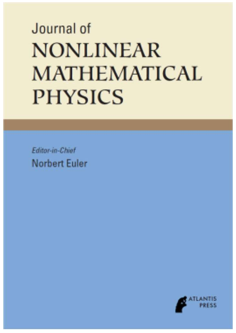

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AN OLD METHOD OF JACOBI TO FIND LAGRANGIANS

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In a recent paper by Ibragimov a method was presented in order to find Lagrangians of certain second-order ordinary differential equations admitting a two-dimensional Lie symmetry algebra. We present a method devised by Jacobi which enables one to derive (many) Lagrangians of any second-order differential equation. The method is based on the search of the Jacobi Last Multipliers for the equations. We exemplify the simplicity and elegance of Jacobi's method by applying it to the same two equations as Ibragimov did. We show that the Lagrangians obtained by Ibragimov are particular cases of some of the many Lagrangians that can be obtained by Jacobi's method.

Keywords: Lagrangian; Jacobi last multiplier; Lie symmetry; Noether symmetry.

1. Introduction

The method of the Jacobi last multiplier [8–12] provides a means to determine an integrating factor of the partial differential equation

$$Af = \sum_{i=1}^{n} a_i(x_1, \dots, x_n) \frac{\partial f}{\partial x_i} = 0 \tag{1.1}$$

or its equivalent associated Lagrange's system

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \cdots = \frac{\mathrm{d}x_n}{a_n}. \tag{1.2}$$

The multiplier M is given by

$$\frac{\partial(f, \omega_1, \omega_2, \dots, \omega_{n-1})}{\partial(x_1, x_2, \dots, x_n)} = MAf, \tag{1.3}$$

where

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = \det \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \\ \dfrac{\partial \omega_1}{\partial x_1} & & \dfrac{\partial \omega_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial \omega_{n-1}}{\partial x_1} & \cdots & \dfrac{\partial \omega_{n-1}}{\partial x_n} \end{bmatrix} = 0 \tag{1.4}$$

and $\omega_1, \ldots, \omega_{n-1}$ are $n-1$ solutions of (1.1) or, equivalently, first integrals of (1.2). Jacobi also proved that M is a solution of the following linear partial differential equation

$$\sum_{i=1}^{n} \frac{\partial(M a_i)}{\partial x_i} = 0 \tag{1.5}$$

or of its equivalent,

$$\sum_{i=1}^{n} a_i \frac{\partial(\log M)}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} = 0. \tag{1.6}$$

In general a different selection of integrals produces another multiplier, $\tilde{M}$. An important property of the last multiplier is that the ratio, $M/\tilde{M}$, is a solution of (1.1), equally a first integral of (1.2). Indeed, if each component of the vector field of the equation of motion is free of the variable associated with that component, i.e. $\partial a_i/\partial x_i = 0$, the last multiplier is a constant.

In its original formulation the method of Jacobi last multiplier required almost complete knowledge of the system, (1.1) or (1.2), under consideration.[a] Since the existence of a solution/first integral is consequent upon the existence of symmetry, an alternative formulation in terms of symmetries was provided by Lie [14,15][Kap 15, §5 in the latter]. A clear treatment of the formulation in terms of solutions/first integrals and symmetries is given by Bianchi [1]. If we know $n-1$ symmetries of (2.1)/(2.2), say

$$\Gamma_i = \sum_{j=1}^{n} \xi_{ij}(x_1, \ldots, x_n)\partial_{x_j}, \quad i = 1, n-1, \tag{1.7}$$

Jacobi Last Multiplier is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where

$$\Delta = \det \begin{bmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{bmatrix}. \tag{1.8}$$

There is an obvious corollary to the results of Jacobi mentioned above. In the case that there exists a constant multiplier, the determinant Δ is a first integral. This result is potentially

[a] Although we should underline that Jacobi himself found last multipliers for several equations without any knowledge of its solutions [8–12].

very useful in the search for first integrals of systems of ordinary differential equations. In particular this feature was put to good use with the Euler–Poinsot system [19] and the Kepler problem [20].

The following relationship between the Jacobi Last Multiplier and the Lagrangian [12,27]

$$\frac{\partial^2 L}{\partial y'^2} = M \tag{1.9}$$

for a one-degree-of-freedom system

$$y'' = f(x, y, y'), \tag{1.10}$$

where the prime denotes differentiation with respect to the independent variable x, is perhaps not widely known although it is certainly not unknown as can be seen from the bibliography in [17]. Given a knowledge of a multiplier, namely a solution of Eq. (1.6), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log M) + \frac{\partial f}{\partial y'} = 0, \tag{1.11}$$

then (1.9) gives a simple recipe for the generation of a Lagrangian. The only possible difficulty is the performance of the double quadrature. Considering the dual nature of the Jacobi Last Multiplier as providing a means to determine both Lagrangians and integrals one is surprised that it has not attracted more attention over the more than one and a half centuries since its introduction. The bibliography of [17] gives a fair indication of its significant applications in the past. In more recent years we have presented the application of Jacobi Last Multiplier to many different problems [17–25].

In a recent paper Ibragimov [7] proposed a practical approach to the resolution of the classical problem of finding the Lagrangian given a second-order ordinary differential equation. In his method Ibragimov introduced the idea of an invariant Lagrangian, and derived Lagrangians of two second-order differential equations after lengthy calculations involving integration of auxiliary differential equations. For the details of the method the interested reader should consult the paper [7].

In this paper we exemplify the simplicity and elegance of the forgotten method devised by Jacobi for finding Lagrangians by applying it to the same two equations as Ibragimov did.[b] Specifically we obtain Jacobi Last Multipliers, and therefore Lagrangians, of the equations

$$y'' = \frac{y'}{y^2} - \frac{1}{xy} \tag{1.12}$$

$$y'' = \mathrm{e}^y - \frac{y'}{x} \tag{1.13}$$

which possess the Lie point symmetries

$$\Gamma_1 = 2x\partial_x + y\partial_y, \quad \Gamma_2 = x^2\partial_x + xy\partial_y \tag{1.14}$$

and

$$\Sigma_1 = x\partial_x - 2\partial_y, \quad \Sigma_2 = x\log(x)\partial_x - 2(1+\log(x))\partial_y, \tag{1.15}$$

[b]Both equations are found in the textbook [6]. The first is example (12.27) on p. 291 and the second is Exercise 12.3 on p. 300.

respectively. We note that both symmetries in (1.14) and in (1.15) generate a Lie's Type III algebra [15], namely a nonabelian and transitive Lie algebra [1].

2. Jacobi Last Multipliers and Lagrangians for (1.12)

The calculation of the Jacobi Last Multiplier requires that the differential equation under consideration be written as a system of first-order equations. Thus (1.12) becomes

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= \frac{u_2}{u_1^2} - \frac{1}{xu_1}, \end{aligned} \tag{2.1}$$

with $u_1 \equiv y$, and $u_2 \equiv y'$. The formula (1.11) for the last multiplier gives a nonlocal $\exp[-\int u_1^{-2}\mathrm{d}x]$ which is not very useful. However, we do have the route, (1.8), through the determinant of the vector field and the two symmetries. Thus we have

$$\Delta_{12} = \det \begin{bmatrix} 1 & u_2 & \dfrac{u_2}{{u_1}^2} - \dfrac{1}{xu_1} \\ x^2 & xu_1 & u_1 - xu_2 \\ 2x & u_1 & -u_2 \end{bmatrix} = -\frac{(xu_1u_2 + x - {u_1}^2)(xu_2 - u_1)}{u_1} \tag{2.2}$$

so that the multiplier is

$$M_{12} = -\frac{u_1}{(xu_1u_2 + x - {u_1}^2)(xu_2 - u_1)}. \tag{2.3}$$

If we integrate M_{12} twice with respect to u_2, then from formula (1.9) we obtain the Lagrangian

$$\begin{aligned} L_{12} = &-\frac{u_1}{x^3}(xu_2 - u_1)\log(xu_2 - u_1) + \frac{xu_1u_2 + x - u_1^2}{x^3}\log(xu_1u_2 + x - u_1^2) \\ &- \frac{1}{x^2} + f_1(x, u_1)u_2 + f_2(x, u_1), \end{aligned} \tag{2.4}$$

where $f_1(x, u_1)$ and $f_2(x, u_1)$ are arbitrary functions of integration. If we substitute (2.4) into the Euler–Lagrangian equation, we obtain the constraint

$$\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial u_1} = \frac{x - u_1^2}{x^3 u_1} \tag{2.5}$$

on the hitherto arbitrary functions f_1 and f_2. This Lagrangian was not found by Ibragimov.

As it was shown in [23, 25], f_1, f_2 are related to the gauge function $g = g(x, u_1)$. In fact, we may assume

$$f_1 = \frac{\partial g}{\partial u_1}, \quad f_2 = \frac{\partial g}{\partial x} + \frac{2x\log(u_1) - u_1^2}{2x^3}, \tag{2.6}$$

namely the arbitrariness in the Lagrangian (2.4) can be expressed as a total time derivative. Such a Lagrangian has been termed "gauge variant" [13] and is notable in that the presence of the arbitrary function g has no effect upon the number of Noether point symmetries [23]. In this respect it could be regarded as part of the boundary term in the way Noether put it in her formulation of her theorem [16]. The class of Lagrangians described by (2.4) is an equivalence class.

We observe that there are two singularities given by

$$xu_1u_2 + x - {u_1}^2 = 0 \quad \text{and} \quad xu_2 - u_1 = 0. \tag{2.7}$$

When we solve these two equations, i.e.:

$$y' = -\frac{1}{y} + \frac{y}{x}, \quad \text{and} \quad y' = \frac{y}{x}, \tag{2.8}$$

we recover the singular solutions of (1.12) associated with the singularities of the Lagrangian (2.4), which are a consequence of the singularities of the last multiplier M_{12}, (2.3).

If we take the Lagrangian (2.4) subject to (2.5) and calculate its Noether point symmetries, we find that there is a single Noether point symmetry which is Γ_2 in (1.14). The corresponding integral is

$$I = \frac{u_1}{xu_1u_2 + x - u_1^2}. \tag{2.9}$$

With an integral and a multiplier we can generate a second multiplier by a reversal of the property that the quotient of two multipliers is an integral. The multiplier is just

$$M_1 = \frac{M_{12}}{I} = -\frac{1}{xu_2 - u_1}. \tag{2.10}$$

Now we can calculate a second Lagrangian from the multiplier (2.10), and find

$$L_1 = \frac{u_1 - xu_2}{x^2}\log(u_1 - xu_2) + \frac{u_2}{x} + f_1(x, u_1)u_2 + f_2(x, u_1), \tag{2.11}$$

with the constraint:

$$\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial u_1} = \frac{u_1^2 + x}{x^2u_1^2} \tag{2.12}$$

or equally in terms of the gauge function $g = g(x, u_1)$

$$f_1 = \frac{\partial g}{\partial u_1}, \quad f_2 = \frac{\partial g}{\partial x} + \frac{x - u_1^2}{u_1x^2}. \tag{2.13}$$

If we take the Lagrangian L_1 in (2.11) and calculate its Noether point symmetries, we find that there is a single Noether point symmetry which is Γ_2 in (1.14). The corresponding integral is I in (2.9).

We can generate many (infinite) different Jacobi Last Multipliers of Eq. (1.12) and consequently many (infinite) different Lagrangians. In fact we may take any function of the first integral I in (2.9) and then its product with either M_{12} or M_1 will generate a new Jacobi Last Multiplier. For example, we obtain the following multiplier:

$$M_2 = M_{12}\frac{-2}{I^2} = 2\frac{xu_1u_2 + x - u_1^2}{u_1(xu_2 - u_1)} \tag{2.14}$$

and consequently Lagrangian:

$$L_2 = -2\frac{u_1 - xu_2}{xu_1}\log(u_1 - xu_2) + \frac{u_2(u_1u_2 - 2)}{u_1} + f_1(x, u_1)u_2 + f_2(x, u_1), \tag{2.15}$$

with the constraint:

$$\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial u_1} = -\frac{2}{u_1^3} \tag{2.16}$$

or equally in terms of the gauge function $g = g(x, u_1)$

$$f_1 = \frac{\partial g}{\partial u_1}, \quad f_2 = \frac{\partial g}{\partial x} - \frac{1}{u_1^2}. \tag{2.17}$$

If we calculate the Noether point symmetries of the Lagrangian L_2 in (2.15), we find that both Γ_1 and Γ_2 in (1.14) are Noether point symmetries. The corresponding integrals are

$$I_1 = \log\left(\frac{u_1^2}{x} - u_1u_2\right) - \frac{1}{u_1^2}(u_1^3u_2 - xu_1^2u_2^2 - 2xu_1u_2 - x), \tag{2.18}$$

and

$$I_2 = \frac{1}{I^2} = \left(\frac{xu_1u_2 + x - u_1^2}{u_1}\right)^2, \tag{2.19}$$

respectively. It is worth noting that both singular solutions obtained in (2.8) correspond to these integrals taking the particular value of zero, namely, when each integral is a configurational invariant [3, 26], we obtain a singular solution.

One of the two Lagrangians derived by Ibragimov [6, Eq. (42), p. 223] for Eq. (1.12) is the following

$$L_{N1} = \frac{1}{xu_1} + \left(\frac{u_1}{x^2} - \frac{u_2}{x}\right)\log\left(\frac{u_1^2}{x} - u_1u_2\right), \tag{2.20}$$

which is a particular case of the Lagrangian L_1 in (2.11) with

$$f_1 = \frac{1}{x}(-\log(u_1) + \log(x) - 1), \quad f_2 = \frac{u_1}{x^2}(\log(u_1) - \log(x)) + \frac{1}{xu_1}. \tag{2.21}$$

The other Lagrangian of Ibragimov [6, Eq. (54), p. 225] is the following

$$L_{N2} = -\frac{1}{u_1^2} + \frac{u_1^2}{x^2} - 2\frac{u_1u_2}{x} + u_2^2 - 2\left(\frac{1}{x} - \frac{u_2}{u_1}\right)\log\left(\frac{u_1^2}{x} - u_1u_2\right), \tag{2.22}$$

which is a particular case of the Lagrangian L_2 in (2.15) with

$$\begin{aligned} f_1 &= -\frac{2}{xu_1}(u_1^2 - x\log(u_1) + x\log(x) - x), \\ f_2 &= -\frac{1}{x^2u_1^2}(x^2 - u_1^4 + 2xu_1^2\log(u_1) - 2xu_1^2\log(x)). \end{aligned} \tag{2.23}$$

3. Jacobi Last Multipliers and Lagrangians for (1.13)

The system of first-order differential equations corresponding to (1.13) is

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\frac{u_2}{x} + \mathrm{e}^{u_1}. \end{aligned} \tag{3.1}$$

In this case the application of formula (1.5) or equivalently (1.11) does produce a multiplier. It is

$$M_0 = x \tag{3.2}$$

from which we obtain the Lagrangian

$$L_0 = \frac{x{u_2}^2}{2} + f_1(x,u_1)u_2 + f_2(x,u_1) \tag{3.3}$$

with the constraint on the two functions of integration being

$$\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial u_1} = -x\mathrm{e}^{u_1}, \tag{3.4}$$

or equally in terms of the gauge function $g = g(x,u_1)$

$$f_1 = \frac{\partial g}{\partial u_1}, \quad f_2 = \frac{\partial g}{\partial x} + x\mathrm{e}^{u_1}. \tag{3.5}$$

This Lagrangian admits one Noether's symmetry namely Σ_1 in (1.15) and yields the following first integral:

$$I_0 = 4xu_2 + {u_2}^2x^2 - 2\mathrm{e}^{u_1}x^2. \tag{3.6}$$

We use the two symmetries Σ_1, Σ_2 in (1.15) and the vector field of the system (3.1) to obtain a second multiplier. The matrix is

$$\mathrm{Mat}_{12} = \begin{bmatrix} 1 & u_2 & -\dfrac{u_2}{x} + \mathrm{e}^{u_1} \\ x\log(x) & -2(1+\log(x)) & -\dfrac{2}{x} - u_2(1+\log(x)) \\ x & -2 & -u_2 \end{bmatrix} \tag{3.7}$$

and the corresponding multiplier is

$$M_{12} = -\frac{x}{4 + 4xu_2 + {u_2}^2x^2 - 2\mathrm{e}^{u_1}x^2}. \tag{3.8}$$

Thus formula (1.9) yields the following Lagrangian

$$\begin{aligned} L_{12} = \frac{1}{x}\log\left(-xu_2 - 2 - \sqrt{2}x\mathrm{e}^{u_1/2}\right) - \frac{1}{2x}\log\left(\frac{xu_2 + 2 + \sqrt{2}x\mathrm{e}^{u_1/2}}{xu_2 + 2 - \sqrt{2}x\mathrm{e}^{u_1/2}}\right) \\ + \frac{\sqrt{2}(xu_2+2)}{4x^2\mathrm{e}^{u_1/2}}\log\left(\frac{xu_2 + 2 + \sqrt{2}x\mathrm{e}^{u_1/2}}{xu_2 + 2 - \sqrt{2}x\mathrm{e}^{u_1/2}}\right) + f_1u_2 + f_2 \end{aligned} \tag{3.9}$$

with the constraint

$$\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial u} = 0 \tag{3.10}$$

or equally in terms of the gauge function $g = g(x, u_1)$

$$f_1 = \frac{\partial g}{\partial u_1}, \quad f_2 = \frac{\partial g}{\partial x}. \tag{3.11}$$

In a curious repetition of the situation with (1.12) we find that the two Lagrangians L_0 in (3.3) and L_{12} in (3.9) have the same Noether point symmetry Σ_1 in (1.15) and lead to what is functionally the same integral I_0 in (3.6). Both Lagrangians were not found by Ibragimov. Indeed Ibragimov did not look for Lagrangians of Eq. (1.13) admitting one Noether point symmetry.

Since we have two multipliers, we can obtain a first integral given by their ratio, namely

$$\frac{M_0}{M_{12}} = -(xu_2 + 2)^2 + 2\mathrm{e}^{u_1}x^2 = -I_0 - 4. \tag{3.12}$$

We can generate many (infinite) different Jacobi Last Multipliers of Eq. (1.13) and consequently many (infinite) different Lagrangians. In fact we may take any function of the first integral I_0 in (3.6) and then its product with either M_{12} or M_0 will generate a new Jacobi Last Multiplier. For example, we obtain the following multiplier:

$$M_2 = -\frac{M_0}{\sqrt{I_0 + 4}} = -\frac{x}{\sqrt{(xu_2 + 2)^2 - 2\mathrm{e}^{u_1}x^2}} \tag{3.13}$$

and consequently the following Lagrangian:

$$\begin{aligned} L_2 = &- \left(u_2 + \frac{2}{x}\right) \log\left(\frac{\sqrt{(xu_2 + 2)^2 - 2\mathrm{e}^{u_1}x^2} + xu_2 + 2}{\sqrt{2}x\mathrm{e}^{u_1/2}}\right) \\ &+ \sqrt{(xu_2 + 2)^2 - 2\mathrm{e}^{u_1}x^2} + f_1(x, u_1)u_2 + f_2(x, u_1), \end{aligned} \tag{3.14}$$

with either the constraint (3.10) or (3.11). This Lagrangian admits both Σ_1 and Σ_2 in (1.15) as Noether point symmetries, and the corresponding integrals are

$$I_1 = \sqrt{I_0 + 4} = \sqrt{(xu_2 + 2)^2 - 2\mathrm{e}^{u_1}x^2}, \tag{3.15}$$

and

$$I_2 = \sqrt{(xu_2 + 2)^2 - 2\mathrm{e}^{u_1}x^2}\log(x) + 2\log\left(\frac{\sqrt{(xu_2 + 2)^2 - 2\mathrm{e}^{u_1}x^2} + xu_2 + 2}{\sqrt{2}x\mathrm{e}^{u_1/2}}\right), \tag{3.16}$$

respectively.

The Lagrangian of Ibragimov [6, Eq. (90), p. 234] is a particular case of the Lagrangian L_2 in (3.14) with the gauge function equal to zero.[c]

[c]In [6] there are some missprints.

The last multiplier M_{12} in (3.8) becomes singular if

$$y' = -\frac{2}{x} \pm \sqrt{2}e^{y/2}. \tag{3.17}$$

Equation (1.13) is satisfied by each of the first-order equations in (3.17) and so we obtain the two singular solutions

$$y = x\left(C \mp \sqrt{2}x\right) \tag{3.18}$$

thereby supplementing the results given in [7].

4. Final Remarks

When one seeks a Lagrangian of an elementary equation, it is usually possible to guess the form of at least one Lagrangian. In the case of not so elementary equations an approach using guesswork is likely to lead to frustration. Consequently any development which can replace guesswork or intuition by a well-defined procedure is to be welcomed. Usually there is a price to pay for the elimination of guesswork.[d] In this paper we have considered two test equations proposed by Ibragimov to illustrate his concept of the use of invariant Lagrangians to provide a new method for the integration of nonlinear equations. We have demonstrated that some quite old knowledge is available for a successful resolution of the same problems. The combination of the concept introduced by Jacobi in his last multiplier, the application by Lie of his ideas of invariance under the transformations generated by continuous groups and the specialization to the Action Integral by Noether provides us with a very powerful and simple tool for the resolution of ordinary differential equations which possess a reasonable amount of symmetry. We have seen in the two examples considered here that they provide richer results when considered from a more classical viewpoint. The Jacobi last multiplier yields more general Lagrangians than those found by Ibragimov, and many more can be generated. One could consider that the combination of Jacobi and Lie gives sufficient material to deal with these equations. In that sense it could be argued that the theorem of Noether is already implicit in the work of Jacobi and Lie. However, we did see that further results were available to us by an application of Noether's Theorem to the information already obtained.

It is important to remark that finding a Jacobi last multiplier of an equation/system does not mean integrability. In fact one can find a Jacobi last multiplier for systems of chaotic regime as well as for integrable equations, say the famous Painlevé equations. In fact, in [2] the method of the Jacobi last multiplier for finding a Lagrangian has been applied to the second-order equations of Painlevé type as given in Ince [4]. For an example of a chaotic system we may consider the Lorenz system:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz \end{aligned} \tag{4.1}$$

[d] Naturally it can be argued that guesswork is a process of looking for some type of symmetry in an intuitive matter.

which possesses a very simple Jacobi last multiplier that can be promptly obtained from Eq. (1.6), i.e.

$$M = \exp[(\sigma + 1 + b)t], \tag{4.2}$$

although the Lorenz system does not admit any point Lie symmetries apart from translation in the independent variable t.

Finally, it may be possible to use the Jacobi last multiplier in order to find Lagrangians for partial differential equations. An hint is given in [5], in which "the knowledge of a symmetry and its corresponding conservation law of a given partial differential equation can be utilized to construct a Lagrangian for the equation".

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