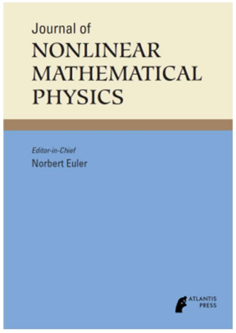

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

Global Analytic First Integrals for the Simplified Multistrain/Two-Stream Model for Tuberculosis and Dengue Fever

Jaume Llibre, Clàudia Valls

Article

Journal of Nonlinear Mathematical Physics, Vol. 16, No. 4 (2009) 505–516

GLOBAL ANALYTIC FIRST INTEGRALS FOR THE SIMPLIFIED MULTISTRAIN/TWO-STREAM MODEL FOR TUBERCULOSIS AND DENGUE FEVER

JAUME LLIBRE[*,‡] and CLÀUDIA VALLS[†,§]

[*]*Department de Matemàtiques*
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Spain

[†]*Departamento de Matemática*
Instituto Superior Técnico
Av. Rovisco Pais 1049-001, Lisboa, Portugal
[‡]*jllibre@mat.uab.cat*
[§]*cvalls@math.ist.utl.pt*

Received 14 April 2009
Accepted 10 June 2009

We provide the complete classification of all global analytic first integrals of the simplified multistrain/two-stream model for tuberculosis and dengue fever that can be written as

$$\dot{x} = x(\beta_1 - b - \gamma_1 - \beta_1 x - (\beta_1 - \nu)y), \quad \dot{y} = y(\beta_2 - b - \gamma_2 - (\beta_2 + \nu)x - \beta_2 y),$$

with $\beta_1, \beta_2, b, \gamma_1, \gamma_2, \nu \in \mathbb{R}$.

Keywords: Analytic first integral; multistrain/two-stream model; tuberculosis; dengue fever.

1. Introduction

The nonlinear ordinary differential equations or simply the differential systems appear in many branches of applied mathematics, physics, and in general in applied sciences. Since generically the differential systems cannot be solved explicitly, the qualitative information provided by the theory of dynamical systems is, in general, the best that one can expect to obtain.

For a two-dimensional differential system the existence of a first integral determines completely its phase portrait, i.e. the description of the domain of definition of the differential system as union of all the orbits or trajectories of the system. To provide the phase portrait of a differential system is the main objective of the qualitative theory of the differential systems. Thus for two-dimensional differential systems one of the main questions is: How to recognize if a given planar differential system has a first integral?

In this paper we characterize for a planar differential system depending on six parameters what are the values of these parameters for which there exists a global analytic first integral. For those systems having such a first integral it is possible to describe their phase portraits, and to understand all their qualitative dynamics. Moreover these integrable systems also provide some information about the dynamics of the nearest systems. Now we shall present the planar differential system whose integrability we shall study in this paper.

Incomplete treatment of patients with infectious tuberculosis may not only lead to relapse but also to the development of antibiotic resistant, which is one of the most serious health problems in our society. In this direction there are some different models. The differential system studied here is the simplified multistrain model of [2] for the transmition of tuberculosis and to the coupled two-stream vector-based model in [3]. This model has been poorly studied up to now, in the sense that only the behavior of the dynamics near the equilibrium points has been well understood, see [3]. Better contributions for understanding the global dynamics of this model has been done in [7] from the viewpoint of symmetry analysis and in [8] using the singularity analysis theory, where the authors identify some combinations of parameters for which the system has a first integral. Our goal is to provide a complete description of all global analytic first integrals that this system exhibits for different sets of values of the parameters.

More precisely, the model which we want to discuss in detail was presented in [9] (see Eq. (14)). This model is

$$\begin{aligned}\dot{x} &= x(-b-\gamma_1+\nu y+\beta_1 z),\\ \dot{y} &= y(-b-\gamma_2-\nu x+\beta_2 z),\\ \dot{z} &= -b(z-1)+\gamma_1 x+\gamma_2 y-(\beta_1 x+\beta_2 y)z,\end{aligned} \tag{1}$$

where β_1 and β_2 represent the infection rates for the two strains in the case of the tuberculosis model and for the two vectors in the Dengue fever model, ν is the common contact rate of infection, b is the common birth and death rate and γ_1 and γ_2 are the recovery rates. This model is a caricature of the system in [2] and has two infections compartments corresponding to the two infectious agents. The variables represent proportions of a constant population which has been scaled to unity, that is, $x+y+z=1$. Then imposing the constraint $z=1-x-y$ to our model (1) we get that the three-dimensional system (1) becomes the two-dimensional system

$$\begin{aligned}\dot{x} &= x(\beta_1-b-\gamma_1-\beta_1 x-(\beta_1-\nu)y),\\ \dot{y} &= y(\beta_2-b-\gamma_2-(\beta_2+\nu)x-\beta_2 y),\end{aligned} \tag{2}$$

in $\mathbb{R}^2$.

Our objective is to characterize the existence of global analytic first integrals of system (2). Here *a global analytic first integral* or simply *an analytic first integral* is a non-constant analytic function $H\colon \mathbb{R}^2 \to \mathbb{R}$ whose domain of definition is the whole $\mathbb{R}^2$, and it is constant on the solutions of system (2). This last assertion means that for any solution $(x(t), y(t))$ of (2) we have that

$$\frac{dH}{dt}(x(t),y(t)) = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = 0.$$

We shall provide a full classification of the existence of global analytic first integrals for system (2).

Our main theorem is the following. The explicit expressions for the global analytic first integrals can be found in the proof of the theorem. Along this paper $\mathbb{N}$ denotes the set of positive integers, $\mathbb{Z}^+$ denotes the set of non-negative integers, $\mathbb{Q}^+$ denotes the set of non-negative rational numbers and $\mathbb{Q}^-$ denotes the set of negative rational numbers.

Theorem 1. *The unique systems* (2) *having a global analytic first integral are the following ones.*

(a) $\beta_1 = 0$, $\beta_2 - b - \gamma_2 \neq 0$ *with*

- (a.1) $\nu = 0$ *and* $b = -\gamma_1$;
- (a.2) $\nu = 0$ *and* $\beta_2 = 0$ *and* $(b+\gamma_1)/(b+\gamma_2) \in \mathbb{Q}^-$;
- (a.3) $\nu \neq 0$, $\beta_2 = 0$, $(b+\gamma_1)/(-b-\gamma_2) \in \mathbb{Q}^+ \setminus \{0\}$;
- (a.4) $\nu(\nu+\beta_2) \neq 0$, $b+\gamma_1 = 0$, $\beta_2 \in \mathbb{Q}^+$.

(b) $\beta_1 = 0$, $\beta_2 - b - \gamma_2 = 0$ *with*

- (b.1) $\nu = 0$ *and* $b = -\gamma_1$;
- (b.2) $\nu = 0$ *and* $\beta_2 = 0$;
- (b.3) $\nu \neq 0$, $b+\gamma_1 = 0$, $\beta_2/\nu \in \mathbb{Q}^+$;
- (b.4) $\nu(b+\gamma_1) \neq 0$, $\beta_2 = 0$.

(c) $\beta_1 \neq 0$, $\beta_2 - b - \gamma_2 = 0$ *with*

- (c.1) $\nu = 0$ *and* $\beta_2 = 0$;
- (c.2) $\nu \neq 0$, $\beta_2 = 0$, *and* $\beta_1 \in \mathbb{Q}^- \cup \{0\}$;
- (c.3) $\beta_2(\beta_2+\nu) \neq 0$, $\beta_1 - b - \gamma_1 = 0$, $\left(\frac{\beta_1}{\beta_2+\nu}, \frac{\beta_1-\nu}{\beta_2}\right) \neq (1,1)$ *and* $\frac{\beta_1-\beta_2-\nu}{\beta_2+\nu}$, $(1-\nu+\beta_1)\frac{\beta_1(\beta_2+\nu-\beta_1)}{\beta_2(\beta_2+\nu)}$ *and* $\frac{\beta_1-\nu}{\beta_2} - \frac{\beta_1}{\beta_2+\nu}$ *have all the same signs.*

(d) $\beta_1(\beta_2 - \beta_1 + \nu)(\beta_2 - b - \gamma_2) \neq 0$, $\gamma_1 = -(b(p+q)+\gamma_2 q)/p$, $\beta_2 = -\beta_1 p/q$, $\nu = \beta_1(qq_1 - pp_1)/(qq_1)$, *with* $p, q, p_1, q_1 \in \mathbb{N}$ *and* $pp_1 - qq_1 \geq 0$.

The proof of Theorem 1 is given in Sec. 2.

2. Proof of Theorem 1

Note that system (2) is a special case of the quadratic Lotka–Volterra systems

$$\begin{aligned} \dot{x} &= x(ax+by+c), \\ \dot{y} &= y(Ax+By+C), \end{aligned} \tag{3}$$

where $a, b, c, A, B, C \in \mathbb{R}$. The existence of global analytic first integrals for system (3) was been studied in [5]. The authors in [5] reduce the study of the 6 parameter family of the quadratic Lotka–Volterra system (3) to the study of 12 subfamilies having 1, 2, 3 or 4 parameters. More precisely, for completeness of the paper, we provide the main results concerning the global analytic integrability of system (3) in the appendix of the paper. The strategy of the proof of Theorem 1 will be as follows: we will put system (2) in one of the subfamilies of system (3) in Theorem 4 and then we will apply the results of Theorem 5.

2.1. *Preliminary results*

In this section we introduce two auxiliary results that will be used through the paper. We write Eq. (2) as the system

$$\dot{x} = f_1(x, y), \quad \dot{y} = f_2(x, y). \tag{4}$$

Let $f(x, y) = (f_1(x, y), f_2(x, y))$. We will denote by $Df(0)$ the Jacobian matrix of system (14) at $(x, y) = (0, 0)$ and by Df the Jacobian matrix of system (14) at an arbitrary point (x, y) that will be explicitly specified.

The following result is due to Poincaré (see [1]) and its proof can be found in [4].

Theorem 2. *Assume that the eigenvalues λ_1 and λ_2 of Df at some singular point $(\bar{x}, \bar{y})$ do not satisfy any resonance condition of the form*

$$\lambda_1 k_1 + \lambda_2 k_2 = 0 \quad \textit{for } k_1, k_2 \in \mathbb{Z}^+ \textit{ with } k_1 + k_2 > 0.$$

Then system (14) *has no global analytic first integrals.*

The following result was proved in [6].

Theorem 3. *Assume that the eigenvalues λ_1 and λ_2 of Df at some singular point $(x, y) = (\bar{x}, \bar{y})$ satisfy that $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Then system* (14) *has no global analytic first integrals if the singular point $(x, y) = (\bar{x}, \bar{y})$ is isolated.*

2.2. *Proof of Theorem 1*

We separate the proof into different cases.

Case 1: If $\beta_1(\beta_2 - \beta_1 + \nu)(\beta_2 - b - \gamma_2) = 0$. We consider different subcases.

Subcase 1.1: If $\beta_1 = 0$, $\beta_2 - b - \gamma_2 \neq 0$ and $-\beta_2 + \beta_1 - \nu$ arbitrary. Then doing the rescaling

$$(x, y, t) \to \left(\alpha x, \beta y, \frac{t}{\beta_2 - b - \gamma_2}\right)$$

in system (2) we obtain the system

$$\begin{aligned} \dot{x} &= x\left(\frac{\nu\beta}{\beta_2 - b - \gamma_2} y - \frac{b + \gamma_1}{\beta_2 - b - \gamma_2}\right), \\ \dot{y} &= y\left(-\frac{(\beta_2 + \nu)\alpha}{\beta_2 - b - \gamma_2} x - \frac{\beta_2\beta}{\beta_2 - b - \gamma_2} y + 1\right). \end{aligned} \tag{5}$$

We consider three subcases.

Subcase 1.1.1: $\nu = 0$. Then system (2) becomes

$$\begin{aligned} \dot{x} &= -\frac{b + \gamma_1}{\beta_2 - b - \gamma_2} x, \\ \dot{y} &= y\left(-\frac{\beta_2\alpha}{\beta_2 - b - \gamma_2} x - \frac{\beta_2\beta}{\beta_2 - b - \gamma_2} y + 1\right) \end{aligned}$$

which is system (lv2) (see Theorem 4). We can take $\alpha = \beta = 1$. In view of Theorem 5, we have a global analytic first integral if and only if:

(1) $b = -\gamma_1$ and in this case a global analytic first integral is $H = x$.
(2) $\beta_2 = 0$ and $\frac{b+\gamma_1}{b+\gamma_2} = -\frac{p}{q}$ with $p, q \in \mathbb{N}$. Then a global analytic first integral is $H = x^q y^p$.

Subcase 1.1.2: $\nu \neq 0$ and $\beta_2 + \nu = 0$. Note that $\beta_2 \neq 0$. Taking $\beta = (\beta_2 - b - \gamma_2)/\nu$, system (5) becomes

$$\dot{x} = x\left(y - \frac{b+\gamma_1}{\beta_2 - b - \gamma_2}\right), \quad \dot{y} = y(y+1),$$

which is system (lv3), after a redefinition of the parameters (see Theorem 4). In view of Theorem 5, and since the coefficient of y^2 in $\dot{y}$ is 1, in this subcase there is no analytic first integral.
Subcase 1.1.3: $\nu(\beta_2 + \nu) \neq 0$. Taking

$$\alpha = -\frac{\beta_2 - b - \gamma_2}{\beta_2 + \nu} \quad \text{and} \quad \beta = \frac{\beta_2 - b - \gamma_2}{\nu},$$

system (5) becomes

$$\dot{x} = x\left(y - \frac{b+\gamma_1}{\beta_2 - b - \gamma_2}\right), \quad \dot{y} = y\left(x - \frac{\beta_2}{\nu}y + 1\right). \tag{6}$$

We consider two subcases.

Subcase 1.1.3.1: $b + \gamma_1 \neq 0$.
We first assume $\frac{b+\gamma_1}{\beta_2 - b - \gamma_2} \notin \mathbb{Q}^+$. The eigenvalues of $Df(0)$ are $-\frac{b+\gamma_1}{\beta_2 - b - \gamma_2}$ and 1. By the hypotheses for any $k_1, k_2 \in \mathbb{Z}^+$ with $k_1 + k_2 > 0$ we have $k_1 - k_2\frac{b+\gamma_1}{\beta_2 - b - \gamma_2} \neq 0$. Thus, by Theorem 2 system (6) has no global analytic first integrals.
We assume that $\frac{b+\gamma_1}{\beta_2 - b - \gamma_2} \in \mathbb{Q}^+ \setminus 0$. We write it as $\frac{b+\gamma_1}{\beta_2 - b - \gamma_2} = p/q$ with $p, q \in \mathbb{N}$. We consider different subcases.

Subcase 1.1.3.1.1: $\beta_2 = 0$. In this case system (6) becomes $\dot{x} = x(y - p/q)$ and $\dot{y} = y(x+1)$. For this system it is easy to check that $e^{q(x-y)}x^q y^p$ is a first integral which is global analytic.
Subcase 1.1.3.1.2: $\frac{\beta_2}{\nu} \notin \mathbb{Q}^+$. The eigenvalues of $Df(0, \nu/\beta_2)$ are $\frac{\nu}{\beta_2} - \frac{p}{q}$ and -1. By the hypotheses for any $k_1, k_2 \in \mathbb{Z}^+$ with $k_1 + k_2 > 0$ we have $-k_1 - k_2(-\frac{\nu}{\beta_2} + \frac{p}{q}) \neq 0$. Thus, by Theorem 2 system (6) has no global analytic first integrals.
Subcase 1.1.3.1.3: $\beta_2/\nu = p_1/q_1$ with $p_1, q_1 \in \mathbb{N}$. Assume now $pp_1 - qq_1 \leq 0$. The eigenvalues of $Df((pp_1 - qq_1)/(qq_1), p/q)$ are

$$\lambda_{1,2} = \frac{-pp_1}{2qq_1} \pm \frac{1}{2}\sqrt{\frac{(pp_1)^2}{(qq_1)^2} + 4\frac{p}{q^2 q_1}(pp_1 - qq_1)}.$$

If $pp_1 - qq_1 = 0$, then the eigenvalues of $Df((pp_1 - qq_1)/(qq_1), p/q)$ are $\lambda_1 = -1$ and $\lambda_2 = 0$. Therefore, since $(x, y) = ((pp_1 - qq_1)/(qq_1), p/q)$ is isolated, by Theorem 3 we get that system (6) has no analytic first integrals.

If $pp_1 - qq_1 < 0$, then the eigenvalues of $Df((pp_1 - qq_1)/(qq_1), p/q)$ satisfy $\lambda_1\lambda_2 = \frac{p(qq_1-pp_1)}{q^2q_1} > 0$. Therefore, for any $k_1, k_2 \in \mathbb{Z}^+$ with $k_1+k_2 > 0$, we have that $\lambda_1 k_1+\lambda_2 k_2 \neq 0$. Thus by Theorem 2 we get that system (6) has no analytic first integrals.

It remains to study the case $pp_1 - qq_1 > 0$. System (6) becomes

$$\dot{x} = x\left(y - \frac{p}{q}\right), \quad \dot{y} = y\left(x - \frac{p_1}{q_1}y + 1\right). \tag{7}$$

We will proceed by contradiction. Assume that $F(x, y)$ is an analytic first integral of system (7). Without loss of generality we can always assume that it has no constant term. Then $F(x, y)$ must satisfy

$$\frac{\partial F}{\partial x}x\left(y - \frac{p}{q}\right) + y\left(x - \frac{p_1}{q_1}y + 1\right)\frac{\partial F}{\partial y} = 0. \tag{8}$$

We write

$$F(x, y) = \sum_{k\geq 0} F_k(y)x^k. \tag{9}$$

We will prove by induction that

$$F_k(y) = 0 \quad \text{for } k \geq 0. \tag{10}$$

Clearly, $F_0(y)$ satisfies (8) restricted to $x = 0$, that is,

$$y\left(-\frac{p_1}{q_1} - y + 1\right)\frac{dF_0}{dy} = 0,$$

and since F has no constant term $F_0(y) = 0$ and (10) is proved for $k = 0$. Now we assume that (10) is true for $k = N - 1$ (with $N \geq 1$) and we will prove it for $k = N$. In view of (9) we have

$$F(x, y) = x^N G(x, y) = x^N \sum_{k\geq 0} F_{N+k}(y)x^k,$$

where $G(x, y)$ satisfies, after simplifying by x^N,

$$N\left(y - \frac{p}{q}\right)G + x\left(y - \frac{p}{q}\right)\frac{\partial G}{\partial x} + y\left(x - \frac{p_1}{q_1}y + 1\right)\frac{\partial G}{\partial y} = 0. \tag{11}$$

Restricting (11) to $x = 0$ (since $G(0, y) = F_N(y)$) we get

$$N\left(y - \frac{p}{q}\right)F_N(y) + y\left(-\frac{p_1}{q_1}y + 1\right)\frac{dF_N}{dy} = 0$$

that is

$$\frac{dF_N}{F_N} = -\frac{N\left(y - \frac{p}{q}\right)}{y\left(-\frac{p_1}{q_1}y + 1\right)}\,dy$$

which yields

$$F_N(y) = K_N y^{\frac{Np}{q}}(q_1 - p_1 y)^{\frac{N}{p_1 q}(qq_1 - pp_1)}, \quad K_N \in \mathbb{R}.$$

Since $qq_1 - pp_1 < 0$ and $F_N(y)$ must be global analytic it follows that $K_N = 0$ and then $F_N(y) = 0$ which concludes the proof of (10).

Subcase 1.1.3.2: $b + \gamma_1 = 0$. We obtain

$$\dot{x} = xy, \quad \dot{y} = y(x - \beta_2 y + 1),$$

which is system (lv4) (see Theorem 4). In view of Theorem 5, it has a global analytic first integral if and only if one of the following two conditions hold:

(1) $\beta_2 = 0$ and a global analytic first integral is $H = xe^{x-y}$;
(2) $\beta_2 = p/q$ with $p, q \in \mathbb{N}$ and a global analytic first integral is $H = x^p(y - (q/p) - [q/(q+p)]x)^q$.

Subcase 1.2: $\beta_1 = 0$, $\beta_2 - b - \gamma_2 = 0$ and $\beta_2 + \nu$ arbitrary. Then doing the rescaling $(x, y, t) \to (\alpha x, \beta y, \gamma t)$ in system (3) we obtain the system

$$\begin{aligned} \dot{x} &= x(\nu\beta\gamma y - (b + \gamma_1)\gamma), \\ \dot{y} &= y(-(\beta_2 + \nu)\alpha\gamma x - \beta_2\beta\gamma y). \end{aligned} \tag{12}$$

We consider three subcases.

Subcase 1.2.1: $\nu = 0$. We take $\alpha = \beta = \gamma = 1$ and we obtain the system

$$\dot{x} = -(b + \gamma_1)x, \quad \dot{y} = -\beta_2 y(x + y),$$

which is a particular case of system (lv2) (see Theorem 4). By Theorem 5 we have a global analytic first integral if and only if:

(1) $b + \gamma_1 = 0$, and a global analytic first integral is $H = x$.
(2) $\beta_2 = 0$ and a global analytic first integral is $H = y$.

Subcase 1.2.2: $\nu \neq 0$ and $b + \gamma_1 = 0$. Taking $\beta = \frac{1}{\nu}$, $\gamma = 1$ and $\alpha = 1$ in Eq. (12) we get

$$\dot{x} = xy, \quad \dot{y} = y\left(-(\beta_2 + \nu)x - \frac{\beta_2}{\nu}y\right),$$

which is a particular case of system (lv5) (see Theorem 4). By Theorem 5, we have a global analytic first integral if and only if one of the following two conditions hold:

(1) $\beta_2 = 0$ and a global analytic first integral is $H = -(\beta_2 + \nu)x - y$.
(2) $\beta_2 = p\nu/q$ with $p, q \in \mathbb{N}$ and a global analytic first integral is $H = x^p(-(\beta_2 + \nu)qx - (p + q)y)^q$.

Subcase 1.2.3: $\nu(b + \gamma_1) \neq 0$. Taking $\beta = -\frac{b+\gamma_1}{\nu}$ and $\gamma = -\frac{1}{b+\gamma_1}$ in Eq. (12) we get

$$\dot{x} = x(y + 1), \quad \dot{y} = y\left(\frac{\beta_2 + \nu}{b + \gamma_1}\alpha x - \frac{\beta_2}{\nu}y\right).$$

We consider two subcases.

Subcase 1.2.3.1: $\beta_2 + \nu = 0$. Then

$$\dot{x} = x(y + 1), \quad \dot{y} = y^2$$

which is a particular case of system (lv6) (see Theorem 4). By Theorem 5 it has no global first integral.

Subcase 1.2.3.2: $\beta_2 + \nu \neq 0$. Taking $\alpha = \frac{b+\gamma_1}{\beta_2+\nu}$ we obtain

$$\dot{x} = x(y+1), \quad \dot{y} = y\left(x - \frac{\beta_2}{\nu} y\right)$$

which is system (lv7) (see Theorem 4). In view of Theorem 5 we have a global analytic first integral if and only if $\beta_2 = 0$ and then $H = ye^{y-x}$.

Subcase 1.3: $\beta_1 \neq 0$, $\beta_2 - b - \gamma_2 = 0$ and $\beta_1 - \beta_2 - \nu$ arbitrary. Then doing the rescaling $(x, y, t) \to (\alpha x, \beta y, \gamma t)$ in system (2) we obtain

$$\begin{aligned}\dot{x} &= x(-\beta_1 \alpha\gamma x + (\nu - \beta_1)\beta\gamma y + (\beta_1 - b - \gamma_1)\gamma),\\ \dot{y} &= y(-(\beta_2 + \nu)\alpha\gamma x - \beta_2 \beta\gamma y).\end{aligned}$$

We consider four different subcases.

Subcase 1.3.1: $\beta_2 + \nu = 0$. Then we take $\alpha = \beta = \gamma = 1$ and we obtain the system

$$\begin{aligned}\dot{x} &= x(-\beta_1 x - (\beta_1 + \beta_2)y + (\beta_1 - b - \gamma_1)),\\ \dot{y} &= -\beta_2 y^2,\end{aligned}$$

which is system (lv6) (see Theorem 4). In view of Theorem 5 it has a global analytic first integral if and only if $\beta_2 = 0$ and in this case a global analytic first integral is $H = y$.

Subcase 1.3.2: $\beta_2 + \nu \neq 0$ and $\beta_2 = 0$. Taking $\alpha = -\frac{1}{\nu}$, $\gamma = 1$ and $\beta = -\frac{1}{\nu}$ we get the system

$$\dot{x} = x\left(\frac{\beta_1}{\nu} x - (\nu - \beta_1)y + (\beta_1 - b - \gamma_1)\right), \quad \dot{y} = xy,$$

which is system (lv8) (see Theorem 4). In view of Theorem 5 we have that it has a global analytic first integrals if and only if one of the following conditions hold:

(1) $\beta_1 = 0$ then a global analytic first integral is $H = y^{b+\gamma_1} e^{x-\nu y}$.
(2) $\beta_1 = -p/q$ with $p, q \in \mathbb{N}$ then a global analytic first integral is

$$H = y^p \left(\frac{p+q}{q}(b + \gamma_1 - \beta_1) + \frac{p}{q} x + \frac{p^2}{q^2} x - \frac{p}{q}(\nu - \beta_1) y\right)^q.$$

Subcase 1.3.3: $\beta_2(\beta_2 + \nu) \neq 0$ and $\beta_1 - b - \gamma_1 = 0$. Taking $\alpha = -\frac{1}{\beta_2+\nu}$, $\gamma = 1$ and $\beta = -\frac{1}{\beta_2}$ we get that system (2) becomes

$$\dot{x} = x\left(\frac{\beta_1}{\beta_2 + \nu} x - \frac{\nu - \beta_1}{\beta_2} y\right), \quad \dot{y} = x(x+y),$$

which is the same as system (lv9) (see Theorem 4). In view of Theorem 5 it has a global analytic first integral if and only if $\left(\frac{\beta_1}{\beta_2+\nu}, \frac{\beta_1-\nu}{\beta_2}\right) \neq (1,1)$ and $\frac{\beta_1-\beta_2-\nu}{\beta_2+\nu}$, $(1-\nu+\beta_1)\frac{\beta_1(\beta_2+\nu-\beta_1)}{\beta_2(\beta_2+\nu)}$ and $\frac{\beta_1-\nu}{\beta_2} - \frac{\beta_1}{\beta_2+\nu}$ have all the same signs, and a global analytic first integral is

$$H = x^{\left|\frac{\beta_1-\beta_2-\nu}{\beta_2+\nu}\right|} y^{\left|\frac{\beta_1(\beta_2+\nu-\beta_1)}{\beta_2(\beta_2+\nu)}\right|} \left(\frac{\beta_1 - \beta_2 - \nu}{\beta_2 + \nu} x + \frac{\beta_2 - \nu + \beta_1}{\beta_2} y\right)^{\left|\frac{\beta_1-\nu}{\beta_2} - \frac{\beta_1}{\beta_2+\nu}\right|}.$$

Subcase 1.3.4: $\beta_2(\beta_2+\nu)(\beta_1-b-\gamma_1)\neq 0$. Taking $\gamma=\frac{1}{\beta_1-b-\gamma_1}$, $\alpha=-\frac{\beta_1-b-\gamma_1}{\beta_2+\nu}$, and $\beta=-\frac{\beta_1-b-\gamma_1}{\beta_2}$, we get

$$\dot{x}=x\left(\frac{\beta_1}{\beta_2+\nu}x+\frac{\beta_1-\nu}{\beta_2}y+1\right),\quad \dot{y}=y(x+y),$$

which is system (lv10) (see Theorem 4). In view of Theorem 5 we have no global analytic first integrals.

Subcase 1.4: $\beta_1\neq 0$, $\beta_2-b-\gamma_2\neq 0$ and $\nu=\beta_1-\beta_2$. Then, doing the rescaling of variables

$$(x,y,t)\rightarrow\left(-\frac{\beta_2-b-\gamma_2}{\beta_1}x,\beta y,(\beta_2-b-\gamma_2)t\right)$$

in system (3) we obtain the system

$$\begin{aligned}\dot{x}&=x\left(x-\frac{\beta_2\beta}{\beta_2-b-\gamma_2}y+\frac{\beta_1-b-\gamma_1}{\beta_2-b-\gamma_2}\right),\\ \dot{y}&=y\left(x-\frac{\beta_2\beta}{\beta_2-b-\gamma_2}y+1\right).\end{aligned}\tag{13}$$

We distinguish the following two subcases.

Subcase 1.4.1: $\beta_2=0$. Then we obtain that system (13) becomes

$$\begin{aligned}\dot{x}&=x\left(x+\frac{\beta_1-b-\gamma_1}{-b-\gamma_2}\right),\\ \dot{y}&=y(x+1)\end{aligned}\tag{14}$$

which is system (lv11) (see Theorem 4), and by Theorem 5 we have no global analytic first integrals.

Subcase 1.4.2: $\beta_2\neq 0$. Taking $\beta=-\frac{\beta_2-b-\gamma_2}{\beta_2}$, system (13) goes over to the system

$$\begin{aligned}\dot{x}&=x\left(x+y+\frac{\beta_1-b-\gamma_1}{\beta_2-b-\gamma_2}\right),\\ \dot{y}&=y(x+y+1).\end{aligned}\tag{15}$$

We first consider the case in which $\frac{\beta_1-b-\gamma_1}{\beta_2-b-\gamma_2}\notin\mathbb{Q}^-$. In this case the eigenvalues of $Df(0)$ are $\frac{\beta_1-b-\gamma_1}{\beta_2-b-\gamma_2}$ and 1. By the hypotheses for any $k_1,k_2\in\mathbb{Z}^+$ with $k_1+k_2>0$, we have $k_1+k_2\frac{\beta_1-b-\gamma_1}{\beta_2-b-\gamma_2}\neq 0$. Thus by Theorem 2 system (15) has no analytic first integrals.

Now we consider the case in which $\beta_1-b-\gamma_1=0$. In this case the eigenvalues of $Df(0)$ are 1 and 0. Therefore, since $(x,y)=(0,0)$ is isolated, by Theorem 3 we get that system (15) has no analytic first integrals.

Finally, we consider the case $\frac{\beta_1-b-\gamma_1}{\beta_2-b-\gamma_2} \in \mathbb{Q}^- \setminus \{0\}$. Let $\frac{\beta_1-b-\gamma_1}{\beta_2-b-\gamma_2} = -p/q \in \mathbb{Q}^-$. Then system (15) becomes

$$\begin{aligned}\dot{x} &= x(x+y-p/q),\\ \dot{y} &= y(-(\beta_2+\nu)x+y+1).\end{aligned} \tag{16}$$

The eigenvalues of $Df(0,-1)$ are $-1-p/q$ and -1. Thus, for any $k_1, k_2 \in \mathbb{Z}^+$ with $k_1+k_2>0$ we have $k_1(-1-p/q)-k_2 \neq 0$. Thus by Theorem 2 system (16) has no analytic first integrals. This concludes the proof of Theorem 1.

Case 2: If $\beta_1(\beta_2-\beta_1+\nu)(\beta_2-b-\gamma_2) \neq 0$, then making the rescaling of the variables

$$(x,y,t) \to \left(\frac{\beta_1-b-\gamma_1}{-\beta_2+\beta_1-\nu}x, \frac{\beta_1-b-\gamma_1}{\beta_2}y, \frac{t}{\beta_1-b-\gamma_1}\right),$$

system (2) becomes

$$\begin{aligned}\dot{x} &= x(-\overline{B}x+(\overline{C}-1)y+1),\\ \dot{y} &= y((1-\overline{B})x-y+\overline{A}),\end{aligned} \tag{17}$$

with

$$\overline{A} = \frac{\beta_2-b-\gamma_2}{\beta_1-b-\gamma_1}, \quad \overline{B} = \frac{-\beta_1}{\beta_2-\beta_1+\nu}, \quad \overline{C} = 1-\frac{\beta_1-\nu}{\beta_2}. \tag{18}$$

We consider two subcases.

Subcase 2.1: $\beta_2(\beta_1-b-\gamma_1) \neq 0$. In this case system (17) is of the form of system (lv1) (see Theorem 4). Then by Theorem 5 it has a global analytic first integral if and only if

$$\overline{A} = -\frac{p}{q}, \quad \overline{C}-1 = \frac{p_1}{q_1} \quad \text{and} \quad \overline{B} = \frac{qq_1}{p(p_1+q_1)},$$

with $p, q, p_1, q_1 \in \mathbb{N}$ and $pp_1 - qq_1 \geq 0$, that is, if and only if (see (18)),

$$\gamma_1 = -\frac{b(p+q)+\gamma_2 q}{p}, \quad \beta_2 = -\frac{\beta_1 p}{q}, \quad \nu = \frac{\beta_1(qq_1-pp_1)}{qq_1}.$$

A global analytic first integral is

$$H = x^{pq_1}y^{qq_1}(pp_1+pq_1-qq_1x+p_1qy+qq_1y)^{pp_1-qq_1}.$$

Subcase 2.2: $\beta_2(\beta_1-b-\gamma_1)=0$. Doing the change of variables $(x,y)\mapsto(y,x)$ it is immediate to check that the expressions of β_1 and β_2 and γ_1 and γ_2 are interchanged. This change of variables pass to a new system satisfying the condition of Case 1. So this case has been studied and we do not obtain new cases of analytic integrability by checking it.

Acknowledgments

The first author has been supported by the grants MCYT/FEDER MTM 2008–03437 and CICYT grant number 2009SGR 410. The second author has been partially supported by FCT through CAMGDS, Lisbon.

Appendix

For the quadratic Lotka–Volterra systems (3) we have the following results (see [5] for their proof).

Theorem 4. *All the quadratic Lotka–Volterra systems* (3) *can be transformed via an affine change of variables and a rescaling of the time to one of the following* 12 *systems*:

$$
\begin{array}{llll}
(\mathrm{lv1}) & \dot{x} = x(ax+by+1), & \dot{y} = y((1+a)x-y+C), & \text{with } a(b+1)C \neq 0.\\
(\mathrm{lv2}) & \dot{x} = cx, & \dot{y} = y(Ax+By+C), & \text{with } A^2+B^2 \neq 0.\\
(\mathrm{lv3}) & \dot{x} = x(y+c), & \dot{y} = y(By+1). & \\
(\mathrm{lv4}) & \dot{x} = xy, & \dot{y} = y(x+By+1). & \\
(\mathrm{lv5}) & \dot{x} = xy, & \dot{y} = y(Ax+By). & \\
(\mathrm{lv6}) & \dot{x} = x(ax+by+c), & \dot{y} = By^2, & \text{with } a^2+b^2+B^2 \neq 0.\\
(\mathrm{lv7}) & \dot{x} = x(y+1), & \dot{y} = y(x+By). & \\
(\mathrm{lv8}) & \dot{x} = x(ax+by+c), & \dot{y} = xy. & \\
(\mathrm{lv9}) & \dot{x} = x(ax+by), & \dot{y} = y(x+y). & \\
(\mathrm{lv10}) & \dot{x} = x(ax+by+1), & \dot{y} = y(x+y). & \\
(\mathrm{lv11}) & \dot{x} = x(x+c), & \dot{y} = y(Ax+1), & \text{with } cA = 0.\\
(\mathrm{lv12}) & \dot{x} = x(x+y+1), & \dot{y} = y(Ax+y+1). &
\end{array}
$$

Theorem 5. *The unique Lotka–Volterra systems* (3) *having a global analytic first integral* $H = H(x,y)$ *are the following ones.*

(a) *Systems* (lv1) *with* $C = -p/q$, $b = p_1/q_1$, $a = -qq_1/(p(p_1+q_1))$ *with* $p, q, p_1, q_1 \in \mathbb{N}$ *and* $pp_1 - qq_1 \geq 0$, *then* $H = x^{pq_1}y^{qq_1}(pp_1 + pq_1 - qq_1x + p_1qy + qq_1y)^{pp_1-qq_1}$.

(b) *Systems* (lv2) *with*:
- (b.1) $c = 0$, *then* $H = x$;
- (b.2) $c > 0$ *and* $B = C = 0$, *then* $H = e^{-Ax}y^c$;
- (b.3) $c < 0$ *and* $B = C = 0$, *then* $H = e^{Ax}y^{-c}$;
- (b.4) $c/C = -p/q$ *and* $B = 0$ *with* $p, q \in \mathbb{N}$, *then* $H = e^{Aqx/C}x^qy^p$.

(c) *Systems* (lv3) *with*:
- (c.1) $c = B = 0$, *then* $H = xe^{-y}$;
- (c.2) *with* $c = 0$, $B = -p/q$ *with* $p, q \in \mathbb{N}$, *then* $H = x^p(q - py)^q$;
- (c.3) $c = -p/q$ *with* $p, q \in \mathbb{N}$ *and* $B = 0$, *then* $H = e^{-qy}x^qy^p$;
- (c.4) $c = -p/q$, $B = -p_1/q_1$ *with* $p, q, p_1, q_1 \in \mathbb{N}$ *and* $qq_1 - pp_1 \geq 0$, *then* $H = x^{p_1q}y^{pp_1}(q_1 - p_1y)^{qq_1-pp_1}$.

(d) *Systems* (lv4) *with*:
- (d.1) $B = 0$, *then* $H = xe^{x-y}$;
- (d.2) $B = -p/q$ *with* $p, q \in \mathbb{N}$, *then* $H = x^p\left(y - \frac{q}{p} - \frac{q}{q+p}x\right)^q$.

(e) *Systems* (lv5) *with*:
- (e.1) $B = 0$, *then* $H = Ax - y$;
- (e.2) $B = -p/q$ *with* $p, q \in \mathbb{N}$, *then* $H = x^p(Aqx - (p+q)y)^q$.

(f) *Systems* (lv6) *with*:
- (f.1) $B = 0$. *The analytic first integral is* y;
- (f.2) $a = b = 0$. *The analytic first integral is* $x^{|B|}$;

(f.3) $a = 0$, $b > 0$ *and* $B < 0$, *then* $H = y^b/x^B$;
(f.4) $a = 0$, $b < 0$ *and* $B > 0$, *then* $H = x^B/y^b$.

(g) *Systems* (lv7) *with* $B = 0$, *then* $H = ye^{y-x}$.

(h) *Systems* (lv8) *with*:

(h.1) $a = 0$, *then* $H = y^{-c}e^{x-by}$;
(h.2) $a = -p/q$ *with* $p, q \in \mathbb{N}$, *then* $H = y^p(-c - \frac{p}{q}c + \frac{p}{q}x + \frac{p^2}{q^2}x - \frac{pb}{q}y)^q$.

(i) *Systems* (lv9) *with* $(a, b) \neq (1, 1)$ *and* $a - 1$, $(1 - b)a$ *and* $b - a$ *have all the same signs*, *then* $H = x^{|a-1|}y^{|a(1-b)|}((a - 1)x + (b - 1)y)^{|b-a|}$.

References

[1] D. V. Anosov and V. I. Arnold, *Dynamical Systems I* (Springer-Verlag, Berlin, 1988).
[2] C. Castillo-Chavez and Z. Feng, To treat or not to treat: The case of tuberculosis, *J. Math. Biol.* **35** (1997) 629–656.
[3] Z. Feng and J. X. Velasco-Hernández, Competitive exclusion in a vector-host model for the dengue fever, *J. Math. Biol.* **35** (1997) 523–544.
[4] S. D. Furta, On non-integrability of general systems of differential equations, *Z. angew. Math. Phys.* **47** (1996) 112–131.
[5] J. Llibre and C. Valls, Global analytic first integrals for the real planar Lotka–Volterra system, *J. Math. Phys.* **48** (2007) 033507.
[6] W. Li, J. Llibre and X. Zhang, Local first integrals of differential systems and diffeomorphisms, *Z. angew. Math. Phys.* **54** (2003) 1–21.
[7] M. C. Nucci and P. G. L. Leach, Lie integrable cases of the simplified multistrain/two-stream model for tuberculosis and dengue fever, *J. Math. Anal. Appl.* **333** (2007) 430–449.
[8] M. C. Nucci and P. G. L. Leach, Singularity analysis and integrability of a simplified multistrain model for the transmission of tuberculosis and dengue fever, *J. Nonlinear Math. Phys.* **15** (2008) 22–34.
[9] P. Van den Driessche and J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Math. Biosci.* **180** (2002) 29–48.