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THE SHAPE OF SOLITON-LIKE SOLUTIONS OF A HIGHER-ORDER KdV EQUATION DESCRIBING WATER WAVES

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We study the solitary wave solutions of a non-integrable generalized KdV equation proposed by Fokas [A. S. Fokas, *Physica D* **87**, 145 (1995)], aiming to describe unidirectional waves in shallow water with greater accuracy than the standard KdV equation. This generalized equation includes higher-order terms in the small parameters α and β, representing respectively the height and inverse width of the wave compared to the thickness of the water sheet. The solitary waves we find have a smaller height and a larger width than the corresponding KdV soliton at the same propagation velocity. Extrapolating these results we conjecture that — in the limit of arbitrarily high order in α and β — the solitary waves will attain a specific, finite height and width as the wave speed c increases.

1. Introduction

The Korteweg–de Vries (KdV) equation, written in a stationary frame as

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + \frac{1}{6}c_0 h_0^2 \eta_{xxx} = 0, \tag{1}$$

represents a first-order approximation in the study of waves of small amplitude and long wavelength propagating in one direction over the surface of a shallow layer of an inviscid

Fig. 1. A solitary wave approaching the beach of Molokai, Hawaii. The lagoon has a constant depth of approximately one metre and is perfectly suited to observe KdV solitons [9].

and incompressible fluid [1–8]. In this equation h_0 denotes the height of the undisturbed fluid layer, $\eta(x,t)$ the elevation of the wave above h_0, and $c_0 = \sqrt{gh_0}$ the nondispersive phase velocity.

A good example of a situation described by the KdV equation is depicted in Fig. 1, in which we see a solitary wave travelling across the lagoon of one of the Hawaiian islands, with an amplitude much smaller than the depth $h_0 = 1\,\mathrm{m}$ of the lagoon. Historically, the first example of a solitary wave of this type was the "Wave of Translation" observed by John Scott Russell in a channel near Edinburgh in 1834 [7, 10, 11], but solitary ripples that are well-described by the KdV equation — if one is willing to ignore the small viscosity and compressibility of water — can be observed whenever a thin sheet of water flows (with velocity exceeding c_0) over a smooth floor. Certainly the most celebrated property of the solitary-wave solutions of Eq. (1) is that they collide elastically: when two of them cross each other, they emerge unscathed from the interaction. It is because of this particle-like property that they are called solitons [12].

We should mention that the study of solitary water waves can also be pursued to some extent at the level of the governing fluid-dynamical equations [13–15]. However, the KdV equation is often preferred — notwithstanding the fact that it is only an approximation — because it admits an in-depth analysis.

Equation (1) can be brought into dimensionless form by a change of variables: $t \to \tau = t/t_c$ (where t_c represents a typical time scale), $x \to \xi = x/\ell$ (with ℓ a typical length scale in the horizontal direction, say the width of the wave) and $\eta \to u = \eta/a$ (with a a typical length scale in the vertical direction, i.e. the height of the wave). Applying this change of variables, and choosing $t_c = \ell/c_0$, we arrive at the following form of the KdV equation:

$$u_\tau + u_\xi + \alpha u u_\xi + \beta u_{\xi\xi\xi} = 0, \tag{2}$$

where $\alpha = \frac{3}{2}a/h_0$ and $\beta = \frac{1}{6}(h_0/\ell)^2$.[a] In the derivation of the KdV equation both α and β are taken to be small compared to 1. Moreover it is usually assumed that $a/h_0 \approx (h_0/\ell)^2$; this is a natural assumption, expressing the balance between nonlinearity and dispersion that plays a crucial role in the solitary wave solution for which the KdV equation is famous [16]. It is not obligatory, however, since a and ℓ are two independent quantities and formal considerations support the view that the KdV equation is a valid approximation to the underlying hydrodynamical equations also outside the regime where a/h_0 and $(h_0/\ell)^2$ have roughly the same value [17].

When α and β are not really small (or when a more accurate description is required), higher-order terms in these parameters should be included. Up to second order the equation is [18–23]

$$u_\tau + u_\xi + \alpha u u_\xi + \beta u_{\xi\xi\xi} + \alpha^2 \rho_1 u^2 u_\xi + \alpha\beta(\rho_2 u u_{\xi\xi\xi} + \rho_3 u_\xi u_{\xi\xi}) + \beta^2 \rho_4 u_{\xi\xi\xi\xi\xi} = 0, \quad (3)$$

where the constant coefficients (determined from the original physical equations for an inviscid, incompressible fluid) are given by $\rho_1 = -\frac{1}{6}, \rho_2 = \frac{5}{3}, \rho_3 = \frac{23}{6}$ and $\rho_4 = \frac{19}{10}$. Unlike the original KdV equation, Eq. (3) with the given values for the coefficients is not integrable and its solitary wave solutions are not true "solitons" since their collisions are not perfectly elastic: During the interaction, a dispersive wave train of small amplitude is generated.[b] It can be shown that exact soliton solutions (with the well-known sech^2 profile) exist for special combinations of the parameters $\{\rho_1, \ldots, \rho_4\}$, but these do *not* include the aforementioned combination for water [24].

The next equation, in which all terms up to third order in α and β are retained, was studied recently by Marchant [25, 26]. He showed that the resulting equation does not in general conserve mass due to the fact that the unidirectional assumption in the derivation (which allows for right-moving waves only) breaks down at this order. To correct this he coupled the third-order equation to a companion equation describing left-moving waves and found that left-moving waves are generated whenever the wave profile is changing, e.g. during the collision of two solitary waves [25]. A single solitary wave, which moves without any change of shape, does not give rise to a left-moving wave.

In the present paper we study the somewhat simpler case when $O(a/h_0) > O((h_0/\ell)^2)$, i.e. when the waves are higher and/or wider than is usually assumed. In this case the terms of order $O(\beta^2), O(\alpha\beta^2)$ and $O(\beta^3)$ may be neglected, and the third-order equation reduces to [27]:

$$u_\tau + u_\xi + \alpha u u_\xi + \beta u_{\xi\xi\xi} + \alpha^2 \rho_1 u^2 u_\xi + \alpha\beta(\rho_2 u u_{\xi\xi\xi} + \rho_3 u_\xi u_{\xi\xi}) + \alpha^3 \sigma_1 u^3 u_\xi$$
$$+ \alpha^2\beta(\sigma_2 u^2 u_{\xi\xi\xi} + \sigma_3 u u_\xi u_{\xi\xi} + \sigma_4 u_\xi^3) = 0. \quad (4)$$

The constant coefficients ρ_1, ρ_2 and ρ_3 have the same values as above, see Eq. (3), and the new ones are given by $\sigma_1 = \frac{1}{8}, \sigma_2 = \frac{7}{18}, \sigma_3 = \frac{79}{36}$ and $\sigma_4 = \frac{45}{36}$ [27]. Again this equation is not

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[a]Note that one could rescale α and β out of Eq. (2) by a further change of variables. However, we choose not to do so because both α and β are essential parameters in our analysis of the higher-order approximations, see Eq. (4).

[b]The solitary wave solutions of Eq. (3) are sometimes called "asymptotic solitons", since the inelastic effects can be shown to be beyond the second-order asymptotic validity of equation [18, 20, 21], i.e., these effects are of order $O(\alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3)$ and higher.

integrable and its solitary wave solutions do not collide precisely elastically. Only for special combinations of the parameters — which do not represent the hydrodynamic situation — does Eq. (4) admit exact soliton solutions with a sech^2 profile [28, 29]. Our main concern is therefore with single solitary-wave solutions and in particular we study how the successive higher-order terms in Eq. (4) alter the shape of these waves.

Since we concentrate on travelling waves moving to the right, we limit ourselves to solutions of the form $u(\xi, \tau) = u(\xi - c\tau)$, where c is the velocity of propagation. The central question we seek to answer is: How do the height and width of the solitary waves depend on the velocity c if terms of increasingly high order are added to the KdV equation? For definiteness we take $\alpha = 0.60$ and $\beta = 0.01$. The choice of α implies that $a/h_0 = \frac{2}{3}\alpha = 0.40$, which is not much smaller than 1, so that there is good reason to include the higher order terms in α. It should be noted that this choice is still within the realistic range for solitary water waves over a horizontal bottom; the maximum value of a/h_0 that can be achieved in practice lies somewhere around 0.55 [30], while the theoretical upper limit is 0.78 [31–33]. (In our equations this means that $u = \eta/a$ should not exceed $0.55h_0/a = 1.375$ or, theoretically, $0.78h_0/a = 1.95$. However, we also study values beyond that to gain extra insight in the mathematical structure of the problem.) The choice of β means that $(h_0/\ell)^2 = 6\beta = 0.06$ in accordance with the assumption $O(a/h_0) > O((h_0/\ell)^2)$, which lies at the basis of Eq. (4).

In Sec. 2 we briefly analyze the solitary wave profile of the KdV equation, i.e. Eq. (4) up to first order in α and β. The shape of the wave in this case is well-known and serves as a benchmark for the subsequent cases. In Sec. 3 we include the terms $O(\alpha^2, \alpha\beta, \alpha^3)$ [i.e. the first line of Eq. (4) as a whole] and in Sec. 4 we turn to the full Eq. (4) by including also the cubic terms $O(\alpha^2\beta)$ of the second line. Finally in Sec. 5 we compare the solitary waves in the three successive approximations and conjecture on the height and width of the waves in the limiting case when terms of arbitrarily high order in α and β are included.

2. First-Order Approximation: The KdV Equation

To make our discussion self-contained, and to demonstrate the method we are going to use, in this section we briefly review the unperturbed KdV equation Eq. (2) or, equivalently, the first line of Eq. (4). Since we are interested in waves travelling to the right, we consider solutions of the form $u(\xi, \tau) = u(\xi - c\tau)$, which do not depend on ξ and τ separately but only on the combined variable $\zeta = \xi - c\tau$. Now, with $u_\xi = u_\zeta \cdot d\zeta/d\xi = u_\zeta$ and $u_\tau = u_\zeta \cdot d\zeta/d\tau = -cu_\zeta$, the KdV equation becomes the following ordinary differential equation (the prime denotes the derivative with respect to ζ):

$$(1 - c)u' + \alpha uu' + \beta u''' = 0. \tag{5}$$

Integrating this equation once yields:

$$(1 - c)u + \frac{1}{2}\alpha u^2 + \beta u'' = 0, \tag{6}$$

where the integration constant (the right-hand side of this equation) is zero because of the vanishing boundary conditions $u(\zeta) = u'(\zeta) = u''(\zeta) = 0$ at $\zeta = \pm\infty$. In order to analyze
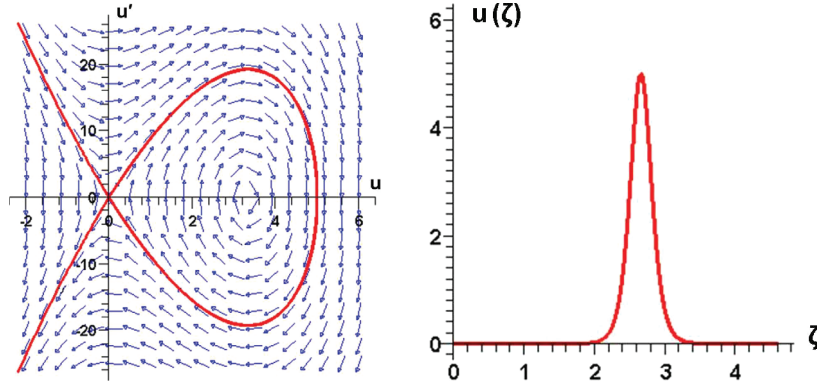
Fig. 2. Left: The (u, u')-plane for Eq. (7) for the values $\alpha = 0.6, \beta = 0.01$ and $c = 2$. The homoclinic orbit emanating from the saddle point $(0,0)$ represents a solitary wave. Right: The corresponding solitary-wave profile $u(\zeta)$.

Eq. (6) by means of a phase-plane analysis, we rewrite this second-order ODE in the form of two first-order ODEs:

$$u' = y$$
$$u'' = y' = -\frac{1}{\beta}\left[(1-c)u + \frac{1}{2}\alpha u^2\right]. \tag{7}$$

The corresponding (u, u') phase-plane is depicted in Fig. 2. We observe two fixed points, a saddle in $(0,0)$ and a centre in $(2\alpha^{-1}(c-1), 0) = (3.33, 0)$. The homoclinic orbit (separatrix) emanating from the $(0,0)$ represents the well-known soliton solution

$$u_{sol}(\xi, \tau) = \frac{3(c-1)}{\alpha}\text{sech}^2\left[\sqrt{\frac{c-1}{4\beta}}(\xi - c\tau)\right]. \tag{8}$$

Note that this solution only exists for $c > 1$, in agreement with our earlier observation that solitary waves occur when the speed exceeds the nondispersive phase velocity $c_0 = \sqrt{gh_0}$, which has been implicitly rescaled to 1 by the nondimensionalization.

The solitary wave solution Eq. (8) divides the unbounded orbits in the outer region of the (u, u')-plane from the periodic orbits within the inner region. These latter solutions are known as the cnoidal waves of the KdV equation [6, 7] and correspond to the Stokes wave solutions of the governing fluid-dynamical equations [2, 7, 34, 35].

3. Second-Order Approximation

We now turn to the next level of approximation by studying Eq. (4) up to the terms of order $O(\alpha^2, \alpha\beta, \alpha^3)$, i.e. the first line of Eq. (4) altogether. It may be noted that we thus mix a term of cubic order (α^3) with the terms of quadratic order in α and β, but this is a logical consequence of the fact that we are working with a relatively large value of $\alpha = 0.60$ and a small value of $\beta = 0.01$. It is for this same reason that we may ignore the quadratic term $\beta^2 \rho_4 u_{\xi\xi\xi\xi}$ [cf. the full second-order KdV equation Eq. (3)] in the present discussion.
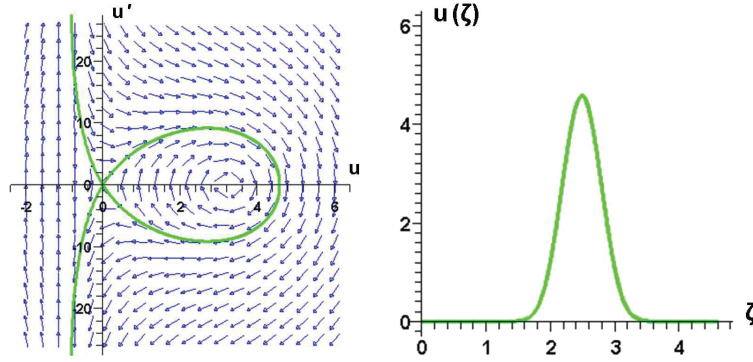
Fig. 3. Left: The (u, u')-plane for the second-order KdV system, Eq. (10), for the same values $\alpha = 0.60$, $\beta = 0.01$ and $c = 2$ as in Fig. 2. Right: The second-order solitary wave profile $u(\zeta)$. It has a smaller height and larger width than the first-order soliton of Fig. 2.

We again assume a travelling-wave solution, $u = u(\xi - c\tau) = u(\zeta)$, and thus turn the original partial differential equation into an ordinary differential equation,

$$(1 - c)u' + \alpha uu' + \beta u''' + \alpha^2 \rho_1 u^2 u' + \alpha\beta(\rho_2 uu''' + \rho_3 u'u'') + \alpha^3 \sigma_1 u^3 u' = 0, \qquad (9)$$

which is readily integrated — just as Eq. (5) in the previous section — and leads to the following system of two first-order ODEs:

$$u' = y$$

$$u'' = y' = \frac{-1}{\beta(1 + \alpha\rho_2 u)} \left\{ (1 - c)u + \frac{1}{2}\alpha u^2 + \frac{1}{3}\alpha^2 \rho_1 u^3 + \frac{1}{2}\alpha\beta(\rho_3 - \rho_2)y^2 + \frac{1}{4}\alpha^3 \sigma_1 u^4 \right\}.$$
$$(10)$$

Solving (10) numerically and plotting the solutions in the (u, u')-plane of Fig. 3, we observe again a homoclinic orbit emanating from the saddle point in $(0, 0)$, representing the solitary wave solution. This homoclinic orbit, however, is much flatter than the corresponding one in the first-order system (cf. Fig. 2) implying that the sides of the solitary wave are now considerably less steep. We further observe that the rightmost point of the homoclinic orbit (where it intersects the u-axis) lies at a somewhat smaller value of u than in the first-order case, meaning that the height of the solitary wave has decreased. These observations are confirmed by the profile of the solitary wave $u(\zeta)$ given in the right-hand plot of Fig. 3. We return to this in the Conclusion.

4. Third-Order Approximation

At the next level of accuracy we consider the full Eq. (4). When we again restrict ourselves to travelling-wave solutions of the form $u = u(\xi - c\tau) = u(\zeta)$, Eq. (4) takes the form of the following ordinary differential equation:

$$(1 - c)u' + \alpha uu' + \beta u''' + \alpha^2 \rho_1 u^2 u' + \alpha\beta(\rho_2 uu''' + \rho_3 u'u'') + \alpha^3 \sigma_1 u^3 u'$$
$$+ \alpha^2\beta(\sigma_2 u^2 u''' + \sigma_3 uu'u'' + \sigma_4 u'^3) = 0. \qquad (11)$$

Equation (11) is, as far as we can tell, not integrable. We therefore transform it directly to the following system of three first-order ODEs,

$$u' = y$$

$$u'' = y' = z$$

$$u''' = z' = \frac{-1}{\beta(1 + \alpha\rho_2 u + \alpha^2\sigma_2 u^2)}\{(1-c)y + \alpha uy + \alpha^2\rho_1 u^2 y$$

$$+ \alpha\beta\rho_3 yz + \alpha^3\sigma_1 u^3 y + \alpha^2\beta\sigma_3 uyz + \alpha^2\beta\sigma_4 y^3\} \tag{12}$$

and solve this system numerically in the (u, u', u'') phase-space. The solution is depicted in Fig. 4.

For all speeds $c > 1$ we again observe a homoclinic orbit — or solitary wave — emanating from the unstable equilibrium point in the origin $(0,0,0)$, which follows a curved surface in the three-dimensional space. In the second plot of Fig. 4 this orbit is projected onto the (u, u')-plane for direct comparison with the results of the previous two approximations. The trend observed in the second-order approximation is retained: The homoclinic orbit
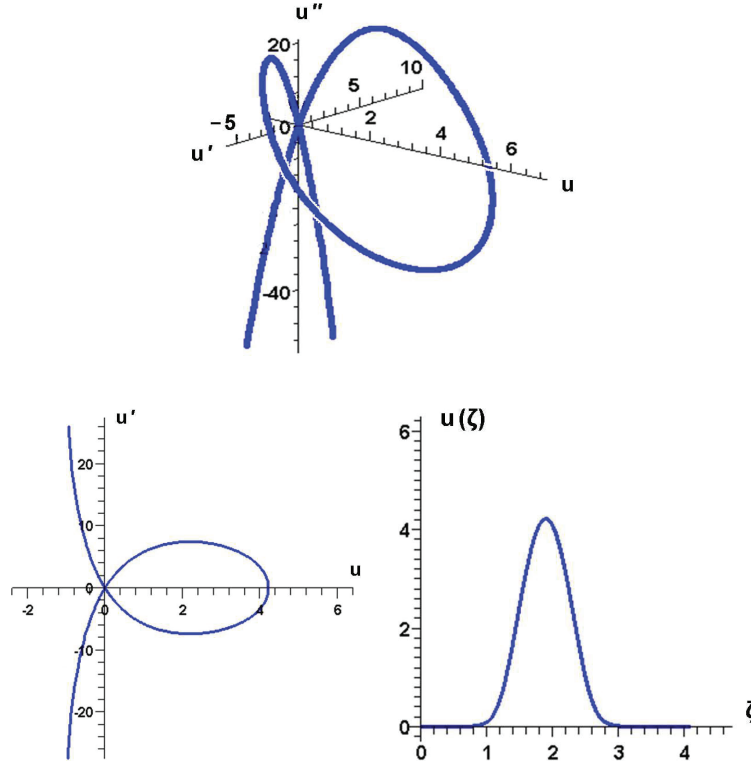

Fig. 4. Top: The (u, u', u'') phase-space for the third-order KdV system Eq. (12) for the same values, $\alpha = 0.60, \beta = 0.01$ and $c = 2$, as in Figs. 2 and 3. The homoclinic orbit corresponding to the solitary-wave solution is seen to lie on a curved surface. Down, left: Projection of the homoclinic orbit on the (u, u')-plane to facilitate the comparison with Figs. 2 and 3. Down, right: The third-order solitary-wave profile $u(\zeta)$. Its height has decreased, and its width increased, with respect to the first- and second-order solutions of the previous figures.

has shrunk in size still further, both in the u- and the u'-directions, corresponding to an increase of the height of the solitary wave and a decrease of its width, respectively.

In the interior region of the homoclinic orbit (following the curved surface in the three-dimensional space) there exist cnoidal waves also in this case. We will discuss these in a forthcoming publication.

5. Conclusion: Comparing the Successive Approximations

In this paper we examined the inclusion of higher-order terms in the small parameters α and β in the KdV equation. The successive approximations preserve the most characteristic solution of the KdV equation, namely the solitary wave, but they do change its shape and also cause the loss of the pure "soliton" property: When two of them meet, they do not emerge from the interaction completely unscathed [24–26]. With respect to the shape, given a certain wave speed, the height of the solitary wave decreases and its width increases, as can be seen clearly in Fig. 5 (left).

It is also interesting to compare three wave profiles for the same height, as in Fig. 5 (right). The second- and third-order approximations are seen to be considerably wider than the sech^2 profile of the first-order approximation [cf. Eq. (8)]. Their velocity differs from that of the first-order wave, but only slightly. Note that in this figure we have chosen a realistic amplitude $u = 1$, corresponding to a wave height $a = 0.4h_0$, well achievable in practice [30]. In the other figures we have *not* restricted ourselves to experimentally attainable values, going also beyond $u = 1.375$, as explained in the Introduction. For these larger values of u (and c) the differences between the solitary waves become much clearer.

Concentrating on the height of the solitary wave, we see that it decreases drastically already as a result of the first correction terms of order $O(\alpha^2, \alpha\beta, \alpha^3)$. Figure 6 illustrates this point. The KdV equation itself (i.e. the first-order approximation indicated by the
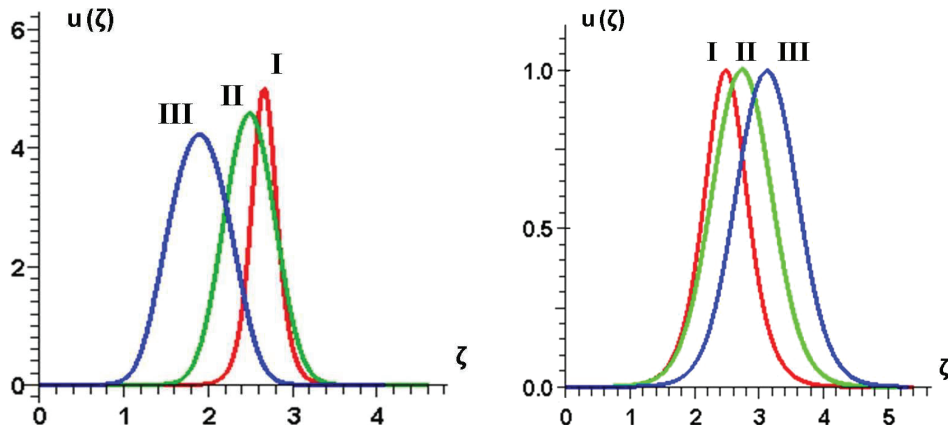

Fig. 5. Comparison of the solitary-wave profiles $u(\zeta)$ for the first-, second- and third-order equations. Left: At one fixed value of the wave speed $c = 2$. The height of the profile diminishes, and its width grows, as the order of the approximation increases. Right: For the same wave amplitude $u = 1$ (corresponding to a slightly different wave speed, $c \approx 1.20$, for each curve). The width of the profiles for the second and third approximations is distinctly larger than that of the $\mathrm{sech}^2(\zeta)$ profile of the first approximation.

symbol I) shows a linear increase of the height, $h(c) \propto (c-1)$, as can also be read from the analytic solution Eq. (8). The second-order approximation causes a sharp decrease of the height and instead of a linear growth we now observe an approximately logarithmic dependence $h(c) \propto \ln c$. The third-order corrections cause a further — albeit less drastic — decrease in the height and make the logarithmic growth even more gradual. On the basis of this we conjecture that the inclusion of corrections of arbitrarily high order in α and β will stop the growth altogether. If this indeed be the case, the height of the solitary waves at high speed c saturates at a definite, finite value.

Likewise the width of the solitary waves as a function of c displays a conspicuous increase as one goes from one approximation to the next. This is illustrated in Fig. 7, where we measure the width w at the half-height of the wave. The behavior is best discerned in terms of the inverse square width w^{-2} depicted in the second plot. This quantity grows linearly in the case of the original KdV equation (I) since $w(c) \propto (c-1)^{-1/2}$ [as can be inferred analytically from Eq. (8)], implying that high-speed solitary waves in this first approximation become infinitesimally thin. This unrealistic feature is corrected at the level of the next approximations, II and III. Indeed we conjecture that the width saturates at a finite nonzero value in the limit for large c when terms of arbitrarily high order in α and β are taken into account.

It would be highly desirable if accurate experiments were performed to test the validity of the results presented in this paper. As far as real water waves are concerned (which are subject to the restriction that a/h_0 cannot exceed 0.55, or equivalently, u cannot grow beyond 1.375, with corresponding wave speeds $c \lesssim 1.3$) the quantity that best lends itself to distinguish between the successive approximations seems to be the width of the profile, as in Figs. 5 (right) and 7.

Further, the method outlined here can be employed to study also other generalizations of the KdV equation, e.g. the ones that take into account vorticity to model wave-current interactions [36, 37], or in fact any other evolution equation possessing travelling wave solutions.
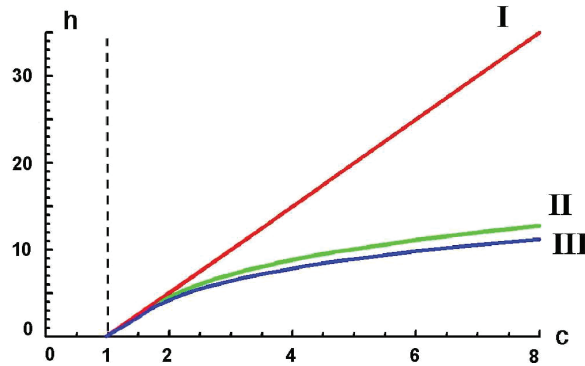

Fig. 6. Height of the solitary wave as a function of its speed c in the three successive approximations. In the first-order approximation I (i.e., the ordinary KdV equation) the height $h(c)$ grows linearly without bound, following the analytic solution Eq. (8). The approximations II and III show the restraining influence of the higher-order terms. Note: The wave amplitudes that can be achieved in water experiments are restricted to the narrow zone $h \leq 1.375$ (or theoretically, $h \leq 1.95$) and corresponding wave velocities $c \lesssim 1.3$ (or $c \lesssim 1.4$), where the curves I, II, and III have hardly separated yet [30–33].
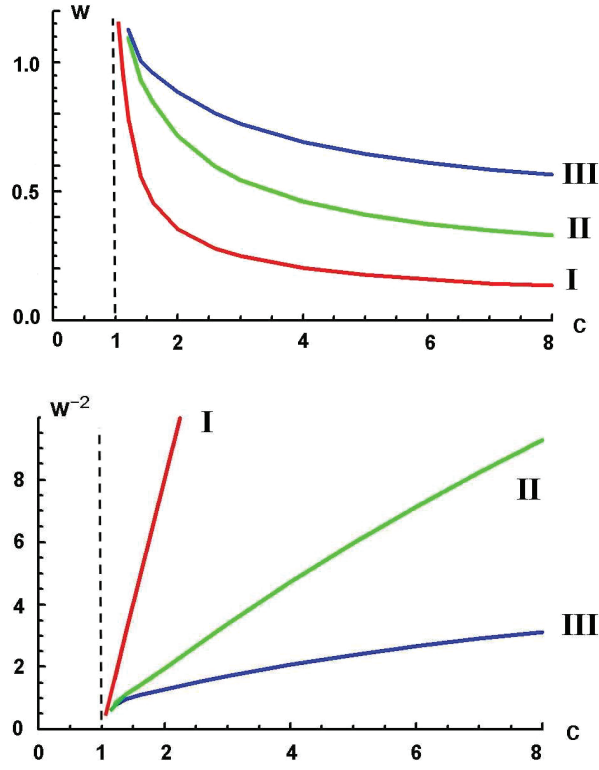
Fig. 7. Top: Width $w(c)$ of the solitary wave as a function of its speed c in the three successive approximations. Bottom: The inverse square width $w^{-2}(c)$, showing a linear increase with respect to c for the first-order KdV equation [in accordance with the analytic solution Eq. (8)] and a much more restricted growth for the second- and third-order approximations.

Acknowledgments

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