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CONDITIONAL LINEARIZABILITY OF FOURTH-ORDER SEMI-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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By the use of geometric methods for linearizing systems of second-order cubically semi-linear ordinary differential equations and the conditional linearizability of third-order quintically semi-linear ordinary differential equations, we extend to the fourth-order by differentiating the third-order conditionally linearizable equation. This yields criteria for conditional linearizability of a class of fourth-order semi-linear ordinary differential equations, which have not been discussed in the literature previously.

Keywords: Conditional linearization; Lie linearization; geometric method; fourth-order ordinary differential equations.

1. Introduction

First-order ordinary differential equations (ODEs) can always be linearized (i.e., converted to linear form) [13] by point transformations [11]. Lie [12] showed that all second-order ODEs that can be converted to linear form must be cubically semi-linear, i.e.

$$y'' + E_3(x, y)y'^3 + E_2(x, y)y'^2 + E_1(x, y)y' + E_0(x, y) = 0, \quad (1)$$

the coefficients E_0 to E_3 satisfying the over-determined integrable system

$$b_x = -\frac{1}{3}E_{1y} + \frac{2}{3}E_{2x} + be - E_0E_3,$$

$$b_y = E_{3x} - b^2 + bE_2 - E_1E_3 + eE_3,$$

$$\begin{aligned} e_x &= E_{0y} + e^2 - eE_1 - bE_0 + E_0E_2, \\ e_y &= \frac{2}{3}E_{1y} - \frac{1}{3}E_{2x} - be + E_0E_3, \end{aligned} \tag{2}$$

where b and e were (for Lie's purposes) auxiliary variables and the suffices x and y refer to partial derivatives. The auxiliary variables arise naturally in the geometric approach mentioned shortly.

Linearization criteria for third-order ODEs were obtained geometrically by Chern [2,3] using contact transformations. He discussed the linearizability of equations reducible to two linear forms. Grebot [7,8] utilized point transformations that were restricted to the projective class for the same purpose. It was algebraically shown [14] that there are three classes of third-order ODES that are linearizable by point transformations. Subsequently Neut and Petitot [21] and later Ibragimov and Meleshko [9], who used the original Lie procedure [12], determined practical invariant criteria for linearizability of third-order ODEs. Linearization can also be achieved by transformations that are other than point [5].

Though little progress was made (as discussed below) towards providing explicit procedures for linearizing systems of ODEs, it was shown [23] that there will generally be multiple classes, even for second-order systems.

The utilization of the connection between the isometry algebra and the symmetries of the system of geodesic equations [1,4] resulted in linearizability criteria being stated for a system of second-order quadratically semi-linear ODEs, of a class that could be regarded as a system of geodesic equations which we call of *geodesic type*. These criteria arose from the requirement that the coefficients in the equations, regarded as Christoffel symbols, yield a zero curvature tensor. For larger dimensional systems, it is possible using computer algebra to associate a metric with the system of geodesics when these criteria are met [6]. The flatness of the metric allows a coordinate transformation to be found from the given metric to a Cartesian form, which yields the linearizing transformations.

The system of n second-order ODEs of geodesic type can be reduced to a system of $(n-1)$ second-order cubically semi-linear ODEs [16]. This is done by utilizing the projection procedure of [1]. The application of this procedure to a system of two dimensions yields a scalar cubically semi-linear ODE that gives rise to the Lie criteria. The procedure applied to a system of three dimensions results in a system of two cubically semi-linear ODEs with extended Lie criteria [16]. The system of ODEs so obtained is maximally symmetric.

Differentiating the quadratically and cubically semi-linear system of ODEs relative to the independent variable gives a system of third-order ODEs. Taking the general class of the scalar third-order ODE one gets linearizability criteria for scalar third-order ODEs [17]. This class is not included in the Neut and Petitot [21] and Ibragimov and Meleshko classes [9]. Though there can be an overlap with the Meleshko class [20], it is found not to be in that either. It may be queried as to how it can be outside the three classes allowed by [14]. The reason is that it is *not* obtained by point transformations but is linearized *conditional* to a second-order ODE being linearizable. In effect its first integral is a linearizable second-order semi-linear ODE. This is, thus, a new type of linearizability — *conditional linearizability subject to a root equation*. The classical procedure of linearizability by point transformations has been further extended to the fourth order [10]. The comparison with the work done in this paper will be given in the final section.

A cubically semi-linear third-order system of the same dimension arises upon differentiation of the quadratically semi-linear system of geodesic type. This is deduced by the replacement of the second-order terms in the differentiated equations by quadratic expressions using the original system. It is also possible to write it as a third-order system with a second-order term multiplied by a first-order term and then followed by a term quadratic in the first derivative. The linearizability criteria for such a system of two dimensions have also been derived [19].

It is clear that the geometrical approach has far-reaching consequences for the linearizability and conditional linearizability of ODEs. In this paper it will be used for stating criteria for the conditional linearizability of fourth-order ODEs in some detail and its utilization for higher orders will also be discussed. Implications of the higher-order conditional linearizability will briefly be touched upon. First a brief summary of the notation used, relevant for our present purposes, will be provided.

2. Notation and Review

Equations of geodesic type for two dimensions are of the form

$$\begin{aligned} x'' &= a(x, y)x'^2 + 2b(x, y)x'y' + c(x, y)y'^2, \\ y'' &= d(x, y)x'^2 + 2e(x, y)x'y' + f(x, y)y'^2. \end{aligned} \quad (3)$$

There are six coefficients and they may be identified with the six Christoffel symbols for a 2-dimensional space,

$$\begin{aligned} \Gamma_{11}^1 &= -a, & \Gamma_{12}^1 &= -b, & \Gamma_{22}^1 &= -c, \\ \Gamma_{11}^2 &= -d, & \Gamma_{12}^2 &= -e, & \Gamma_{22}^2 &= -f. \end{aligned} \quad (4)$$

Requiring that the space corresponding to these Christoffel symbols be flat

$$R_{jkl}^i = \Gamma_{jl,k}^i - \Gamma_{jk,l}^i + \Gamma_{mk}^i \Gamma_{jl}^m - \Gamma_{ml}^i \Gamma_{jk}^m = 0, \quad (5)$$

yields the linearizability conditions

$$\begin{aligned} a_y - b_x + be - cd &= 0, & b_y - c_x + (ac - b^2) + (bf - ce) &= 0, \\ d_y - e_x - (ae - bd) - (df - e^2) &= 0, & (b + f)_x = (a + e)_y, \end{aligned} \quad (6)$$

which provide the metric coefficients through

$$\begin{aligned} p_x &= -2(ap + dq), & q_x &= -bp - (a + e)q - dr, & r_x &= -2(bq + er), \\ p_y &= -2(bp + eq), & q_y &= -cp - (b + f)q - er, & r_y &= -2(cq + fr). \end{aligned} \quad (7)$$

The compatibility of these equations is guaranteed by the curvature tensor being zero. The metric coefficients being obtained, the linearizing transformation is available as a coordinate transformation and its inverse can then be computed.

The above system of two equations can be projected to the scalar cubically semi-linear ODE

$$y'' + cy'^3 - (f - 2b)y'^2 + (a - 2e)y' - d = 0, \quad (8)$$

which is clearly of the form (1) for suitable values of the coefficients in the former equation. It is here that one sees the “auxiliary variables” arising naturally. Note that there are only

four coefficients here as against the six Christoffel symbols and for the original system of two equations. This degeneracy allows for extra classes when this procedure is applied to larger systems of ODEs but does not interfere with the uniqueness for the scalar equation [6]. We shall write $g = f - 2b$ and $h = a - 2e$ for convenience. Again, the coordinate transformations can be obtained, but there is freedom of choice of g and h that can lead to more or less convenient coordinate transformations. The linearizability criteria can then be given in the form [22]

$$\begin{aligned} 3(ch)_x + 3dc_y - 2gg_x - gh_y - 3c_{xx} - 2g_{xy} - h_{yy} &= 0, \\ 3(dg)_y + 3cd_x - 2hh_y - hg_x - 3d_{yy} - 2h_{xy} - g_{xx} &= 0. \end{aligned} \quad (9)$$

We shall be using this form.

Equation (8) can be differentiated to give rise to the third-order equation of the general form

$$y''' + (A_2y'^2 - A_1y' + A_0)y'' + B_4y'^4 - B_3y'^3 + B_2y'^2 - B_1y' + B_0 = 0, \quad (10)$$

which is linear in the second derivative, but with a coefficient quadratic in the first derivative. The coefficients of this equation are identified with the original equation as

$$c = A_2/3, \quad g = A_1/2, \quad h = A_0, \quad (11)$$

$$d = - \int B_2 dx + k(y) = \int (B_1 - A_{0x}) dy + l(x). \quad (12)$$

The constant of integration is fixed by compatibility with the metric tensor, provided the compatibility conditions

$$B_4 = A_{2y}/3, \quad B_3 = A_{1y}/2 - A_{1x}/3, \quad B_2 = A_{0y} - A_{1x}/2, \quad (13)$$

are satisfied and the linearizability conditions (6) hold. The coordinate transformations are obtainable and result in the solution.

The third-order ODE (10) is a total derivative. Though it may not be obvious that a given ODE is an exact derivative, the equation in this form may appear trivial in some sense. One can invoke (8) and insert it in (10) to replace the second derivative term. We then obtain the third-order ODE quintically semi-linear in the first derivative, viz.

$$y''' - \alpha y'^5 + \beta y'^4 - \gamma y'^3 + \delta y'^2 - \epsilon y' + \phi = 0, \quad (14)$$

where

$$\alpha = 3c^2, \quad \beta = 5cg + c_y, \quad \gamma = 4ch + 2g^2 + g_y - c_x, \quad \delta = 3cd + 3gh + h_y - g_x, \quad (15)$$

with the compatibility conditions

$$\epsilon = 2dg + h^2 + d_y - h_x, \quad \phi = dh - d_x. \quad (16)$$

This equation is not an exact derivative. Hence, it provides a non-trivial utilization of the procedure given. The coefficients can be inverted to obtain the original coefficients and thus the metric coefficients which yields the solution.

We use the above formulation to obtain the linearizability criteria for fourth-order equations in the next section.

3. Conditional Linearizability Criteria for Fourth-Order ODEs

On differentiating (14) for the scalar third order ODE, writing the independent variable as x and the dependent variable as y , we get

$$\begin{aligned} y''' - (5\alpha y'^4 - 4\beta y'^3 + 3\gamma y'^2 - 2\delta y' + \epsilon)y'' - \alpha y'^6 + (\beta_y - \alpha_x)y'^5 - (\gamma_y - \beta_x)y'^4 \\ + (\delta_y - \gamma_x)y'^3 - (\epsilon_y - \delta_x)y'^2 + (\phi_x - \epsilon_x)y' + \phi_x = 0. \end{aligned} \quad (17)$$

The general form of this equation is

$$\begin{aligned} y''' - (A_4 y'^4 - A_3 y'^3 + A_2 y'^2 - A_1 y' + A_0)y'' \\ - B_6 y'^6 + B_5 y'^5 - B_4 y'^4 + B_3 y'^3 - B_2 y'^2 + B_1 y' - B_0 = 0, \end{aligned} \quad (18)$$

subject to the identification of coefficients

$$\begin{aligned} c = \sqrt{A_4/15}, \quad g = (A_3 - 4c_y)/20c, \quad h = (A_2 - 6g^2 - 3g_y + 3c_x)/12c, \\ d = (A_1 - 6gh - 2h_y + 2g_x)/6c, \end{aligned} \quad (19)$$

with the constraints

$$\begin{aligned} B_6 = A_4/5, \quad B_5 = A_{3y}/4 - A_{4x}/5, \quad B_4 = A_{2y}/3 - A_{3x}/4, \quad B_3 = A_{1y}/2 - A_{2x}/3, \\ B_2 = A_{0y} - A_{1x}/2, \quad B_1 = dh_y + hd_y - d_{xy} - A_{0x}/2, \quad B_0 = dh_x + hd_x - d_{xx}, \\ A_0 = 2gd + h^2 + d_y - h_x. \end{aligned} \quad (20)$$

In this form it is a total derivative and may be regarded as trivial in some sense.

Replacing the second derivative by the first derivative expressions from (8) we get the fourth order ODE semi-linear in the first derivative in the seventh degree

$$y''' + P_7 y'^7 - P_6 y'^6 + P_5 y'^5 - P_4 y'^4 + P_3 y'^3 - P_2 y'^2 + P_1 y' - P_0 = 0, \quad (21)$$

which is *not* a total derivative. For consistency of the identification of coefficients

$$\begin{aligned} c = (P_7/15)^{1/3}, \quad g = P_6/35c^2 - 2c_y/7c, \\ h = P_5/27c^2 - 26g^2/27c + c_x/3c - gc_y/3c^2 - 8g_y/27c - c_{yy}/27c^2, \\ d = P_4/21c^2 - 38gh/21c - 2g^3/7c^2 + 8gc_x/21c^2 - 8hc_y/21c^2 \\ + g_x/3c - gg_y/3c^2 - 2h_y/7c + 2c_{xy}/21c^2 - g_{yy}/21c^2, \end{aligned} \quad (22)$$

so that all four coefficients are explicitly given. We now have four differential constraints

$$\begin{aligned} P_3 &= 28cdg + 13ch^2 + 12g^2h - 3(h + d)c_x + 4dc_y - (2g + 3h)g_x + (3h + 2d)g_y \\ &\quad - (c + 3g)h_x + 2(g + h)h_y - 3cd_x + (c + 2g)d_y + g_{xx} - 2h_{xy} + d_{yy}, \\ P_2 &= 18chd + 8g^2d + 7gh^2 - 6dc_x - 5hg_x + 4dg_y - 4gh_x + 4hh_y - 3cd_x + 2gd_y \\ &\quad + g_{xx} - 2h_{xy} + d_{yy}, \\ P_1 &= 6cd^2 - 8ghd + h^3 - 4dg_x - 3hh_x + 3dh_y - 2gd_x + 2hd_y + h_{xx} - 2d_{xy}, \\ P_0 &= 2gd^2 + h^2d - 2dh_x - hd_x + dd_y + d_{xx}, \end{aligned} \quad (23)$$

apart from the earlier stated linearizability conditions (9).

Instead of starting with the quintically semi-linear third order ODE, we can start with the form involving the second derivative (10). In this case we get

$$\begin{aligned} y''' + (A_2y'^2 - A_1y' + A_0)y''' + (B_1y' - B_0)y''^2 + (C_3y'^3 - C_2y'^2 + C_1y' - C_0)y'' \\ - (D_5y'^5 - D_4y'^4 + D_3y'^3 - D_2y'^2 + D_1y' - D_0) = 0, \end{aligned} \quad (24)$$

which is semi-linear involving the third derivative with a coefficient quadratic in the first derivative, second derivative squared with a coefficient linear in the first derivative, and otherwise quintic in the first derivative. It is a total derivative with the identification

$$c = A_2/3 = B_1/6, \quad g = A_1/2 = B_0/2, \quad h = A_0, \quad (25)$$

the differential constraints

$$\begin{aligned} C_3 = 7A_{2y}/3, \quad C_2 = 5A_{1y}/2 - 2A_{2x}, \quad C_1 = 3A_{0y} - 2A_{1x}, \\ D_5 = A_{2yy}/3, \quad D_4 = A_{1yy}/2 - 2A_{2xy}/3, \quad D_3 = A_{0yy}/2 - A_{1xy} + A_{2xx}/3, \end{aligned} \quad (26)$$

and the additional constraints that yield d :

$$C_0 = d_y - 2A_{0x}, \quad D_0 = d_{xx}, \quad (27)$$

which gives the value as a single and a double integral to be solved simultaneously

$$d = \int (C_0 + 2A_{0x})dy + k(x) = \int \left(\int D_0 dx \right) dy + l(y)x + m(y), \quad (28)$$

where k, l, m are arbitrary functions of the respective variables. We have the additional requirements for D_1 and D_2 ,

$$D_1 + 2C_{0x} + 3A_{0xx} = 0, \quad D_2 - C_{0y} - A_{1xx}/2 = 0. \quad (29)$$

One can now use the third order equation (10) to replace the third derivative term in (24) and obtain the equation in a form quadratically semi-linear in the second derivative that has a coefficient quadratically semi-linear in the first derivative for the higher order, quartic in the first derivative for the one linear in the second derivative and otherwise of sixth order in the first derivative,

$$\begin{aligned} y''' + (Q_1y' - Q_0)y''^2 - (R_4y'^4 - R_3y'^3 + R_2y'^2 - R_1y' + R_0)y'' \\ - (S_6y'^6 - S_5y'^5 + S_4y'^4 - S_3y'^3 + S_2y'^2 - S_1y' + S_0) = 0, \end{aligned} \quad (30)$$

with the identification of the first three coefficients given by

$$c = Q_1/6, \quad g = Q_0/2, \quad h = (R_2 - Q_0^2 + Q_{1x} - 5Q_{0y}/2)/Q_1. \quad (31)$$

Since the coefficient h is very complicated we shall use it as a shorthand for the above expression. The fourth coefficient is given by the solution of

$$d = \int (R_0 - h^2 + 2h_x)dy + k(x). \quad (32)$$

In the subsequent equations replacing d_y by the integrand above and writing d_x (as there is no convenient expression for it), the following constraint equations must hold

$$\begin{aligned} R_4 &= Q_1^2/4, \quad R_3 = Q_1Q_0 + 7Q_{1y}/6, \quad R_1 = 2Q_0h + h^2 + 3h_y - 2Q_{0x}, \\ S_6 &= Q_1Q_{1y}/12, \quad S_5 = -Q_1Q_{1x}/36 + Q_0Q_{1y}/6 + Q_1Q_{0y}/4 + Q_{1yy}/6, \\ S_4 &= -Q_0Q_{1x}/6 + hQ_{1y}/6 - Q_1Q_{0x}/4 + Q_0Q_{0y}/2 + Q_1h_y/2 - Q_{1xy}/3 + Q_{0yy}/2, \\ S_3 &= Q_1(R_0 - h^2 - h_x)/2 - hQ_1/6 - Q_0Q_{0x}/2 + hQ_{0y}/2 + Q_0h_y + Q_{1xx}/6 - Q_{0xy} + h_{yy}, \\ S_2 &= Q_1d_x/2 + Q_0(R_0 - h^2 + h_x) - hh_y - hQ_{0x}/2 + R_{0y} + Q_{0xx}/2, \\ S_1 &= h(R_0 - h^2 + 5h_x) - 2R_{0x} - Q_0d_x - 3h_{xx}, \quad S_0 = hd_x - d_{xx}, \end{aligned} \quad (33)$$

along with the linearization criteria. This is not a total derivative. Though looking more messy, it is perfectly usable.

Yet another form that is not a total derivative can be obtained from (24) by using (8) to replace the second derivative term. In this case we get

$$\begin{aligned} y''' + (A_2y'^2 - A_1y' + A_0)y''' \\ + B_7y'^7 - B_6y'^6 + B_5y'^5 - B_4y'^4 + B_3y'^3 - B_2y'^2 + B_1y' - B_0 = 0, \end{aligned} \quad (34)$$

with the identification of all four coefficients given by

$$\begin{aligned} c &= \sqrt{A_2/3}, \quad g = A_1/2, \quad h = A_0, \\ d &= \left(B_4 - 8cA_1A_0 - \frac{1}{4}A_1^3 + 3A_1c_x - 7A_0c_y + 2cA_{1x} - 5A_1A_{1y}/4 \right. \\ &\quad \left. - 3cA_{0x} + 2c_{xy} - \frac{1}{2}A_{1yy} \right) / 4A_2, \end{aligned} \quad (35)$$

where we have largely used the symbol c rather than A_2 to avoid the extra complications due to a square root and its differentiation. Further, though d is given algebraically, the expression is too complicated to use conveniently. The constraint equations are

$$\begin{aligned} B_7 &= 6c^3, \quad B_6 = 7c(A_1 + c_y), \\ B_5 &= 3A_2A_0 + 5cA_1^2/2 - 6cc_x + 7A_1c_y/2 + 5cA_{1y}/2 + c_{yy}, \\ B_3 &= 8cA_1d + 6cA_0^2 + A_1^2A_0 - 6A_0c_x + 7dc_y - A_1A_{1x} + 5A_0A_{1y}/2 \\ &\quad - 2cA_{0x} + 3A_1A_{0y}/2 + cd_y + c_{xx} - A_{1xy} + A_{0yy}, \\ B_2 &= 12cA_0d + A_1^2d + A_1A_0^2 - 6dc_x - 2A_0A_{1x} + 5dA_{1y}/2 - A_1A_{0x}) \\ &\quad + 3A_0A_{0y} + A_1d_y/2 + A_{1xx}/2 - 2A_{0xy} + d_{yy}, \\ B_1 &= 6cd^2 + 2A_1A_0d - 4dA_{1x} - 2A_0A_{0x} + 3dA_{0y} + A_0d_y + A_{0xx} - 2d_{xy}, \\ B_0 &= d(A_1d + d_y - A_{0x}) + d_{xx}, \end{aligned} \quad (36)$$

along with the linearization conditions. Again, this seems very messy but it has the advantage that all four coefficients are identified.

We could have one other procedure, to use both Eqs. (8) and (10) to replace the second and third derivatives. Since this only involves the first derivative to the same power as before, this will not yield a new class. Thus we have the following theorems.

Theorem 1. *Equation (21) is conditionally linearizable with the identifications (22) if the constraints (23) and the linearizability criteria (9) are satisfied, where ($c \neq 0$), with $h = a - 2e, g = f - 2b$, after requiring consistency of the Christoffel symbols with the deduced metric coefficients.*

It is to be noted that the four linearizability conditions (6) are stated in terms of the 6 coefficients a, \dots, f and not the four coefficients c, d, g, h . Thus there is degeneracy in the choices available. Any choices of a and e for a given h , or f and b for a given g , compatible with the metric coefficient relations (7) are permissible. For each such choice we would get corresponding conditional linearizability conditions. Instead, we use the conditions (9) to check linearizability and exploit the freedom of choice to construct a convenient metric.

Theorem 2. *Equation (30) is conditionally linearizable with the identifications (31) and (32) if the constraints (33) along with the linearizability criteria (9) are satisfied, after requiring consistency of the Christoffel symbols with the deduced metric coefficients.*

Theorem 3. *Equation (34) is conditionally linearizable with the identifications (35) if the constraints (36) and the linearizability criteria (9) are satisfied, after requiring consistency of the Christoffel symbols with the deduced metric coefficients.*

Note that there are two other equations, (18) and (24), that are total derivatives of linearizable equations with appropriate identifications, that are also linearizable subject to the corresponding constraints and conditions. We do not stress on these because they may be regarded, in some sense, as trivial.

4. Examples

In this section we present some examples of fourth-order equations that can be conditionally linearized by our procedure. Note that conditional linearizability is always subject to the linearizability of the second-order root equation (8).

1. The equation

$$y''' - (18y'^2/y^2 - 16y'/y + k^2 - 5l)y'' + 12y'^4/y^3 - 8ky'^3/y^2 + kly' = 0, \quad (37)$$

is a fourth-order total derivative of the form of (18). However, it is clearly not trivial to spot this fact by looking at the equation. It satisfies the required constraints (20) and linearizability conditions (9). Note that the coefficients of this equation violate the conditions of Theorem 2 of [10] and thus it is not linearizable in the sense of [10].

2. The equation

$$y''' - 24y'^4/y^3 + 33ky'^3/y^2 + (28l - 10k^2)y'^2/y + (k^2 - 35l)ky'/2 - (k^2 - 5l)ly = 0, \quad (38)$$

which is a fourth-order equation that is not a total derivative, of the form (21) and is conditionally linearizable as it satisfies the constraints (23) and the linearizability conditions (9). The coefficients violate the conditions of Theorem 2 of [10].

3. The equation

$$y''' - (4y'/y - k)y''' - 4y''^2/y + (10y'^2/y^2 - l)y'' - 4y'^4/y^3 = 0, \quad (39)$$

which is a total derivative of the form of (24) and may be more easily identified as a total derivative. It is conditionally linearizable as it satisfies the constraints (26) and (29) and the linearizability conditions (9). The coefficients here too violate the conditions of Theorem 2 of [10].

4. The equation

$$\begin{aligned} y''' - 4y''^2/y^4 - (6y'^2/y^2 - 8ky'/y + k^2 + l)y'' + 4y'^4/y^3 \\ - 2ky'^3/y^2 - 4kly'^2/y + kly' = 0, \end{aligned} \quad (40)$$

which is of the form (30) is conditionally linearizable and satisfies the constraints (33) and the linearizability conditions (9). It is not a total derivative. The coefficients do not satisfy the conditions of Theorem 2 of [10].

5. The equation

$$y''' - (4y'y - k)y''' + 4y'^4/y^3 + 3ky'^3/y^2 + (4l - k^2)y'^2/y - 7kly'/2 - 3l^2y = 0, \quad (41)$$

which is not a total derivative and is of the form (34) satisfying the constraints (36) and the linearizability criteria (9). As such it is conditionally linearizable. The coefficients violate the conditions of Theorem 2 of [10].

Though it is not apparent from looking at the equations they come from differentiating the conditionally linearizable third-order equation

$$y''' - 6y'^3/y^2 + 8ky'^2/y - (k^2 - 5l)y' + kly = 0, \quad (42)$$

and the total derivative conditionally linearizable equation

$$y''' - (4y'/y - k)y'' + 2y'^3/y^2 - ly' = 0, \quad (43)$$

(in some cases subject to conditions of some equation holding) which both come from differentiating the second-order linearizable equation

$$y'' - 2y'^2/y + ky'/2 + ly = 0, \quad (44)$$

and in one case applying a condition. As such, their solutions necessarily have two arbitrary constants but may not have more. It may be noted that these equations do not have any explicit dependence on x . That symmetry being guaranteed only one more needs to be looked for to find the solution by symmetry analysis. Using the procedure of writing the equation as of geodesic type, we can write down the solution directly.

The fact that all of these fourth-order equations have a common root becomes obvious from our analysis right in the beginning, as the identification of the four coefficients in each case gives

$$c = 0, \quad g = 2, \quad h = k/2, \quad d = -l. \quad (45)$$

6. The fourth-order ODE

$$y''' + (3xy'^2 + 2/x)y''' + 6xy'y''^2 + (6y'^2 - 4/x^2)y'' + 4y'/x^3 = 0, \quad (46)$$

is of the form of (24) and is hence a total derivative (though this fact is not obvious by inspection), with the coefficients given by (25), (27) and (28). It satisfies the constraints (26) and (29) and the linearizability criteria (9). As such it is conditionally linearizable. Note that this equation is not of the form (8) of [10]. Therefore the results of [10] are not applicable.

7. The fourth-order ODE

$$y''''' + 6xy'y''^2 - (9x^2y'^4 + 6y'^2 + 8/x^2)y'' + 4y'/x^3 = 0, \quad (47)$$

is of the form of (30) and hence is not a total derivative, with the coefficients given by (31) and (32) and satisfies the constraints (33) and the linearizability criteria (9). As such it is conditionally linearizable. Note that this equation too is not of the form (8) of [10].

8. The fourth-order ODE

$$y''''' + (3xy'^2 + 2/x)y'''' + 6x^3y'^7 + 18y'^5 + 16y'^3/x + 12y'/x^3 = 0, \quad (48)$$

is of the form of (34), and hence is not a total derivative, with the coefficients given by (35) and satisfying the constraints (36) and the linearizability criteria (9). As such it is conditionally linearizable. This equation is clearly not of the form (8) of [10].

9. The fourth-order ODE

$$y''''' - (15x^2y'^4 + 21y'^2 + 6/x^2)y'' - 6xy'^5 + 12y'/x^3 = 0, \quad (49)$$

is of the form of (18) and hence is a total derivative, with the coefficients given by (19), satisfying the constraints (20) and the linearizability criteria (9). As such it is conditionally linearizable. This equation is not of the form (8) of [10].

It is again apparent that even the total derivative equation is not obviously so, and that the other equations being linearizable would not be clear by inspection. The coefficients of examples (6) to (10) are all $c = x, h = 2/x, g = d = 0$ and the *root equation* is the geodesic equation for flat space in polar coordinates. Thus the linearizing transformation is simply the conversion from polar to Cartesian coordinates.

It will be noticed that in the above examples either x or y are missing from the coefficients. This makes the application of the identification and constraints relatively trivial. We end with a couple of non-trivial examples, in which both arise.

10. The fourth-order ODE

$$\begin{aligned} y''''' - 15x^3y'^7/y^6 - 15xy'^6/y^5 + 39y'^5/y^4 + 39y'^4/xy^3 - 36y'^3/xy^2 \\ - 36y'^2/x^2y + 24y'/x^3 = 0, \end{aligned} \quad (50)$$

is of the form of (21), and hence is not a total derivative. Its coefficients are given by (22) and come out to be $c = -x/y^2, g = 1/y, h = 2/x, d = 0$. They satisfy the constraints (23) and linearizability criteria (9). As such this equation is conditionally linearizable. Note that this equation is not of the form (8) of [10] and thus not linearizable in that sense. In fact it

can be derived by differentiating the conditionally linearizable third-order equation [17]

$$y''' - 3x^2y'^5/y^4 - 3xy'^4/y^3 + 6y'^3/y^2 + 6y'^2/xy - 6y'/x^2 = 0, \quad (51)$$

writing the root equation from the coefficients, and using it to substitute the y'' arising from the differentiation. The solution is

$$Axy + Bx/y = 1, \quad (52)$$

where A and B are arbitrary constant real numbers.

11. The fourth-order ODE

$$\begin{aligned} y''''' - (6xy'/y^2 + 2/y)y''^2 - (9x^2y'^4/y^4 - 2xy'^3/y^3 - 7y'^2/y^2 - 8y'/xy + 8/x^2)y'' \\ + 6x^2y'^6/y^5 - 2xy'^5/y^4 - 2y'^4/y^3 - 6y'^3/xy^2 - 4y'^2/x^2y + 8y'/x^3 = 0, \end{aligned} \quad (53)$$

is of the form of (30). Its coefficients are given by (31) and (32) and are the same as of the above example. Hence they share a common root equation. This equation arises by differentiating the root equation twice and using its first derivative to replace the y''' . It satisfies the constraints (33) and the linearizability criteria (9) and is conditionally linearizable, yielding the same linear equation and possessing the same solution as of the previous example. Here too we point out that this equation is not of the form (8) of [10].

5. Concluding Remarks

We have provided a procedure to determine the conditional linearizability of some fourth-order scalar semi-linear ODEs. We have written them as five classes. There could have been other ways of getting to the five classes, e.g. by first differentiating and then replacing or first replacing and then differentiating. Since, the general form of both procedures would be the same they cannot be regarded as different. Again, we could have used a lower-order equation to *partially* replace terms in the higher-order equation. For example, we could have used the second-order equation to replace the quadratic term in the second derivative in (51) but retain the linear term as it is. These are not *really independent* in some sense. In this sense there are five independent classes. Two of these may be regarded as trivial as they are total derivatives. However, a glance at the equations will show that even these are not so easy to identify as total derivatives. The other three classes are *not* total derivatives. All can be thought of as arising from the same second-order linearizable differential equation that satisfies the Lie conditions, by double differentiation, but conditional to the original differential equation and in some cases the differentiated differential equation. As such, the linearizability classes are *non-classical* and would not lie in the three classical classes reducible to the fourth-order ODEs $y''''' + \alpha(x)y = 0$, where $\alpha = 0$, *constant* or a function of x . These are guaranteed four arbitrary constants while our solutions are only guaranteed *two*.

It is worth comparing the fourth order conditional linearizability with a second order root equation, the fourth order Lie linearizability [10] and the conditional linearizability of a fourth-order ODE with a third order root equation [18]. In the first case we have

two constants guaranteed, which correspond to the root equation and there could be other solutions that would not satisfy the root equation. In the case of Lie linearizability there are four arbitrary constants guaranteed and in the case of conditional linearizability with a third-order root equation there are three guaranteed, corresponding to the root equation, and a fourth solution may exist which would not satisfy the root equation. More specifically, it is found that the conditionally linearizable equations with a second-order root equation do *not* overlap with the Lie linearizable or the conditionally linearizable ODEs with a third-order root equation.

We call the underlying second-order equation the root equation, and the specific fourth-order equations *forms* of the *similar* fourth-order equations. It is clear that all forms arising from a common second-order equation will form an equivalence class. As such, all similar fourth-order equations will have two solutions in common but may have other, different solutions. The question arises whether these equivalence classes of conditionally linearizable fourth-order semi-linear ODEs are disjoint and can be used to decompose the space of those ODEs that are linearizable of this type. In this context, it seems unlikely that this be the case because there would be some of them that are linearizable and have two solutions in common with others. As such, we conjecture that the set of linearizable fourth-order ODEs with common root equations will not be decomposable into disjoint classes.

It is worth stressing that as can be seen from the examples, our procedure provides the solutions of the conditionally linearizable semi-linear equations of our classes for any order that is identified. It should not be too difficult to prepare an algebraic code for identification of coefficients, the constraints and the linearizability conditions for the various classes. It would be worth while to get the algebraic codes prepared for at least some of the higher-order classes.

It is worth noting that there is only one class of linearizable second-order semi-linear ODEs (as proved by Lie) and two classes of third-order ODEs, one of which is a total derivative and the other is conditionally linearizable. For the fourth-order we find *five* of which two are total derivatives and the other three are not. Can this procedure be carried further? The answer was found to be positive. The fifth-order will have one form that only involves the first derivatives apart from the fifth derivative and is of ninth-order in them. It is obvious that here all four coefficients will be easily identified, but now instead of the four constraints (22) we will have *six* constraints. Similarly, for the sixth-order the corresponding class will have the first derivative to the eleventh power and will have eight constraints and so on. Of course the usual linearizability conditions (6) would also have to be met.

One could also extend conditional linearizability to systems of semi-linear ODEs having a second-order cubically semi-linear system of root equations. Again, one can extend it to higher-order classically linearizable root equations if the linearizability conditions for the higher-order equations are given [18]. It is hoped that the extension to higher orders may lead to a classification scheme for nonlinear ODEs that yields the general solution. However, one cannot so far put the two extensions together. The reason is that the extension to the system depends on the use of the system of geodesic equations. *This is not available for the higher-order equations.* It would be of immense interest to find some way of extending the methods for systems to higher-order root equations. The geometric method used for second-order ODEs provided a constructive procedure that allowed one to obtain the linearization transformations and consequently the solution of the linearizable equations.

This fact guarantees that we find at least two of the linearly independent solutions of the conditionally linearizable higher-order ODEs (or system of ODEs) with a second-order (system of) root equation(s). The advantage is lost for higher-order root equations. Again, it can be seen that it would be of great interest to find methods to extend the results for higher-order root equations. This has been done in [17].

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