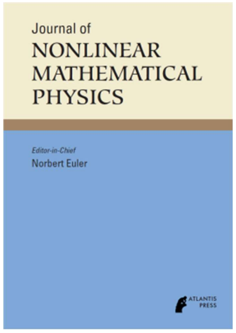

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LAGRANGIANS FOR EQUATIONS OF PAINLEVÉ TYPE BY MEANS OF THE JACOBI LAST MULTIPLIER

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We apply the method of Jacobi Last Multiplier to the fifty second-order ordinary differential equations of Painlevé type as given in Ince in order to obtain a Lagrangian and consequently solve the inverse problem of Calculus of Variations for those equations. The easiness and straightforwardness of Jacobi's method is underlined.

Keywords: Ordinary differential equations; equations of Painlevé type; Lagrangians; Jacobi Last Multiplier.

1. Introduction

It is well known that a Lagrangian always exists for any second-order ordinary differential equation (ODE) [17]. What is less known is that the knowledge of a Jacobi Last Multiplier [4, 6, 7] always yields such a Lagrangian [8, 17], which means that one can always solve the inverse problem of the Calculus of Variations by the multiplier.

Even less known is the link between Lie symmetries and Jacobi Last Multiplier. In fact Lie found a mathematical connection between the namesake symmetries and Last Multipliers, as we can read in [10, 9].

Moreover a Multiplier can be directly derived, as it is shown in the next sections. So, while one is often forced to face an odyssey of lengthy and nonobvious mathematical steps to obtain a Lagrangian (or, equivalently, an Hamiltonian), with the Last Multiplier one derives the Lagrangian in a very easy and straightforward manner.

We use this powerful means of calculus to confront fifty second-order ordinary differential equations of Painlevé type as given in [3] and get Lagrangians for these equations. Hamiltonians are long known for the so-called six Painlevé equations [11, 16]. As stated in [16], "the Hamiltonian structure for the six Painlevé equations can be defined in a canonical way by the theory of isomonodromic deformations of linear differential equations or by geometric properties of the foliations associated with Painlevé equations". The scope of our work is to show that the method of the Jacobi Last Multiplier accomplishes the same without any knowledge of either monodromy or foliation properties, and how Jacobi's method can be extended to get new results. Moreover, Jacobi's method does not require any *a priori* assumption on the Lagrangian such as rationality [16].

As far as we know nobody has ever searched for the Lagrangians (Hamiltonians) of all the fifty equations of Painlevé type as given in Ince.

This paper is organized in the following way. In Sec. 2 we recall the definition and many properties of the Jacobi Last Multiplier. Then in Sec. 3 after recalling the connection between the Jacobi Last Multiplier and the Lagrangians of second-order ordinary differential equations, we seek the Lagrangians of the fifty second-order ordinary differential equations of Painlevé type in [3]. In Sec. 4 we conclude with a final remark.

2. Jacobi Last Multiplier

We present a new method devised by Jacobi to derive Lagrangians of any second-order differential equation: it consists in finding the Last Multiplier for the equation.

In the Winter of 1842–1843 at the University of Berlin, Jacobi delivered a series of lectures on Dynamics [8]. Of the 38 lectures published posthumously several were devoted to what he had previously termed "a new principle of analytical mechanics" [4] which suggests that Jacobi was very hopeful about his new discovery in mechanics, although it has never been particularly popular as a tool in the solution of problems in mechanics. Jacobi termed this new development "the Last Multiplier" and this name has persisted.

This method provides a means [5–7] to determine all the solutions of the partial differential equation,

$$Af = \sum_{i=1}^{n} a_i(x_1, x_2 \cdots x_n) \frac{\partial f}{\partial x_i} = 0, \tag{2.1}$$

or its equivalent associated Lagrange's system,

$$\frac{dx_1}{a_1} = \frac{dx_2}{a_2} = \cdots = \frac{dx_n}{a_n}. \tag{2.2}$$

One can get the Multiplier in many ways. Firstly, if one knows all but one of the solutions, then the Last Multiplier is given by

$$\frac{\partial(f, \omega_1, \omega_2 \cdots \omega_{n-1})}{\partial(x_1, x_2 \cdots x_n)} = MAf, \tag{2.3}$$

where

$$\frac{\partial(f, \omega_1, \omega_2 \cdots \omega_{n-1})}{\partial(x_1, x_2 \cdots x_n)} = \det \begin{pmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \\ \dfrac{\partial \omega_1}{\partial x_1} & \cdots & \dfrac{\partial \omega_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \omega_{n-1}}{\partial x_1} & \cdots & \dfrac{\partial \omega_{n-1}}{\partial x_n} \end{pmatrix} = 0 \tag{2.4}$$

and $\omega_1, \omega_2 \cdots \omega_{n-1}$ are the solutions of (2.1) or, that is the same, independent first integrals of (2.2). So M is a function of the variables $(x_1 \cdots x_n)$ and depends upon the $n-1$ solutions chosen, in the sense that it varies as they vary. The essential properties of the Jacobi Last Multiplier are:

(a) If one selects a different set of $n-1$ independent solutions $\eta_1, \eta_2 \cdots \eta_{n-1}$ for the equation $Af = 0$, then the corresponding Last Multiplier N is linked to M by the relationship:

$$N = M \frac{\partial(\eta_1, \eta_2 \cdots \eta_{n-1})}{\partial(\omega_1, \omega_2 \cdots \omega_{n-1})}. \tag{2.5}$$

(b) Given a nonsingular transformation of variables,

$$\tau : \quad (x_1, x_2 \cdots x_n) \to (x'_1, x'_2 \cdots x'_n), \tag{2.6}$$

the Multiplier M' of $A'f = 0$ is given by

$$M' = M \frac{\partial(x'_1, x'_2 \cdots x'_n)}{\partial(x_1, x_2 \cdots x_n)}, \tag{2.7}$$

where M obviously comes from the $n-1$ solutions of $Af = 0$ corresponding to those chosen for $A'f = 0$ through the inverse transformation τ^{-1}.

(c) One can prove that every Multiplier M is a solution of the linear partial differential equation,

$$\frac{\partial(Ma_1)}{\partial x_1} + \frac{\partial(Ma_2)}{\partial x_2} + \cdots + \frac{\partial(Ma_n)}{\partial x_n} = 0, \tag{2.8}$$

or of its equivalent,

$$\sum_{i=1}^{n} a_i \frac{\partial(\log M)}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} = 0; \tag{2.9}$$

vice versa every solution M of this equation is a Jacobi Last Multiplier.

(d) If one knows two Jacobi Last Multipliers, M_1 and M_2, of Eq. (2.1), then their ratio is a solution ω of (2.1) or a first integral of (2.2). Naturally the ratio may be quite trivial, namely a constant. Vice versa the product of a Multiplier M_1 times any solution ω yields another Last Multiplier, $M_2 = M_1\omega$.

Since the existence of a solution/first integral is consequent upon the existence of a symmetry, an alternative formulation in terms of symmetries was provided by Lie [10]. A clear treatment of the formulation in terms of solutions/first integrals and symmetries is given by Bianchi [1]. If we know $n-1$ symmetries of (2.1)/(2.2), say

$$\Gamma_i = \sum_{j=1}^{n} \xi_{ij}(x_1, x_2 \cdots x_n)\partial_{x_j}, \quad i = 1 \cdots n-1, \tag{2.10}$$

then Jacobi's Last Multiplier is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where

$$\Delta = \det \begin{pmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & \cdots & \xi_{1,n} \\ \vdots & \ddots & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{pmatrix}. \tag{2.11}$$

An obvious corollary of these features is that, if there exists a constant Multiplier, the determinant is a first integral. This result is potentially very useful in the search for first integrals of systems of ordinary differential equations.

3. A Direct Route to Lagrangians

We now present the connection between the Jacobi Last Multiplier and a Lagrangian of an ordinary differential equation of the second order, such as

$$\ddot{y} = F(t, y, \dot{y}). \tag{3.1}$$

From property (c) of the previous section we deduce that we can write

$$\frac{d}{dt}(\log M) + \frac{\partial F}{\partial \dot{y}} = 0 \Rightarrow M = \exp\left[-\int \frac{\partial F}{\partial \dot{y}} dt\right] \tag{3.2}$$

that is to be satisfied by the Multiplier which we are going to find. It was demonstrated [8,17] that

$$M = \frac{\partial^2 L}{\partial \dot{y}^2}, \tag{3.3}$$

where $L = L(t, y, \dot{y})$ is the Lagrangian sought. This means that, if we know one Last Multiplier, then we can obtain L by two successive quadratures:

$$L = \int\left(\int M d\dot{y}\right) d\dot{y} + f_1(t, y)\dot{y} + f_2(t, y) \tag{3.4}$$

with f_1 and f_2 arbitrary functions.

Since every pair of Lagrangians which differ only by a total derivative with respect to time of a differentiable function give rise to the same equations, we can define classes of equivalence among Lagrangians apart from the total derivative of an arbitrary function called a *gauge function*. Thereby we can set [14] a gauge function $g(t, y)$ such that:

$$f_1 = \frac{\partial g}{\partial y} \tag{3.5}$$

$$f_2 = \frac{\partial g}{\partial t} + f_3(t, y). \tag{3.6}$$

Hence we get the Lagrangian

$$L = \int\left(\int M d\dot{y}\right) d\dot{y} + \frac{dg}{dt} + f_3. \tag{3.7}$$

Now g is the arbitrary gauge function, but we do not have to forget that the initial equation (3.1) is derivable by this Lagrangian and so, using the equation of Lagrange:

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) + \frac{\partial L}{\partial y} = 0, \tag{3.8}$$

this must give back the ODE we began with. This means that f_3 is not arbitrary at all, but it has to satisfy the Lagrange's equation (3.8).

Using the above mentioned properties of the Last Multiplier and the comments we made and with the help of interactive programs in REDUCE, we sought the Lagrangians of the fifty second-order ordinary differential equations of Painlevé type [3]. Each of them can be written as a second-degree polynomial in $\dot{y}$ with coefficients analytical in t and y:

$$\ddot{y} + A(y)\dot{y}^2 + B(t,y)\dot{y} + C(t,y) = 0. \tag{3.9}$$

The equation defining the Last Multiplier (3.2)

$$\frac{d}{dt}(\log M) - 2A(y)\dot{y} - B(t,y) = 0 \Rightarrow M = \exp\left[\int (2A(y)\dot{y} + B(t,y))dt\right] \tag{3.10}$$

simplifies very much when B does not depend on y. This is the case for many equations. From the fifty equations — among which we find the six famous Painlevé equations, those which define the Painlevé transcendents [16] — we emphasize some particularly easy to treat.

We begin with a simple case, Equation I in [3]:

$$I \quad \ddot{y} = 0. \tag{3.11}$$

Here

$$\frac{d}{dt}(\log M) = 0 \tag{3.12}$$

so we can take $M = 1$ and obviously get

$$L = \frac{\dot{y}^2}{2} + \frac{dg}{dt}$$

since f_3 is found to be identically null. Again for Equation II:

$$II \quad \ddot{y} = 6y^2$$

we get $M = 1$ and

$$L = \frac{\dot{y}^2}{2} + f_3 + \frac{dg}{dt}.$$

Substituting this Lagrangian into (3.8) and taking into consideration Equation II yield

$$f_3 = 2y^3.$$

Proceeding the same way we obtain a Lagrangian for other Painlevé equations with $M = 1$. Precisely, we have

$$III \quad \ddot{y} = 6y^2 + \frac{1}{2} \qquad L = \frac{\dot{y}^2}{2} + 2y^3 + \frac{y}{2} + \frac{dg}{dt};$$

$$IV \quad \ddot{y} = 6y^2 + t \qquad L = \frac{\dot{y}^2}{2} + 2y^3 + ty + \frac{dg}{dt};$$

$$\begin{array}{lll} VII & \ddot{y} = 2y^3 & L = \dfrac{\dot{y}^2}{2} + \dfrac{y^4}{2} + \dfrac{dg}{dt}; \\ VIII & \ddot{y} = 2y^3 + \beta y + \gamma & L = \dfrac{\dot{y}^2}{2} + \dfrac{1}{2}(y^4 + 2\gamma y + \beta y^2) + \dfrac{dg}{dt}; \\ IX & \ddot{y} = 2y^3 + ty + \gamma & L = \dfrac{\dot{y}^2}{2} + \dfrac{1}{2}(y^4 + 2\gamma y + ty^2) + \dfrac{dg}{dt}. \end{array}$$

Another nontrivial, but also simple, example is given by Equation VI

$$VI \quad \ddot{y} = -3y\dot{y} - y^3 + q(\dot{y} + y^2)$$

with q an arbitrary function of time. Here

$$\frac{d}{dt}(\log M) - 3y + q = 0,$$

but, since

$$y = q - \frac{d}{dt}\log(\dot{y} + y^2),$$

we can use it to calculate M and find that

$$M = \frac{\exp\left[2\int q dt\right]}{(\dot{y} + y^2)^3},$$

hence

$$L = \frac{\exp\left[2\int q dt\right]}{2(\dot{y} + y^2)} + \frac{dg}{dt}$$

and so on for many other equations, as it was shown in [13].

While integrating with respect to the variable t in (3.10), one can put $\dot{y}dt = dy$, namely it is sufficient to change variable of integration in order to get the Multiplier. For example:

$$XI \quad \ddot{y} = \frac{\dot{y}^2}{y}$$

$$\frac{d}{dt}(\log M) + \frac{2\dot{y}}{y} = 0 \Rightarrow -\log M = \int 2\frac{\dot{y}}{y}dt = \int 2\frac{dy}{y} = 2\log y,$$

so $M = 1/y^2$ and

$$L = \frac{\dot{y}^2}{2y^2} + \frac{dg}{dt}.$$

In the same way:

$$\begin{array}{lll} XII & \ddot{y} = \dfrac{\dot{y}^2}{y} + \alpha y^3 + \beta y^2 + \gamma + \dfrac{\delta}{y} & L = \dfrac{\dot{y}^2}{2y^2} + \dfrac{\alpha}{2}y^2 + \beta y - \dfrac{\delta}{2y^2} - \dfrac{\gamma}{y} + \dfrac{dg}{dt}; \\ XIII & \ddot{y} = \dfrac{\dot{y}^2}{y} - \dfrac{\dot{y}}{t} + \dfrac{\alpha y^2 + \beta}{t} + \gamma y^3 + \dfrac{\delta}{y} & L = \dfrac{t\dot{y}^2}{2y^2} + \alpha y - \dfrac{\beta}{y} - \dfrac{\delta t}{2y^2} + \dfrac{\gamma t y^2}{2} + \dfrac{dg}{dt}. \end{array}$$

Similarly Equation XVII,

$$XVII \quad \ddot{y} = \frac{(m-1)\dot{y}^2}{my},$$

gives

$$L = \frac{\dot{y}^2 y^{\frac{2}{m}}}{2y^2} + \frac{dg}{dt}.$$

For some other equations the Multiplier is simply $M = 1/y$, such as:

$$XVIII \quad \ddot{y} = \frac{\dot{y}^2}{2y} + 4y^2 \qquad L = \frac{\dot{y}^2}{2y} + 2y^2 + \frac{dg}{dt};$$

$$XIX \quad \ddot{y} = \frac{\dot{y}^2}{2y} + 4y^2 + 2y \qquad L = \frac{\dot{y}^2}{2y} + 2y^2 + 2y + \frac{dg}{dt};$$

$$XX \quad \ddot{y} = \frac{\dot{y}^2}{2y} + 4y^2 + 2ty \qquad L = \frac{\dot{y}^2}{2y} + 2y^2 + 2ty + \frac{dg}{dt};$$

$$XXIX \quad \ddot{y} = \frac{\dot{y}^2}{2y} + \frac{3y^3}{2} \qquad L = \frac{\dot{y}^2}{2y} + \frac{y^3}{2} + \frac{dg}{dt};$$

$$XXX \quad \ddot{y} = \frac{\dot{y}^2}{2y} + \frac{3y^3}{2} + 4\alpha y^2 + 2\beta y - \frac{\gamma^2}{2y} \qquad L = \frac{\dot{y}^2}{2y} + 2\alpha y^2 + 2\beta y + \frac{\gamma^2}{2y} + \frac{y^3}{2} + \frac{dg}{dt};$$

$$XXXI \quad \ddot{y} = \frac{\dot{y}^2}{2y} + \frac{3y^3}{2} + 4ty^2 + 2(t^2-\alpha)y - \frac{\beta^2}{2y} \qquad L = \frac{\dot{y}^2}{2y} + 2y(t^2-\alpha) + \frac{y^3}{2} + 2ty^2 + \frac{\beta^2}{2y} + \frac{dg}{dt};$$

$$XXXII \quad \ddot{y} = \frac{\dot{y}^2}{2y} - \frac{1}{2y} \qquad L = \frac{\dot{y}^2}{2y} + \frac{1}{2y} + \frac{dg}{dt};$$

$$XXXIII \quad \ddot{y} = \frac{\dot{y}^2}{2y} + 4y^2 + \alpha y - \frac{1}{2y} \qquad L = \frac{\dot{y}^2}{2y} + \alpha y + 2y^2 + \frac{1}{2y} + \frac{dg}{dt};$$

$$XXXIV \quad \ddot{y} = \frac{\dot{y}^2}{2y} + 4\alpha y^2 - ty - \frac{1}{2y} \qquad L = \frac{\dot{y}^2}{2y} + 2\alpha y^2 - ty + \frac{1}{2y} + \frac{dg}{dt};$$

or $M = y^{-\frac{3}{2}}$ for equations:

$$XXI \quad \ddot{y} = \frac{3\dot{y}^2}{4y} + 3y^2 \qquad L = \frac{\dot{y}^2\sqrt{y}}{2y^2} + 2y\sqrt{y} + \frac{dg}{dt};$$

$$XXII \quad \ddot{y} = \frac{3y\dot{y}^2}{4y} - 1 \qquad L = \frac{\dot{y}^2}{2y^{\frac{3}{2}}} + \frac{2\sqrt{y}}{y} + \frac{dg}{dt};$$

$$XXIII \quad \ddot{y} = \frac{3\dot{y}^2}{4y} + 3y^2 + \alpha y + \beta \qquad L = \frac{\dot{y}^2}{2y^{\frac{3}{2}}} + 2(y^2 + \alpha y - \beta)\frac{\sqrt{y}}{y} + \frac{dg}{dt}.$$

As another example take Equation XXXVII, namely

$$\ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\dot{y}^2,$$

for which the condition (3.10) for the Multiplier becomes

$$\frac{d}{dt}(\log M) = -2\dot{y}\left(\frac{1}{2y} + \frac{1}{y-1}\right).$$

One gets after integration

$$M = \frac{1}{y(y-1)^2}$$

and consequently

$$L = \frac{\dot{y}^2}{2y(y-1)^2} + \frac{dg}{dt}.$$

Similarly we find:

$$XXXVIII \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\dot{y}^2 + y(y-1)\left(\alpha(y-1) + \beta\frac{y-1}{y^2} + \frac{\gamma}{y-1} + \frac{\delta}{(y-1)^2}\right)$$

$$L = \frac{\dot{y}^2}{2y(y-1)^2} + \frac{\alpha(2y+1)}{2} - \frac{\beta(y+2)}{2y} - \frac{\gamma(y+1)}{2(y-1)} - \frac{\delta}{2(y-1)^2} + \frac{dg}{dt};$$

$$XXXIX \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\dot{y}^2 - \frac{\dot{y}}{t} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}$$

$$L = \frac{t\dot{y}^2}{2y(y-1)^2} + \frac{1}{2t}\left(\alpha(2y+1) - \frac{t\gamma(y+1)}{y-1} - \frac{\beta(y+2)}{y} - \frac{\delta t^2(y^2+1)}{(y-1)^2}\right) + \frac{dg}{dt}.$$

An equation for which $M = [y(y-1)]^{-\frac{4}{3}}$ is Equation XLI:

$$XLI \quad \ddot{y} = \frac{2}{3}\left(\frac{1}{y} + \frac{1}{y-1}\right)\dot{y}^2 \quad L = \frac{\dot{y}^2}{2(y(y-1))^{\frac{4}{3}}} + \frac{dg}{dt}.$$

With the Multiplier $M = [y(y-1)]^{-\frac{3}{2}}$ we have

$$XLIII \quad \ddot{y} = \frac{3}{4}\left(\frac{1}{y} + \frac{1}{y-1}\right)\dot{y}^2 \quad L = \frac{\dot{y}^2}{2(y(y-1))^{\frac{3}{2}}} + \frac{dg}{dt};$$

$$XLIV \quad \ddot{y} = \frac{3}{4}\left(\frac{1}{y} + \frac{1}{y-1}\right)\dot{y}^2 + y(y-1)\left(\frac{\alpha}{y} + \frac{\beta}{y-1} + 2\gamma(y-1)\right)$$

$$L = \frac{\dot{y}^2}{2(y(y-1))^{\frac{3}{2}}} - 2\gamma\log\left(\sqrt{y-1} + \sqrt{y}\right) + 2\alpha\left(1 + \sqrt{\frac{y-1}{y}}\right)$$

$$-2\beta\left(1 + \sqrt{\frac{y}{y-1}}\right) + 2\gamma\sqrt{y(y-1)} + \frac{dg}{dt}.$$

Finally $M = [y(y-1)(y-\alpha)]^{-1}$ is the Multiplier of Equation XLIX:

$$XLIX \quad \ddot{y} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-\alpha}\right)\dot{y}^2 + (y^2-y)(y-\alpha)\left(\beta + \frac{\gamma}{y^2} + \frac{\delta}{(y-1)^2} + \frac{\varepsilon}{(y-\alpha)^2}\right)$$

$$L = \frac{\dot{y}^2}{2y(y-1)(y-\alpha)} + \beta y + \beta\frac{\alpha}{\alpha+1} - \frac{\gamma}{y} - \frac{\gamma}{\alpha+1} - \frac{\delta(y+\alpha)}{(y-1)(\alpha+1)} + \frac{\epsilon(y+1)}{(\alpha+1)(\alpha-y)} + \frac{dg}{dt}.$$

A particular class of ODEs was investigated by Jacobi himself [6, 15], and contains a class of equations studied by Euler [2]:

$$\ddot{y} + \frac{1}{2}\frac{\partial\phi}{\partial y}(t,y)\dot{y}^2 + \frac{\partial\phi}{\partial t}(t,y)\dot{y} + D(t,y) = 0. \tag{3.13}$$

Jacobi derived that the Last Multiplier is

$$M = \exp[\phi(t,y)].$$

Combining (3.9) with (3.13) yields

$$\begin{cases} A(y) = \dfrac{1}{2}\dfrac{\partial\phi}{\partial y} \\ B(t,y) = \dfrac{\partial\phi}{\partial t}. \end{cases}$$

This system has solution only if

$$2\frac{\partial A}{\partial t} = \frac{\partial B}{\partial y}.$$

Apart from the equations in which the term B does not depend on y, only Equation L belongs to this case:

$$L \quad \ddot{y} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\dot{y}^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\dot{y} + \frac{y(y-1)(y-t)}{2t^2(t-1)^2}\left(\alpha - \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} - \frac{(\delta-1)t(t-1)}{(y-t)^2}\right).$$

Following Jacobi we obtain the Multiplier

$$M = \frac{t(t-1)}{y(y-1)(y-t)}$$

and consequently the Lagrangian:

$$L = \frac{t(t-1)\dot{y}^2}{2y(y-1)(y-t)} + \frac{(yt+y+t)\alpha}{2(t^2-1)t} + \frac{(y+t+1)\beta}{2(t^2-1)y} - \frac{\gamma(y+t)}{2(y-1)(t+1)t} + \frac{\delta t(y+1)}{2(y-t)(t+1)t} - \frac{(y+1)}{2(y-t)(t+1)} + \frac{dg}{dt}.$$

In order to find the remaining Lagrangians, we now look for Lie symmetries and use the nonlocal Jacobi Last Multiplier to raise (and then lower) the order of the equation by one as shown in [12]. In particular we consider this autonomous equation (q is taken to be constant):

$$\ddot{y} = -y\dot{y} + y^3 - 12qy$$

that is a subcase of Equation X:

$$X \quad \ddot{y} = -y\dot{y} + y^3 - 12qy + 12\dot{q}.$$

We apply the following transformation [12] as suggested by the equation of the Jacobi Last Multiplier (3.10):

$$y = \frac{2\dot{w}}{w}$$

and so we get

$$\dddot{w} = \frac{\dot{w}}{w^2}(\ddot{w}w + 4\dot{w}^2 - 12qw^2).$$

Since this equation is autonomous we have the trivial symmetry $\Gamma = \partial_t$ and we can use it to reduce the order by one and therefore get

$$s'' = \frac{4s}{x^2} + \frac{s'}{x} - 12\frac{q}{s} - \frac{s'^2}{s},$$

where $w = x$ and $\dot{w} = s(x)$. Finally we get for this last equation the Lagrangian:

$$L = \frac{s^2 s'^2}{2x} + \frac{s^2}{x^3}(s^2 - 6qx^2) + \frac{dG}{dx}(x, s).$$

Consequently a Lagrangian can be found if the equation under study is changed by a suitable transformation of dependent and independent variables by means of the Jacobi Last Multiplier.

4. Final Remark

We have exemplified the use of Jacobi Last Multiplier in the search of Lagrangians for a second-order ordinary differential equation. Several distinguished authors keep inventing (and publishing) their own new method in order to find Lagrangians for a single second-order ordinary differential equation. None of them acknowledges Jacobi Last Multiplier, yet each of their method can be shown to derive from the Jacobi Last Multiplier and its Eq. (3.2).

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References

[1] L. Bianchi, *Lezioni sulla Teoria dei Gruppi Continui Finiti di Trasformazioni* (Enrico Spoerri, Pisa, 1918).

[2] L. Euler, *(E366) Institutionum calculi integralis, Volumen Secundum* (Petropoli impensis academiae imperialis scientiarum, Petersburg, 1769) available at www.eulerarchive.org.

[3] E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).

[4] C. G. J. Jacobi, Sur un noveau principe de la mécanique analytique, *Comptes Rendus du Académie des Sciences de Paris* **15** (1842) 202–205.

[5] C. G. J. Jacobi, Sul principio dell'ultimo moltiplicatore e suo uso come nuovo principio generale di meccanica, *Giornale Arcadico di Scienze, Lettere ed Arti* **99** (1844) 129–146.

[6] C. G. J. Jacobi, Theoria novi multiplicatoris systemati æquationum differentialum vulgarium applicandi: Pars I, *J. Reine Angew. Math.* **27** (1844) 199–268.

[7] C. G. J. Jacobi, Theoria novi multiplicatoris systemati æquationum differentialum vulgarium applicandi: Pars II, *J. Reine Angew. Math.* **29** (1845) 213–279 and 333–376.

[8] C. G. J. Jacobi, *Vorlesungen über Dynamik. Nebst fünf hinterlassenen Abhandlungen desselben herausgegeben von A Clebsch* (Druck und Verlag von Georg Reimer, Berlin, 1886).

[9] S. Lie, Veralgemeinerung und neue Verwerthung der Jacobischen Multiplicator-Theorie, *Fordhandlinger i Videnokabs–Selshabet i Christiania* (1874) 255–274.

[10] S. Lie, *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen* (Teubner, Leipzig, 1912).

[11] J. Malmquist, Sur les équations différentielles de second ordre dont l'intégrale générale a ses points critiques fixes, *Ark. Mat. Astr. Fys.* **17** (1922/23) l–89.

[12] M. C. Nucci, Jacobi last multiplier and Lie symmetries: A novel application of an old relationship *J. Nonlinear Math. Phys.* **12** (2005) 284–304.

[13] M. C. Nucci, Lie symmetries of a Painlevé-type equation without Lie symmetries, *J. Nonlinear Math. Phys.* **15** (2008) 205–211.

[14] M. C. Nucci and P. G. L. Leach, Gauge variant symmetries for the Schrödinger equation *IL Nuovo Cimento B* **123** (2008) 93–101.

[15] M. C. Nucci and K. M. Tamizhmani, Lagrangians for dissipative nonlinear oscillators: The method of Jacobi Last Multiplier arXiv:0809.0022v1 [math-ph] (2008).

[16] K. Okamoto, On the τ-function of the Painlevé equations, *Physica D* **2** (1981) 525–535.

[17] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge University Press, Cambridge, 1988, First published 1904).