (iii) $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{-1, \frac{3}{2}})$ is spanned by the non-trivial cocycle $\Upsilon_{-1, \frac{3}{2}}$ corresponding to the cocycle $\Phi_{-1, \frac{3}{2}}^{-1} \circ \Pi \circ (C_{-\frac{1}{2}, \frac{3}{2}} - 3C_{-1, 2})$ via its restriction to $\mathbf{vect}(1)$.
- (iv) $H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{-\frac{3}{2}, 2}) = H_{\text{diff}}^1(\mathcal{K}(1); \mathfrak{D}_{-2, \frac{5}{2}}) = 0$. □

5. Deformation Theory and Cohomology

Deformation theory of Lie algebra homomorphisms was first considered with only one-parameter of deformation [14, 23, 26]. Recently, deformations of Lie (super)algebras with multi-parameters were intensively studied (see, e.g., [1, 3, 5, 6, 24, 25]). Here we give an outline of this theory.

5.1. Infinitesimal deformations and the first cohomology

Let $\rho_0 : \mathfrak{g} \rightarrow \text{End}(V)$ be an action of a Lie superalgebra \mathfrak{g} on a vector superspace V and let \mathfrak{h} be a subalgebra of \mathfrak{g} . When studying \mathfrak{h} -trivial deformations of the \mathfrak{g} -action ρ_0 , one usually starts with *infinitesimal* deformations:

$$\rho = \rho_0 + t \Upsilon, \quad (5.1)$$

where $\Upsilon : \mathfrak{g} \rightarrow \text{End}(V)$ is a linear map vanishing on \mathfrak{h} and t is a formal parameter with $p(t) = p(\Upsilon)$. The homomorphism condition

$$[\rho(x), \rho(y)] = \rho([x, y]), \quad (5.2)$$

where $x, y \in \mathfrak{g}$, is satisfied in order 1 in t if and only if Υ is a \mathfrak{h} -relative 1-cocycle. That is, the map Υ satisfies

$$(-1)^{p(x)p(\Upsilon)}[\rho_0(x), \Upsilon(y)] - (-1)^{p(y)(p(x)+p(\Upsilon))}[\rho_0(y), \Upsilon(x)] - \Upsilon([x, y]) = 0.$$

Moreover, two \mathfrak{h} -trivial infinitesimal deformations $\rho = \rho_0 + t \Upsilon_1$, and $\rho = \rho_0 + t \Upsilon_2$, are equivalent if and only if $\Upsilon_1 - \Upsilon_2$ is \mathfrak{h} -relative coboundary:

$$(\Upsilon_1 - \Upsilon_2)(x) = (-1)^{p(x)p(A)}[\rho_0(x), A] := \delta A(x),$$

where $A \in \text{End}(V)^{\mathfrak{h}}$ and δ stands for differential of cochains on \mathfrak{g} with values in $\text{End}(V)$ (see, e.g., [15, 23]). So, the space $H^1(\mathfrak{g}, \mathfrak{h}; \text{End}(V))$ determines and classifies infinitesimal deformations up to equivalence. If $\dim H^1(\mathfrak{g}, \mathfrak{h}; \text{End}(V)) = m$, then choose 1-cocycles $\Upsilon_1, \dots, \Upsilon_m$ representing a basis of $H^1(\mathfrak{g}, \mathfrak{h}; \text{End}(V))$ and consider the infinitesimal deformation

$$\rho = \rho_0 + \sum_{i=1}^m t_i \Upsilon_i, \quad (5.3)$$

where t_1, \dots, t_m are independent parameters with $p(t_i) = p(\Upsilon_i)$.

Since we are interested in the $\mathfrak{osp}(1|2)$ -trivial deformations of the $\mathcal{K}(1)$ -action on \mathfrak{S}_{β}^n , we consider the space $H_{\text{diff}}^1(\mathcal{K}(1), \mathfrak{osp}(1|2); \text{End}(\mathfrak{S}_{\beta}^n))$ spanned by the classes $\Upsilon_{\lambda, \lambda + \frac{k}{2}}$, where $k = 3, 4, 5$ and $2(\beta - \lambda) \in \{k, k+1, \dots, 2n\}$ for generic β . Any infinitesimal $\mathfrak{osp}(1|2)$ -trivial deformation of the $\mathcal{K}(1)$ -module \mathfrak{S}_{β}^n is then of the form

$$\tilde{\mathfrak{L}}_{X_F} = \mathfrak{L}_{X_F} + \mathfrak{L}_{X_F}^{(1)}, \quad (5.4)$$

where \mathfrak{L}_{X_F} is the Lie derivative of \mathfrak{S}_{β}^n along the vector field X_F defined by (2.2), and

$$\mathfrak{L}_{X_F}^{(1)} = \sum_{\lambda} \sum_{k=3,4,5} t_{\lambda, \lambda + \frac{k}{2}} \Upsilon_{\lambda, \lambda + \frac{k}{2}}(X_F), \quad (5.5)$$

where the $t_{\lambda, \lambda + \frac{k}{2}}$ are independent parameters with $p(t_{\lambda, \lambda + \frac{k}{2}}) = p(\Upsilon_{\lambda, \lambda + \frac{k}{2}})$ and $2(\beta - \lambda) \in \{k, k+1, \dots, 2n\}$.

5.2. Integrability conditions and deformations over supercommutative algebras

Consider the supercommutative associative superalgebra with unity $\mathbb{C}[[t_1, \dots, t_m]]$ and consider the problem of integrability of infinitesimal deformations. Starting with the infinitesimal deformation (5.3), we look for a formal series

$$\rho = \rho_0 + \sum_{i=1}^m t_i \Upsilon_i + \sum_{i,j} t_i t_j \rho_{ij}^{(2)} + \dots, \quad (5.6)$$

where the higher order terms $\rho_{ij}^{(2)}, \rho_{ijk}^{(3)}, \dots$ are linear maps from \mathfrak{g} to $\text{End}(V)$ with $p(\rho_{ij}^{(2)}) = p(t_i t_j), p(\rho_{ijk}^{(3)}) = p(t_i t_j t_k), \dots$ such that the map

$$\rho : \mathfrak{g} \rightarrow \mathbb{C}[[t_1, \dots, t_m]] \otimes \text{End}(V), \quad (5.7)$$

satisfies the homomorphism condition (5.2).

Quite often the above problem has no solution. Following [14] and [1], we will impose extra algebraic relations on the parameters t_1, \dots, t_m . Let \mathcal{R} be an ideal in $\mathbb{C}[[t_1, \dots, t_m]]$ generated by some set of relations, and we can speak about deformations with base $\mathcal{A} = \mathbb{C}[[t_1, \dots, t_m]]/\mathcal{R}$, (for details, see [14]). The map (5.7) sends \mathfrak{g} to $\mathcal{A} \otimes \text{End}(V)$.

Setting

$$\varphi_t = \rho - \rho_0, \quad \rho^{(1)} = \sum t_i \Upsilon_i, \quad \rho^{(2)} = \sum t_i t_j \rho_{ij}^{(2)}, \dots,$$

we can rewrite the relation (5.2) in the following way:

$$[\varphi_t(x), \rho_0(y)] + [\rho_0(x), \varphi_t(y)] - \varphi_t([x, y]) + \sum_{i,j>0} [\rho^{(i)}(x), \rho^{(j)}(y)] = 0. \quad (5.8)$$

The first three terms are $(\delta\varphi_t)(x, y)$. For arbitrary linear maps $\gamma_1, \gamma_2 : \mathfrak{g} \rightarrow \text{End}(V)$, consider the standard *cup-product*: $\llbracket \gamma_1, \gamma_2 \rrbracket : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \text{End}(V)$ defined by:

$$\llbracket \gamma_1, \gamma_2 \rrbracket(x, y) = (-1)^{p(\gamma_2)(p(\gamma_1)+p(x))} [\gamma_1(x), \gamma_2(y)] + (-1)^{p(\gamma_1)p(x)} [\gamma_2(x), \gamma_1(y)]. \quad (5.9)$$

The relation (5.8) becomes now equivalent to:

$$\delta\varphi_t + \frac{1}{2} \llbracket \varphi_t, \varphi_t \rrbracket = 0, \quad (5.10)$$

Expanding (5.10) in power series in t_1, \dots, t_m , we obtain the following equation for $\rho^{(k)}$:

$$\delta\rho^{(k)} + \frac{1}{2} \sum_{i+j=k} \llbracket \rho^{(i)}, \rho^{(j)} \rrbracket = 0. \quad (5.11)$$

The first non-trivial relation $\delta\rho^{(2)} + \frac{1}{2} \llbracket \rho^{(1)}, \rho^{(1)} \rrbracket = 0$ gives the first obstruction to integration of an infinitesimal deformation. Thus, considering the coefficient of $t_i t_j$, we get

$$\delta\rho_{ij}^{(2)} + \frac{1}{2} \llbracket \Upsilon_i, \Upsilon_j \rrbracket = 0. \quad (5.12)$$

It is easy to check that for any two 1-cocycles γ_1 and $\gamma_2 \in Z^1(\mathfrak{g}, \mathfrak{h}; \text{End}(V))$, the bilinear map $[[\gamma_1, \gamma_2]]$ is a \mathfrak{h} -relative 2-cocycle. The relation (5.12) is precisely the condition for this cocycle to be a coboundary. Moreover, if one of the cocycles γ_1 or γ_2 is a \mathfrak{h} -relative coboundary, then $[[\gamma_1, \gamma_2]]$ is a \mathfrak{h} -relative 2-coboundary. Therefore, we naturally deduce that the operation (5.9) defines a bilinear map:

$$H^1(\mathfrak{g}, \mathfrak{h}; \text{End}(V)) \otimes H^1(\mathfrak{g}, \mathfrak{h}; \text{End}(V)) \rightarrow H^2(\mathfrak{g}, \mathfrak{h}; \text{End}(V)). \quad (5.13)$$

All the obstructions lie in $H^2(\mathfrak{g}, \mathfrak{h}; \text{End}(V))$ and they are in the image of $H^1(\mathfrak{g}, \mathfrak{h}; \text{End}(V))$ under the cup-product.

5.3. Equivalence

Two deformations, ρ and ρ' of a \mathfrak{g} -module V over \mathcal{A} are said to be *equivalent* (see [14]) if there exists an inner automorphism Ψ of the associative superalgebra $\mathcal{A} \otimes \text{End}(V)$ such that

$$\Psi \circ \rho = \rho' \quad \text{and} \quad \Psi(\mathbb{I}) = \mathbb{I},$$

where \mathbb{I} is the unity of the superalgebra $\mathcal{A} \otimes \text{End}(V)$.

The following notion of miniversal deformation is fundamental. It assigns to a \mathfrak{g} -module V a canonical commutative associative algebra \mathcal{A} and a canonical deformation over \mathcal{A} . A deformation (5.6) over \mathcal{A} is said to be *miniversal* if

- (i) for any other deformation ρ' with base (local) \mathcal{A}' , there exists a homomorphism $\psi : \mathcal{A}' \rightarrow \mathcal{A}$ satisfying $\psi(1) = 1$, such that

$$\rho = (\psi \otimes \text{Id}) \circ \rho'.$$

- (ii) under notation of (i), if ρ is infinitesimal, then ψ is unique.

If ρ satisfies only the condition (i), then it is called *versal*. This definition does not depend on the choice 1-cocycles $\Upsilon_1, \dots, \Upsilon_m$ representing a basis of $H^1(\mathfrak{g}, \mathfrak{h}; \text{End}(V))$.

The miniversal deformation corresponds to the smallest ideal \mathcal{R} . We refer to [14] for a construction of miniversal deformations of Lie algebras and to [1] for miniversal deformations of \mathfrak{g} -modules. Superization of these results is immediate: by the Sign Rule.

6. Integrability Conditions

In this section we obtain the integrability conditions for the infinitesimal deformation (5.4).

Proposition 6.1. *The second-order integrability conditions of the infinitesimal deformation (5.4) are the following:*

$$t_{\lambda, \lambda + \frac{5}{2}} t_{\lambda + \frac{5}{2}, \lambda + 5} = 0, \quad \text{where } 2(\beta - \lambda) \in \{10, \dots, 2n\}. \quad (6.1)$$

To prove Proposition 6.1, we need the following lemmas:

Lemma 6.1. *Consider a linear differential operator $b : \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda, \mu}$. If b satisfies*

$$\delta(b)(X, Y) = b(X) = 0 \quad \text{for all } X \in \mathfrak{osp}(1|2),$$

then b is a supertransvectant.

Proof. For all $X, Y \in \mathcal{K}(1)$ we have

$$\delta(b)(X, Y) := (-1)^{p(X)p(b)} \mathfrak{L}_X^{\lambda, \mu}(b(Y)) - (-1)^{p(Y)(p(X)+p(b))} \mathfrak{L}_Y^{\lambda, \mu}(b(X)) - b([X, Y]).$$

Since $\delta(b)(X, Y) = b(X) = 0$ for all $X \in \mathfrak{osp}(1|2)$ we deduce that

$$(-1)^{p(X)p(b)} \mathfrak{L}_X^{\lambda, \mu}(b(Y)) - b([X, Y]) = 0.$$

Thus, the map b is $\mathfrak{osp}(1|2)$ -invariant. \square

Lemma 6.2. *The map $B_{\lambda, \lambda+5} = \llbracket \Upsilon_{\lambda+\frac{5}{2}, \lambda+5}, \Upsilon_{\lambda, \lambda+\frac{5}{2}} \rrbracket$ is a non-trivial $\mathfrak{osp}(1|2)$ -relative 2-cocycle for $\lambda \neq 0, -1, -\frac{7}{2}, -\frac{9}{2}$.*

Proof. First, observe that for $\lambda = -1, -\frac{7}{2}$, the map $B_{\lambda, \lambda+5}$ is not defined (see Theorem 4.1). The map $B_{\lambda, \lambda+5}$ is the cup-product of two $\mathfrak{osp}(1|2)$ -relative 1-cocycles, so, $B_{\lambda, \lambda+5}$ is a $\mathfrak{osp}(1|2)$ -relative 2-cocycle: $B_{\lambda, \lambda+5} \in \mathbb{Z}^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \lambda+5})$. This 2-cocycle is trivial if and only if it is the coboundary of a linear differential operator

$$b_{\lambda, \lambda+5} : \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda, \lambda+5}$$

vanishing on $\mathfrak{osp}(1|2)$. Consider $b_{\lambda, \lambda+5}$ as a bilinear map $\mathfrak{F}_{-1} \otimes \mathfrak{F}_{\lambda} \rightarrow \mathfrak{F}_{\lambda+5}$. So, according to Lemma 6.1 and Theorem 3.1, the operator $b_{\lambda, \lambda+5}$ coincides (up to a scalar factor) with the supertransvectant $\mathfrak{J}_6^{-1, \lambda}$. But, by a direct computation, we have, up to a multiple

$$\begin{aligned} B_{\lambda, \lambda+5}(X_{g_1}, X_{g_2})(F\alpha^\lambda) &= (g_1^{(4)} g_2^{(3)} - g_1^{(3)} g_2^{(4)})(2\lambda f_0 - (2\lambda + 9)f_1\theta)\alpha^{\lambda+5}, \\ \delta(\mathfrak{J}_6^{-1, \lambda})(X_{g_1}, X_{g_2})(F\alpha^\lambda) &= (g_1^{(3)} g_2^{(4)} - g_1^{(4)} g_2^{(3)}) \left(\frac{\lambda(2\lambda + 3)(\lambda^2 + 6\lambda + 8)}{9} f_0 \right. \\ &\quad \left. + \frac{(2\lambda + 1)(\lambda + 3)(4\lambda^2 + 28\lambda + 45)}{36} f_1\theta \right) \alpha^{\lambda+5}, \end{aligned}$$

where $g_1, g_2 \in \mathbb{K}[x]$ and $F = f_0 + f_1\theta \in \mathbb{K}[x, \theta]$. Therefore, the restrictions of the maps $B_{\lambda, \lambda+5}$ and $\delta(\mathfrak{J}_6^{-1, \lambda})$ to $\mathbf{vect}(1) \times \mathbf{vect}(1)$ are linearly dependant if and only if

$$\lambda(\lambda + 1)(2\lambda + 7)(2\lambda + 9)(4\lambda + 9) = 0.$$

Thus, the maps $B_{\lambda, \lambda+5}$ and $\delta(\mathfrak{J}_6^{-1, \lambda})$ are linearly independent for $\lambda \neq 0, -1, -\frac{7}{2}, -\frac{9}{2}, -\frac{9}{4}$. Besides, we check that the maps $B_{-\frac{9}{4}, \frac{11}{4}}$ and $\delta(\mathfrak{J}_6^{-1, -\frac{9}{4}})$ are also linearly independent although their restrictions to $\mathbf{vect}(1) \times \mathbf{vect}(1)$ are linearly dependant. Finally, for $\lambda = 0, -\frac{9}{2}$, we check that $B_{\lambda, \lambda+5}$ coincides (up to a scalar factor) with $\delta(\mathfrak{J}_6^{-1, \lambda})$. This completes the proof. \square

Remark 6.1. The map $B_{\lambda, \lambda+5} : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda, \lambda+5}$ is a non-trivial 2-cocycle, so, $H_{\text{diff}}^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \lambda+5}) \neq 0$ while $H_{\text{diff}}^2(\mathbf{vect}(1), \mathfrak{sl}(2); \mathfrak{D}_{\lambda, \lambda+5}) = 0$ for generic l (see [7]). Hence, for the second cohomology, the analog of Lemma 4.1 is not true.

Proof of Proposition 6.1. Assume that the infinitesimal deformation (5.4) can be integrated to a formal deformation:

$$\tilde{\mathfrak{L}}_{X_F} = \mathfrak{L}_{X_F} + \mathfrak{L}_{X_F}^{(1)} + \mathfrak{L}_{X_F}^{(2)} + \cdots.$$

The homomorphism condition gives, for the term $\mathfrak{L}_{\lambda,\mu,\lambda',\mu'}^{(2)}$ in $t_{\lambda,\mu}t_{\lambda',\mu'}$, the following equation

$$\delta(\mathfrak{L}_{\lambda,\mu,\lambda',\mu'}^{(2)}) = -[\Upsilon_{\lambda,\mu}, \Upsilon_{\lambda',\mu'}]. \quad (6.2)$$

For arbitrary λ , the right-hand side of (6.2) yields the following 2-cocycles:

$$\begin{aligned} B_{\lambda,\lambda+3} &= [\Upsilon_{\lambda+\frac{3}{2},\lambda+3}, \Upsilon_{\lambda,\lambda+\frac{3}{2}}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda,\lambda+3}, \\ B_{\lambda,\lambda+\frac{7}{2}} &= [\Upsilon_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}}, \Upsilon_{\lambda,\lambda+\frac{3}{2}}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda,\lambda+\frac{7}{2}}, \\ \tilde{B}_{\lambda,\lambda+\frac{7}{2}} &= [\Upsilon_{\lambda+2,\lambda+\frac{7}{2}}, \Upsilon_{\lambda,\lambda+2}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda,\lambda+\frac{7}{2}}, \\ B_{\lambda,\lambda+4} &= [\Upsilon_{\lambda+\frac{3}{2},\lambda+4}, \Upsilon_{\lambda,\lambda+\frac{3}{2}}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda,\lambda+4}, \\ \tilde{B}_{\lambda,\lambda+4} &= [\Upsilon_{\lambda+\frac{5}{2},\lambda+4}, \Upsilon_{\lambda,\lambda+\frac{5}{2}}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda,\lambda+4}, \\ \bar{B}_{\lambda,\lambda+4} &= [\Upsilon_{\lambda+2,\lambda+4}, \Upsilon_{\lambda,\lambda+2}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda,\lambda+4}, \\ B_{\lambda,\lambda+\frac{9}{2}} &= [\Upsilon_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}}, \Upsilon_{\lambda,\lambda+\frac{5}{2}}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda,\lambda+\frac{9}{2}}, \\ \tilde{B}_{\lambda,\lambda+\frac{9}{2}} &= [\Upsilon_{\lambda+2,\lambda+\frac{9}{2}}, \Upsilon_{\lambda,\lambda+2}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda,\lambda+\frac{9}{2}}, \\ B_{\lambda,\lambda+5} &= [\Upsilon_{\lambda+\frac{5}{2},\lambda+5}, \Upsilon_{\lambda,\lambda+\frac{5}{2}}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda,\lambda+5}. \end{aligned} \quad (6.3)$$

The necessary integrability conditions for the second-order terms $\mathfrak{L}^{(2)}$ are that each 2-cocycle $B_{\lambda,\lambda+k}$, where $2k = 6, 7, 8, 9, 10$, must be a coboundary of a linear differential operator $b_{\lambda,\lambda+k} : \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda,\lambda+k}$, vanishing on $\mathfrak{osp}(1|2)$. More precisely, as in the proof of Lemma 6.2, the operator $b_{\lambda,\lambda+k}$ coincides (up to a scalar factor) with the supertransvectant $\mathfrak{J}_{k+1}^{-1,\lambda}$. Clearly,

$$\tilde{B}_{\lambda,\lambda+4} = B_{\lambda,\lambda+4} = 3\bar{B}_{\lambda,\lambda+4}, \quad B_{\lambda,\lambda+\frac{7}{2}} = -\tilde{B}_{\lambda,\lambda+\frac{7}{2}}, \quad B_{\lambda,\lambda+\frac{9}{2}} = -\tilde{B}_{\lambda,\lambda+\frac{9}{2}}$$

and, by a direct computation, we check that

$$\begin{aligned} B_{\lambda,\lambda+3}(X_{G_1}, X_{G_2})(F\alpha^\lambda) &= (-2(-1)^{p(G_1)}\bar{\eta}(G_1'')\bar{\eta}(G_2'')F)\alpha^{\lambda+3}, \\ B_{\lambda,\lambda+\frac{7}{2}}(X_{G_1}, X_{G_2})(F\alpha^\lambda) &= \left(\frac{2\lambda}{3}((-1)^{p(G_1)}G_1^{(3)}\bar{\eta}(G_2'') - \bar{\eta}(G_1'')G_2^{(3)})F \right. \\ &\quad \left. + 2(-1)^{p(G_2)}\bar{\eta}(G_1'')\bar{\eta}(G_2'')\bar{\eta}(F) \right) \alpha^{\lambda+\frac{7}{2}}, \end{aligned}$$

$$\begin{aligned}
B_{\lambda, \lambda+4}(X_{G_1}, X_{G_2})(F\alpha^\lambda) &= (-2\lambda(-1)^{p(G_1)}(\bar{\eta}(G_1^{(3)})\bar{\eta}(G_2'') + \bar{\eta}(G_1'')\bar{\eta}(G_2^{(3)}))F \\
&\quad + (-1)^{p(G_2)}((-1)^{p(G_1)}\bar{\eta}(G_1'')G_2^{(3)} - G_1^{(3)}\bar{\eta}(G_2''))\bar{\eta}(F) \\
&\quad + 6(-1)^{p(G_1)}\bar{\eta}(G_1'')\bar{\eta}(G_2'')F')\alpha^{\lambda+4}, \\
B_{\lambda, \lambda+\frac{9}{2}}(X_{G_1}, X_{G_2})(F\alpha^\lambda) &= \left(\frac{2\lambda}{3}(2\lambda+5)((-1)^{p(G_1)}G_1^{(3)}\bar{\eta}(G_2^{(3)}) - \bar{\eta}(G_1^{(3)})G_2^{(3)})F \right. \\
&\quad + 2\lambda(\bar{\eta}(G_1'')G_2^{(4)} - (-1)^{p(G_1)p(G_2)}\bar{\eta}(G_2'')G_1^{(4)})F \\
&\quad + (2\lambda+1)(-1)^{p(G_2)}(\bar{\eta}(G_1'')\bar{\eta}(G_2^{(3)}) + \bar{\eta}(G_1^{(3)})\bar{\eta}(G_2''))\bar{\eta}(F) \\
&\quad + (2\lambda+1)(\bar{\eta}(G_1'')G_2^{(3)} - (-1)^{p(G_1)}G_1^{(3)}\bar{\eta}(G_2''))F' \\
&\quad \left. - 6(-1)^{p(G_2)}\bar{\eta}(G_1'')\bar{\eta}(G_2'')\bar{\eta}(F') \right) \alpha^{\lambda+\frac{9}{2}},
\end{aligned}$$

where $G_1, G_2, F \in \mathbb{K}[x, \theta]$. Besides, we can see that

$$\begin{aligned}
\zeta_\lambda B_{\lambda, \lambda+3} &= \delta(\mathfrak{J}_4^{-1, \lambda}), \quad \text{where } \zeta_\lambda = \frac{\lambda(2\lambda+5)}{4} \binom{2\lambda+3}{2}, \\
\alpha_\lambda B_{\lambda, \lambda+\frac{7}{2}} &= \delta(\mathfrak{J}_{\frac{9}{2}}^{-1, \lambda}), \quad \text{where } \alpha_\lambda = \frac{6\lambda+9}{4} \binom{2\lambda+4}{3}, \\
\beta_\lambda B_{\lambda, \lambda+4} &= \delta(\mathfrak{J}_5^{-1, \lambda}), \quad \text{where } \beta_\lambda = \frac{2\lambda^2+7\lambda+2}{6} \binom{2\lambda+4}{2}, \\
\gamma_\lambda B_{\lambda, \lambda+\frac{9}{2}} &= \delta(\mathfrak{J}_{\frac{11}{2}}^{-1, \lambda}), \quad \text{where } \gamma_\lambda = \frac{3\lambda+6}{2} \binom{2\lambda+5}{3}.
\end{aligned}$$

Now, by Lemma 6.2, $B_{\lambda, \lambda+5}$ is a non-trivial $\mathfrak{osp}(1|2)$ -relative 2-cocycle, so, its coefficient must vanish, that is, we get the first set of necessary integrability conditions:

$$t_{\lambda, \lambda+\frac{5}{2}} t_{\lambda+\frac{5}{2}, \lambda+5} = 0, \quad \text{where } 2(\beta - \lambda) \in \{10, \dots, 2n\}. \quad (6.4)$$

Equations (6.4) are the unique integrability conditions for the 2nd order term $\mathfrak{L}^{(2)}$. Under these conditions, the second-order term $\mathfrak{L}^{(2)}$ can be given by

$$\begin{aligned}
\mathfrak{L}^{(2)} &= - \sum_{\lambda} \zeta_\lambda^{-1} t_{\lambda+\frac{3}{2}, \lambda+3} t_{\lambda, \lambda+\frac{3}{2}} \mathfrak{J}_4^{-1, \lambda} \\
&\quad - \sum_{\lambda} \alpha_\lambda^{-1} (t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} t_{\lambda, \lambda+\frac{3}{2}} - t_{\lambda+2, \lambda+\frac{7}{2}} t_{\lambda, \lambda+2}) \mathfrak{J}_{\frac{9}{2}}^{-1, \lambda} \\
&\quad - \sum_{\lambda} \beta_\lambda^{-1} \left(t_{\lambda+\frac{3}{2}, \lambda+4} t_{\lambda, \lambda+\frac{3}{2}} + t_{\lambda+\frac{5}{2}, \lambda+4} t_{\lambda, \lambda+\frac{5}{2}} + \frac{1}{3} t_{\lambda+2, \lambda+4} t_{\lambda, \lambda+2} \right) \mathfrak{J}_5^{-1, \lambda} \\
&\quad - \sum_{\lambda} \gamma_\lambda^{-1} (t_{\lambda+\frac{5}{2}, \lambda+\frac{9}{2}} t_{\lambda, \lambda+\frac{5}{2}} - t_{\lambda+2, \lambda+\frac{9}{2}} t_{\lambda, \lambda+2}) \mathfrak{J}_{\frac{11}{2}}^{-1, \lambda}. \quad \square
\end{aligned}$$

To compute the third term $\mathfrak{L}^{(3)}$, we need the following two lemmas which we can check by a direct computation with the help of *Maple*.

Lemma 6.3.

$$\begin{aligned}
(1) \quad & \xi_{\lambda}^{-1} \delta(\mathfrak{J}_{\frac{11}{2}}^{-1, \lambda}) = (\zeta_{\lambda}^{-1} [\Upsilon_{\lambda+3, \lambda+\frac{9}{2}}, \mathfrak{J}_4^{-1, \lambda}] + \zeta_{\lambda+\frac{3}{2}}^{-1} [\mathfrak{J}_4^{-1, \lambda+\frac{3}{2}}, \Upsilon_{\lambda, \lambda+\frac{3}{2}}]), \\
(2) \quad & \alpha_{\lambda+\frac{3}{2}}^{-1} [\mathfrak{J}_{\frac{9}{2}}^{-1, \lambda+\frac{3}{2}}, \Upsilon_{\lambda, \lambda+\frac{3}{2}}] = \epsilon_{1, \lambda} \alpha_{\lambda}^{-1} [\Upsilon_{\lambda+\frac{7}{2}, \lambda+5}, \mathfrak{J}_{\frac{9}{2}}^{-1, \lambda}] + \epsilon_{2, \lambda} \zeta_{\lambda}^{-1} [\Upsilon_{\lambda+3, \lambda+5}, \mathfrak{J}_4^{-1, \lambda}] \\
& \quad + \epsilon_{3, \lambda} \delta(\mathfrak{J}_6^{-1, \lambda}), \\
(3) \quad & \beta_{\lambda+\frac{3}{2}}^{-1} [\mathfrak{J}_5^{-1, \lambda+\frac{3}{2}}, \Upsilon_{\lambda, \lambda+\frac{3}{2}}] = \epsilon_{4, \lambda} \beta_{\lambda}^{-1} [\Upsilon_{\lambda+4, \lambda+\frac{11}{2}}, \mathfrak{J}_5^{-1, \lambda}] + \epsilon_{5, \lambda} \alpha_{\lambda}^{-1} [\Upsilon_{\lambda+\frac{7}{2}, \lambda+\frac{11}{2}}, \mathfrak{J}_{\frac{9}{2}}^{-1, \lambda}] \\
& \quad + \epsilon_{6, \lambda} \alpha_{\lambda+2}^{-1} [\mathfrak{J}_{\frac{9}{2}}^{-1, \lambda+2}, \Upsilon_{\lambda, \lambda+2}], \\
(4) \quad & \alpha_{\lambda}^{-1} [\Upsilon_{\lambda+\frac{7}{2}, \lambda+6}, \mathfrak{J}_{\frac{9}{2}}^{-1, \lambda}] = \epsilon_{7, \lambda} \beta_{\lambda+2}^{-1} [\mathfrak{J}_5^{-1, \lambda+2}, \Upsilon_{\lambda, \lambda+2}] + \epsilon_{8, \lambda} \alpha_{\lambda+\frac{5}{2}}^{-1} [\mathfrak{J}_{\frac{9}{2}}^{-1, \lambda+\frac{5}{2}}, \Upsilon_{\lambda, \lambda+\frac{5}{2}}] \\
& \quad + \epsilon_{9, \lambda} \delta(\mathfrak{J}_7^{-1, \lambda}), \\
(5) \quad & \gamma_{\lambda+2}^{-1} [\mathfrak{J}_{\frac{11}{2}}^{-1, \lambda+2}, \Upsilon_{\lambda, \lambda+2}] = \epsilon_{10, \lambda} \beta_{\lambda}^{-1} [\Upsilon_{\lambda+4, \lambda+\frac{13}{2}}, \mathfrak{J}_5^{-1, \lambda}] + \epsilon_{11, \lambda} \beta_{\lambda+\frac{5}{2}}^{-1} [\mathfrak{J}_5^{-1, \lambda+\frac{5}{2}}, \Upsilon_{\lambda, \lambda+\frac{5}{2}}] \\
& \quad + \epsilon_{12, \lambda} \gamma_{\lambda}^{-1} [\Upsilon_{\lambda+\frac{9}{2}, \lambda+\frac{13}{2}}, \mathfrak{J}_{\frac{11}{2}}^{-1, \lambda}],
\end{aligned}$$

where

$$\begin{aligned}
\epsilon_{1, \lambda} &= \frac{(2\lambda + 11)(2\lambda + 9)(\lambda + 2)}{2(\lambda + 3)(2\lambda^2 + 3\lambda - 17)}, \\
\epsilon_{2, \lambda} &= \frac{15(2\lambda + 5)(\lambda + 4)}{2(\lambda + 3)(2\lambda^2 + 3\lambda - 17)}, \\
\epsilon_{3, \lambda} &= \frac{48}{\lambda(\lambda + 3)(2\lambda + 3)(2\lambda + 5)(2\lambda^2 + 3\lambda - 17)}, \\
\epsilon_{4, \lambda} &= -\frac{(2\lambda + 9)(2\lambda + 3)(2\lambda^2 + 7\lambda + 2)}{(2\lambda + 7)(2\lambda + 1)(2\lambda^2 + 13\lambda + 17)}, \\
\epsilon_{5, \lambda} &= -\frac{3(2\lambda + 9)(2\lambda + 3)^2}{2(2\lambda + 7)(2\lambda + 1)(2\lambda^2 + 13\lambda + 17)}, \\
\epsilon_{6, \lambda} &= -\frac{3(2\lambda + 7)}{2(2\lambda^2 + 13\lambda + 17)}, \\
\epsilon_{7, \lambda} &= -\frac{5(6\lambda^2 + 33\lambda + 17)(2\lambda^2 + 15\lambda + 24)}{2(\lambda + 5)(\lambda + 2)(2\lambda - 3)(2\lambda^2 + 13\lambda + 13)}, \\
\epsilon_{8, \lambda} &= \frac{(2\lambda + 9)(2\lambda + 5)(\lambda + 7)(2\lambda^2 + 11\lambda + 4)}{2(\lambda + 5)(2\lambda - 3)(\lambda + 2)(2\lambda^2 + 13\lambda + 13)}, \\
\epsilon_{9, \lambda} &= \frac{60}{(2\lambda + 3)(2\lambda - 3)(\lambda + 3)(\lambda + 4)(2\lambda^2 + 13\lambda + 13)},
\end{aligned}$$

$$\begin{aligned}
\epsilon_{10,\lambda} &= -\frac{(\lambda+5)(\lambda+2)(2\lambda^2+7\lambda+2)}{9(\lambda+4)^2(\lambda+1)}, \\
\epsilon_{11,\lambda} &= -\frac{(2\lambda^2+17\lambda+32)}{9(\lambda+4)}, \\
\epsilon_{12,\lambda} &= -\frac{(\lambda+2)^2(\lambda+5)}{(\lambda+4)^2(\lambda+1)}, \\
\xi_\lambda &= \frac{3}{16}\lambda(\lambda+4)(2\lambda+3)(2\lambda+5) \binom{2\lambda+5}{3}.
\end{aligned}$$

Lemma 6.4. *Each of the following systems is linearly independent*

- (1) $([\Upsilon_{\lambda+\frac{7}{2},\lambda+5}, \mathfrak{J}_{\frac{9}{2}}^{-1,\lambda}], [\Upsilon_{\lambda+3,\lambda+5}, \mathfrak{J}_4^{-1,\lambda}], [\mathfrak{J}_4^{-1,\lambda+2}, \Upsilon_{\lambda,\lambda+2}], \delta(\mathfrak{J}_6^{-1,\lambda})),$
- (2) $([\Upsilon_{\lambda+4,\lambda+\frac{11}{2}}, \mathfrak{J}_5^{-1,\lambda}], [\Upsilon_{\lambda+\frac{7}{2},\lambda+\frac{11}{2}}, \mathfrak{J}_{\frac{9}{2}}^{-1,\lambda}], [\mathfrak{J}_{\frac{9}{2}}^{-1,\lambda+2}, \Upsilon_{\lambda,\lambda+2}], [\Upsilon_{\lambda+3,\lambda+\frac{11}{2}}, \mathfrak{J}_4^{-1,\lambda}], [\mathfrak{J}_4^{-1,\lambda+\frac{5}{2}}, \Upsilon_{\lambda,\lambda+\frac{5}{2}}], \delta(\mathfrak{J}_{\frac{13}{2}}^{-1,\lambda})),$
- (3) $([\Upsilon_{\lambda+\frac{9}{2},\lambda+6}, \mathfrak{J}_{\frac{11}{2}}^{-1,\lambda}], [\mathfrak{J}_{\frac{11}{2}}^{-1,\lambda+\frac{3}{2}}, \Upsilon_{\lambda,\lambda+\frac{3}{2}}], [\Upsilon_{\lambda+4,\lambda+6}, \mathfrak{J}_5^{-1,\lambda}], [\mathfrak{J}_5^{-1,\lambda+2}, \Upsilon_{\lambda,\lambda+2}], [\mathfrak{J}_{\frac{9}{2}}^{-1,\lambda+\frac{5}{2}}, \Upsilon_{\lambda,\lambda+\frac{5}{2}}], \delta(\mathfrak{J}_7^{-1,\lambda})),$
- (4) $([\Upsilon_{\lambda+4,\lambda+\frac{13}{2}}, \mathfrak{J}_5^{-1,\lambda}], [\mathfrak{J}_5^{-1,\lambda+\frac{5}{2}}, \Upsilon_{\lambda,\lambda+\frac{5}{2}}], [\Upsilon_{\lambda+\frac{9}{2},\lambda+\frac{13}{2}}, \mathfrak{J}_{\frac{11}{2}}^{-1,\lambda}], \delta(\mathfrak{J}_{\frac{15}{2}}^{-1,\lambda})),$
- (5) $([\Upsilon_{\lambda+\frac{9}{2},\lambda+7}, \mathfrak{J}_{\frac{11}{2}}^{-1,\lambda}], [\mathfrak{J}_{\frac{11}{2}}^{-1,\lambda+\frac{5}{2}}, \Upsilon_{\lambda,\lambda+\frac{5}{2}}], \delta(\mathfrak{J}_8^{-1,\lambda})).$

Now, we are in position to exhibit the 3rd order integrability conditions.

Proposition 6.2. *The 3rd order integrability conditions of the infinitesimal deformation (5.4) are the following:*

(a) For $2(\beta - \lambda) \in \{10, \dots, 2n\}$:

$$\begin{aligned}
t_{\lambda,\lambda+\frac{3}{2}}(\epsilon_{1,\lambda}t_{\lambda+3,\lambda+5}t_{\lambda+\frac{3}{2},\lambda+3} + (1 - \epsilon_{1,\lambda})t_{\lambda+\frac{7}{2},\lambda+5}t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}}) &= 0, \\
t_{\lambda,\lambda+\frac{3}{2}}(\epsilon_{2,\lambda}t_{\lambda+\frac{7}{2},\lambda+5}t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}} - (1 + \epsilon_{2,\lambda})t_{\lambda+3,\lambda+5}t_{\lambda+\frac{3}{2},\lambda+3}) &= 0, \\
t_{\lambda+\frac{7}{2},\lambda+5}t_{\lambda+2,\lambda+\frac{7}{2}}t_{\lambda,\lambda+2} &= 0.
\end{aligned}$$

(b) For $2(\beta - \lambda) \in \{11, \dots, 2n\}$:

$$\begin{aligned}
t_{\lambda+4,\lambda+\frac{11}{2}}t_{\lambda+\frac{5}{2},\lambda+4}t_{\lambda,\lambda+\frac{5}{2}} &= t_{\lambda,\lambda+\frac{3}{2}}t_{\lambda+3,\lambda+\frac{11}{2}}t_{\lambda+\frac{3}{2},\lambda+3} = 0, \\
t_{\lambda+4,\lambda+\frac{11}{2}}(3(1 + \epsilon_{4,\lambda})t_{\lambda+\frac{3}{2},\lambda+4}t_{\lambda,\lambda+\frac{3}{2}} + t_{\lambda+2,\lambda+4}t_{\lambda,\lambda+2}) \\
&\quad + \epsilon_{4,\lambda}t_{\lambda,\lambda+\frac{3}{2}}t_{\lambda+\frac{7}{2},\lambda+\frac{11}{2}}t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}} = 0, \\
t_{\lambda+\frac{7}{2},\lambda+\frac{11}{2}}\left(\left(1 + \frac{\epsilon_{5,\lambda}}{3}\right)t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}}t_{\lambda,\lambda+\frac{3}{2}} - t_{\lambda+2,\lambda+\frac{7}{2}}t_{\lambda,\lambda+2}\right) \\
&\quad + \epsilon_{5,\lambda}t_{\lambda+4,\lambda+\frac{11}{2}}t_{\lambda+\frac{3}{2},\lambda+4}t_{\lambda,\lambda+\frac{3}{2}} = 0,
\end{aligned}$$

$$\begin{aligned} & \epsilon_{6,\lambda} t_{\lambda,\lambda+\frac{3}{2}} \left(t_{\lambda+4,\lambda+\frac{11}{2}} t_{\lambda+\frac{3}{2},\lambda+4} + \frac{1}{3} t_{\lambda+\frac{7}{2},\lambda+\frac{11}{2}} t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}} \right) \\ & + t_{\lambda,\lambda+2} (t_{\lambda+\frac{7}{2},\lambda+\frac{11}{2}} t_{\lambda+2,\lambda+\frac{7}{2}} - t_{\lambda+4,\lambda+\frac{11}{2}} t_{\lambda+2,\lambda+4}) = 0. \end{aligned}$$

(c) For $2(\beta - \lambda) \in \{12, \dots, 2n\}$:

$$\begin{aligned} & t_{\lambda+\frac{9}{2},\lambda+6} (t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}} t_{\lambda,\lambda+\frac{5}{2}} - t_{\lambda+2,\lambda+\frac{9}{2}} t_{\lambda,\lambda+2}) = 0, \\ & t_{\lambda,\lambda+\frac{3}{2}} (t_{\lambda+4,\lambda+6} t_{\lambda+\frac{3}{2},\lambda+4} - t_{\lambda+\frac{7}{2},\lambda+6} t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}}) = 0, \\ & t_{\lambda+4,\lambda+6} (3 t_{\lambda+\frac{3}{2},\lambda+4} t_{\lambda,\lambda+\frac{3}{2}} + 3 t_{\lambda+\frac{5}{2},\lambda+4} t_{\lambda,\lambda+\frac{5}{2}} + t_{\lambda+2,\lambda+4} t_{\lambda,\lambda+2}) = 0, \\ & t_{\lambda,\lambda+2} \left((1 - \epsilon_{7,\lambda}) t_{\lambda+\frac{7}{2},\lambda+6} t_{\lambda+2,\lambda+\frac{7}{2}} + t_{\lambda+\frac{9}{2},\lambda+6} t_{\lambda+2,\lambda+\frac{9}{2}} + \frac{1}{3} t_{\lambda+4,\lambda+6} t_{\lambda+2,\lambda+4} \right) \\ & + \epsilon_{7,\lambda} t_{\lambda+\frac{7}{2},\lambda+6} t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}} t_{\lambda,\lambda+\frac{3}{2}} = 0, \\ & \epsilon_{8,\lambda} t_{\lambda+\frac{7}{2},\lambda+6} (t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}} t_{\lambda,\lambda+\frac{3}{2}} - t_{\lambda+2,\lambda+\frac{7}{2}} t_{\lambda,\lambda+2}) \\ & + (t_{\lambda+4,\lambda+6} t_{\lambda+\frac{5}{2},\lambda+4} - t_{\lambda+\frac{9}{2},\lambda+6} t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}}) t_{\lambda,\lambda+\frac{5}{2}} = 0. \end{aligned}$$

(d) For $2(\beta - \lambda) \in \{13, \dots, 2n\}$:

$$\begin{aligned} & t_{\lambda+4,\lambda+\frac{13}{2}} \left(t_{\lambda+\frac{3}{2},\lambda+4} t_{\lambda,\lambda+\frac{3}{2}} + t_{\lambda+\frac{5}{2},\lambda+4} t_{\lambda,\lambda+\frac{5}{2}} + \left(\frac{1}{3} - \epsilon_{10,\lambda} \right) t_{\lambda+2,\lambda+4} t_{\lambda,\lambda+2} \right) \\ & + \epsilon_{10,\lambda} t_{\lambda,\lambda+2} t_{\lambda+\frac{9}{2},\lambda+\frac{13}{2}} t_{\lambda+2,\lambda+\frac{9}{2}} = 0, \\ & t_{\lambda,\lambda+\frac{5}{2}} \left(t_{\lambda+4,\lambda+\frac{13}{2}} t_{\lambda+\frac{5}{2},\lambda+4} + \frac{1}{3} t_{\lambda+\frac{9}{2},\lambda+\frac{13}{2}} t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}} \right) \\ & + \epsilon_{11,\lambda} t_{\lambda,\lambda+2} (t_{\lambda+\frac{9}{2},\lambda+\frac{13}{2}} t_{\lambda+2,\lambda+\frac{9}{2}} - t_{\lambda+4,\lambda+\frac{13}{2}} t_{\lambda+2,\lambda+4}) = 0, \\ & t_{\lambda+\frac{9}{2},\lambda+\frac{13}{2}} (t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}} t_{\lambda,\lambda+\frac{5}{2}} + (\epsilon_{12,\lambda} - 1) t_{\lambda+2,\lambda+\frac{9}{2}} t_{\lambda,\lambda+2}) \\ & - \epsilon_{12,\lambda} t_{\lambda,\lambda+2} t_{\lambda+4,\lambda+\frac{13}{2}} t_{\lambda+2,\lambda+4} = 0. \end{aligned}$$

(e) For $2(\beta - \lambda) \in \{14, \dots, 2n\}$:

$$t_{\lambda+\frac{9}{2},\lambda+7} t_{\lambda+2,\lambda+\frac{9}{2}} t_{\lambda,\lambda+2} = t_{\lambda,\lambda+\frac{5}{2}} t_{\lambda+\frac{9}{2},\lambda+7} t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}} = 0.$$

Proof. Considering again the homomorphism condition, we compute the 3rd order term $\mathfrak{L}^{(3)}$ which is a solution of the Maurer–Cartan equation:

$$\delta(\mathfrak{L}^{(3)}) = -\frac{1}{2}(\llbracket \mathfrak{L}^{(1)}, \mathfrak{L}^{(2)} \rrbracket + \llbracket \mathfrak{L}^{(2)}, \mathfrak{L}^{(1)} \rrbracket). \quad (6.5)$$

The right-hand side of (6.5) together with Eq. (6.1) yield the following maps:

$$\begin{aligned} \Omega_{\lambda,\lambda+\frac{9}{2}} &= \varphi_1(t) \llbracket \Upsilon_{\lambda+3,\lambda+\frac{9}{2}}, \mathfrak{J}_4^{-1,\lambda} \rrbracket + \psi_1(t) \llbracket \mathfrak{J}_4^{-1,\lambda+\frac{3}{2}}, \Upsilon_{\lambda,\lambda+\frac{3}{2}} \rrbracket : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda,\lambda+\frac{9}{2}}, \\ \Omega_{\lambda,\lambda+5} &= \varphi_2(t) \llbracket \Upsilon_{\lambda+\frac{7}{2},\lambda+5}, \mathfrak{J}_{\frac{9}{2}}^{-1,\lambda} \rrbracket + \psi_2(t) \llbracket \mathfrak{J}_{\frac{9}{2}}^{-1,\lambda+\frac{3}{2}}, \Upsilon_{\lambda,\lambda+\frac{3}{2}} \rrbracket : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda,\lambda+5}, \end{aligned}$$

$$\begin{aligned}
\tilde{\Omega}_{\lambda, \lambda+5} &= \tilde{\varphi}_2(t) [\Upsilon_{\lambda+3, \lambda+5}, \mathfrak{J}_4^{-1, \lambda}] + \tilde{\psi}_2(t) [\mathfrak{J}_4^{-1, \lambda+2}, \Upsilon_{\lambda, \lambda+2}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda, \lambda+5}, \\
\Omega_{\lambda, \lambda+\frac{11}{2}} &= \varphi_3(t) [\Upsilon_{\lambda+4, \lambda+\frac{11}{2}}, \mathfrak{J}_5^{-1, \lambda}] + \psi_3(t) [\mathfrak{J}_5^{-1, \lambda+\frac{3}{2}}, \Upsilon_{\lambda, \lambda+\frac{3}{2}}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda, \lambda+\frac{11}{2}}, \\
\tilde{\Omega}_{\lambda, \lambda+\frac{11}{2}} &= \tilde{\varphi}_3(t) [\Upsilon_{\lambda+\frac{7}{2}, \lambda+\frac{11}{2}}, \mathfrak{J}_5^{-1, \lambda}] + \tilde{\psi}_3(t) [\mathfrak{J}_5^{-1, \lambda+2}, \Upsilon_{\lambda, \lambda+2}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda, \lambda+\frac{11}{2}}, \\
\overline{\Omega}_{\lambda, \lambda+\frac{11}{2}} &= \overline{\varphi}_3(t) [\Upsilon_{\lambda+3, \lambda+\frac{11}{2}}, \mathfrak{J}_4^{-1, \lambda}] + \overline{\psi}_3(t) [\mathfrak{J}_4^{-1, \lambda+\frac{5}{2}}, \Upsilon_{\lambda, \lambda+\frac{5}{2}}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda, \lambda+\frac{11}{2}}, \\
\Omega_{\lambda, \lambda+6} &= \varphi_4(t) [\Upsilon_{\lambda+\frac{9}{2}, \lambda+6}, \mathfrak{J}_{\frac{11}{2}}^{-1, \lambda}] + \psi_4(t) [\mathfrak{J}_{\frac{11}{2}}^{-1, \lambda+\frac{3}{2}}, \Upsilon_{\lambda, \lambda+\frac{3}{2}}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda, \lambda+6}, \\
\tilde{\Omega}_{\lambda, \lambda+6} &= \tilde{\varphi}_4(t) [\Upsilon_{\lambda+4, \lambda+6}, \mathfrak{J}_5^{-1, \lambda}] + \tilde{\psi}_4(t) [\mathfrak{J}_5^{-1, \lambda+2}, \Upsilon_{\lambda, \lambda+2}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda, \lambda+6}, \\
\overline{\Omega}_{\lambda, \lambda+6} &= \overline{\varphi}_4(t) [\Upsilon_{\lambda+\frac{7}{2}, \lambda+6}, \mathfrak{J}_{\frac{9}{2}}^{-1, \lambda}] + \overline{\psi}_4(t) [\mathfrak{J}_{\frac{9}{2}}^{-1, \lambda+\frac{5}{2}}, \Upsilon_{\lambda, \lambda+\frac{5}{2}}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda, \lambda+6}, \\
\Omega_{\lambda, \lambda+\frac{13}{2}} &= \varphi_5(t) [\Upsilon_{\lambda+4, \lambda+\frac{13}{2}}, \mathfrak{J}_5^{-1, \lambda}] + \psi_5(t) [\mathfrak{J}_5^{-1, \lambda+\frac{5}{2}}, \Upsilon_{\lambda, \lambda+\frac{5}{2}}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda, \lambda+\frac{13}{2}}, \\
\tilde{\Omega}_{\lambda, \lambda+\frac{13}{2}} &= \tilde{\varphi}_5(t) [\Upsilon_{\lambda+\frac{9}{2}, \lambda+\frac{13}{2}}, \mathfrak{J}_{\frac{11}{2}}^{-1, \lambda}] + \tilde{\psi}_5(t) [\mathfrak{J}_{\frac{11}{2}}^{-1, \lambda+2}, \Upsilon_{\lambda, \lambda+2}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda, \lambda+\frac{13}{2}}, \\
\Omega_{\lambda, \lambda+7} &= \varphi_6(t) [\Upsilon_{\lambda+\frac{9}{2}, \lambda+7}, \mathfrak{J}_{\frac{11}{2}}^{-1, \lambda}] + \psi_6(t) [\mathfrak{J}_{\frac{11}{2}}^{-1, \lambda+\frac{5}{2}}, \Upsilon_{\lambda, \lambda+\frac{5}{2}}] : \mathcal{K}(1) \otimes \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda, \lambda+7},
\end{aligned}$$

where

$$\begin{aligned}
\varphi_1(t) &= \zeta_{\lambda}^{-1} t_{\lambda+3, \lambda+\frac{9}{2}} t_{\lambda+\frac{3}{2}, \lambda+3} t_{\lambda, \lambda+\frac{3}{2}}, \\
\psi_1(t) &= \zeta_{\lambda+\frac{3}{2}}^{-1} t_{\lambda+3, \lambda+\frac{9}{2}} t_{\lambda+\frac{3}{2}, \lambda+3} t_{\lambda, \lambda+\frac{3}{2}}, \\
\varphi_2(t) &= \alpha_{\lambda}^{-1} t_{\lambda+\frac{7}{2}, \lambda+5} (t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} t_{\lambda, \lambda+\frac{3}{2}} - t_{\lambda+2, \lambda+\frac{7}{2}} t_{\lambda, \lambda+2}), \\
\psi_2(t) &= \alpha_{\lambda+\frac{3}{2}}^{-1} (t_{\lambda+3, \lambda+5} t_{\lambda+\frac{3}{2}, \lambda+3} - t_{\lambda+\frac{7}{2}, \lambda+5} t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}}) t_{\lambda, \lambda+\frac{3}{2}}, \\
\tilde{\varphi}_2(t) &= \zeta_{\lambda}^{-1} t_{\lambda+3, \lambda+5} t_{\lambda+\frac{3}{2}, \lambda+3} t_{\lambda, \lambda+\frac{3}{2}}, \\
\tilde{\psi}_2(t) &= \zeta_{\lambda+2}^{-1} t_{\lambda+\frac{7}{2}, \lambda+5} t_{\lambda+2, \lambda+\frac{7}{2}} t_{\lambda, \lambda+2}, \\
\varphi_3(t) &= \beta_{\lambda}^{-1} t_{\lambda+4, \lambda+\frac{11}{2}} \left(t_{\lambda+\frac{3}{2}, \lambda+4} t_{\lambda, \lambda+\frac{3}{2}} + t_{\lambda+\frac{5}{2}, \lambda+4} t_{\lambda, \lambda+\frac{5}{2}} + \frac{1}{3} t_{\lambda+2, \lambda+4} t_{\lambda, \lambda+2} \right), \\
\psi_3(t) &= \beta_{\lambda+\frac{3}{2}}^{-1} t_{\lambda, \lambda+\frac{3}{2}} \left(t_{\lambda+3, \lambda+\frac{11}{2}} t_{\lambda+\frac{3}{2}, \lambda+3} + t_{\lambda+4, \lambda+\frac{11}{2}} t_{\lambda+\frac{3}{2}, \lambda+4} + \frac{1}{3} t_{\lambda+\frac{7}{2}, \lambda+\frac{11}{2}} t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} \right), \\
\tilde{\varphi}_3(t) &= \alpha_{\lambda}^{-1} t_{\lambda+\frac{7}{2}, \lambda+\frac{11}{2}} (t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} t_{\lambda, \lambda+\frac{3}{2}} - t_{\lambda+2, \lambda+\frac{7}{2}} t_{\lambda, \lambda+2}), \\
\tilde{\psi}_3(t) &= \alpha_{\lambda+2}^{-1} t_{\lambda, \lambda+2} (t_{\lambda+\frac{7}{2}, \lambda+\frac{11}{2}} t_{\lambda+2, \lambda+\frac{7}{2}} - t_{\lambda+4, \lambda+\frac{11}{2}} t_{\lambda+2, \lambda+4}), \\
\overline{\varphi}_3(t) &= \zeta_{\lambda}^{-1} t_{\lambda+3, \lambda+\frac{11}{2}} t_{\lambda+\frac{3}{2}, \lambda+3} t_{\lambda, \lambda+\frac{3}{2}}, \\
\overline{\psi}_3(t) &= \zeta_{\lambda+\frac{5}{2}}^{-1} t_{\lambda+4, \lambda+\frac{11}{2}} t_{\lambda+\frac{5}{2}, \lambda+4} t_{\lambda, \lambda+\frac{5}{2}}, \\
\varphi_4(t) &= \gamma_{\lambda}^{-1} t_{\lambda+\frac{9}{2}, \lambda+6} (t_{\lambda+\frac{5}{2}, \lambda+\frac{9}{2}} t_{\lambda, \lambda+\frac{5}{2}} - t_{\lambda+2, \lambda+\frac{9}{2}} t_{\lambda, \lambda+2}),
\end{aligned}$$

$$\begin{aligned}
\psi_4(t) &= \gamma_{\lambda+\frac{3}{2}}^{-1} (t_{\lambda+4,\lambda+6} t_{\lambda+\frac{3}{2},\lambda+4} - t_{\lambda+\frac{7}{2},\lambda+6} t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}}) t_{\lambda,\lambda+\frac{3}{2}}, \\
\tilde{\varphi}_4(t) &= \beta_{\lambda}^{-1} t_{\lambda+4,\lambda+6} \left(t_{\lambda+\frac{3}{2},\lambda+4} t_{\lambda,\lambda+\frac{3}{2}} + t_{\lambda+\frac{5}{2},\lambda+4} t_{\lambda,\lambda+\frac{5}{2}} + \frac{1}{3} t_{\lambda+2,\lambda+4} t_{\lambda,\lambda+2} \right), \\
\tilde{\psi}_4(t) &= \beta_{\lambda+2}^{-1} t_{\lambda,\lambda+2} \left(t_{\lambda+\frac{7}{2},\lambda+6} t_{\lambda+2,\lambda+\frac{7}{2}} + t_{\lambda+\frac{9}{2},\lambda+6} t_{\lambda+2,\lambda+\frac{9}{2}} + \frac{1}{3} t_{\lambda+4,\lambda+6} t_{\lambda+2,\lambda+4} \right), \\
\overline{\varphi}_4(t) &= \alpha_{\lambda}^{-1} t_{\lambda+\frac{7}{2},\lambda+6} (t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}} t_{\lambda,\lambda+\frac{3}{2}} - t_{\lambda+2,\lambda+\frac{7}{2}} t_{\lambda,\lambda+2}), \\
\overline{\psi}_4(t) &= \alpha_{\lambda+\frac{5}{2}}^{-1} (t_{\lambda+4,\lambda+6} t_{\lambda+\frac{5}{2},\lambda+4} - t_{\lambda+\frac{9}{2},\lambda+6} t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}}) t_{\lambda,\lambda+\frac{5}{2}}, \\
\varphi_5(t) &= \beta_{\lambda}^{-1} t_{\lambda+4,\lambda+\frac{13}{2}} \left(t_{\lambda+\frac{3}{2},\lambda+4} t_{\lambda,\lambda+\frac{3}{2}} + t_{\lambda+\frac{5}{2},\lambda+4} t_{\lambda,\lambda+\frac{5}{2}} + \frac{1}{3} t_{\lambda+2,\lambda+4} t_{\lambda,\lambda+2} \right), \\
\psi_5(t) &= \beta_{\lambda+\frac{5}{2}}^{-1} t_{\lambda,\lambda+\frac{5}{2}} \left(t_{\lambda+4,\lambda+\frac{13}{2}} t_{\lambda+\frac{5}{2},\lambda+4} + \frac{1}{3} t_{\lambda+\frac{9}{2},\lambda+\frac{13}{2}} t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}} \right), \\
\tilde{\varphi}_5(t) &= \gamma_{\lambda}^{-1} t_{\lambda+\frac{9}{2},\lambda+\frac{13}{2}} (t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}} t_{\lambda,\lambda+\frac{5}{2}} - t_{\lambda+2,\lambda+\frac{9}{2}} t_{\lambda,\lambda+2}), \\
\tilde{\psi}_5(t) &= \gamma_{\lambda+2}^{-1} t_{\lambda,\lambda+2} (t_{\lambda+\frac{9}{2},\lambda+\frac{13}{2}} t_{\lambda+2,\lambda+\frac{9}{2}} - t_{\lambda+4,\lambda+\frac{13}{2}} t_{\lambda+2,\lambda+4}), \\
\varphi_6(t) &= \gamma_{\lambda}^{-1} t_{\lambda+\frac{9}{2},\lambda+7} (t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}} t_{\lambda,\lambda+\frac{5}{2}} - t_{\lambda+2,\lambda+\frac{9}{2}} t_{\lambda,\lambda+2}), \\
\psi_6(t) &= \gamma_{\lambda+\frac{5}{2}}^{-1} t_{\lambda,\lambda+\frac{5}{2}} t_{\lambda+\frac{9}{2},\lambda+7} t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}}.
\end{aligned}$$

Now, the same arguments, as in the proof of Proposition 6.1, show that we must have:

$$\begin{aligned}
\Omega_{\lambda,\lambda+\frac{9}{2}} &= \omega_1(t) \delta(\mathfrak{J}_{\frac{11}{2}}^{-1,\lambda}), \\
\Omega_{\lambda,\lambda+5} + \tilde{\Omega}_{\lambda,\lambda+5} &= \omega_2(t) \delta(\mathfrak{J}_6^{-1,\lambda}), \\
\Omega_{\lambda,\lambda+\frac{11}{2}} + \tilde{\Omega}_{\lambda,\lambda+\frac{11}{2}} + \overline{\Omega}_{\lambda,\lambda+\frac{11}{2}} &= \omega_3(t) \delta(\mathfrak{J}_{\frac{13}{2}}^{-1,\lambda}), \\
\Omega_{\lambda,\lambda+6} + \tilde{\Omega}_{\lambda,\lambda+6} + \overline{\Omega}_{\lambda,\lambda+6} &= \omega_4(t) \delta(\mathfrak{J}_7^{-1,\lambda}), \\
\Omega_{\lambda,\lambda+\frac{13}{2}} + \tilde{\Omega}_{\lambda,\lambda+\frac{13}{2}} &= \omega_5(t) \delta(\mathfrak{J}_{\frac{15}{2}}^{-1,\lambda}), \\
\Omega_{\lambda,\lambda+7} &= \omega_6(t) \delta(\mathfrak{J}_8^{-1,\lambda}),
\end{aligned}$$

where $\omega_1, \dots, \omega_5$ are some functions. So, by Lemma 6.3 and Lemma 6.4, we obtain for the nonzero $\varphi_i(t), \tilde{\varphi}_i(t), \overline{\varphi}_i, \psi_i(t), \tilde{\psi}_i(t), \overline{\psi}_i(t)$ and $\omega_i(t)$ the following relation:

$$\begin{aligned}
\alpha_{\lambda} \varphi_2(t) + \epsilon_{1,\lambda} \alpha_{\lambda+\frac{3}{2}} \psi_2(t) &= 0, & \zeta_{\lambda} \tilde{\varphi}_2(t) + \epsilon_{2,\lambda} \alpha_{\lambda+\frac{3}{2}} \psi_2(t) &= 0, \\
\beta_{\lambda} \varphi_3(t) + \epsilon_{4,\lambda} \beta_{\lambda+\frac{3}{2}} \psi_3(t) &= 0, & \alpha_{\lambda} \tilde{\varphi}_3(t) + \epsilon_{5,\lambda} \beta_{\lambda+\frac{3}{2}} \psi_3(t) &= 0, \\
\alpha_{\lambda+2} \tilde{\psi}_3(t) + \epsilon_{6,\lambda} \beta_{\lambda+\frac{3}{2}} \psi_3(t) &= 0, & \beta_{\lambda+2} \tilde{\psi}_4(t) + \epsilon_{7,\lambda} \alpha_{\lambda} \overline{\varphi}_4(t) &= 0, \\
\alpha_{\lambda+\frac{5}{2}} \overline{\psi}_4(t) + \epsilon_{8,\lambda} \alpha_{\lambda} \overline{\varphi}_4(t) &= 0, & \beta_{\lambda} \varphi_5(t) + \epsilon_{10,\lambda} \gamma_{\lambda+2} \tilde{\psi}_5(t) &= 0, \\
\beta_{\lambda+\frac{5}{2}} \psi_5(t) + \epsilon_{11,\lambda} \gamma_{\lambda+2} \tilde{\psi}_5(t) &= 0, & \gamma_{\lambda} \tilde{\varphi}_5(t) + \epsilon_{12,\lambda} \gamma_{\lambda+2} \tilde{\psi}_5(t) &= 0, \\
\omega_4(t) = \epsilon_{9,\lambda} \alpha_{\lambda} \overline{\varphi}_4(t), & & \omega_2(t) = \epsilon_{3,\lambda} \alpha_{\lambda+\frac{3}{2}} \psi_2(t), \\
\omega_1(t) = \xi_{\lambda}^{-1} \zeta_{\lambda} \varphi_1(t) = \xi_{\lambda+\frac{3}{2}}^{-1} \zeta_{\lambda+\frac{3}{2}} \psi_1(t). & &
\end{aligned}$$

Therefore, we get the necessary integrability conditions for $\mathfrak{L}^{(3)}$. Under these conditions, the third-order term $\mathfrak{L}^{(3)}$ can be given by:

$$\begin{aligned}\mathfrak{L}^{(3)} &= \sum_{\lambda} \xi_{\lambda}^{-1} t_{\lambda+3, \lambda+\frac{9}{2}} t_{\lambda+\frac{3}{2}, \lambda+3} t_{\lambda, \lambda+\frac{3}{2}} \mathfrak{J}_{\frac{11}{2}}^{-1, \lambda} \\ &+ \sum_{\lambda} \epsilon_{3, \lambda} (t_{\lambda+3, \lambda+5} t_{\lambda+\frac{3}{2}, \lambda+3} - t_{\lambda+\frac{7}{2}, \lambda+5} t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}}) t_{\lambda, \lambda+\frac{3}{2}} \mathfrak{J}_6^{-1, \lambda} \\ &+ \sum_{\lambda} \epsilon_{9, \lambda} t_{\lambda+\frac{7}{2}, \lambda+6} (t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} t_{\lambda, \lambda+\frac{3}{2}} - t_{\lambda+2, \lambda+\frac{7}{2}} t_{\lambda, \lambda+2}) \mathfrak{J}_7^{-1, \lambda}. \quad \square\end{aligned}$$

Proposition 6.3. *The 4th order integrability conditions of the infinitesimal deformation (5.4) are the following:*

(a) For $2(\beta - \lambda) \in \{12, \dots, 2n\}$:

$$t_{\lambda+\frac{9}{2}, \lambda+6} t_{\lambda+3, \lambda+\frac{9}{2}} t_{\lambda+\frac{3}{2}, \lambda+3} t_{\lambda, \lambda+\frac{3}{2}} = 0.$$

(b) For $2(\beta - \lambda) \in \{13, \dots, 2n\}$:

$$\begin{aligned}t_{\lambda+5, \lambda+\frac{13}{2}} t_{\lambda+\frac{7}{2}, \lambda+5} t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} t_{\lambda, \lambda+\frac{3}{2}} &= 0, \\ t_{\lambda+\frac{3}{2}, \lambda+3} t_{\lambda, \lambda+\frac{3}{2}} t_{\lambda+5, \lambda+\frac{13}{2}} t_{\lambda+3, \lambda+5} &= 0, \\ t_{\lambda+\frac{9}{2}, \lambda+\frac{13}{2}} t_{\lambda+3, \lambda+\frac{9}{2}} t_{\lambda+\frac{3}{2}, \lambda+3} t_{\lambda, \lambda+\frac{3}{2}} &= 0.\end{aligned}$$

(c) For $2(\beta - \lambda) \in \{14, \dots, 2n\}$:

$$\begin{aligned}t_{\lambda+\frac{9}{2}, \lambda+7} t_{\lambda+3, \lambda+\frac{9}{2}} t_{\lambda+\frac{3}{2}, \lambda+3} t_{\lambda, \lambda+\frac{3}{2}} &= 0, \\ t_{\lambda, \lambda+\frac{3}{2}} t_{\lambda+\frac{3}{2}, \lambda+3} t_{\lambda+5, \lambda+7} t_{\lambda+3, \lambda+5} &= 0, \\ t_{\lambda, \lambda+2} t_{\lambda+2, \lambda+\frac{7}{2}} t_{\lambda+\frac{11}{2}, \lambda+7} t_{\lambda+\frac{7}{2}, \lambda+\frac{11}{2}} &= 0, \\ t_{\lambda, \lambda+\frac{3}{2}} t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} t_{\lambda+\frac{11}{2}, \lambda+7} t_{\lambda+\frac{7}{2}, \lambda+\frac{11}{2}} &= 0, \\ t_{\lambda, \lambda+\frac{3}{2}} t_{\lambda+5, \lambda+7} t_{\lambda+\frac{7}{2}, \lambda+5} t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} &= 0, \\ t_{\lambda+\frac{11}{2}, \lambda+7} t_{\lambda+4, \lambda+\frac{11}{2}} (3t_{\lambda+\frac{3}{2}, \lambda+4} t_{\lambda, \lambda+\frac{3}{2}} + t_{\lambda+2, \lambda+4} t_{\lambda, \lambda+2}) &= 0.\end{aligned}$$

(d) For $2(\beta - \lambda) \in \{15, \dots, 2n\}$:

$$\begin{aligned}t_{\lambda, \lambda+\frac{3}{2}} t_{\lambda+\frac{3}{2}, \lambda+3} t_{\lambda+5, \lambda+\frac{15}{2}} t_{\lambda+3, \lambda+5} &= 0, \\ t_{\lambda, \lambda+\frac{3}{2}} t_{\lambda+5, \lambda+\frac{15}{2}} t_{\lambda+\frac{7}{2}, \lambda+5} t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} &= 0, \\ t_{\lambda, \lambda+\frac{5}{2}} t_{\lambda+\frac{5}{2}, \lambda+4} t_{\lambda+6, \lambda+\frac{15}{2}} t_{\lambda+4, \lambda+6} &= 0, \\ t_{\lambda+\frac{11}{2}, \lambda+\frac{15}{2}} t_{\lambda+\frac{7}{2}, \lambda+\frac{11}{2}} (t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} t_{\lambda, \lambda+\frac{3}{2}} - t_{\lambda+2, \lambda+\frac{7}{2}} t_{\lambda, \lambda+2}) &= 0, \\ t_{\lambda+\frac{11}{2}, \lambda+\frac{15}{2}} t_{\lambda+4, \lambda+\frac{11}{2}} (3t_{\lambda+\frac{3}{2}, \lambda+4} t_{\lambda, \lambda+\frac{3}{2}} + t_{\lambda+2, \lambda+4} t_{\lambda, \lambda+2}) &= 0, \\ t_{\lambda+6, \lambda+\frac{15}{2}} t_{\lambda+\frac{7}{2}, \lambda+6} (t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} t_{\lambda, \lambda+\frac{3}{2}} - t_{\lambda+2, \lambda+\frac{7}{2}} t_{\lambda, \lambda+2}) &= 0.\end{aligned}$$

(e) For $2(\beta - \lambda) \in \{16, \dots, 2n\}$:

$$\begin{aligned}
& \epsilon_{14,\lambda} t_{\lambda+6,\lambda+8} t_{\lambda+\frac{7}{2},\lambda+6} (t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}} t_{\lambda,\lambda+\frac{3}{2}} - t_{\lambda+2,\lambda+\frac{7}{2}} t_{\lambda,\lambda+2}) \\
& + t_{\lambda+\frac{13}{2},\lambda+8} t_{\lambda+\frac{9}{2},\lambda+\frac{13}{2}} (t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}} t_{\lambda,\lambda+\frac{5}{2}} - t_{\lambda+2,\lambda+\frac{9}{2}} t_{\lambda,\lambda+2}) = 0, \\
& \epsilon_{16,\lambda} t_{\lambda+6,\lambda+8} t_{\lambda+\frac{7}{2},\lambda+6} (t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}} t_{\lambda,\lambda+\frac{3}{2}} - t_{\lambda+2,\lambda+\frac{7}{2}} t_{\lambda,\lambda+2}) \\
& + t_{\lambda,\lambda+2} t_{\lambda+\frac{11}{2},\lambda+8} (t_{\lambda+\frac{7}{2},\lambda+\frac{11}{2}} t_{\lambda+2,\lambda+\frac{7}{2}} - t_{\lambda+4,\lambda+\frac{11}{2}} t_{\lambda+2,\lambda+4}) = 0, \\
& ((1 + \epsilon_{15,\lambda}) t_{\lambda+6,\lambda+8} t_{\lambda+\frac{7}{2},\lambda+6} - t_{\lambda+\frac{11}{2},\lambda+8} t_{\lambda+\frac{7}{2},\lambda+\frac{11}{2}}) \\
& \times (t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}} t_{\lambda,\lambda+\frac{3}{2}} - t_{\lambda+2,\lambda+\frac{7}{2}} t_{\lambda,\lambda+2}) = 0, \\
& \epsilon_{13,\lambda} t_{\lambda+6,\lambda+8} t_{\lambda+\frac{7}{2},\lambda+6} (t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}} t_{\lambda,\lambda+\frac{3}{2}} - t_{\lambda+2,\lambda+\frac{7}{2}} t_{\lambda,\lambda+2}) \\
& + t_{\lambda+\frac{11}{2},\lambda+8} t_{\lambda+4,\lambda+\frac{11}{2}} \left(t_{\lambda+\frac{3}{2},\lambda+4} t_{\lambda,\lambda+\frac{3}{2}} + \frac{1}{3} t_{\lambda+2,\lambda+4} t_{\lambda,\lambda+2} \right) \\
& + t_{\lambda+\frac{13}{2},\lambda+8} t_{\lambda+4,\lambda+\frac{13}{2}} \left(t_{\lambda+\frac{3}{2},\lambda+4} t_{\lambda,\lambda+\frac{3}{2}} + t_{\lambda+\frac{5}{2},\lambda+4} t_{\lambda,\lambda+\frac{5}{2}} + \frac{1}{3} t_{\lambda+2,\lambda+4} t_{\lambda,\lambda+2} \right) = 0.
\end{aligned}$$

(f) For $2(\beta - \lambda) \in \{17, \dots, 2n\}$:

$$\begin{aligned}
& t_{\lambda,\lambda+\frac{5}{2}} t_{\lambda+6,\lambda+\frac{17}{2}} t_{\lambda+4,\lambda+6} t_{\lambda+\frac{5}{2},\lambda+4} = 0, \\
& t_{\lambda+\frac{13}{2},\lambda+\frac{17}{2}} t_{\lambda+\frac{9}{2},\lambda+\frac{13}{2}} (t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}} t_{\lambda,\lambda+\frac{5}{2}} - t_{\lambda+2,\lambda+\frac{9}{2}} t_{\lambda,\lambda+2}) = 0, \\
& t_{\lambda+6,\lambda+\frac{17}{2}} t_{\lambda+\frac{7}{2},\lambda+6} (t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}} t_{\lambda,\lambda+\frac{3}{2}} - t_{\lambda+2,\lambda+\frac{7}{2}} t_{\lambda,\lambda+2}) = 0, \\
& t_{\lambda+\frac{13}{2},\lambda+\frac{17}{2}} t_{\lambda+4,\lambda+\frac{13}{2}} (3 t_{\lambda+\frac{3}{2},\lambda+4} t_{\lambda,\lambda+\frac{3}{2}} + 3 t_{\lambda+\frac{5}{2},\lambda+4} t_{\lambda,\lambda+\frac{5}{2}} + t_{\lambda+2,\lambda+4} t_{\lambda,\lambda+2}) = 0.
\end{aligned}$$

(g) For $2(\beta - \lambda) \in \{18, \dots, 2n\}$:

$$t_{\lambda+\frac{13}{2},\lambda+9} t_{\lambda+\frac{9}{2},\lambda+\frac{13}{2}} (t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}} t_{\lambda,\lambda+\frac{5}{2}} - t_{\lambda+2,\lambda+\frac{9}{2}} t_{\lambda,\lambda+2}) = 0.$$

To prove Proposition 6.3, we need the following two lemmas which we can check by a direct computation with the help of *Maple*.

Lemma 6.5. *We have*

$$\begin{aligned}
\epsilon_{9,\lambda} [\Upsilon_{\lambda+6,\lambda+8}, \mathfrak{J}_7^{-1,\lambda}] &= \epsilon_{13,\lambda} \beta_{\lambda+4}^{-1} \beta_{\lambda}^{-1} [\mathfrak{J}_5^{-1,\lambda+4}, \mathfrak{J}_5^{-1,\lambda}] + \epsilon_{14,\lambda} \alpha_{\lambda+\frac{9}{2}}^{-1} \gamma_{\lambda}^{-1} [\mathfrak{J}_{\frac{9}{2}}^{-1,\lambda+\frac{9}{2}}, \mathfrak{J}_{\frac{11}{2}}^{-1,\lambda}] \\
&+ \epsilon_{15,\lambda} \alpha_{\lambda}^{-1} \gamma_{\lambda+\frac{7}{2}}^{-1} [\mathfrak{J}_{\frac{11}{2}}^{-1,\lambda+\frac{7}{2}}, \mathfrak{J}_{\frac{9}{2}}^{-1,\lambda}] + \epsilon_{16,\lambda} \epsilon_{9,\lambda+2} [\mathfrak{J}_7^{-1,\lambda+2}, \Upsilon_{\lambda,\lambda+2}] \\
&+ \epsilon_{17,\lambda} \delta(\mathfrak{J}_9^{-1,\lambda}),
\end{aligned}$$

where

$$\begin{aligned}
& \epsilon_{9,\lambda} (2\lambda + 3)(2\lambda + 5)(2\lambda + 9)(\lambda + 2)(\lambda + 3)(\lambda + 5)(2\lambda^2 + 7\lambda + 2) \\
& \times (2\lambda^2 + 23\lambda + 62)(16\lambda^4 + 240\lambda^3 + 1034\lambda^2 + 1005\lambda + 300) \\
\epsilon_{13,\lambda} &= \frac{}{36(\lambda + 4)(2\lambda + 7)(32\lambda^6 + 784\lambda^5 + 7156\lambda^4 + 29576\lambda^3 + 53961\lambda^2 + 40281\lambda + 11760)},
\end{aligned}$$

$$\begin{aligned}
\epsilon_{14,\lambda} &= \frac{\epsilon_{9,\lambda} (2\lambda + 3)(2\lambda + 5)(2\lambda - 5)(2\lambda + 9)(\lambda - 4)(\lambda + 2)(\lambda + 7)(\lambda + 9)}{(\lambda + 4)(32\lambda^6 + 784\lambda^5 + 7156\lambda^4 + 29576\lambda^3 + 53961\lambda^2 + 40281\lambda + 11760)}, \\
\epsilon_{15,\lambda} &= -\frac{\epsilon_{9,\lambda} (2\lambda - 3)(2\lambda + 1)(2\lambda + 3)(2\lambda + 6)(2\lambda + 23)(\lambda + 2)(\lambda + 5)(\lambda + 10)}{(2\lambda + 7)(32\lambda^6 + 784\lambda^5 + 7156\lambda^4 + 29576\lambda^3 + 53961\lambda^2 + 40281\lambda + 11760)}, \\
\epsilon_{16,\lambda} &= -\frac{\epsilon_{9,\lambda} (2\lambda + 3)(\lambda + 2)(32\lambda^6 + 656\lambda^5 + 4756\lambda^4 + 14104\lambda^3 + 14901\lambda^2 + 7059\lambda + 240)}{\epsilon_{9,\lambda+2}(2\lambda + 11)(\lambda + 6)(32\lambda^6 + 784\lambda^5 + 7156\lambda^4 + 29576\lambda^3 + 53961\lambda^2 + 40281\lambda + 11760)}, \\
\epsilon_{17,\lambda} &= -\frac{\epsilon_{9,\lambda} (2\lambda + 5)(2\lambda + 9)(2\lambda + 6)(\lambda + 5)(16\lambda^4 + 240\lambda^3 + 1034\lambda^2 + 1005\lambda + 420)}{(2\lambda + 11)(2\lambda + 7)(\lambda + 4)(\lambda + 6)(32\lambda^6 + 784\lambda^5 + 7156\lambda^4 + 29576\lambda^3 + 53961\lambda^2 + 40281\lambda + 11760)}.
\end{aligned}$$

Lemma 6.6. *Each of the following systems is linearly independent*

- (1) $(\delta(\mathfrak{J}_7^{-1,\lambda}), \zeta_{\lambda+3}^{-1}\zeta_{\lambda}^{-1}[\mathfrak{J}_4^{-1,\lambda+3}, \mathfrak{J}_4^{-1,\lambda}] + \xi_{\lambda}^{-1}[\Upsilon_{\lambda+\frac{9}{2},\lambda+6}, \mathfrak{J}_{\frac{11}{2}}^{-1,\lambda}] + \xi_{\lambda+\frac{3}{2}}^{-1}[\mathfrak{J}_{\frac{11}{2}}^{-1,\lambda+\frac{3}{2}}, \Upsilon_{\lambda,\lambda+\frac{3}{2}}]),$
- (2) $(\delta(\mathfrak{J}_{\frac{15}{2}}^{-1,\lambda}), [\mathfrak{J}_4^{-1,\lambda+\frac{7}{2}}, \mathfrak{J}_{\frac{9}{2}}^{-1,\lambda}], [\Upsilon_{\lambda+5,\lambda+\frac{13}{2}}, \mathfrak{J}_6^{-1,\lambda}], [\Upsilon_{\lambda+\frac{9}{2},\lambda+\frac{13}{2}}, \mathfrak{J}_{\frac{11}{2}}^{-1,\lambda}], \zeta_{\lambda}^{-1}\alpha_{\lambda+3}^{-1}[\mathfrak{J}_{\frac{9}{2}}^{-1,\lambda+3}, \mathfrak{J}_4^{-1,\lambda}] + \epsilon_{3,\lambda+\frac{3}{2}}[\mathfrak{J}_6^{-1,\lambda+\frac{3}{2}}, \Upsilon_{\lambda,\lambda+\frac{3}{2}}]),$
- (3) $(\delta(\mathfrak{J}_8^{-1,\lambda}), [\mathfrak{J}_{\frac{9}{2}}^{-1,\lambda+\frac{7}{2}}, \mathfrak{J}_{\frac{9}{2}}^{-1,\lambda}], [\mathfrak{J}_4^{-1,\lambda+4}, \mathfrak{J}_5^{-1,\lambda}], [\mathfrak{J}_5^{-1,\lambda+3}, \mathfrak{J}_4^{-1,\lambda}], [\mathfrak{J}_6^{-1,\lambda+2}, \Upsilon_{\lambda,\lambda+2}], [\Upsilon_{\lambda+\frac{9}{2},\lambda+7}, \mathfrak{J}_{\frac{11}{2}}^{-1,\lambda}], [\Upsilon_{\lambda+5,\lambda+7}, \mathfrak{J}_6^{-1,\lambda}]),$
- (4) $(\delta(\mathfrak{J}_{\frac{17}{2}}^{-1,\lambda}), [\mathfrak{J}_5^{-1,\lambda+\frac{7}{2}}, \mathfrak{J}_{\frac{9}{2}}^{-1,\lambda}], [\mathfrak{J}_{\frac{9}{2}}^{-1,\lambda+4}, \mathfrak{J}_5^{-1,\lambda}], [\mathfrak{J}_{\frac{11}{2}}^{-1,\lambda+3}, \mathfrak{J}_4^{-1,\lambda}], [\Upsilon_{\lambda+6,\lambda+\frac{15}{2}}, \mathfrak{J}_7^{-1,\lambda}], [\mathfrak{J}_6^{-1,\lambda+\frac{5}{2}}, \Upsilon_{\lambda,\lambda+\frac{5}{2}}], \epsilon_{3,\lambda}[\Upsilon_{\lambda+5,\lambda+\frac{15}{2}}, \mathfrak{J}_6^{-1,\lambda}] + \epsilon_{9,\lambda+\frac{3}{2}}[\mathfrak{J}_7^{-1,\lambda+\frac{3}{2}}, \Upsilon_{\lambda,\lambda+\frac{3}{2}}]),$
- (5) $(\delta(\mathfrak{J}_9^{-1,\lambda}), [\mathfrak{J}_5^{-1,\lambda+4}, \mathfrak{J}_5^{-1,\lambda}], [\mathfrak{J}_{\frac{9}{2}}^{-1,\lambda+\frac{9}{2}}, \mathfrak{J}_{\frac{11}{2}}^{-1,\lambda}], [\mathfrak{J}_{\frac{11}{2}}^{-1,\lambda+\frac{7}{2}}, \mathfrak{J}_{\frac{9}{2}}^{-1,\lambda}], [\mathfrak{J}_7^{-1,\lambda+2}, \Upsilon_{\lambda,\lambda+2}]),$
- (6) $(\delta(\mathfrak{J}_{\frac{19}{2}}^{-1,\lambda}), [\mathfrak{J}_5^{-1,\lambda+\frac{9}{2}}, \mathfrak{J}_{\frac{11}{2}}^{-1,\lambda}], [\mathfrak{J}_{\frac{11}{2}}^{-1,\lambda+4}, \mathfrak{J}_5^{-1,\lambda}], [\mathfrak{J}_7^{-1,\lambda+\frac{5}{2}}, \Upsilon_{\lambda,\lambda+\frac{5}{2}}], [\Upsilon_{\lambda+6,\lambda+\frac{17}{2}}, \mathfrak{J}_7^{-1,\lambda}]),$
- (7) $(\delta(\mathfrak{J}_{10}^{-1,\lambda}), [\mathfrak{J}_{\frac{11}{2}}^{-1,\lambda+\frac{9}{2}}, \mathfrak{J}_{\frac{11}{2}}^{-1,\lambda}]).$

Proof of Proposition 6.3. The fourth order integrability conditions of the infinitesimal deformation (5.4) follow from Lemma 6.5 and Lemma 6.6 together with Proposition 6.1 and Proposition 6.2 and arguments similar to those from the proof of Proposition 6.2. Under these conditions, the fourth-order term $\mathfrak{L}^{(4)}$ can be given by:

$$\mathfrak{L}^{(4)} = -\epsilon_{17,\lambda} t_{\lambda+6,\lambda+8} t_{\lambda+\frac{7}{2},\lambda+6} (t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}} t_{\lambda,\lambda+\frac{3}{2}} - t_{\lambda+2,\lambda+\frac{7}{2}} t_{\lambda,\lambda+2}) \mathfrak{J}_9^{-1,\lambda}. \quad \square$$

Proposition 6.4. *The 5th order integrability conditions of the infinitesimal deformation (5.4) are the following:*

(a) For $2(\beta - \lambda) \in \{19, \dots, 2n\}$:

$$t_{\lambda+8,\lambda+\frac{19}{2}} t_{\lambda+6,\lambda+8} t_{\lambda+\frac{7}{2},\lambda+6} (t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}} t_{\lambda,\lambda+\frac{3}{2}} - t_{\lambda+2,\lambda+\frac{7}{2}} t_{\lambda,\lambda+2}) = 0,$$

(b) For $2(\beta - \lambda) \in \{20, \dots, 2n\}$:

$$\begin{aligned} t_{\lambda+8, \lambda+10} t_{\lambda+6, \lambda+8} t_{\lambda+\frac{7}{2}, \lambda+6} (t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} t_{\lambda, \lambda+\frac{3}{2}} - t_{\lambda+2, \lambda+\frac{7}{2}} t_{\lambda, \lambda+2}) &= 0, \\ t_{\lambda, \lambda+2} t_{\lambda+8, \lambda+10} t_{\lambda+\frac{11}{2}, \lambda+8} (t_{\lambda+\frac{7}{2}, \lambda+\frac{11}{2}} t_{\lambda+2, \lambda+\frac{7}{2}} - t_{\lambda+4, \lambda+\frac{11}{2}} t_{\lambda+2, \lambda+4}) &= 0, \end{aligned}$$

(c) For $2(\beta - \lambda) \in \{21, \dots, 2n\}$:

$$\begin{aligned} t_{\lambda+8, \lambda+\frac{21}{2}} t_{\lambda+6, \lambda+8} t_{\lambda+\frac{7}{2}, \lambda+6} (t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} t_{\lambda, \lambda+\frac{3}{2}} - t_{\lambda+2, \lambda+\frac{7}{2}} t_{\lambda, \lambda+2}) &= 0, \\ t_{\lambda+8, \lambda+\frac{21}{2}} t_{\lambda+\frac{13}{2}, \lambda+8} t_{\lambda+\frac{9}{2}, \lambda+\frac{13}{2}} (t_{\lambda+\frac{5}{2}, \lambda+\frac{9}{2}} t_{\lambda, \lambda+\frac{5}{2}} - t_{\lambda+2, \lambda+\frac{9}{2}} t_{\lambda, \lambda+2}) &= 0, \\ t_{\lambda, \lambda+\frac{5}{2}} t_{\lambda+\frac{17}{2}, \lambda+\frac{21}{2}} t_{\lambda+6, \lambda+\frac{17}{2}} t_{\lambda+\frac{9}{2}, \lambda+6} t_{\lambda+\frac{5}{2}, \lambda+\frac{9}{2}} &= 0. \end{aligned}$$

To prove Proposition 6.4, we need the following lemma which we can check by a direct computation.

Lemma 6.7. *Each of the following systems is linearly independent:*

- (1) $(\delta(\mathfrak{J}_{\frac{21}{2}}^{-1, \lambda}), \alpha_{\lambda+6}^{-1} \epsilon_{9, \lambda} [\mathfrak{J}_{\frac{9}{2}}^{-1, \lambda+6}, \mathfrak{J}_7^{-1, \lambda}] - \epsilon_{17, \lambda} [\Upsilon_{\lambda+8, \lambda+\frac{19}{2}}, \mathfrak{J}_9^{-1, \lambda}]),$
- (2) $(\delta(\mathfrak{J}_{11}^{-1, \lambda}), [\mathfrak{J}_9^{-1, \lambda+2}, \Upsilon_{\lambda, \lambda+2}], \epsilon_{17, \lambda} [\Upsilon_{\lambda+8, \lambda+10}, \mathfrak{J}_9^{-1, \lambda}] + \frac{1}{3} \beta_{\lambda+6}^{-1} \epsilon_{9, \lambda} [\mathfrak{J}_5^{-1, \lambda+6}, \mathfrak{J}_7^{-1, \lambda}]),$
- (3) $(\delta(\mathfrak{J}_{\frac{23}{2}}^{-1, \lambda}), [\mathfrak{J}_7^{-1, \lambda+\frac{9}{2}}, \mathfrak{J}_{\frac{11}{2}}^{-1, \lambda}], [\mathfrak{J}_9^{-1, \lambda+\frac{5}{2}}, \Upsilon_{\lambda, \lambda+\frac{5}{2}}], \epsilon_{9, \lambda} \gamma_{\lambda+6}^{-1} [\mathfrak{J}_{\frac{11}{2}}^{-1, \lambda+6}, \mathfrak{J}_7^{-1, \lambda}] - \epsilon_{17, \lambda} [\Upsilon_{\lambda+8, \lambda+\frac{21}{2}}, \mathfrak{J}_9^{-1, \lambda}]).$

Proof of Proposition 6.4. Using the same arguments as in proof of Proposition 6.2 together with Lemma 6.7, Proposition 6.2 and Proposition 6.3, we get the necessary integrability conditions for $\mathfrak{L}^{(5)}$. Under these conditions, it can be easily checked that $\delta(\mathfrak{L}^{(m)}) = 0$ for $m = 5, 6, 7, 8$. \square

The main result in this section is the following theorem.

Theorem 6.1. *The conditions given in Propositions 6.1, 6.2, 6.3, 6.4 are necessary and sufficient for the integrability of the infinitesimal deformation (5.4). Moreover, any formal $\mathfrak{osp}(1|2)$ -trivial deformation of the $\mathcal{K}(1)$ -module \mathfrak{S}_{β}^n is equivalent to a polynomial one of degree ≤ 4 .*

Proof. Of course these conditions are necessary. Now, we show that these conditions are sufficient. The solution $\mathfrak{L}^{(m)}$ of the Maurer–Cartan equation is defined up to a 1-cocycle and it has been shown in [14, 1] that different choices of solutions of the Maurer–Cartan equation correspond to equivalent deformations. Thus, we can always reduce $\mathfrak{L}^{(m)}$, for $m = 5, 6, 7, 8$, to zero by equivalence. Then, by recurrence, the terms $\mathfrak{L}^{(m)}$, for $m \geq 9$, satisfy the equation $\delta(\mathfrak{L}^{(m)}) = 0$ and can also be reduced to the identically zero map. \square

Remark 6.2. There are no integrability conditions of any infinitesimal $\mathfrak{osp}(1|2)$ -trivial deformation of the $\mathcal{K}(1)$ -module \mathfrak{S}_{β}^n if $n < 5$. In this case, any formal $\mathfrak{osp}(1|2)$ -trivial deformation is equivalent to its infinitesimal part.

7. Examples

We study formal $\mathfrak{osp}(1|2)$ -trivial deformations of $\mathcal{K}(1)$ -modules $\mathfrak{S}_{\lambda+n}^n$ for some $n \in \frac{1}{2}\mathbb{N}$ and for arbitrary generic $\lambda \in \mathbb{K}$. For $n < 5$, each of these deformations is equivalent to its infinitesimal one, without any integrability condition.

Example 7.1. The $\mathcal{K}(1)$ -module $\mathfrak{S}_{\lambda+5}^5$.

Proposition 7.1. *The $\mathcal{K}(1)$ -module $\mathfrak{S}_{\lambda+5}^5$ admits six formal $\mathfrak{osp}(1|2)$ -trivial deformations with 18 independent parameters. These deformations are polynomial of degree 3.*

Proof. In this case, any $\mathfrak{osp}(1|2)$ -trivial deformation is given by

$$\tilde{\mathfrak{L}}_{X_F} = \mathfrak{L}_{X_F} + \mathfrak{L}_{X_F}^{(1)} + \mathfrak{L}_{X_F}^{(2)} + \mathfrak{L}_{X_F}^{(3)}, \quad (7.1)$$

where

$$\begin{aligned} \mathfrak{L}^{(1)} = & t_{\lambda, \lambda+\frac{3}{2}} \Upsilon_{\lambda, \lambda+\frac{3}{2}} + t_{\lambda, \lambda+2} \Upsilon_{\lambda, \lambda+2} + t_{\lambda, \lambda+\frac{5}{2}} \Upsilon_{\lambda, \lambda+\frac{5}{2}} + t_{\lambda+\frac{1}{2}, \lambda+2} \Upsilon_{\lambda+\frac{1}{2}, \lambda+2} \\ & + t_{\lambda+\frac{1}{2}, \lambda+\frac{5}{2}} \Upsilon_{\lambda+\frac{1}{2}, \lambda+\frac{5}{2}} + t_{\lambda+\frac{1}{2}, \lambda+3} \Upsilon_{\lambda+\frac{1}{2}, \lambda+3} + t_{\lambda+1, \lambda+\frac{5}{2}} \Upsilon_{\lambda+1, \lambda+\frac{5}{2}} + t_{\lambda+1, \lambda+3} \Upsilon_{\lambda+1, \lambda+3} \\ & + t_{\lambda+1, \lambda+\frac{7}{2}} \Upsilon_{\lambda+1, \lambda+\frac{7}{2}} + t_{\lambda+\frac{3}{2}, \lambda+3} \Upsilon_{\lambda+\frac{3}{2}, \lambda+3} + t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} \Upsilon_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} + t_{\lambda+\frac{3}{2}, \lambda+4} \Upsilon_{\lambda+\frac{3}{2}, \lambda+4} \\ & + t_{\lambda+2, \lambda+\frac{7}{2}} \Upsilon_{\lambda+2, \lambda+\frac{7}{2}} + t_{\lambda+2, \lambda+4} \Upsilon_{\lambda+2, \lambda+4} + t_{\lambda+2, \lambda+\frac{9}{2}} \Upsilon_{\lambda+2, \lambda+\frac{9}{2}} + t_{\lambda+\frac{5}{2}, \lambda+4} \Upsilon_{\lambda+\frac{5}{2}, \lambda+4} \\ & + t_{\lambda+\frac{5}{2}, \lambda+\frac{9}{2}} \Upsilon_{\lambda+\frac{5}{2}, \lambda+\frac{9}{2}} + t_{\lambda+\frac{5}{2}, \lambda+5} \Upsilon_{\lambda+\frac{5}{2}, \lambda+5} + t_{\lambda+3, \lambda+\frac{9}{2}} \Upsilon_{\lambda+3, \lambda+\frac{9}{2}} + t_{\lambda+3, \lambda+5} \Upsilon_{\lambda+3, \lambda+5} \\ & + t_{\lambda+\frac{7}{2}, \lambda+5} \Upsilon_{\lambda+\frac{7}{2}, \lambda+5}, \end{aligned}$$

$$\begin{aligned} \mathfrak{L}^{(2)} = & - \sum_{\mu} \zeta_{\mu}^{-1} t_{\mu+\frac{3}{2}, \mu+3} t_{\mu, \mu+\frac{3}{2}} \mathfrak{J}_4^{-1, \mu} \\ & - \sum_{\nu} \alpha_{\nu}^{-1} (t_{\nu+\frac{3}{2}, \nu+\frac{7}{2}} t_{\nu, \nu+\frac{3}{2}} - t_{\nu+2, \nu+\frac{7}{2}} t_{\nu, \nu+2}) \mathfrak{J}_{\frac{9}{2}}^{-1, \nu} \\ & - \sum_{\varepsilon} \beta_{\varepsilon}^{-1} \left(t_{\varepsilon+\frac{3}{2}, \varepsilon+4} t_{\varepsilon, \varepsilon+\frac{3}{2}} + t_{\varepsilon+\frac{5}{2}, \varepsilon+4} t_{\varepsilon, \varepsilon+\frac{5}{2}} + \frac{1}{3} t_{\varepsilon+2, \varepsilon+4} t_{\varepsilon, \varepsilon+2} \right) \mathfrak{J}_5^{-1, \varepsilon} \\ & - \sum_{\ell} \gamma_{\ell}^{-1} (t_{\ell+\frac{5}{2}, \ell+\frac{9}{2}} t_{\ell, \ell+\frac{5}{2}} - t_{\ell+2, \ell+\frac{9}{2}} t_{\ell, \ell+2}) \mathfrak{J}_{\frac{11}{2}}^{-1, \ell}, \end{aligned}$$

$$\begin{aligned} \mathfrak{L}^{(3)} = & \sum_{\ell} \xi_{\ell}^{-1} t_{\ell+3, \ell+\frac{9}{2}} t_{\ell+\frac{3}{2}, \ell+3} t_{\ell, \ell+\frac{3}{2}} \mathfrak{J}_{\frac{11}{2}}^{-1, \ell} \\ & + \epsilon_{3, \lambda} (t_{\lambda+3, \lambda+5} t_{\lambda+\frac{3}{2}, \lambda+3} - t_{\lambda+\frac{7}{2}, \lambda+5} t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}}) t_{\lambda, \lambda+\frac{3}{2}} \mathfrak{J}_6^{-1, \lambda} \end{aligned}$$

with $\mu \in \{\lambda, \lambda + \frac{1}{2}, \lambda + 1, \lambda + \frac{3}{2}, \lambda + 2\}$, $\nu \in \{\lambda, \lambda + \frac{1}{2}, \lambda + 1, \lambda + \frac{3}{2}\}$, $\varepsilon \in \{\lambda, \lambda + \frac{1}{2}, \lambda + 1\}$ and $\ell \in \{\lambda, \lambda + \frac{1}{2}\}$. The following equations

$$t_{\lambda, \lambda+\frac{5}{2}} t_{\lambda+\frac{5}{2}, \lambda+5} = 0, \quad (7.2)$$

$$t_{\lambda, \lambda+\frac{3}{2}} (\epsilon_{1, \lambda} t_{\lambda+3, \lambda+5} t_{\lambda+\frac{3}{2}, \lambda+3} + (1 - \epsilon_{1, \lambda}) t_{\lambda+\frac{7}{2}, \lambda+5} t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}}) = 0, \quad (7.3)$$

$$t_{\lambda, \lambda+\frac{3}{2}} (\epsilon_{2, \lambda} t_{\lambda+\frac{7}{2}, \lambda+5} t_{\lambda+\frac{3}{2}, \lambda+\frac{7}{2}} - (1 + \epsilon_{2, \lambda}) t_{\lambda+3, \lambda+5} t_{\lambda+\frac{3}{2}, \lambda+3}) = 0, \quad (7.4)$$

$$t_{\lambda+\frac{7}{2}, \lambda+5} t_{\lambda+2, \lambda+\frac{7}{2}} t_{\lambda, \lambda+2} = 0 \quad (7.5)$$

are the integrability conditions of the infinitesimal deformation. The formal deformations with the greatest number of independent parameters are those corresponding to $t_{\lambda, \lambda + \frac{5}{2}} t_{\lambda + \frac{5}{2}, \lambda + 5} = t_{\lambda + \frac{7}{2}, \lambda + 5} t_{\lambda + 2, \lambda + \frac{7}{2}} t_{\lambda, \lambda + 2} = t_{\lambda, \lambda + \frac{3}{2}} = 0$. So, we must kill at least three parameters and there are six choices. Thus, there are only six deformations with eighteen independent parameters. Of course, there are many formal deformations with less than eighteen independent parameters. The deformation $\tilde{\mathfrak{L}}_{X_F} = \mathfrak{L}_{X_F} + \mathfrak{L}_{X_F}^{(1)} + \mathfrak{L}_{X_F}^{(2)} + \mathfrak{L}_{X_F}^{(3)}$, is the miniversal $\mathfrak{osp}(1|2)$ -trivial deformation of $\mathfrak{S}_{\lambda + \frac{5}{2}}^5$ with base $\mathcal{A} = \mathbb{C}[t]/\mathcal{R}$, where $t = (t_{\lambda, \lambda + \frac{3}{2}}, \dots)$ is the family of all parameters given in the expression of $\mathfrak{L}^{(1)}$ and \mathcal{R} is the ideal generated by the left-hand sides of (7.2)–(7.5). \square

Example 7.2. The $\mathcal{K}(1)$ -module $\mathfrak{S}_{\lambda + \frac{11}{2}}^{\frac{11}{2}}$.

Proposition 7.2. *The $\mathcal{K}(1)$ -module $\mathfrak{S}_{\lambda + \frac{11}{2}}^{\frac{11}{2}}$ admits 36 $\mathfrak{osp}(1|2)$ -trivial deformations with 17 independent parameters. These deformations are polynomial of degree 3.*

Proof. Any $\mathfrak{osp}(1|2)$ -trivial deformation of $\mathfrak{S}_{\lambda + \frac{11}{2}}^{\frac{11}{2}}$ is given by

$$\tilde{\mathfrak{L}}_{X_F} = \mathfrak{L}_{X_F} + \mathfrak{L}_{X_F}^{(1)} + \mathfrak{L}_{X_F}^{(2)} + \mathfrak{L}_{X_F}^{(3)}, \quad (7.6)$$

where

$$\begin{aligned} \mathfrak{L}^{(1)} = & t_{\lambda, \lambda + \frac{3}{2}} \Upsilon_{\lambda, \lambda + \frac{3}{2}} + t_{\lambda, \lambda + 2} \Upsilon_{\lambda, \lambda + 2} + t_{\lambda, \lambda + \frac{5}{2}} \Upsilon_{\lambda, \lambda + \frac{5}{2}} + t_{\lambda + \frac{1}{2}, \lambda + 2} \Upsilon_{\lambda + \frac{1}{2}, \lambda + 2} \\ & + t_{\lambda + \frac{1}{2}, \lambda + \frac{5}{2}} \Upsilon_{\lambda + \frac{1}{2}, \lambda + \frac{5}{2}} + t_{\lambda + \frac{1}{2}, \lambda + 3} \Upsilon_{\lambda + \frac{1}{2}, \lambda + 3} + t_{\lambda + 1, \lambda + \frac{5}{2}} \Upsilon_{\lambda + 1, \lambda + \frac{5}{2}} + t_{\lambda + 1, \lambda + 3} \Upsilon_{\lambda + 1, \lambda + 3} \\ & + t_{\lambda + 1, \lambda + \frac{7}{2}} \Upsilon_{\lambda + 1, \lambda + \frac{7}{2}} + t_{\lambda + \frac{3}{2}, \lambda + 3} \Upsilon_{\lambda + \frac{3}{2}, \lambda + 3} + t_{\lambda + \frac{3}{2}, \lambda + \frac{7}{2}} \Upsilon_{\lambda + \frac{3}{2}, \lambda + \frac{7}{2}} + t_{\lambda + \frac{3}{2}, \lambda + 4} \Upsilon_{\lambda + \frac{3}{2}, \lambda + 4} \\ & + t_{\lambda + 2, \lambda + \frac{7}{2}} \Upsilon_{\lambda + 2, \lambda + \frac{7}{2}} + t_{\lambda + 2, \lambda + 4} \Upsilon_{\lambda + 2, \lambda + 4} + t_{\lambda + 2, \lambda + \frac{9}{2}} \Upsilon_{\lambda + 2, \lambda + \frac{9}{2}} + t_{\lambda + \frac{5}{2}, \lambda + 4} \Upsilon_{\lambda + \frac{5}{2}, \lambda + 4} \\ & + t_{\lambda + \frac{5}{2}, \lambda + \frac{9}{2}} \Upsilon_{\lambda + \frac{5}{2}, \lambda + \frac{9}{2}} + t_{\lambda + \frac{5}{2}, \lambda + 5} \Upsilon_{\lambda + \frac{5}{2}, \lambda + 5} + t_{\lambda + 3, \lambda + \frac{9}{2}} \Upsilon_{\lambda + 3, \lambda + \frac{9}{2}} + t_{\lambda + 3, \lambda + 5} \Upsilon_{\lambda + 3, \lambda + 5} \\ & + t_{\lambda + 3, \lambda + \frac{11}{2}} \Upsilon_{\lambda + 3, \lambda + \frac{11}{2}} + t_{\lambda + \frac{7}{2}, \lambda + 5} \Upsilon_{\lambda + \frac{7}{2}, \lambda + 5} + t_{\lambda + \frac{7}{2}, \lambda + \frac{11}{2}} \Upsilon_{\lambda + \frac{7}{2}, \lambda + \frac{11}{2}} \\ & + t_{\lambda + 4, \lambda + \frac{11}{2}} \Upsilon_{\lambda + 4, \lambda + \frac{11}{2}}, \end{aligned}$$

$$\begin{aligned} \mathfrak{L}^{(2)} = & - \sum_{\mu} \zeta_{\mu}^{-1} t_{\mu + \frac{3}{2}, \mu + 3} t_{\mu, \mu + \frac{3}{2}} \mathfrak{J}_4^{-1, \mu} \\ & - \sum_{\nu} \alpha_{\nu}^{-1} (t_{\nu + \frac{3}{2}, \nu + \frac{7}{2}} t_{\nu, \nu + \frac{3}{2}} - t_{\nu + 2, \nu + \frac{7}{2}} t_{\nu, \nu + 2}) \mathfrak{J}_{\frac{9}{2}}^{-1, \nu} \\ & - \sum_{\varepsilon} \beta_{\varepsilon}^{-1} \left(t_{\varepsilon + \frac{3}{2}, \varepsilon + 4} t_{\varepsilon, \varepsilon + \frac{3}{2}} + t_{\varepsilon + \frac{5}{2}, \varepsilon + 4} t_{\varepsilon, \varepsilon + \frac{5}{2}} + \frac{1}{3} t_{\varepsilon + 2, \varepsilon + 4} t_{\varepsilon, \varepsilon + 2} \right) \mathfrak{J}_5^{-1, \varepsilon} \\ & - \sum_{\ell} \gamma_{\ell}^{-1} (t_{\ell + \frac{5}{2}, \ell + \frac{9}{2}} t_{\ell, \ell + \frac{5}{2}} - t_{\ell + 2, \ell + \frac{9}{2}} t_{\ell, \ell + 2}) \mathfrak{J}_{\frac{11}{2}}^{-1, \ell}, \end{aligned}$$

$$\begin{aligned} \mathfrak{L}^{(3)} = & \sum_{\ell} \xi_{\ell}^{-1} t_{\ell + 3, \ell + \frac{9}{2}} t_{\ell + \frac{3}{2}, \ell + 3} t_{\ell, \ell + \frac{3}{2}} \mathfrak{J}_{\frac{11}{2}}^{-1, \ell} \\ & + \sum_{\iota} \epsilon_{3, \iota} (t_{\iota + 3, \iota + 5} t_{\iota + \frac{3}{2}, \iota + 3} - t_{\iota + \frac{7}{2}, \iota + 5} t_{\iota + \frac{3}{2}, \iota + \frac{7}{2}}) t_{\iota, \iota + \frac{3}{2}} \mathfrak{J}_6^{-1, \iota} \end{aligned}$$

with $\mu \in \{\lambda, \lambda + \frac{1}{2}, \lambda + 1, \lambda + \frac{3}{2}, \lambda + 2, \lambda + \frac{5}{2}\}$, $\nu \in \{\lambda, \lambda + \frac{1}{2}, \lambda + 1, \lambda + \frac{3}{2}, \lambda + 2\}$, $\epsilon \in \{\lambda, \lambda + \frac{1}{2}, \lambda + 1, \lambda + \frac{3}{2}\}$, $\ell \in \{\lambda, \lambda + \frac{1}{2}, \lambda + 1\}$ and $\iota \in \{\lambda, \lambda + \frac{1}{2}\}$. The integrability conditions of this infinitesimal deformation vanishing of the following polynomials, where in the first four lines $\mu \in \{\lambda, \lambda + \frac{1}{2}\}$:

$$\begin{aligned}
& t_{\mu, \mu + \frac{5}{2}} t_{\mu + \frac{5}{2}, \mu + 5}, \\
& t_{\mu + \frac{7}{2}, \mu + 5} t_{\mu + 2, \mu + \frac{7}{2}} t_{\mu, \mu + 2}, \\
& t_{\mu, \mu + \frac{3}{2}} (\epsilon_{1, \mu} t_{\mu + 3, \mu + 5} t_{\mu + \frac{3}{2}, \mu + 3} + (1 - \epsilon_{1, \mu}) t_{\mu + \frac{7}{2}, \mu + 5} t_{\mu + \frac{3}{2}, \mu + \frac{7}{2}}), \\
& t_{\mu, \mu + \frac{3}{2}} (\epsilon_{2, \mu} t_{\mu + \frac{7}{2}, \mu + 5} t_{\mu + \frac{3}{2}, \mu + \frac{7}{2}} - (1 + \epsilon_{2, \mu}) t_{\mu + 3, \mu + 5} t_{\mu + \frac{3}{2}, \mu + 3}), \\
& t_{\lambda, \lambda + \frac{3}{2}} t_{\lambda + 3, \lambda + \frac{11}{2}} t_{\lambda + \frac{3}{2}, \lambda + 3}, \\
& t_{\lambda + 4, \lambda + \frac{11}{2}} t_{\lambda + \frac{5}{2}, \lambda + 4} t_{\lambda, \lambda + \frac{5}{2}}, \\
& t_{\lambda + 4, \lambda + \frac{11}{2}} (3(1 + \epsilon_{4, \lambda}) t_{\lambda + \frac{3}{2}, \lambda + 4} t_{\lambda, \lambda + \frac{3}{2}} + t_{\lambda + 2, \lambda + 4} t_{\lambda, \lambda + 2}) + \epsilon_{4, \lambda} t_{\lambda, \lambda + \frac{3}{2}} t_{\lambda + \frac{7}{2}, \lambda + \frac{11}{2}} t_{\lambda + \frac{3}{2}, \lambda + \frac{7}{2}}, \\
& t_{\lambda + \frac{7}{2}, \lambda + \frac{11}{2}} \left(\left(1 + \frac{\epsilon_{5, \lambda}}{3}\right) t_{\lambda + \frac{3}{2}, \lambda + \frac{7}{2}} t_{\lambda, \lambda + \frac{3}{2}} - t_{\lambda + 2, \lambda + \frac{7}{2}} t_{\lambda, \lambda + 2} \right) + \epsilon_{5, \lambda} t_{\lambda + 4, \lambda + \frac{11}{2}} t_{\lambda + \frac{3}{2}, \lambda + 4} t_{\lambda, \lambda + \frac{3}{2}}, \\
& \epsilon_{6, \lambda} t_{\lambda, \lambda + \frac{3}{2}} \left(t_{\lambda + 4, \lambda + \frac{11}{2}} t_{\lambda + \frac{3}{2}, \lambda + 4} + \frac{1}{3} t_{\lambda + \frac{7}{2}, \lambda + \frac{11}{2}} t_{\lambda + \frac{3}{2}, \lambda + \frac{7}{2}} \right) \\
& + t_{\lambda, \lambda + 2} (t_{\lambda + \frac{7}{2}, \lambda + \frac{11}{2}} t_{\lambda + 2, \lambda + \frac{7}{2}} - t_{\lambda + 4, \lambda + \frac{11}{2}} t_{\lambda + 2, \lambda + 4}).
\end{aligned} \tag{7.7}$$

These deformations are with 24 parameters $t_{\mu, \nu}$ which are subject to conditions (7.7). Obviously, we can construct many $\mathfrak{osp}(1|2)$ -trivial deformation of $\mathfrak{S}_{\lambda + \frac{11}{2}}$ with independent parameters. But, to have the greatest number of independent parameters, we see that we must kill at least seven parameters, that is, we put

$$t_{\lambda, \lambda + \frac{3}{2}} = t_{\lambda + \frac{1}{2}, \lambda + 2} = t_{\lambda, \lambda + 2} = 0 \quad \text{and} \quad t_{\mu, \mu + \frac{5}{2}} t_{\mu + \frac{5}{2}, \mu + 5} = t_{\mu + \frac{7}{2}, \mu + 5} t_{\mu + 2, \mu + \frac{7}{2}} t_{\mu, \mu + 2} = 0$$

where $\mu = \lambda$ or $\lambda + \frac{1}{2}$. So, there are 36 possible choices of such parameters. \square

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