Symmetries of Hamiltonian Equations and Λ-Constants of Motion

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We consider symmetries and perturbed symmetries of canonical Hamiltonian equations of motion. Specifically we consider the case in which the Hamiltonian equations exhibit a Λ-symmetry under some Lie point vector field. After a brief survey of the relationships between standard symmetries and the existence of first integrals, we recall the definition and the properties of Λ-symmetries. We show that in the presence of a Λ-symmetry for the Hamiltonian equations, one can introduce the notion of "Λ-constant of motion". The presence of a Λ-symmetry leads also to a nice and useful reduction of the form of the equations. We then consider the case in which the Hamiltonian problem is deduced from a Λ-invariant Lagrangian. We illustrate how the Lagrangian Λ-invariance is transferred into the Hamiltonian context and show that the Hamiltonian equations are Λ-symmetric. We also compare the "partial" (Lagrangian) reduction of the Euler–Lagrange equations with the reduction which can be obtained for the Hamiltonian equations. Several examples illustrate and clarify the various situations.

Keywords: Hamiltonian equations of motion; Lie point symmetries; Λ-symmetries; Λ-constants of motion; Λ-invariant Lagrangians; reduction procedures.

1. Introduction

In this paper we consider symmetries and perturbed symmetries of canonical Hamiltonian equations of motion. More specifically we consider the case in which the Hamiltonian equations of motion exhibit a Λ-symmetry under some Lie point vector field. After a brief survey (Sec. 2) of the relationships between standard "exact" symmetries of the equations and the existence of first integrals (constants of motion or conserved quantities), we recall the definitions of Λ-symmetry for a system of first-order ordinary differential equations and their properties. We show (Sec. 3) that in the presence of a Λ-symmetry for the Hamiltonian equations of motion, one can introduce the notion of "Λ-constant of motion" in a well-defined way. Under some circumstances the presence of a Λ-symmetry leads also to a nice and useful reduction of the form of the equations. We then consider in some detail (Sec. 4) the case in which the Hamiltonian problem is deduced from a Lagrangian which is Λ-invariant. We carefully illustrate how the Lagrangian Λ-invariance is transferred into the

Hamiltonian context and show that the Hamiltonian equations of motion are Λ-symmetric. We also compare the "partial" (Lagrangian) reduction of the Euler–Lagrange equations with the reduction which can be obtained for the Hamiltonian equations of motion. Several examples illustrate and clarify the various situations.

2. Symmetries and First Integrals of Hamiltonian Equations of Motion

This section is devoted to fix our notations and more importantly to provide a brief survey of some facts and properties concerning standard "exact" Lie point symmetries of canonical Hamiltonian equations of motion. Although these properties are essentially standard, the aim of this presentation is to allow an easier exposition of the case of approximate (or perturbed) symmetries of the equations, which is the main argument of this paper.

Let $u = u(t) \in \mathbf{R}^{2n}$ (or in some open domain $\Omega \subset \mathbf{R}^{2n}$), with $u \equiv (q(t), p(t))$, let J be the standard symplectic matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the $(n \times n)$ identity matrix, and let us write

$$\nabla = \nabla_u \equiv (\nabla_q, \nabla_p) \equiv \left(\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_n}, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n} \right).$$

The canonical Hamiltonian equations of motion are then

$$\dot{u} = J\nabla H = F(u, t), \tag{2.1}$$

where $H = H(q, p, t)$ is the given (smooth) Hamiltonian.

Given a vector field

$$X = \varphi_\alpha(u, t) \frac{\partial}{\partial q_\alpha} + \psi_\alpha(u, t) \frac{\partial}{\partial p_\alpha} + \tau(u, t) \frac{\partial}{\partial t} \equiv \Phi \cdot \nabla_u + \tau \, \partial_t \tag{2.2}$$

(sum over $\alpha = 1, \ldots, n$), where clearly $\Phi \equiv (\varphi, \psi)$ and the dot stands here for the scalar product in $\mathbf{R}^{2n}$, we want to look for the conditions ensuring that X is the generator of a (Lie point) symmetry for the Hamiltonian equations of motion (2.1). This is a classical problem, which has been considered since long time, possibly in different forms and also in connection with the similar problem for Newtonian equations (see e.g. [11, 12, 16, 37]).

Using standard techniques (see e.g. [5, 24, 35, 36, 39]), one easily obtains that the vector field X is the generator of a symmetry (shortly: a symmetry) for the Hamiltonian equations of motion if and only if

$$[F, \Phi]_a + \partial_t \Phi_a - (D_t \tau) F_a - \tau \, \partial_t F_a = 0 \quad (a = 1, \ldots, 2n), \tag{2.3}$$

where D_t is the total derivative with respect to t and

$$[F, \Phi]_a \equiv F_b \nabla_{u_b} \Phi_a - \Phi_b \nabla_{u_b} F_a \quad (\text{sum over } b = 1, \ldots, 2n).$$

Writing the symmetry condition (2.3) in the equivalent but more explicitly form

$$D_t\varphi_\alpha - \frac{\partial H}{\partial p_\alpha}D_t\tau - \varphi_\beta\frac{\partial^2 H}{\partial q_\beta \partial p_\alpha} - \psi_\beta\frac{\partial^2 H}{\partial p_\alpha \partial p_\beta} - \tau\frac{\partial^2 H}{\partial t \partial p_\alpha} = 0 \tag{2.4}$$

$$D_t\psi_\alpha + \frac{\partial H}{\partial q_\alpha}D_t\tau + \varphi_\beta\frac{\partial^2 H}{\partial q_\beta \partial q_\alpha} + \psi_\beta\frac{\partial^2 H}{\partial q_\alpha \partial p_\beta} + \tau\frac{\partial^2 H}{\partial t \partial q_\alpha} = 0, \tag{2.5}$$

one can verify by means of direct calculations that (again, the sum over $\alpha = 1, \ldots, n$ is understood)

$$0 = \frac{\partial}{\partial q_\alpha}(\alpha\text{-th equation in } (2.4)) + \frac{\partial}{\partial p_\alpha}(\alpha\text{-th equation in } (2.5))$$

$$= D_t\left(\frac{\partial \varphi_\alpha}{\partial q_\alpha} + \frac{\partial \psi_\alpha}{\partial p_\alpha} - D_t\tau + \frac{\partial \tau}{\partial t}\right).$$

One then deduces from this equation that, if X is a symmetry for the Hamiltonian equations of motion, the quantity $S = S(q, p, t)$ defined by

$$S \equiv \frac{\partial \varphi_\alpha}{\partial q_\alpha} + \frac{\partial \psi_\alpha}{\partial p_\alpha} - D_t\tau + \frac{\partial \tau}{\partial t} = \nabla \cdot \Phi - \{H, \tau\} = \nabla \cdot \widetilde{\Phi}, \tag{2.6}$$

where $\widetilde{\Phi} \equiv (\widetilde{\varphi}, \widetilde{\psi})$ and $\widetilde{\varphi} = \varphi - \tau\nabla_p H$, $\widetilde{\psi} = \psi + \tau\nabla_q H$, is a constant of motion (first integral or conserved quantity) for the problem (2.1), i.e.

$$D_t S = 0. \tag{2.7}$$

Unfortunately it can happen that S turns out to be identically 0 or a constant. We then distinguish various cases:

(i) Let X be any vector field (2.2) and *assume* that there is a "generating function" $G = G(u, t)$ such that

$$\frac{\partial G}{\partial p_\alpha} = \varphi_\alpha - \tau\frac{\partial H}{\partial p_\alpha} = \widetilde{\varphi}_\alpha \quad \frac{\partial G}{\partial q_\alpha} = -\psi_a - \tau\frac{\partial H}{\partial q_\alpha} = -\widetilde{\psi}_\alpha \quad \text{or} \quad \widetilde{\Phi} = J\nabla G. \tag{2.8}$$

Then $\widetilde{\Phi}$ is divergence free, $\nabla \cdot \widetilde{\Phi} = 0$, and so in this case $S \equiv 0$. In turn it is not difficult to verify using (2.8) that

$$J\nabla(D_t G) \equiv \text{l.h.s. of the symmetry conditions } (2.4), (2.5).$$

Therefore, a vector field X which admits a generating function G satisfying (2.8) is a symmetry for the Hamiltonian equations of motion if and only if

$$\nabla(D_t G) = 0 \quad \text{or} \quad D_t G = g(t), \tag{2.9}$$

i.e., G is a first integral, possibly apart from an additional time-dependent term. This corresponds of course to a completely standard case (cf. [5, 35]; see also [15] for a different approach to the searching for first integrals and their relation with symmetry properties[a]).

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[a]It can be noted that all examples of symmetries given in [15] which admit a first integral belong to case (i) and those with no first integral belong to case (ii) below.

We just remark here that in many physically relevant cases one has $\tau = \tau(t)$. Then instead of (2.8) it is enough to require the existence of a function G_0 such that $\varphi_\alpha = \partial G_0/\partial p_\alpha, \psi_\alpha = -\partial G_0/\partial q_\alpha$, and G is then $G = G_0 - \tau H$, whereas the existence of G_0 requires that

$$\frac{\partial \varphi_\alpha}{\partial p_\beta} = \frac{\partial \varphi_\beta}{\partial p_\alpha} \quad \frac{\partial \psi_\alpha}{\partial q_\beta} = \frac{\partial \psi_\beta}{\partial q_\alpha} \quad \frac{\partial \varphi_\alpha}{\partial q_\beta} = -\frac{\partial \psi_\beta}{\partial p_\alpha}.$$

(ii) It is clear that, if $S = \text{const} \neq 0$, where S is the quantity defined in (2.6), then G does not exist. On the other hand, as we have seen, condition (2.8) implies $S = 0$, but clearly the converse is not true (apart from the case $n = 1$, trivially). This means that, although $S = 0$, it can happen that the function G does not exist (not even locally) even if X is a symmetry for the Hamiltonian equations of motion.

(iii) If X is a symmetry for the Hamiltonian equations of motion with $S \neq \text{const}$, G does not exist, of course. However, we have shown that in this case S provides a first integral: $D_t S = 0$. Examples of this situation can be obtained just multiplying a symmetry vector field X (*not* necessarily belonging to the case (i)) by any first integral K:

$$X_1 = KX.$$

This vector field X_1 is another symmetry for the Hamiltonian equations of motion (cf. [12]) and the corresponding quantity S_1, evaluated according to (2.6), is *not* a constant. This S_1, being a first integral, is the generating function of a *new* symmetry vector field Y_1 according to the standard rule

$$Y_1 = (\nabla_p S_1)\nabla_q - (\nabla_q S_1)\nabla_p = \varphi_1 \nabla_q + \psi_1 \nabla_p.$$

Notice that, if one starts with a vector field X belonging to case (i) (with its generating function G), the new first integral S_1 produced in this way by $X_1 = KX$ is related to K and G, according to

$$S_1 = \nabla_q(K\varphi) + \nabla_p(K\psi) = \{K, G\}$$

(we are assuming here that $\tau = 0$) and the new symmetry Y_1 generated by S_1 is just

$$Y_1 = [X, X_K], \quad \text{where } X_K = (\nabla_p K)\nabla_q - (\nabla_q K)\nabla_p.$$

We give very simple examples to illustrate the various situations described above.

Example 1. Let $H = (p^2 + q^2)/2$ (in $n = 1$ degree of freedom): the vector field $X = q(\partial/\partial q) + p(\partial/\partial p)$ is the well-known scaling symmetry for the Hamiltonian equations of motion, but $S = 2$, and no first integral related to this symmetry can be found. Quite trivially, $X_1 = (q^3 + qp^2)\partial/\partial q + (q^2 p + p^3)\partial/\partial p$ is a symmetry for the Hamiltonian equations of motion and $S_1 = 8H$ is a first integral: this is an example of case (iii). As another example for case (ii), let $H = (p_1^2 + q_1^2)/2 + (p_2^2 + q_2^2)/2$ in $n = 2$ degrees of freedom; the vector field $X = q_1(\partial/\partial q_1) + p_1(\partial/\partial p_1) - q_2(\partial/\partial q_2) - p_2(\partial/\partial p_2)$ is a symmetry for the equations and $S = 0$, but this X does not determine any first integral.

Finally, observing that

$$\frac{\partial H}{\partial p_\alpha}\frac{\partial}{\partial q_\alpha} - \frac{\partial H}{\partial q_\alpha}\frac{\partial}{\partial p_\alpha} + \frac{\partial}{\partial t} \equiv 0$$

along all the solutions of (2.1), one can safely replace X with the equivalent vector field $\widetilde{X}$ given by

$$\widetilde{X} = X - \tau \frac{\partial H}{\partial p_\alpha}\frac{\partial}{\partial q_\alpha} + \tau \frac{\partial H}{\partial q_\alpha}\frac{\partial}{\partial p_\alpha} - \tau \frac{\partial}{\partial t} = \widetilde{\varphi}\nabla_q + \widetilde{\psi}\nabla_p = \widetilde{\Phi} \cdot \nabla_u \ .$$

This amounts exactly to replace X with its evolutionary form $\widetilde{X}$; notice, however, that in this case $\widetilde{X}$ is still a Lie point symmetry. So it is not restrictive to assume $\tau = 0$ as we will do hereafter.

3. Perturbed Symmetries and Λ-Constants of Motion

We now pass to consider the main point of this paper, namely the case of approximate (or perturbed) symmetries of the Hamiltonian equations of motion. More specifically we consider the case in which the Hamiltonian equations of motion exhibit a Λ-symmetry under some vector field X.

We briefly recall the notion of λ-symmetry (with lower case λ), introduced in 2001 by Muriel and Romero [26, 27]. It is a well-known property that, if an ordinary differential equation admits a (standard) Lie point-symmetry, then the order of the equation can be lowered by one (see e.g. [35]). The idea of λ-symmetries consists of introducing a suitable modification, in terms of a given C^∞ function λ, of the prolongation rules of the vector field in such a way that this lowering procedure still works, even in the absence of standard Lie symmetries and even if λ-symmetries are not symmetries in the proper sense, as they do not map in general solutions into solutions.

It should be remarked that the case $\lambda = 0$ corresponds to standard symmetries. In this sense one can think of λ-symmetries as "perturbations" of the exact symmetries.

Several applications and extensions of λ-symmetries have been proposed: see e.g. [9, 13, 14, 17, 21, 22, 28, 30, 32, 33, 38, 40]. They also admit a deep interpretation by means of nontrivial geometrical language and are related to symmetries of different nature (symmetries of integral-exponential type, hidden and potential symmetries, nonlocal symmetries, and solvable structures as well): see e.g.[b] [1–4, 6–8, 18, 19, 23, 25, 31, 34]. For a very recent, fairly complete and updated survey, see [20].

In the case of first-order ordinary differential equations, as is the case of Hamiltonian equations of motion and of dynamical systems in general, standard Lie symmetries cannot lower the order of the equations, but they can provide a "reduction" of the complexity of the system, or — more precisely — a reduction of the number of the variables involved (see [35, Ch. 2, Theorem 2.66]). One can obtain a similar (although obviously not identical) result also introducing the idea of Λ-symmetries for this case, as we describe.

Let a system of canonical Hamiltonian equations of motion (2.1) be given and let X be a vector field; denoting by Λ a $(2n \times 2n)$ matrix of C^∞ functions depending on $t, q, \dot{q}, p, \dot{p}$, we define the first Λ-prolongation $X_\Lambda^{(1)}$ of the vector field X according to

$$X_\Lambda^{(1)} = X^{(1)} + (\Lambda\Phi)_a \frac{\partial}{\partial \dot{u}_a} = X + (D_t\Phi_a + (\Lambda\Phi)_a)\frac{\partial}{\partial \dot{u}_a}, \tag{3.1}$$

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[b]We quote the papers which, to our knowledge, are more or less directly related to the idea of λ-symmetries.

where the sum over $a = 1, \ldots, 2n$ is understood and $X^{(1)}$ is the standard first prolongation. We say that the system $\dot{u} = F(t, u)$ is Λ-symmetric under X if

$$X_\Lambda^{(1)}(\dot{u} - F)|_{\dot{u}=F} = 0. \qquad (3.2)$$

This condition becomes explicitly

$$[F, \Phi]_a + \frac{\partial \Phi_a}{\partial t} = -(\Lambda \Phi)_a \qquad (3.3)$$

and is to be compared with (2.3) (putting $\tau = 0$). Clearly all results obtained for generic dynamical systems in the presence of a Λ-symmetry [10, 29], which concern the reduction properties for the equations (see below Theorem 1), are still true for Hamiltonian equations of motion. However, one can also expect that some special results hold for the case of Hamiltonian systems, mainly related to the presence of first integrals, along the lines discussed in the previous section. Before dealing with this aspect and in order to provide an easier presentation, we summarize — in the form appropriate for our case — the general reduction properties which hold for generic dynamical systems.

Firstly, one has to introduce $2n$ "symmetry-adapted coordinates" w_a with the property of being invariant under X, i.e., $Xw_a = 0$ (note that they are independent of Λ). One of these is clearly the time t, which can be still used as the independent variable. As $(2n+1)$-th variable, which is called z, we take the coordinate "along the action of X", i.e. such that $Xz = 1$. The equations then take the form

$$\dot{w}_j = W_j(t, w, z) \quad (j = 1, \ldots, 2n - 1) \qquad (3.4a)$$

$$\dot{z} = Z(t, w, z). \qquad (3.4b)$$

The reduction of these equations is obtained if some of their right-hand sides W_j, Z are independent of z. In the case in which X is a standard symmetry for the equations (i.e. if $\Lambda \equiv 0$), then $\dot{w}_j$ and $\dot{z}$ are automatically first-order invariants under $X^{(1)}$ and are then independent of z. If $\Lambda \neq 0$, this is no longer true, but the following result can be shown.

Theorem 1 [10, 29]. *The explicit dependence on z of the r.h.s. W_j, Z of Eq. (3.4) is governed by the formulas $(j = 1, \ldots, 2n - 1; a = 1; \ldots, 2n)$*

$$\frac{\partial W_j}{\partial z} = \frac{\partial w_j}{\partial q_a}(\Lambda \Phi)_a = M_j \qquad \frac{\partial Z}{\partial z} = \frac{\partial z}{\partial q_a}(\Lambda \Phi)_a = M_{2n}. \qquad (3.5)$$

If for some j one has $M_j = 0$, then $\dot{w}_j$ is still invariant and the r.h.s of the corresponding equation does not contain z. If in particular the matrix Λ is such that

$$\Lambda \Phi = \lambda \Phi, \qquad (3.6)$$

where λ is a (scalar) function, then all W_j are independent of z (Z is independent of z if X is a standard symmetry for the equations).

It can be interesting to examine how the Λ-symmetry of the equations is transformed when these are written in terms of the variables w_j, z, as done in (3.4). Using the tilde to

indicate that we are here working with these variables, we have

$$\widetilde{X} = \frac{\partial}{\partial z} = \widetilde{X}^{(1)} \quad \text{or} \quad \widetilde{\Phi} = (0, \dots, 0, 1)$$

and as a consequence only the last column $\widetilde{\Lambda}_{a,2n}$ of $\widetilde{\Lambda}$ is relevant (recall that Λ is not uniquely defined, see [10]). Applying the condition (3.3) which expresses the Λ-symmetry to Eq. (3.4) and using (3.5), we deduce

$$\frac{\partial W_j}{\partial z} = \widetilde{\Lambda}_{j,2n} = M_j \qquad \frac{\partial Z}{\partial z} = \widetilde{\Lambda}_{2n,2n} = M_{2n}$$

and then

$$\widetilde{X}_{\Lambda}^{(1)} = \frac{\partial}{\partial z} + M_j \frac{\partial}{\partial \dot{w}_j} + M_{2n} \frac{\partial}{\partial \dot{z}}.$$

In the case in which (3.6) is satisfied (cf. [29]) one has $M_j = 0$ and $M_{2n} = \lambda$.

We now consider the Hamiltonian structure of our equations. Firstly, when standard symmetries of the Hamiltonian equations of motion are replaced by Λ-symmetries, one has to look for the presence of (approximate, in some sense to be defined) first integrals.

According to the discussion in Sec. 2, we consider separately the two cases (i) and (iii). We can state the following result, which can be easily obtained by means of direct calculations and by comparison with (3.3).

Theorem 2. *Let $X = \varphi \nabla_q + \psi \nabla_p$ be any vector field admitting a generating function G such that $\varphi = \nabla_p G, \psi = -\nabla_q G$, i.e.,*

$$X = (J\nabla)G \cdot \nabla,$$

and let $F = J\nabla H$. Writing $\dot{G}$ instead of $D_t G = \partial G/\partial t + \{G, H\}$, one has the relation

$$J\nabla \dot{G} = [F, \Phi] + \frac{\partial \Phi}{\partial t}. \tag{3.7}$$

Then, if X is a Λ-symmetry for the Hamiltonian equations of motion, combining (3.7) with (3.3), one gets

$$\nabla(\dot{G}) = J\Lambda\Phi = J\Lambda \, J\nabla G. \tag{3.8}$$

Similarly, if X is such that the quantity S defined in (2.6) is not a constant, then

$$\dot{S} = -\nabla(\Lambda\Phi). \tag{3.9}$$

Equations (3.8) and (3.9), to be compared with (2.9) and resp. (2.7), express the "deviation" from the exact conservation of the quantity G (resp. S) as a consequence of the "breaking" of the exact invariance of the Hamiltonian equations of motion under X due to the presence of the matrix Λ. We can say that in this case G (resp. S) is a "Λ-*constant of motion*".

Notice that G can be certainly (and conveniently) chosen as one of the variables w_j introduced before; now assuming that $\Lambda\Phi = \lambda\Phi$, as in second part of Theorem 1, an

interesting situation can occur in which the Λ-conserved quantity G satisfies a "separate" equation involving only G itself:

Corollary 1. *If $\Lambda\Phi = \lambda\Phi$, where λ is a scalar function, then*

$$\nabla(\dot{G}) = -\lambda\nabla G$$

and, if in addition $\lambda = \lambda(G)$, Eq. (3.4) take the form

$$\dot{w}_\ell = W_\ell(t, w_\ell, G) \quad (\ell = 1, \ldots, 2n - 2)$$
$$\dot{G} = \gamma(t, G)$$
$$\dot{z} = Z(t, w_\ell, G, z)$$

(apart from a possible additional time-dependent term in the equation for G, as in (2.9)).

We now give some examples.

Example 2. This is a quite simple example, which is particularly useful as an illustration of the results. Let $n = 2$ be the number of degrees of freedom and

$$H = -(q_1 p_2 + q_2 p_1) H_0(q_1 + q_2) + H'(q_1, q_2, p_1 - p_2),$$

where H_0 and H' are arbitrary functions of the specified arguments. Let X be the vector field

$$X = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2},$$

with Λ given by

$$\Lambda = \mathrm{diag}(0, 0, 1, 1).$$

The first Λ-prolongation is then

$$X_\Lambda^{(1)} = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2} + \frac{\partial}{\partial \dot{p}_1} + \frac{\partial}{\partial \dot{p}_2}$$

and it is a simple exercise to verify that the Hamiltonian equations of motion, which can be easily written, are Λ-symmetric (but not symmetric) under X. This can be performed either verifying condition (3.3) or directly checking that the Hamiltonian equations of motion satisfy (3.2). The Λ-conserved quantity is $G = q_1 + q_2$ and the hypotheses of Corollary 1 are satisfied. Taking indeed the variables $w_1 = q_1 - q_2, w_2 = p_1 - p_2, w_3 = G = q_1 + q_2$ and with $z = p_1 + p_2$, the equations become

$$\dot{w}_1 = w_1 H_0(G) + W_1'(w_1, w_2, G)$$
$$\dot{w}_2 = -w_2 H_0(G) + W_2'(w_1, w_2, G)$$
$$\dot{G} = -G H_0(G)$$
$$\dot{z} = z H_0(G) + Z'(w_1, w_2, G),$$

where W', Z' are some suitable functions, in complete agreement with Corollary 1.

If in the above example we assume for instance $H_0 = 1$, the equation for G is solved by $G = G_0 \exp(-t)$, which implies that

$$G_0 \equiv (q_1 + q_2) \exp(t)$$

is trivially a time-dependent first integral: $D_t((q_1 + q_2) \exp(t)) = 0$, as can be directly confirmed. This fact admits an obvious generalization:

Corollary 2. *In the same assumptions as in Corollary 1, inverting the solution $G = G(t, G_0)$ of the equation for G, in the form $G_0 = \Gamma(t, G)$, one has that*

$$\Gamma = \Gamma(t, G(t, q, p))$$

is a time-dependent first integral of the Hamiltonian equations of motion: $D_t\Gamma = 0$.

Actually this can be viewed as a special case of a much more general situation examined in [40].

Example 3. This is a more elaborate example, where some different situations can occur. Let the Hamiltonian be given by, with $n = 2$ and $q_1, q_2 > 0$,

$$H = \frac{1}{2}q_1^2 p_1^2 \log q_1 + \frac{1}{2}q_2^2 p_2^2 \log q_2 + H'(q_1 p_1, q_2 p_2, q_1/q_2)$$

and let X be the vector field

$$X = q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} - p_1 \frac{\partial}{\partial p_1} - p_2 \frac{\partial}{\partial p_2}.$$

When we introduce the X-invariant variables

$$w_1 = q_1 p_2, \quad w_2 = q_2 p_2, \quad w_3 = q_1/q_2,$$

the resulting equations of motion can be written

$$\dot{q}_1 = q_1^2 p_1 \log q_1 + q_1 \frac{\partial H'}{\partial w_1}$$

$$\dot{q}_2 = q_2^2 p_2 \log q_2 + q_2 \frac{\partial H'}{\partial w_2}$$

$$\dot{p}_1 = -q_1 p_1^2 \log q_1 - \frac{1}{2}q_1 p_1^2 - p_1 \frac{\partial H'}{\partial w_1} - \frac{1}{q_2} \frac{\partial H'}{\partial w_3}$$

$$\dot{p}_2 = -q_2 p_2^2 \log q_2 - \frac{1}{2}q_2 p_2^2 - p_2 \frac{\partial H'}{\partial w_2} + \frac{q_1}{q_2^2} \frac{\partial H'}{\partial w_3}.$$

These are Λ-symmetric under X, with Λ given by

$$\Lambda = \mathrm{diag}(q_1 p_1, q_2 p_2, q_1 p_1, q_2 p_2)$$

and the first Λ-prolongation is

$$X_\Lambda^{(1)} = X + (\dot{q}_1 + q_1^2 p_1)\frac{\partial}{\partial \dot{q}_1} + (\dot{q}_2 + q_2^2 p_2)\frac{\partial}{\partial \dot{q}_2} - (p_1 + q_1 p_1^2)\frac{\partial}{\partial \dot{p}_1} - (p_2 + q_2 p_2^2)\frac{\partial}{\partial \dot{p}_2}.$$

In terms of the coordinates $w_j(t)$ and putting $z = \log q_1$ we obtain

$$\dot{w}_1 = -\frac{1}{2}w_1^2 - w_3\frac{\partial H'}{\partial w_3}$$

$$\dot{w}_2 = -\frac{1}{2}w_2^2 + w_3\frac{\partial H'}{\partial w_3}$$

$$\dot{w}_3 = w_1 w_3 z - w_2 w_3 z + w_3 \left(\frac{\partial H'}{\partial w_1} - \frac{\partial H'}{\partial w_2}\right) + w_2 w_3 \log w_3$$

$$\dot{z} = z w_1 + \frac{\partial H'}{\partial w_1}.$$

Note that, in this example, $\Lambda\Phi \neq \lambda\Phi$ and indeed the above equations do not assume the "completely reduced" form as in Corollary 1; in particular, the Λ-conserved quantity G, which is given by $G = w_1 + w_2$, satisfies the equation

$$\dot{G} = -\frac{1}{2}(w_1^2 + w_2^2)$$

which has not the "separate" form $\dot{G} = \gamma(t, G)$, but satisfies (3.8), as expected. Also we see that the equations for w_1 and w_2 do not contain z, in agreement with Theorem 1. If, for instance, H' has the form $H' = \log w_3 H''(w_1, w_2)$, then a separate subsystem for the two variables w_1 and w_2 would be obtained. We can also modify the definition of Λ inserting a "small" real coefficient ε, i.e., $\Lambda_\varepsilon = \varepsilon\Lambda$, to emphasize the idea that this Λ_ε may be considered a perturbation of the (standard) symmetry X. The equation for G, for instance, is changed into

$$\dot{G} = -\varepsilon\frac{1}{2}(w_1^2 + w_2^2).$$

This could allow one to perform some perturbative calculations: if $\varepsilon \ll 1$, then

$$\dot{G} \simeq 0, \quad \dot{w}_1 \simeq -\dot{w}_2, \quad \dot{z} \simeq \frac{\partial H'}{\partial w_1}$$

and so on.

Example 4. This is a simple example (with $n = 1$) in which the vector field belongs to case (iii). The vector field $X = q^2 p \partial/\partial q$ is a (standard) symmetry for the equations $\dot{q} = -q$, $\dot{p} = p$, and for this X one has $S = 2qp$ which is trivially a first integral for these equations. However, the vector field X is also a Λ-symmetry for the Hamiltonian equations of motion

$$\dot{q} = -\varepsilon q \log p - q \qquad \dot{p} = \varepsilon p \log p + p - \varepsilon p$$

with $H = -qp + \varepsilon qp - \varepsilon qp \log p$, $p > 0$ and with $\Lambda = \varepsilon \operatorname{diag}(1,0)$. It can be immediately checked that

$$\dot{S} = -2\varepsilon qp = -\nabla(\Lambda\Phi)$$

in agreement with (3.9).

4. When a Λ-Symmetry is Inherited by a Λ-Invariant Lagrangian

4.1. Λ-*symmetry of the Hamiltonian equations of motion*

A specially interesting case of the problem examined in the previous section occurs when the Hamiltonian problem is deduced from a Lagrangian which is Λ-invariant [13,33]. Considering for concreteness only first-order Lagrangians:

$$\mathcal{L} = \mathcal{L}(t, q_\alpha, \dot{q}_\alpha) \quad (\alpha = 1, \dots, n),$$

we recall that such a Lagrangian is Λ-invariant under the vector field

$$X^{(\mathcal{L})} = \varphi_\alpha(t, q)\frac{\partial}{\partial q_\alpha} = \varphi\nabla_q \tag{4.1}$$

if there is an $(n \times n)$ matrix $\Lambda^{(\mathcal{L})}(t, q, \dot{q})$ such that

$$(X_\Lambda^{(\mathcal{L})})^{(1)}(\mathcal{L}) = 0, \tag{4.2}$$

where $(X_\Lambda^{(\mathcal{L})})^{(1)}$ is the first Λ-prolongation of X defined by

$$(X_\Lambda^{(\mathcal{L})})^{(1)} = \varphi_\alpha\frac{\partial}{\partial q_\alpha} + (D_t\varphi_\alpha + (\Lambda^{(\mathcal{L})}\varphi)_\alpha)\frac{\partial}{\partial \dot{q}_\alpha}.$$

Examples of Λ-invariant Lagrangians are given in $[13, 30, 33]$, where also the consequences of Λ-invariance on Noether's theorem are discussed.

We now introduce the Hamiltonian $H(t, q, p)$ from the given Lagrangian. The first step is to extend the vector field $X^{(\mathcal{L})}$ and the $(n \times n)$ matrix $\Lambda^{(\mathcal{L})}$ to a vector field X and a $(2n \times 2n)$ matrix Λ acting on the $2n$ variables q, p. Next one has to check if and how the $\Lambda^{(\mathcal{L})}$ invariance of the Lagrangian is transferred into symmetry properties of the Hamiltonian equations of motion.

We start with the standard situation where the Lagrangian is exactly invariant (i.e. $\Lambda^{(\mathcal{L})} = 0$) under some $X^{(\mathcal{L})}$. According to Noether theorem, the quantity

$$P(t, q, \dot{q}) = \varphi_\alpha\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}$$

is a constant of the motion, $D_tP = 0$. Introducing the coordinates $p_\alpha = \partial\mathcal{L}/\partial\dot{q}_\alpha$, express P as a function of t, q and p and denote by G this expression: this notation is motivated by the fact that $G = \varphi_\alpha p_\alpha$ is indeed the generating function of the vector field

$$X = \varphi_\alpha\frac{\partial}{\partial q_\alpha} - p_\beta\frac{\partial \varphi_\beta}{\partial q_\alpha}\frac{\partial}{\partial p_\alpha} \tag{4.3}$$

which is a (standard) symmetry for the corresponding Hamiltonian equations of motion.

In the case of $\Lambda^{(\mathcal{L})}$-invariance of the Lagrangian under some $X^{(\mathcal{L})}$, we have to deduce how the coordinates p are transformed: the infinitesimal transformations of q and $\dot{q}$ under

the action of $(X_\Lambda^{(\mathcal{L})})^{(1)}$ are

$$\delta q_\alpha = \varepsilon\varphi_\alpha \quad \delta\dot{q}_\alpha = \varepsilon(D_t\varphi_\alpha + (\Lambda^{(\mathcal{L})}\varphi)_\alpha)$$

and as a consequence

$$\delta p_\alpha = \varepsilon\left(\frac{\partial p_\alpha}{\partial q_\beta}\varphi_\beta + \frac{\partial p_\alpha}{\partial \dot{q}_\beta}(D_t\varphi_\beta + (\Lambda^{(\mathcal{L})}\varphi)_\beta)\right) = \varepsilon\psi_\alpha$$

using the notation ψ_α as in (2.2). Thanks to the definition of the variables p, to the Euler–Lagrange equations and still writing $G = \varphi_\alpha p_\alpha$, the functions ψ_α can be rewritten as

$$\psi_\alpha = \frac{\partial}{\partial \dot{q}_\alpha}\left(D_tG + (\Lambda^{(\mathcal{L})}\varphi)_\beta\frac{\partial\mathcal{L}}{\partial\dot{q}_\beta}\right) - \frac{\partial\Lambda_{\beta\gamma}^{(\mathcal{L})}}{\partial\dot{q}_\alpha}\varphi_\gamma\frac{\partial\mathcal{L}}{\partial\dot{q}_\beta} - p_\beta\frac{\partial\varphi_\beta}{\partial q_\alpha}. \tag{4.4}$$

It has been shown [13] that if the Lagrangian is $\Lambda^{(\mathcal{L})}$-invariant under $X^{(\mathcal{L})}$ or equivalently if (4.2) is satisfied, then

$$D_tG + (\Lambda^{(\mathcal{L})}\varphi)\frac{\partial\mathcal{L}}{\partial\dot{q}} \equiv D_t(\varphi_\alpha p_\alpha) + \Lambda_{\alpha\beta}^{(\mathcal{L})}\varphi_\alpha p_\beta = 0. \tag{4.5}$$

We assume in addition that, as usually happens, the matrix $\Lambda^{(\mathcal{L})}$ does not depend upon $\dot{q}$ (the case of a possible dependence upon $\dot{q}$ is briefly considered at the end of the paper): then (4.4) becomes finally

$$\psi_\alpha = -p_\beta\frac{\partial\varphi_\beta}{\partial q_\alpha}$$

and the coefficients functions ψ_α are independent of Λ; more importantly,

$$G = \varphi_\alpha p_\alpha$$

is still the generating function of a vector field X (which justifies our notation also in the case $\Lambda^{(\mathcal{L})} \neq 0$), coinciding with the standard symmetry case (4.3).

The next step deals with the Hamiltonian equations of motion and the searching for their symmetry. It is well known that Euler–Lagrange equations coming from a $\Lambda^{(\mathcal{L})}$-invariant Lagrangian do *not* exhibit in general Λ-symmetry. In contrast we want to show that one can extend the $(n \times n)$ matrix $\Lambda^{(\mathcal{L})}$ to a $(2n \times 2n)$ matrix Λ in such a way that the Hamiltonian equations of motion are Λ-symmetric. To obtain this extension we differentiate (4.5) with respect to $\nabla \equiv (\nabla_{q_\alpha}, \nabla_{p_\alpha})$ and compare this result with (3.8). Recalling also that $\Phi \equiv (\varphi_\alpha, \psi_\alpha)$ we easily obtain that Λ has the form

$$\Lambda = \begin{pmatrix} \Lambda^{(\mathcal{L})} & 0 \\ -\dfrac{\partial\Lambda^{(\mathcal{L})}}{\partial q_\alpha}p_\gamma & \Lambda^{(2)} \end{pmatrix}, \tag{4.6}$$

where $\Lambda^{(2)}$ must satisfy (Λ is not uniquely defined, as already remarked)

$$\Lambda_{\alpha\beta}^{(2)}\frac{\partial\varphi_\gamma}{\partial q_\beta} = \Lambda_{\gamma\beta}^{(\mathcal{L})}\frac{\partial\varphi_\beta}{\partial q_\alpha}.$$

It is now straightforward to verify that the Hamiltonian equations of motion are indeed Λ-symmetric under the vector field X given in (4.3) and with the above matrix Λ.

We summarize:

Theorem 3. *If a first-order Lagrangian $\mathcal{L}$ is $\Lambda^{(\mathcal{L})}$-invariant under a vector field $X^{(\mathcal{L})}$ with a matrix $\Lambda^{(\mathcal{L})}$ not depending on $\dot{q}$, then the corresponding Hamiltonian equations of motion are Λ-symmetric under the vector field X defined in (4.3) with Λ given in (4.6); accordingly, the quantity $G = \varphi_\alpha p_\alpha$ is Λ-constant of motion.*

Example 5. The Lagrangian (with $n = 2$)

$$\mathcal{L} = \frac{1}{2}\left(\frac{\dot{q}_1}{q_1} - q_1\right)^2 + \frac{1}{2}(\dot{q}_1 - q_1\dot{q}_2)^2 \exp(-2q_2) + q_1 \exp(-q_2)$$

is $\Lambda^{(\mathcal{L})}$-invariant under the vector field

$$X^{(\mathcal{L})} = q_1\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}$$

with

$$\Lambda^{(\mathcal{L})} = \mathrm{diag}(q_1, q_1).$$

It is a simple exercise to verify that the corresponding Hamiltonian equations of motion

$$\dot{q}_1 = q_1^2 p_1 + q_1^2 + q_1 p_2$$

$$\dot{q}_2 = \frac{p_2}{q_1^2}\exp(2q_2) + q_1 p_1 + q_1 + p_2$$

$$\dot{p}_1 = -q_1 p_1^2 - 2q_1 p_1 + \frac{p_2^2}{q_1^3}\exp(2q_2) - p_1 p_2 - p_2 + \exp(-q_2)$$

$$\dot{p}_2 = -\frac{p_2^2}{q_1^2}\exp(2q_2) - q_1 \exp(-q_2)$$

are Λ-symmetric under

$$X = q_1\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} - p_1\frac{\partial}{\partial p_1}$$

according to (4.3) and with Λ given by

$$\Lambda = \begin{pmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_1 & 0 & 0 \\ -p_1 & -p_2 & q_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

according to (4.6). The transformation of the Hamiltonian equations of motion in terms of the X-invariant coordinates $w_1 = q_1 \exp(-q_2), w_2 = q_1 p_1, w_3 = p_2$ is left to the reader; we only point out that the generating function of the above vector field $G = w_2 + w_3$ satisfies the Λ-conservation rule

$$\dot{G} = -q_1 G$$

in agreement with (3.8).

We conclude this section with the following result, which is obtained combining Corollary 1 with (4.6).

Corollary 3. *If* $\Lambda^{(\mathcal{L})}\varphi = c\varphi$, *where* c *is a constant, then also* $\Lambda\Phi = c\Phi$ *and the "complete" reduction of the Hamiltonian equations of motion as in Corollary 1 is ensured.*

4.2. *Partial (Lagrangian) reduction of the Hamiltonian equations*

Any vector field of the form $X = \varphi_\alpha \partial/\partial q_\alpha$, as in Eq. (4.1), admits n 0-order invariants $\eta_\alpha(t, q)$ (including the time t), i.e.

$$Xt = X\eta_r = 0 \quad (r = 1, \ldots, n-1)$$

and n other first-order differential invariants $\theta_\alpha(t, q, \dot{q})$

$$X^{(1)}\theta_\alpha = 0 \quad (\alpha = 1, \ldots, n),$$

where $X^{(1)}$ is the first standard prolongation of X. Actually one can choose as first-order invariants just the $n-1$ functions $\dot{\eta}_r$ (which are automatically invariants — we have already mentioned and used this fact in Theorem 1) and another independent invariant θ. The important result, as shown in [26, 29], is that the same is true even if the first (standard) prolongation $X^{(1)}$ is replaced by a first Λ-prolongation, *provided that* the $(n \times n)$ matrix Λ satisfies the condition

$$\Lambda\varphi = \lambda\varphi, \quad \text{where } \varphi \equiv (\varphi_1, \ldots, \varphi_n), \tag{4.7}$$

with λ a scalar function. It can be noticed that this a purely "algebraic" property, not related to any dynamics (Lagrangian, Hamiltonian etc.).

Let now be given a Lagrangian and assume that it is invariant (or *also* Λ-invariant: we are explicitly assuming from now that (4.7) is satisfied) under a vector field $X = X^{(\mathcal{L})}$, then it must be a function of the above $2n + 1$ invariants $t, \eta_r, \dot{\eta}_r$ and θ. The Euler–Lagrange equation for the variable θ is therefore simply

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0 \tag{4.8}$$

and this is a first-order equation which provides, as is well known [33], a "partial" reduction of the equations, meaning that it produces in general only a particular set of solutions (this is true both for exactly and for $\Lambda^{(\mathcal{L})}$-invariant Lagrangians).

We are now interested in $\Lambda^{(\mathcal{L})}$-invariant Lagrangians: having introduced the corresponding Hamiltonian together with its vector field X and the $(2n \times 2n)$ matrix Λ as explained in Sec. 4.1, we can compare the above "partial" (Lagrangian) reduction with the reduced form of the resulting Λ-symmetric Hamiltonian equations of motion as said in Sec. 3. This is well illustrated by the following example.

Example 6. The Lagrangian, in $n = 2$ degrees of freedom and $q_1 > 0$,

$$\mathcal{L} = \frac{1}{2}\left(\frac{\dot{q}_1}{q_1} - \log q_1\right)^2 + \frac{1}{2}\left(\frac{\dot{q}_1}{q_1} + \frac{\dot{q}_2}{q_2}\right)^2$$

is $\Lambda^{(\mathcal{L})}$-invariant under

$$X^{(\mathcal{L})} = q_1 \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial q_2}$$

with $\Lambda^{(\mathcal{L})} = \mathrm{diag}\,(1,1)$ which satisfies (4.7) and also the hypothesis of Corollary 3. We can choose the invariants, apart from t,

$$\eta = q_1 q_2, \quad \dot{\eta} = \dot{q}_1 q_2 + q_1 \dot{q}_2, \quad \theta = \frac{\dot{q}_1}{q_1} - \log q_1.$$

Writing the Lagrangian in terms of these, we obtain

$$\widetilde{\mathcal{L}} = \frac{1}{2}\theta^2 + \frac{1}{2}\frac{\dot{\eta}^2}{\eta^2}$$

and the Euler–Lagrange equation for θ, $\partial\widetilde{\mathcal{L}}/\partial\theta = \theta = 0$, produces the particular solution

$$\dot{q}_1 = q_1 \log q_1.$$

We now introduce the corresponding Hamiltonian:

$$H = \frac{1}{2}q_1^2 p_1^2 + q_2^2 p_2^2 + (q_1 p_1 - q_2 p_2)\log q_1 - q_1 q_2 p_1 p_2.$$

It is simple to verify (we omit to give detailed calculations) that the Hamiltonian equations of motion are Λ-symmetric under the vector field

$$X = q_1 \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial q_2} - p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2}$$

with $\Lambda = \mathrm{diag}\,(1,1,1,1)$ in agreement with (4.3) and (4.6). The invariants under this X are

$$w_1 = q_1 q_2, \quad w_2 = q_1 p_1, \quad w_3 = q_2 p_2$$

and X is generated by $G = w_2 - w_3$. All conditions of Corollary 1 are satisfied, and, as expected, a "complete" reduction is obtained: indeed, if we choose $z = \log q_1$, the equations for w_1, w_2, G and z are

$$\dot{w}_1 = w_1 w_3$$
$$\dot{w}_2 = w_3 - w_2$$
$$\dot{G} = -G$$
$$\dot{z} = z + w_2 - w_3.$$

We note that the above "partial" (Lagrangian) solution $\theta = 0$ corresponds here to the special case $\dot{z} = z$, $w_2 = w_3 = c =$const, $\dot{w}_1 = cw_1$.

4.3. *When Λ depends on $\dot{q}$*

We finally consider the case in which the given Lagrangian is Λ-invariant under a vector field $X^{(\mathcal{L})}$ with a matrix $\Lambda^{(\mathcal{L})}$ which depends also on $\dot{q}$. Thanks to Eqs. (4.4) and (4.5), we see that the extension of $X^{(\mathcal{L})}$ to a vector field X for the Hamiltonian is still possible

and the resulting vector field X is a symmetry for the Hamiltonian equations of motion, as expected, but it does not admit a generating function G. Then our above procedure cannot in general be performed and only the results stated in Theorem 1 remain valid. This case is described by the following example, which also provides another example of the Λ-conservation of the quantity S (see Eq. (2.6)) according to (3.9).

Example 7. The Lagrangian (in $n = 1$ degree of freedom)

$$\mathcal{L} = \frac{1}{2}\left(\frac{\dot{q}}{q} + 1\right)^2 \exp(-2q)$$

is Λ-invariant under

$$X^{(\mathcal{L})} = q\frac{\partial}{\partial q} \quad \text{with } \Lambda^{(\mathcal{L})} = q + \dot{q}.$$

According to (4.4) and (4.5) one has $\psi = -qp - p$ and then

$$X = q\frac{\partial}{\partial q} - (qp + p)\frac{\partial}{\partial p}$$

which does not admit a generating function. The Hamiltonian equations of motion are

$$\dot{q} = q^2 p \exp(2q) - q \quad \dot{p} = -qp^2 \exp(2q) - q^2 p^2 \exp(2q) + p$$

and it is not difficult to verify that these are Λ-symmetric under the above vector field X with[c]

$$\Lambda = \begin{pmatrix} q + \dot{q} & 0 \\ -p & q + \dot{q} \end{pmatrix}.$$

It can be remarked that the quantity S defined in (2.6), which is in this case $S = -q$, satisfies $\dot{S} = -\nabla(\Lambda\Phi)$ as in (3.9) and is indeed a Λ-constant of motion. We can also introduce the X-invariant coordinate $w = qp\exp(q)$ (which is independent of Λ, as already remarked) and write the equations in terms of w and $z = q$: we get

$$\dot{w} = -zw \quad \dot{z} = -z + zw\exp(z)$$

which are quite simpler than the initial ones. They can also be compared with the equations one would obtain by means of the partial (Lagrangian) reduction, as discussed in Sec. 4.2. Writing indeed the Lagrangian as a function of the invariant $\theta = (\dot{q}/q)\exp(-q) + \exp(-q)$ under $X^{(\mathcal{L})}$, one has $\widetilde{\mathcal{L}} = \theta^2/2$ and the condition (4.8) produces the particular solution

$$\dot{z} = -z \qquad w = 0$$

or $\dot{q} = q, p = 0$.

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[c]It can be noted that in this example the matrix Λ has the form of Eq. (4.6) although the proof of (4.6) requires that the vector field X admits a generating function G.

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