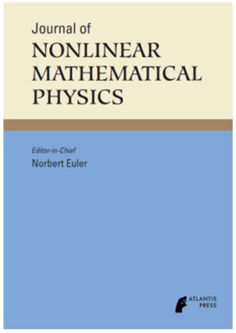

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GLOBAL SOLVABILITY OF A FRAGMENTATION-COAGULATION EQUATION WITH GROWTH AND RESTRICTED COAGULATION

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We consider a fragmentation-coagulation equation with growth, where the nonlinear coagulation term, introduced in O. Arino and R. Rudnicki [2], is designed to model processes in which only a part of particles in the aggregates is capable of coalescence. We introduce various growth models, describing both biological and inorganic processes, and discuss their effect on the generation of the linear growth-fragmentation semigroup. Once its existence has been established, the solution of the full nonlinear problem follows by showing that the coagulation term is globally Lipschitz.

Keywords: Fragmentation-coagulation equation; semigroups; structured population model; global solutions.

1. Introduction

Fragmentation and coagulation processes occur in many fields of applied sciences and engineering. We mention polymerization and depolimerization phenomena in chemical engineering, agglutination and splitting of blood cells, formation and splitting of aerosol droplets or evolution of phytoplankton aggregates as some of the recently analyzed cases, see e.g. [1, 2, 9, 13]. Fragmentation and coagulation processes, modeled here by integral operators, often are accompanied by growth or decay of aggregates e.g. by surface deposition or dissolution in chemical applications, see e.g. [10] or by birth or division processes in biological ones, see e.g. [1, 16]. In this paper we only consider growth processes which are modeled by a first order partial differential operator which, depending upon the characteristics, may require a boundary condition at the left end of the size domain. Our analysis covers a wide range of boundary behaviors including the most challenging case in which the smallest monomers having been shed from larger aggregates re-enter the system as "newborn" aggregates. This leads to nonlocal boundary conditions of the McKendrick type, see [1, 8]. Another novelty of the model is the coagulation kernel which takes into account that not all particles in an aggregate have the same ability to combine with particles of other aggregates which results in a "damped" coagulation process. Similar kernels were introduced by one of the authors in [2] to model evolution of phytoplankton, but may describe a variety of similar phenomena in nonbiological applications. An important observation is that the kernels of this type have better mathematical properties and thus allow for an extension of earlier results of e.g. [8] to more general fragmentation and growth operators.

Further, following [1, 8] we allow that the smallest size/mass of a monomer may be $x_0 > 0$ and thus no particle of mass smaller than x_0 can appear in the system due to fragmentation and also the smallest mass of an aggregate after coagulation is $2x_0$. Since, however, particles can increase their mass in a continuous way by, e.g. surface deposition, the mass of an aggregate can be in principle any real number $x \geq x_0$ and thus it is modeled by a continuous variable. This extension of the model also creates interesting technical challenges.

2. Description of the Model and Assumptions

We describe the dynamics of an ensemble of particles using the aggregate density function $u(t,x)$. Here $x \in (x_0, \infty)$ is a variable that represents the size, or mass, of the aggregate, the variable t represents time and $u(t,x)$ is the concentration of aggregates of size x at time t. We assume that for each $t \geq 0$ the function $x \mapsto u(t,x)$ is from the space

$$X = L_1((x_0,\infty),\, x\,dx) = \left\{\psi :\ \|\psi\|_{\mathrm{def}} = \int_{x_0}^{\infty} x|\psi(x)|dx < \infty\right\}. \tag{2.1}$$

As we mentioned in the Introduction, the aggregates may die (or vanish from the system), grow, may split or join together. We begin with a description of the fragmentation process.

The fragmentation operator is given by

$$[\mathcal{F}u](x) := -a(x)u(x) + \int_{x+x_0}^{\infty} a(y)b(x|y)u(y)dy, \tag{2.2}$$

where a is the fragmentation rate, satisfying $0 \leq a \in L_{\infty,\mathrm{loc}}((x_0,\infty))$. The mass x distribution of daughter aggregates after fragmentation of a parent of mass y is denoted by $b(x|y)$. As discussed in the Introduction, particles of mass less than $2x_0$ cannot fragment, hence we assume $a(x) = 0$ for $0 < x < 2x_0$. Similarly, $b(x|y) = 0$ for $y < x + x_0$ and $x < x_0$. We assume that mass is conserved in each fragmentation event so that b must satisfy

$$\int_{x_0}^{y-x_0} xb(x|y)dx = y \tag{2.3}$$

for each $y > 2x_0$.

Remark 1. The space X is chosen in a natural way because $\int_{x_0}^{\infty} x|\psi(x)|dx$ is the total mass of the ensemble and (2.3) ensures that the fragmentation process is formally conservative, see e.g. [4, p. 200]. We note that a simpler standard space $L_1((x_0,\infty),dx)$ cannot be used if one wants to consider fragmentation problems with unbounded coefficients as then the fragmentation operator does not generate a strongly continuous semigroup. Indeed, considering pure fragmentation with $x_0 = 0$, by [17], the function

$$u(t,x) = e^{-xt}\left(f(x) + \int_x^\infty f(y)[2t + t^2(y-x)]dy\right)$$

is the solution to a pure fragmentation equation with $a(x) = x$ and $b(x|y) = 2/y$ and the initial condition f. Calculating the norm of the solution in $L_1((0,\infty),dx)$ and taking into account that the solution is nonnegative for nonnegative initial data we get

$$\begin{aligned}
&\|u(1,\cdot)\|_{L_1((0,\infty),dx)} \\
&\quad\geq \int_0^\infty \left(e^{-x}\int_x^\infty f(y)(y-x)dy\right)dx = \int_0^\infty f(y)\left(\int_0^y e^{-x}(y-x)dx\right)dy \\
&\quad= \int_0^\infty f(y)e^{-y}\left(\int_0^y e^z z dz\right)dy = \int_0^\infty f(y)(y-1+e^{-y})dy
\end{aligned}$$

which is finite only if $f \in X$. We note that the necessity to work in X creates some constraints in the growth part of the model which are discussed below. On the other hand, the model of coagulation adopted in this paper behaves well in X, contrary to a more standard approach, see e.g. [7, 8].

Next we discuss the coagulation process. Here, as in [2], we assume that only a part of the aggregates has the competence to join. For instance this could be due to the fact that aggregates may be composed of different types of particles only part of which have the necessary devices to attach to others, or are active. We introduce the coefficient of competence, $g(x)$, which is assumed to be a positive and bounded function. The number of particles in all aggregates that at time t are implicated in the coagulation process is given by:

$$J(t) := \int_{x_0}^\infty z g(z) u(t,z) dz$$

and

$$j(t,x) := \frac{x g(x) u(t,x)}{J(t)}$$

is the fraction of particles in size-x aggregates competent for the coagulation process with respect to the total population of particles in aggregates which are prone to join. In terms of the quantities introduced so far we can express the rate at which cells are forming aggregates of size x as

$$J(t)\chi_U(x)\int_{x_0}^{x-x_0} j(t,x-y)j(t,y)dy,$$

where $\chi_U(x)$ is the characteristic function of the interval $U = [2x_0,\infty)$, introduced to account for the fact that no aggregate of mass smaller than $2x_0$ can be formed in the coagulation process. The limits of integration ensure that no particle of mass smaller than x_0 takes part in the process. Hence the coagulation operator is given by

$$\begin{aligned}[\mathcal{C}u](t,x) &= J(t)x^{-1}\chi_U(x)\int_{x_0}^{x-x_0} j(t,x-y)j(t,y)dy - g(x)u(t,x)\\ &= \chi_U(x)\frac{\displaystyle\int_{x_0}^{x-x_0} yg(y)u(t,y)(x-y)g(x-y)u(t,x-y)dy}{\displaystyle x\int_{x_0}^{\infty} zg(z)u(t,z)dz} - g(x)u(t,x).\end{aligned} \tag{2.4}$$

Proposition 1. *The coagulation model described by* (2.4) *is formally conservative.*

Proof. Since the total mass is given by $M(t) = \int_{x_0}^{\infty} xu(t,x)dx$, we have to show that

$$\frac{d}{dt}M(t) = \frac{d}{dt}\int_{x_0}^{\infty} xu(t,x)dx = \int_{x_0}^{\infty} x\frac{\partial}{\partial t}u(t,x)dx = 0$$

for nonnegative solutions u. Because $g \in L_\infty((x_0,\infty))$, it is enough to prove that

$$\begin{aligned}&\int_{x_0}^{\infty} \chi_U(x)\int_{x_0}^{x-x_0} yg(y)u(t,y)(x-y)g(x-y)u(t,x-y)dydx\\ &\quad = \left(\int_{x_0}^{\infty} xg(x)u(t,x)dx\right)^2.\end{aligned} \tag{2.5}$$

By the Fubini theorem,

$$\begin{aligned}&\int_{x_0}^{\infty} \chi_U(x)\left(\int_{x_0}^{x-x_0} yg(y)u(t,y)(x-y)g(x-y)u(t,x-y)dy\right)dx\\ &\quad = \int_{x_0}^{\infty} yg(y)u(t,y)\left(\int_{y+x_0}^{\infty} (x-y)g(x-y)u(t,x-y)dx\right)dy\\ &\quad = \left(\int_{x_0}^{\infty} yg(y)u(t,y)dy\right)^2.\end{aligned}$$

□

Next we introduce the other two processes incorporated into the model. In biological applications, the death process is modeled by exponential decay with size-dependent death rate d. The same model is used in the inorganic case when the fraction $d(x)$ of x-sized aggregates can be removed from the system by a natural or artificial mechanism. We assume that $0 \le d \in L_\infty((x_0,\infty))$.

Aggregates may grow as a result of divisions of cells in biological applications or, for instance, by surface deposition in chemical ones. The growth rate is denoted by r. We note that the range of rs we have to accommodate is rather wide: In biological applications typically we have $r(x) \sim x$ as growth is proportional to the number of particles (cells) in the aggregate. On the other hand, if the surface deposition rate is proportional to the surface area and the mass is proportional to the cube of the typical dimension of the aggregate, then $r(x) \sim x^{2/3}$. Thus we assume that r is a nonnegative function and

$$r \in \mathrm{AC}((x_0,\infty)) \cap X_\infty, \tag{2.6}$$

where $r \in \mathrm{AC}((x_0,\infty))$ means that r is absolutely continuous on each compact subinterval of (x_0,∞) and X_∞ is the dual space of X:

$$X_\infty = \left\{\psi;\ \|\psi\|_\infty := \mathrm{ess}\sup_{x_0 \le x < \infty} x^{-1}|\psi(x)|\right\}$$

so that the duality pairing is given by

$$\langle \psi, \omega \rangle = \int_{x_0}^{\infty} \psi(x)\omega(x)dx. \tag{2.7}$$

The growth operator is given by

$$[\mathcal{G}u](x) = -[r(x)u(x)]_x.$$

A vital role in the analysis of the model also is played by the integrability of $1/r(x)$ at x_0, see [4,8]. If $1/r(x)$ is not integrable at x_0, then the characteristics of $\mathcal{G}$ do not reach the line $x = x_0$ and there is no need to prescribe any boundary condition at x_0. On the other hand, if $1/r(x)$ is integrable at x_0, the characteristics do reach the line $x = x_0$ and therefore the boundary condition becomes crucial for the investigation of uniqueness. General boundary conditions considered in this paper are

$$\lim_{x \to x_0^+} r(x)u(t,x) = \int_{x_0}^{\infty} \beta(y)u(t,y)dy, \tag{2.8}$$

where $0 \le \beta \in X_\infty$. If $\beta \equiv 0$, then we have the standard no-influx condition. If, however, $\beta(y) \neq 0$, then (2.8) describes the rate at which an aggregate of size y sheds monomers of the smallest "zero" size which then re-enter the system as new aggregates and start to grow.

Remark 2. We note that, by the Gronwall inequality, assumption (2.6) ensures that the characteristics of the $\mathcal{G}$ are defined for all $t > 0$. This property is crucial for the generation of a strongly continuous semigroup by $\mathcal{G}$ in X (but not necessarily in $L_1((x_0,\infty),dx)$) as we demonstrate now. Consider the case with $x_0 = 0$ and the growth equation

$$u_t(t,x) = -(x^2 u)_x, \quad u(0,x) = u_0(x), \quad x > 0. \tag{2.9}$$

Since $1/x^2$ is not integrable at 0, there is no need to impose any boundary conditions. Standard calculations show that characteristics are given by $x(t) = \xi/(1-\xi t)$, $\xi > 0$ and we see that they blow up at $t = 1/\xi$. The solution is given by

$$u(t,x) = \frac{1}{(1+xt)^2}\, u_0\left(\frac{x}{1+xt}\right). \tag{2.10}$$

The solution is defined for all $t, x > 0$, the process is dissipative in $L_1((0,\infty))$:

$$\int_0^\infty \frac{1}{(1+xt)^2} u_0\left(\frac{x}{1+xt}\right) dx = \int_0^{1/t} u_0(\xi) d\xi \leq \int_0^\infty u_0(\xi) d\xi$$

and, indeed, it can be proved that (2.10) defines a strongly continuous semigroup in $L_1((0,\infty))$. On the other hand the norm of the solution in X is given by

$$\int_0^\infty \frac{x}{(1+xt)^2} u_0\left(\frac{x}{1+xt}\right) dx = \int_0^{1/t} \frac{\xi u_0(\xi)}{1-\xi t} d\xi$$

and the right-hand side has a nonintegrable singularity at $\xi = 1/t$ which shows that (2.10) cannot define a semigroup in X. Physically, the first norm expresses the total number of particles and the second the total mass of the ensemble. The equation describes a growth process which results in a redistribution of mass among particles, but which does not introduce new particles into the system. Thus the number of particles should not grow. However, the mass of particles of initial mass larger than $1/t$ reaches infinity at times shorter than t resulting in the blow-up of the total mass of the ensemble. These "infinite" particles are no longer in the system after time t which explains the first integral — at time t the particles of original mass larger than $1/t$ cease to be counted resulting in the process being non-conservative.

Taking into account all mechanisms described above, we arrive at the full evolution equation

$$\begin{aligned} u_t(t,x) = {} & -[r(x)u(t,x)]_x - d(x)u(t,x) - g(x)u(t,x) \\ & - a(x)u(t,x) + \int_{x+x_0}^\infty a(y)b(x|y)u(t,y)dy \\ & + \chi_U(x) \frac{\displaystyle\int_{x_0}^{x-x_0} yg(y)u(t,y)(x-y)g(x-y)u(t,x-y)dy}{\displaystyle x\int_{x_0}^\infty zg(z)u(t,z)dz}. \end{aligned} \tag{2.11}$$

This nonlinear integro differential equation is supplemented with the initial condition

$$u(0,x) = u_0(x), \tag{2.12}$$

where $u_0 \in X$, and with boundary condition (2.8):

$$\lim_{x\to x_0^+} r(x)u(t,x) = \int_{x_0}^\infty \beta(y)u(t,y)dy, \tag{2.13}$$

if $1/r(x)$ is integrable at $x = x_0$.

3. Analysis of the Linear Part

Firstly, we provide results concerning solvability of the linear part of (2.11) as it is crucial for analyzing the full equation.

We analyze this system in the space X. In what follows we denote by $\mathcal{T}$ and $\mathcal{B}$ the expressions appearing on the right-hand side of Eq. (2.11); that is,

$$[\mathcal{T}\psi](x) = -[r(x)\psi(x)]_x - q(x)\psi(x) \tag{3.1}$$

where $q = a + d + g$, and

$$[\mathcal{B}\psi](x) = \int_{x+x_0}^{\infty} a(y)b(x|y)\psi(y)dy. \tag{3.2}$$

The expressions $\mathcal{T}$ and $\mathcal{B}$ are defined on measurable and finite almost everywhere functions ψ for which they make pointwise (almost everywhere) sense.

3.1. *The case of r^{-1} nonintegrable at x_0*

Denote by T the realization of $\mathcal{T}$ (defined via (3.1)) on the domain

$$D(T) = \{\psi \in X;\ q\psi \in X,\ \ r\psi \in AC((x_0,\infty)) \text{ and } (r\psi)_x \in X\}. \tag{3.3}$$

Further let B be the realization of $\mathcal{B}$ (see (3.2)) on the domain $D(B) = D(T) = \{\psi \in X;\ q\psi \in X,\ \ r\psi \in AC((x_0,\infty))$ and $(r\psi)_x \in X\}$.

The corresponding Cauchy problem is

$$\begin{aligned} u_t(t) &= [T + B]u(t) \quad t > 0 \\ u(0) &= u_0, \end{aligned} \tag{3.4}$$

where

$$[(T+B)\psi](x) = -[r(x)\psi(x)]_x - q(x)\psi(x) + \int_{x+x_0}^{\infty} a(y)b(x|y)\psi(y)dy$$

for $\psi \in D(T)$. Recall that the norm of r in X_∞ satisfies $\|r\|_\infty < +\infty$.

Theorem 1. (i) *There is an extension G of the operator $T + B$ which generates a positive semigroup $(S_G(t))_{t\geq 0}$ that satisfies*

$$\|S_G(t)\| \leq e^{(\|r\|_\infty - \kappa)t}, \quad t > 0,$$

where $\kappa = \inf_{x\in(x_0,\infty)}(d(x) + g(x))$.

(ii) *If q is continuous on some open interval $(x_0, x_0 + \eta)$, $\eta > 0$ with $\lim_{x\to x_0} q(x) = \lim_{x\to x_0} a(x) + d(x) + g(x) < +\infty$ and the growth rate r is differentiable at x_0, then $G = \overline{T + B}$ and thus the semigroup $(S_G(t))_{t\geq 0}$ satisfies*

$$\frac{d}{dt}\|S_G(t)u_0\|_X = \int_{x_0}^{\infty} (r(x) - d(x))S_G(t)u_0(x)dx \tag{3.5}$$

for any $0 \leq u_0 \in D(G)$.

Proof. This theorem is a slight generalization of results in [4, Sec. 9.3] proved for the case $x_0 = 0$ and in which it was additionally assumed that r was differentiable at $x = 0$ with $r'(0) > 0$. However, we note that allowing x_0 to be nonzero introduces only cosmetic changes in the proofs, see [14]. Moreover in the proof of the generation result in *op.cit.* this condition was used only to ensure that $1/r$ was not integrable at $x = 0$ (see [4, Eq. (9.102)]). Here it is the latter that is assumed to cover a larger class of growth rates (e.g. $r(x) \sim x^2$ close to zero).

In the proof of the second part we note that in [4, Theorem 9.31] we only used differentiability of r at zero and it was irrelevant whether the derivative was positive or zero. $\square$

Remark 3. We note that Theorem 1 covers the typical case in biological applications in which $r(x) = \tilde{r}x$ describes growth at the rate proportional to the number of cells in the aggregate.

3.2. *Case of r^{-1} integrable at x_0*

The approach in this section is analogous to the work of [8] in which the abstract space $X_{0,1} = L_1((x_0, \infty), (1 + x)dx)$ is used to cater for a different coagulation kernel. Working in $X_{0,1}$ also required the fragmentation rate to grow at most linearly for $x \to \infty$. Working in the bigger space X, we can extend the work of [8] to general fragmentation rate kernels, specified under Eq. (2.2). Details of calculations can be found in [14].

Firstly, we define T_β to be $\mathcal{T}$ restricted to

$$D(T_\beta) = \left\{ \psi \in D(T) : \lim_{x \to x_0^+} r(x)\psi(x) = \int_{x_0}^{\infty} \beta(y)\psi(y)dy \right\}, \tag{3.6}$$

where $D(T)$ was defined in (3.3). Defining, as before, B to be the restriction of $\mathcal{B}$ to $D(T_\beta)$ we obtain the corresponding growth fragmentation equation:

$$\begin{aligned} u_t(t) &= [T_\beta + B]u(t) \quad t > 0, \\ u(0) &= u_0. \end{aligned} \tag{3.7}$$

The result in this case is similar to Theorem 1:

Theorem 2. (i) *There is an extension G_β of the operator $T_\beta + B$ which generates a positive semigroup $(S_{G_\beta}(t))_{t \geq 0}$ that satisfies*

$$\|S_{G_\beta}(t)\| \leq e^{(\|r\|_\infty + x_0\|\beta\|_\infty - \kappa)t}, \quad t > 0. \tag{3.8}$$

(ii) *If q is bounded at $x = x_0$, then $G_\beta = \overline{T_\beta + B}$ and thus the semigroup $(S_{G_\beta}(t))_{t \geq 0}$ satisfies*

$$\frac{d}{dt}\|S_{G_\beta}(t)u_0\|_X = \int_{x_0}^{\infty} (r(x) + x_0\beta(x) - (d(x) + g(x))S_{G_\beta}(t)u_0(x)dx \tag{3.9}$$

for any $0 \leq u_0 \in D(G_\beta)$.

Proof. As we noted above, the proof of this result is similar to that in [8] and detailed calculations can be found in [14]. Therefore here we restrict ourselves to deriving estimate (3.8) which is rather untypical in the case $\beta \neq 0$ and differs from [8]. We obtain it by finding the estimates of the relevant resolvents. The general solution of the resolvent equation for

$\mathcal{T}$ is of the form

$$u(x) = [\widetilde{R}_\lambda \psi](x) + c\frac{e^{-\lambda \widetilde{R}(x) - \widetilde{Q}(x)}}{r(x)}, \tag{3.10}$$

where c is a suitable scalar,

$$[\widetilde{R}_\lambda \psi](x) = \frac{e^{-\lambda \widetilde{R}(x) - \widetilde{Q}(x)}}{r(x)} \int_{x_0}^{x} e^{\lambda \widetilde{R}(y) + \widetilde{Q}(y)} \psi(y) dy, \tag{3.11}$$

and

$$\widetilde{R}(x) = \int_{x_0}^{x} \frac{ds}{r(s)}, \quad \text{and} \quad \widetilde{Q}(x) = \int_{x_0}^{x} \frac{q(s)}{r(s)} ds. \tag{3.12}$$

Recall that T_0 denotes T_β with $\beta = 0$, that is with zero boundary conditions. Then it is well known, see e.g. [8, Lemma 2.2], that $\widetilde{R}_\lambda$ defines the resolvent $R(\lambda, T_0)$ of $(T_0, D(T_0))$ for $\lambda > \|r\|_\infty$ and the following estimate is satisfied:

$$\|R(\lambda, T_0)\| \leq \frac{1}{\lambda - \|r\|_\infty}. \tag{3.13}$$

Next we set $\kappa := x_0\|\beta\|_\infty + \|r\|_\infty$ and turn our attention to the problem with $\beta \neq 0$. The solution u of the resolvent equation $\lambda u - T_\beta u = \psi$ of the operator T_β satisfies

$$u(x) = \widetilde{R}_\lambda \psi(x) + \frac{\varepsilon_\lambda(x)}{r(x)} \langle \beta, u \rangle,$$

where $\varepsilon_\lambda(x) := \exp[-\lambda \widetilde{R}(x) - \widetilde{Q}(x)]$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing (2.7). The above equation can be solved for $\langle \beta, u \rangle$ yielding

$$\langle \beta, u \rangle = \frac{\langle \beta, \widetilde{R}_\lambda \psi \rangle}{1 - \langle \beta, r^{-1} \varepsilon_\lambda \rangle},$$

provided $\langle \beta, r^{-1}\varepsilon_\lambda \rangle \neq 1$. We assume that this condition holds. Then

$$u(x) = \widetilde{R}_\lambda \psi(x) + [\Phi_{\lambda,\beta}(\widetilde{R}_\lambda \psi)](x), \tag{3.14}$$

where $\lambda > \|r\|_\infty$ and $\Phi_{\lambda,\beta}\psi = \frac{\varepsilon_\lambda}{r} \frac{\langle \beta, \psi \rangle}{(1 - \langle \beta, r^{-1}\varepsilon_\lambda \rangle)}$. As in [11], $\Phi_{\lambda,\beta}$ is a bounded compact (since rank one) operator on X such that $I + \Phi_{\lambda,\beta}$ is invertible with inverse

$$(I + \Phi_{\lambda,\beta})^{-1}\phi = \phi - \frac{\varepsilon_\lambda}{r} \langle \beta, \phi \rangle, \quad \phi \in X,$$

and clearly $(I + \Phi_{\lambda,\beta})(D(T_0)) = D(T_\beta)$. Thus

$$R(\lambda, T_\beta) = R(\lambda, T_0) + \Phi_{\lambda,\beta} R(\lambda, T_0). \tag{3.15}$$

Next we notice that

$$\int_x^\infty \frac{e^{-\lambda \widetilde{R}(s) - \widetilde{Q}(s)}}{r(s)} s ds \leq \int_x^\infty \frac{e^{-\lambda \widetilde{R}(s)}}{r(s)} s ds \leq \frac{1}{\lambda - \|r\|_\infty} x e^{-\lambda \widetilde{R}(x)}$$

for $\lambda > \|r\|_\infty$, see [8, Lemma 2.1]. Hence

$$\langle \beta, r^{-1}\varepsilon_\lambda\rangle \leq \frac{\|\beta\|_\infty}{\lambda - \|r\|_\infty} x_0,$$

where we used $R(x_0) = 0$. Thus for $\lambda > \|r\|_\infty + \|\beta\|_\infty x_0$ the resolvent of T_β satisfies

$$\begin{aligned}\|R(\lambda, T_\beta)\psi\| &\leq \|R(\lambda, T_0)\psi\| \left(1 + \frac{x_0\|\beta\|_\infty}{(\lambda - \|r\|_\infty)(1 - \|\beta\|_\infty \frac{x_0}{\lambda - \|r\|_\infty})}\right)\\ &\leq \|R(\lambda, T_0)\psi\| \left(\frac{\lambda - \|r\|_\infty}{\lambda - \|r\|_\infty - \|\beta\|_\infty x_0}\right) \leq \|\psi\| \left(\frac{1}{\lambda - \|r\|_\infty - \|\beta\|_\infty x_0}\right).\end{aligned}$$

Next we prove that

$$D(T_\beta) \subseteq D(B). \tag{3.16}$$

Indeed let $\psi \in D(T_\beta)$. By definition of $D(T_\beta)$ it is clear that $a\psi \in X$. It follows that

$$\begin{aligned}\|B\psi\| &= \int_{x_0}^{\infty} \left| \int_{x+x_0}^{\infty} a(y)b(x|y)\psi(y)dy \right| x\,dx\\ &\leq \int_{2x_0}^{\infty} |\psi(y)|a(y) \left(\int_{x_0}^{y-x_0} b(x|y)x\,dx\right) dy\\ &\leq \int_{x_0}^{\infty} |\psi(y)|a(y)y\,dy = \|a\psi\| < \infty,\end{aligned}$$

where we used (2.3). Therefore $\psi \in D(B)$.

Finally we show that, by a left shift, the operator $T_\beta + B$ can be transformed to a dissipative operator. More precisely we prove that for any $\psi \in D(T_\beta)_+$ the operator $\widetilde{T}_\beta + B := T_\beta - \kappa I + B$ on $D(T_\beta)$ satisfies

$$\int_{x_0}^{\infty} [(\widetilde{T}_\beta + B)(\psi)](x)x\,dx \leq 0.$$

Let $\psi \in D(T_\beta)_+$. Integrating the growth term by parts for any $x_0 < x' < x'' < \infty$, we get

$$\int_{x'}^{x''} \partial_x[r(x)\psi(x)]x\,dx = x''r(x'')\psi(x'') - x'r(x')\psi(x') - \int_{x'}^{x''} r(x)\psi(x)dx.$$

Because $(r\psi)_x \in X$, the left-hand side converges to $\int_{x_0}^{\infty} \partial_x[r(x)\psi(x)]x\,dx$ and

$$x'r(x')\psi(x') \to x_0 \int_{x_0}^{\infty} \beta(x)\psi(x)dx$$

as $x' \to x_0$ and $x'' \to \infty$. Since $r \in X_\infty$, $r\psi$ is integrable on (x_0, ∞) and so the last integral on the right-hand side converges to $\int_{x_0}^{\infty} r(x)u(x)dx$. Then it follows that $x''r(x'')\psi(x'')$ converges to a limit l as $x'' \to \infty$. Suppose that $l \neq 0$. Then $r(x)\psi(x) \geq \nu x^{-1}$ for some

$\nu > 0$ and large enough x, which contradicts the integrability of $r\psi$. Hence

$$\lim_{x''\to\infty} x'' r(x'')\psi(x'') = 0.$$

Therefore

$$\int_0^\infty (T_\beta\psi + B\psi)x\,dx = x_0\int_{x_0}^\infty \beta(x)\psi(x)dx + \int_{x_0}^\infty r(x)\psi(x)dx \\ - \int_{x_0}^\infty [d+g](x)\psi(x)x\,dx \tag{3.17}$$

which gives dissipativity of $\widetilde{T}_\beta$. Then a standard application of the substochastic semigroup theory, [4, Sec. 9.3] ends the proof of (i).

Part (ii) is proved as in [8] and, in particular, (3.9) follows as (3.17) can be extended to all $u \in D(G_\beta)$. □

Remark 4. Two comments are in place here.

Firstly, Theorems 2 and 1 differ in one technical detail in the assumptions of the second part which provides characterization of the generator. This difference is indeed related to the behavior of $1/r$ close to $x = x_0$: if $1/r$ is not integrable, then any antiderivative $R(x) = \int_\alpha^x r^{-1}(x)dx$, $\alpha > x_0$, diverges to $-\infty$ as $x \to x_0$ and the behavior of the solution at $x = x_0$ must be controled by imposing more regularity on the coefficients (compare Lemma 2.6 of [8], pertaining to the case of integrable $1/r$ the statement of which for non-integrable $1/r$ follows from a technically more involved argument in [4, Theorem 9.32]).

Secondly, it is interesting to note that for the case $x_0 = 0$ the boundary condition does not enter into the semigroup estimates (3.8) unlike in the $L_1((0,\infty), dx)$ and $X_{0,1}$ cases. This is due to the fact that new particles entering through the boundary $x = 0$ have zero mass and thus they are not "visible" by the norm which only gives the mass of the ensemble (compare [8, Lemma 2.4]).

Remark 5. Theorem 2 covers the cases in which $r(x) = \tilde{r}x^{2/3}$ which describes growth of aggregates through deposition of a substance on the surface of them. The boundary conditions may depend upon a particular model.

4. Global Solution of the Fragmentation-Coagulation Equation with Growth

The combined mortality, coagulation and mass growth fragmentation equation is

$$\begin{aligned} \frac{du}{dt}(t) &= [T + B + N]u(t) \\ u(0) &= u_0, \end{aligned} \tag{4.1}$$

where N is the realization of the expression

$$[\mathcal{N}\psi](x) = \chi_U(x)\frac{\displaystyle\int_{x_0}^{x-x_0} y\mathrm{g}(y)\psi(y)(x-y)\mathrm{g}(x-y)\psi(x-y)dy}{\displaystyle x\int_{x_0}^\infty z\mathrm{g}(z)\psi(z)dz} \tag{4.2}$$

for nonzero ψ and $\mathcal{N}0 = 0$, on the space X. Since the linear semigroups $(S_G(t))_{t\geq 0}$ and $(S_{G_\beta}(t))_{t\geq 0}$ are positive, we work in the positive cone of X, denoted by X_+. To prove that (4.1) has a mild solution which is global in time we proceed in a standard way converting it to the integral equation

$$u(t) = S_{\mathsf{G}}(t)u_0 + \int_0^t S_{\mathsf{G}}(t-s)N[u(s)]ds, \quad t \geq 0, \tag{4.3}$$

where $(S_{\mathsf{G}}(t))_{t\geq 0}$ is the semigroup generated by either G or G_β: the distinction is irrelevant at this moment. The proof uses the following result which is of independent interest.

Lemma 4.1. *Assume that X is a normed space and $A : \Omega \to \Omega$, where Ω is a convex set in X. If there is L such that for any $x \in \Omega$ there is an open (in the induced topology) neighborhood N_x of x such that $\|Ax - Ay\| \leq L\|x - y\|$ for any $y \in N_x$, then it is globally Lipschitz continuous on Ω:*

$$\|Ax - Ay\| \leq L\|x - y\|$$

for any $x, y \in \Omega$.

Proof. Take $x, y \in \Omega$ and consider the segment $\overline{xy} = \{z \in \Omega; z = x + t(y-x),\ 0 \leq t \leq 1\}$. Consider an open covering of $\overline{xy}$ by N_z with $z \in \overline{xy}$. Since $\overline{xy}$ is compact, we can select a finite subcover $N_{z_1}, \ldots, N_{z_k}$ ordered in such a way that if $z_i = x + t_i(y-x)$ then $t_i < t_k$ for $i < k$. Then denoting $z_0 = x$ and $z_{k+1} = y$

$$\begin{aligned} \|Ax - Ay\| &\leq \sum_{i=0}^{k} \|Az_i - Az_{i+1}\| \leq L\sum_{i=0}^{k} \|z_i - z_{i+1}\| = L\|x-y\|\sum_{i=0}^{k}(t_{i+1} - t_i) \\ &= L\|x - y\|, \end{aligned}$$

where we used $t_0 = 0$ and $t_{k+1} = 1$. □

Lemma 4.2. *The operator N is globally Lipschitz continuous on the set X_+.*

Proof. We use the following notation: $\Theta\psi(x) := xg(x)\psi(x)$ and $\alpha(\psi) = \int_{x_0}^\infty \Theta\psi(x)dx$. Then N can be expressed as $N\psi(x) = \chi_U(x)\frac{(\Theta\psi * \Theta\psi)(x)}{x\alpha(\psi)}$, where $\psi \in X_+\setminus\{0\}$ and $(\Theta\psi * \Theta\psi)(x) = \int_{x_0}^{x-x_0} \Theta\psi(y)\Theta\psi(x-y)dy$. Fix a function $\psi_0 \in X_+\setminus\{0\}$, set $c = \operatorname{ess\,sup}\{g(x) :\ x_0 < x < \infty\}$ and $\varepsilon = \alpha(\psi_0)c^{-1}$. Let ψ be any function from $X_+\setminus\{0\}$ such that $\|\psi - \psi_0\| \leq \varepsilon$. Then

$$\alpha(\psi) = \alpha(\psi_0) + \alpha(\psi - \psi_0) \leq 2\alpha(\psi_0). \tag{4.4}$$

Note that by the linearity of α and properties of the convolution $\star$ we have

$$N\psi - N\psi_0 = \left[\frac{(\Theta\psi * \Theta\psi)\alpha(\psi_0 - \psi)}{x\alpha(\psi_0)\alpha(\psi)} + \frac{\Theta(\psi + \psi_0) * \Theta(\psi - \psi_0)}{x\alpha(\psi_0)}\right]. \tag{4.5}$$

It follows that

$$\|N\psi - N\psi_0\| \le \frac{\alpha(|\psi_0-\psi|)\int_{x_0}^{\infty}(\Theta\psi * \Theta\psi)(x)dx}{\alpha(\psi_0)\alpha(\psi)} + \frac{\int_{x_0}^{\infty}[\Theta(\psi+\psi_0) * |\Theta(\psi-\psi_0)|](x)dx}{\alpha(\psi_0)}. \tag{4.6}$$

Because $\int_{x_0}^{\infty}(\Theta\psi * \Theta\psi)(x)dx = [\int_{x_0}^{\infty}(\Theta\psi)(x)dx]^2 = [\alpha(\psi)]^2$ and

$$\int_{x_0}^{\infty}[\Theta(\psi+\psi_0) * |\Theta(\psi-\psi_0)|](x)dx = \alpha(\psi+\psi_0)\alpha(|\psi-\psi_0|),$$

the inequality (4.6) yields

$$\begin{aligned}\|N\psi - N\psi_0\| &\le \frac{\alpha(\psi)\alpha(|\psi_0-\psi|)}{\alpha(\psi_0)} + \frac{\alpha(\psi+\psi_0)\alpha(|\psi-\psi_0|)}{\alpha(\psi_0)} \le 5\alpha(|\psi-\psi_0|) \\ &\le 5c\|\psi-\psi_0\|,\end{aligned} \tag{4.7}$$

where we used linearity of α and applied (4.4). Furthermore, by (2.5),

$$\|N\psi - N0\| = \|N\psi\| \le \int_0^{\infty} xg(x)\psi(x)dx \le c\|\psi\| \le 5c\|\psi\|$$

for any $\psi \in X_+$ and Lemma 4.1 ensures that N satisfies global Lipschitz condition on X_+. □

Theorem 3. *Let* G *be either* G *or* G_β. *If* $u_0 \in X_+$, *then the Cauchy problem*

$$u_t(t) = \mathsf{G}[u(t)] + N[u(t)], \quad u(0) = u_0, \tag{4.8}$$

has a unique global mild solution.

Proof. Rewriting (4.8) in the form of integral equation (4.3) we see that, since N is non-negative, X_+ is an invariant space for the operator $u \mapsto \int_0^t S_{\mathsf{G}}(t-s)N[u(s)]ds$. Thus global solvability follows in a standard way since X_+ is a complete metric space as a closed subspace of a Banach space, see [15, Theorem 6.1.2]. □

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