Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852

ISSN (Print): 1402-9251

Journal Home Page: <https://www.atlantis-press.com/journals/jnmp>

A Group Classification of a System of Partial Differential Equations Modeling Flow in Collapsible Tubes

M. Molati, F. M. Mahomed, C. Wafo Soh

To cite this article: M. Molati, F. M. Mahomed, C. Wafo Soh (2009) A Group Classification of a System of Partial Differential Equations Modeling Flow in Collapsible Tubes, Journal of Nonlinear Mathematical Physics 16:S1, 179–208, DOI: <https://doi.org/10.1142/S1402925109000406>

To link to this article: <https://doi.org/10.1142/S1402925109000406>

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 16, Suppl. (2009) 179–208

© M. Molati, F. M. Mahomed & C. Wafo Soh

A GROUP CLASSIFICATION OF A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS MODELING FLOW IN COLLAPSIBLE TUBES

M. MOLATI^{*,†} and F. M. MAHOMED[†]

*Centre for Differential Equations, Continuum Mechanics and Applications
School of Computational and Applied Mathematics
University of the Witwatersrand, Johannesburg, Private Bag 3
Wits 2050, South Africa*

^{*}*m.molati@gmail.com*

[†]*Fazal.Mahomed@wits.ac.za*

C. WAFO SOH

*Department of Mathematics, Jackson State University
P. O. Box 17610, MS 39217, USA
wafosoh@yahoo.com*

Received 30 August 2009

Accepted 30 September 2009

The purpose of this work is to perform group classification of a coupled system of partial differential equations (PDEs) modeling a flow in collapsible tubes. This system of PDEs contains unknown functions of the dependent variables whose forms are specified via the classification with respect to subalgebras of real three and four-dimensional Lie algebras.

Keywords: Group classification; Lie algebra; collapsible tube.

Mathematics Subject Classification 2000: 35Q35, 58D19, 76M60, 76Z05

1. Introduction

It is well known in physiological, clinical or surgical applications that the flow of a fluid changes the geometry of the collapsible tube through which it is flowing. The phenomenon of a fluid flow through collapsible tubes is complex due to the fluid flow interaction with the motion of the tube wall. Owing to this complexity much of the research (analytical and numerical investigations [9, 16, 18, 28], just to mention a few) have been based on one-dimensional and two-dimensional models. However, a complete understanding and comprehension of the complex dynamics involved in the flow through a collapsible tube requires a three-dimensional approach (experimental investigations and three-dimensional

[†]Permanent Address: Department of Mathematics and Computer Science, National University of Lesotho, P. O. Roma 180, Lesotho.

numerical simulations). Due to the extensive computational resources needed, currently there are limited three-dimensional numerical simulations for fluid flow through collapsible tubes [20].

There are a number of physiological, clinical and surgical applications of fluid flow through collapsible channels, for example: blood flow in arteries or veins during sphygmomanometry; urethral flow during micturition; ureteral flow during peristaltic pumping; airflow in the bronchial airways during forced expiration [4, 16] and in other circumstances which are less usual, such as life-sustaining and life-threatening situations. Bertram [4] gives a recent review of respiratory cases of flow in collapsed tubes and deformed airways, analytical and numerical models, experimental studies and current controversial issues on the subject.

When the original system contains arbitrary parameters or functions, the consistency conditions of the determining equations provide a means to specify their forms. This is the essence of the group classification method proposed by Ovsiannikov [26]. The importance of group classification stems from the fact that many models in application contain parameters or functions which cannot be determined from any known physical law. It is more desirable to perform the complete group classification, but in some problems of interest the analysis of the determining equations can be quite challenging, in such a case one can make use of the method of preliminary group classification [11, 12]. The results of the preliminary group classification could then be used to obtain solutions for the known models. However, when the determining equations are difficult to analyze an approach based on abstract Lie algebras of low dimension discussed by Basarab–Horwath *et al.* [3] can be used. The above mentioned approaches require the computation of equivalence group (i.e., classification is performed up to equivalence group). Mahomed and Leach [19] used similar ideas to perform classification with respect to low dimensional Lie algebras for ordinary differential equations (ODEs). The symmetry classification can also be performed without using the equivalence transformations, i.e., the direct Lie’s approach can be employed provided the analysis of the determining equations which include those of the arbitrary elements (compatibility conditions) could be handled without much difficulty [22, 30].

The outline of this paper is as follows. In Sec. 2 the model to be investigated is presented. In Sec. 3 we determine the generators of the equivalence group for the model. In Sec. 4 we carry out the classification with respect to subalgebras of real three and four-dimensional Lie algebras and finally, summarize our findings.

2. Collapsible Tube Model

We consider one-dimensional flow through a collapsible tube [10, 16], governed by the equations:

- Conservation of mass

$$\alpha_t + (\alpha u)_x = 0, \quad (2.1)$$

where subscripts t and x denote time and spatial derivatives respectively.

- Conservation of momentum

$$\rho(u_t + uu_x) = -p_x - R(\alpha, u), \quad (2.2)$$

where ρ is the density of fluid assumed to be constant and $R(\alpha, u) > 0$ is a term representing distributed frictional losses.

- Pressure-area relation

$$p - p_e = \mathcal{P}(\alpha). \quad (2.3)$$

The tube law (2.3) can be accompanied by more terms to represent additional physical effects, i.e.,

$$p - p_e = \mathcal{P}(\alpha) - T\alpha_{xx} + D\alpha_t + M\alpha_{tt}, \quad (2.4)$$

in which $T \geq 0$ and α_{xx} approximate the effect of longitudinal tension and curvature of the tube wall respectively. The constant $D \geq 0$ represents the viscous damping in the wall and $M \geq 0$ represents wall inertia.

The dependent variables $\alpha(t, x)$, $u(t, x)$ and $p(t, x)$ are the cross-sectional area of the tube, cross-sectional axial velocity and cross-sectional internal pressure respectively. The independent variable x measures the tube length and t the time. The nonlinear tube law function $\mathcal{P}(\alpha)$ can be computed using thin-shell theory, but a quite similar function is assumed for some models in applications [1, 6, 23]. The form of the term representing viscous effects, $R(\alpha, u)$ depends on the type of flow considered. Equations (2.1)–(2.3) must be solved subject to appropriate initial and boundary conditions.

Many features of the systems of PDEs describing compressible gas and shallow water flows [2, 17, 29] also arise in collapsible tube flow. These include models for steady and unsteady flows. In the sequel we investigate the system of Eqs. (2.1)–(2.3) from the Lie group theory point view. Similar systems of equations have been studied in [7, 21, 24, 26] and the references therein using this approach.

3. Equivalence Generators

An equivalence transformation of a system is a change of both dependent and independent variables into new dependent and independent variables taking the original system into a system of the same form [11–15]. We rewrite Eqs. (2.1)–(2.3) as the system

$$\begin{aligned} \alpha_t + \alpha u_x + u \alpha_x &= 0, \\ \rho(u_t + uu_x) + \alpha_x \mathcal{P}_\alpha + R(\alpha, u) &= 0. \end{aligned} \quad (3.1)$$

Since the independent variables t and x do not appear explicitly in the above system (3.1), the principal Lie algebra for the system is spanned by at least two operators (to be established later)

$$X_1 = \frac{\partial}{\partial t} \quad \text{and} \quad X_2 = \frac{\partial}{\partial x}. \quad (3.2)$$

We seek for the generator of equivalence group having the form

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial \alpha} + \eta^2 \frac{\partial}{\partial u} + \mu^1 \frac{\partial}{\partial \mathcal{P}} + \mu^2 \frac{\partial}{\partial R}, \quad (3.3)$$

where

$$\xi^i = \xi^i(t, x, \alpha, u), \quad \eta^i = \eta^i(t, x, \alpha, u), \quad \mu^i = \mu^i(t, x, \alpha, u, \mathcal{P}, R), \quad i = 1, 2.$$

The operator (3.3) is the generator of the equivalence group provided it is admitted by the extended system

$$\alpha_t + \alpha u_x + u \alpha_x = 0, \quad (3.4)$$

$$\rho(u_t + u u_x) + \alpha_x \mathcal{P}_\alpha + R = 0, \quad (3.5)$$

$$\mathcal{P}_t = 0, \quad \mathcal{P}_x = 0, \quad \mathcal{P}_u = 0, \quad R_t = 0, \quad R_x = 0. \quad (3.6)$$

The invariance conditions for system (3.4)–(3.6) require the following prolongation of the operator (3.3)

$$\begin{aligned} \tilde{Y} = Y &+ \zeta_1^1 \frac{\partial}{\partial \alpha_t} + \zeta_2^1 \frac{\partial}{\partial \alpha_x} + \zeta_1^2 \frac{\partial}{\partial u_t} + \zeta_1^2 \frac{\partial}{\partial u_x} + \omega_1^1 \frac{\partial}{\partial \mathcal{P}_t} + \omega_2^1 \frac{\partial}{\partial \mathcal{P}_x} \\ &+ \omega_3^1 \frac{\partial}{\partial \mathcal{P}_\alpha} + \omega_4^1 \frac{\partial}{\partial \mathcal{P}_u} + \omega_1^2 \frac{\partial}{\partial R_t} + \omega_2^2 \frac{\partial}{\partial R_x}. \end{aligned} \quad (3.7)$$

The variables ζ_j^i and ω_k^i are given by the prolongation formulae

$$\begin{aligned} \zeta_1^1 &= D_t(\eta^1) - \alpha_t D_t(\xi^1) - \alpha_x D_t(\xi^2) \\ &\equiv \eta_t^1 + \alpha_t \eta_\alpha^1 + u_t \eta_u^1 - \alpha_t(\xi_t^1 + \alpha_t \xi_\alpha^1 + u_t \xi_u^1) - \alpha_x(\xi_t^2 + \alpha_t \xi_\alpha^2 + u_t \xi_u^2), \end{aligned} \quad (3.8)$$

$$\begin{aligned} \zeta_2^1 &= D_x(\eta^1) - \alpha_t D_x(\xi^1) - \alpha_x D_x(\xi^2) \\ &\equiv \eta_x^1 + \alpha_x \eta_\alpha^1 + u_x \eta_u^1 - \alpha_t(\xi_x^1 + \alpha_x \xi_\alpha^1 + u_x \xi_u^1) - \alpha_x(\xi_x^2 + \alpha_x \xi_\alpha^2 + u_x \xi_u^2), \end{aligned} \quad (3.9)$$

$$\begin{aligned} \zeta_1^2 &= D_t(\eta^2) - u_t D_t(\xi^1) - u_x D_t(\xi^2) \\ &\equiv \eta_t^2 + \alpha_t \eta_\alpha^2 + u_t \eta_u^2 - \alpha_t(\xi_t^1 + \alpha_t \xi_\alpha^1 + u_t \xi_u^1) - \alpha_x(\xi_t^2 + \alpha_t \xi_\alpha^2 + u_t \xi_u^2), \end{aligned} \quad (3.10)$$

$$\begin{aligned} \zeta_2^2 &= D_x(\eta^2) - u_t D_x(\xi^1) - u_x D_x(\xi^2) \\ &\equiv \eta_x^2 + \alpha_x \eta_\alpha^2 + u_x \eta_u^2 - \alpha_t(\xi_x^1 + \alpha_x \xi_\alpha^1 + u_x \xi_u^1) - \alpha_x(\xi_x^2 + \alpha_x \xi_\alpha^2 + u_x \xi_u^2), \end{aligned} \quad (3.11)$$

and

$$\omega_1^1 = \tilde{D}_t(\mu^1) - \mathcal{P}_\alpha \tilde{D}_t(\eta^1), \quad (3.12a)$$

$$\omega_2^1 = \tilde{D}_x(\mu^1) - \mathcal{P}_\alpha \tilde{D}_x(\eta^1), \quad (3.12b)$$

$$\omega_3^1 = \tilde{D}_\alpha(\mu^1) - \mathcal{P}_\alpha \tilde{D}_\alpha(\eta^1), \quad (3.12c)$$

$$\omega_4^1 = \tilde{D}_u(\mu^1) - \mathcal{P}_\alpha \tilde{D}_u(\eta^1), \quad (3.12d)$$

$$\omega_1^2 = \tilde{D}_t(\mu^2) - R_\alpha \tilde{D}_t(\eta^1) - R_u \tilde{D}_t(\eta^2), \quad (3.12e)$$

$$\omega_2^2 = \tilde{D}_x(\mu^2) - R_\alpha \tilde{D}_x(\eta^1) - R_u \tilde{D}_x(\eta^2), \quad (3.12f)$$

respectively, where

$$D_t = \frac{\partial}{\partial t} + \alpha_t \frac{\partial}{\partial \alpha} + u_t \frac{\partial}{\partial u}, \quad D_x = \frac{\partial}{\partial x} + \alpha_x \frac{\partial}{\partial \alpha} + u_x \frac{\partial}{\partial u}$$

are the total derivative operators, whereas

$$\tilde{D}_t = \frac{\partial}{\partial t}, \quad \tilde{D}_x = \frac{\partial}{\partial x}, \quad \tilde{D}_\alpha = \frac{\partial}{\partial \alpha} + \mathcal{P}_\alpha \frac{\partial}{\partial \mathcal{P}} + R_\alpha \frac{\partial}{\partial R}, \quad \tilde{D}_u = \frac{\partial}{\partial u} + R_u \frac{\partial}{\partial R}$$

are the total derivative operators for the extended system (3.4)–(3.6). Acting the prolonged operator (3.7) on Eqs. (3.4)–(3.6) yields the determining equations

$$\zeta_1^1 + u_x \eta^1 + \alpha \zeta_2^2 + \alpha_x \eta^2 + u \zeta_2^1 = 0, \quad (3.13)$$

$$\rho(\zeta_1^2 + u_x \eta^2 + u \zeta_2^2) + \mathcal{P}_\alpha \zeta_2^1 + \alpha_x \omega_3^1 + \mu^2 = 0, \quad (3.14)$$

$$\omega_1^1 = 0, \quad \omega_2^1 = 0, \quad \omega_4^1 = 0, \quad \omega_1^2 = 0, \quad \omega_2^2 = 0. \quad (3.15)$$

First we consider Eq. (3.15) together with Eqs. (3.12a), (3.12b), (3.12d)–(3.12f). Assuming that the arbitrary elements and their derivatives are arbitrary functions we obtain

$$\eta_t^1 = \eta_x^1 = \eta_u^1 = 0, \quad \eta_t^2 = \eta_x^2 = 0, \quad \mu_t^1 = \mu_x^1 = \mu_u^1 = \mu_R^1 = 0, \quad \mu_t^2 = \mu_x^2 = 0.$$

Therefore,

$$\eta^1 = \eta^1(\alpha), \quad \eta^2 = \eta^2(\alpha, u), \quad \mu^1 = \mu^1(\alpha, \mathcal{P}), \quad \mu^2 = \mu^2(\alpha, u, \mathcal{P}, R). \quad (3.16)$$

Next we consider Eq. (3.13). Taking into account Eqs. (3.16), we substitute (3.8), (3.9) and (3.11) into Eq. (3.13). This is followed by equating to zero the coefficients of $\mathcal{P}_\alpha$ and the terms without $\mathcal{P}_\alpha$. Thereafter, separating by α_x , u_x and their powers we obtain

$$\xi_x^1 = 0, \quad \xi_u^2 + \alpha \xi_\alpha^1 - u \xi_u^1 = 0, \quad (3.17a)$$

$$\eta^1 - \alpha \eta_\alpha^1 + \alpha(\eta_u^2 - \xi_x^2 + \xi_t^1) = 0, \quad \eta^2 + \alpha \eta_\alpha^2 - u(\xi_x^2 - \xi_t^1) - \xi_t^2 = 0. \quad (3.17b)$$

Now we consider Eq. (3.14). We substitute (3.9), (3.10), (3.11) and (3.12c) into Eq. (3.14) taking into account Eqs (3.16). By equating to zero the coefficients of $\mathcal{P}_\alpha$ and $\mathcal{P}_\alpha^2$ and separating by α_x , α_x^2 we obtain

$$\xi_u^1 = 0, \quad \mu_{\mathcal{P}}^1 = \eta_u^2 + \xi_x^2 - \xi_t^1, \quad \xi_\alpha^2 - u \xi_\alpha^1 = 0. \quad (3.18)$$

Separating by α_x , u_x and their powers, the terms without $\mathcal{P}_\alpha$ results in

$$\mu_\alpha^1 = 0, \quad \rho[\eta^2 - \alpha \eta_\alpha^2 - u(\xi_x^2 - \xi_t^1) - \xi_t^2] + (\xi_u^2 - \alpha \xi_\alpha^1)R = 0, \quad \mu^2 = (\eta_u^2 - \xi_t^1)R. \quad (3.19)$$

The first equations in (3.17a) and (3.18) imply that $\xi^1 = \xi^1(t, \alpha)$. Hence, the second equation in (3.17a) reduces to

$$\xi_u^2 + \alpha \xi_\alpha^1 = 0. \quad (3.20)$$

In summary, we have

$$\begin{aligned} \xi^1 &= \xi^1(t, \alpha), \quad \xi^2 = \xi^2(t, x, \alpha, u), \quad \eta^1 = \eta^1(\alpha), \quad \eta^2 = \eta^2(\alpha, u), \\ \mu^1 &= \mu^1(\mathcal{P}), \quad \mu^2 = \mu^2(\alpha, u, R) \end{aligned} \quad (3.21)$$

and

$$\xi_\alpha^2 - u\xi_\alpha^1 = 0, \quad (3.22)$$

$$\xi_u^2 + \alpha\xi_\alpha^1 = 0, \quad (3.23)$$

$$\eta^1 - \alpha\eta_\alpha^1 + \alpha(\eta_u^2 - \xi_x^2 + \xi_t^1) = 0, \quad (3.24)$$

$$\eta^2 + \alpha\eta_\alpha^2 - u(\xi_x^2 - \xi_t^1) - \xi_t^2 = 0, \quad (3.25)$$

$$\rho[\eta^2 - \alpha\eta_\alpha^2 - u(\xi_x^2 - \xi_t^1) - \xi_t^2] + (\xi_u^2 - \alpha\xi_\alpha^1)R = 0, \quad (3.26)$$

$$\mu_{\mathcal{P}}^1 - \eta_u^2 - \xi_x^2 + \xi_t^1 = 0, \quad (3.27)$$

$$\mu^2 - (\eta_u^2 - \xi_t^1)R = 0. \quad (3.28)$$

Solving Eqs (3.22)–(3.28) in view of (3.21) yields the general solution of the determining equations (3.13)–(3.15) given by

$$\xi^1 = C_1 t + C_2, \quad (3.29)$$

$$\xi^2 = C_3 x + C_4 t + C_5, \quad (3.30)$$

$$\eta^1 = C_6 \alpha, \quad (3.31)$$

$$\eta^2 = (C_3 - C_1)u + C_4, \quad (3.32)$$

$$\mu^1 = 2(C_3 - C_1)\mathcal{P} + C_7, \quad (3.33)$$

$$\mu^2 = (C_3 - 2C_1)R, \quad (3.34)$$

where $C_1, C_2, \dots, C_7$ are the arbitrary constants. Thus, the generator of equivalence group is

$$\begin{aligned} Y = & (C_1 t + C_2) \frac{\partial}{\partial t} + [C_3 x + C_4 t + C_5] \frac{\partial}{\partial x} + C_6 \alpha \frac{\partial}{\partial \alpha} + [(C_3 - C_1)u + C_4] \frac{\partial}{\partial u} \\ & + [2(C_3 - C_1)\mathcal{P} + C_7] \frac{\partial}{\partial \mathcal{P}} + (C_3 - 2C_1)R \frac{\partial}{\partial R}. \end{aligned} \quad (3.35)$$

Therefore, the underlying system (3.1) has a seven-dimensional equivalence Lie algebra spanned by the operators

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} - 2\mathcal{P} \frac{\partial}{\partial \mathcal{P}} - 2R \frac{\partial}{\partial R}, \\ Y_4 &= x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + 2\mathcal{P} \frac{\partial}{\partial \mathcal{P}} + R \frac{\partial}{\partial R}, \quad Y_5 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad Y_6 = \alpha \frac{\partial}{\partial \alpha}, \quad Y_7 = \frac{\partial}{\partial \mathcal{P}}. \end{aligned} \quad (3.36)$$

The group of equivalence transformations include the following discrete transformations:

$$t \rightarrow -t, \quad \alpha \rightarrow -\alpha, \quad u \rightarrow -u, \quad (3.37a)$$

$$t \rightarrow -t, \quad u \rightarrow -u, \quad \mathcal{P} \rightarrow -\mathcal{P}, \quad R \rightarrow -R, \quad (3.37b)$$

$$\alpha \rightarrow -\alpha. \quad (3.37c)$$

We use the theorem on projections of equivalence Lie algebras [12,13,15] to find the *principal Lie algebra* (Lie algebra of the maximal group) admitted by system (3.1). Consider the

following projections of the equivalence generator (3.3):

$$X = \text{pr}_{(t,x,\alpha,u)}(Y) \equiv \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial \alpha} + \eta^2 \frac{\partial}{\partial u}, \quad (3.38a)$$

$$Z = \text{pr}_{(\alpha,u,\mathcal{P},R)}(Y) \equiv \eta^1 \frac{\partial}{\partial \alpha} + \eta^2 \frac{\partial}{\partial u} + \mu^1 \frac{\partial}{\partial \mathcal{P}} + \mu^2 \frac{\partial}{\partial R}, \quad (3.38b)$$

where $\text{pr}_{(t,x,\alpha,u)}$ denote projection on the space (t, x, α, u) and so is $\text{pr}_{(\alpha,u,\mathcal{P},R)}$ on the space $(\alpha, u, \mathcal{P}, R)$.

An operator X belongs to the principal Lie algebra if and only if

$$Z = \text{pr}_{(\alpha,u,\mathcal{P},R)}(Y) = 0. \quad (3.39)$$

According to (3.35) and (3.38b), Eq. (3.39) is written

$$C_6 \alpha \frac{\partial}{\partial \alpha} + [(C_3 - C_1)u + C_4] \frac{\partial}{\partial u} + [2(C_3 - C_1)\mathcal{P} + C_7] \frac{\partial}{\partial \mathcal{P}} + (C_3 - 2C_1)R \frac{\partial}{\partial R} = 0.$$

Thus,

$$C_1 = 0, \quad C_3 = 0, \quad C_4 = 0, \quad C_6 = 0, \quad C_7 = 0.$$

Hence, in view of (3.35) and (3.38a)

$$Y = C_2 \frac{\partial}{\partial t} + C_5 \frac{\partial}{\partial x},$$

and we conclude that the principal Lie algebra of (3.1) is two-dimensional and spanned by operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}. \quad (3.40)$$

The equivalence transformations (3.36) for system (3.1) can also be written in the form

$$\bar{t} = \sigma_1 t + \gamma_1, \quad \bar{x} = \sigma_2 t + \sigma_3 x + \gamma_2, \quad \bar{\alpha} = \sigma_4 \alpha, \quad \bar{u} = \sigma_5 u + \gamma_3, \quad (3.41a)$$

$$\bar{\mathcal{P}} = \sigma_5^2 \mathcal{P} + \gamma_4, \quad \bar{R} = \frac{1}{2} \sigma_5 R, \quad (3.41b)$$

where $\sigma_1, \sigma_3, \sigma_4, \sigma_5 > 0$.

4. Determining Equations for Classification

According to Lie's algorithm [5, 25, 26], the infinitesimal generator of the maximal symmetry group admitted by (3.1) is

$$X = \xi^1(t, x, \alpha, u) \frac{\partial}{\partial t} + \xi^2(t, x, \alpha, u) \frac{\partial}{\partial x} + \eta^1(t, x, \alpha, u) \frac{\partial}{\partial \alpha} + \eta^2(t, x, \alpha, u) \frac{\partial}{\partial u} \quad (4.1)$$

if and only if the invariance conditions of system (3.1) are

$$\begin{aligned} \tilde{X}(\alpha_t + \alpha u_x + u \alpha_x) &= 0, \\ \tilde{X}(\rho(u_t + u u_x) + \alpha_x \mathcal{P}_\alpha + R(\alpha, u)) &= 0, \end{aligned} \quad (4.2)$$

where

$$\tilde{X} = X + \zeta_1^1 \frac{\partial}{\partial \alpha_t} + \zeta_2^1 \frac{\partial}{\partial \alpha_x} + \zeta_1^2 \frac{\partial}{\partial u_t} + \zeta_2^2 \frac{\partial}{\partial u_x}$$

is the prolongation of the operator (4.1). The variables ζ_j^i are as given in Eqs. (3.8)–(3.11).

The coefficients ξ^i 's and η^i 's of the symmetry generator (4.1) do not involve the derivatives of the dependent variables. Thus, we can separate (4.2) with respect to the derivatives of the dependent variables and their powers to obtain an overdetermined system of linear homogeneous PDEs (determining equations). The following determining equations were generated manually and confirmed by the software package **YaLie** [8]:

$$u\xi_u^1 - \xi_u^2 - \alpha\xi_\alpha^1 = 0, \quad (4.3)$$

$$\rho(\eta_t^1 + u\eta_x^1 + \alpha\eta_x^2) + (\alpha\xi_x^1 - \eta_u^1)R = 0, \quad (4.4)$$

$$\alpha\eta_u^2 + \eta^1 - \alpha\eta_\alpha^1 - \alpha\xi_x^2 + 2\alpha u\xi_x^1 + \alpha\xi_t^1 = 0, \quad (4.5)$$

$$\rho(u\xi_t^1 + u^2\xi_x^1 - \xi_t^2 - u\xi_x^2 + \eta^2 + \alpha\eta_\alpha^2) + (\alpha\xi_x^1 - \eta_u^1)\mathcal{P}' = 0, \quad (4.6)$$

$$\xi_u^1\mathcal{P}' + \rho(\xi_\alpha^2 - u\xi_\alpha^1) = 0, \quad (4.7)$$

$$\rho^2(\eta_t^2 + u\eta_x^2) + \rho[\eta_x^1\mathcal{P}' + \eta^1R_\alpha + \eta^2R_u + (\xi_t^1 + u\xi_x^1 - \eta_u^2)R] - \xi_u^1R^2 = 0, \quad (4.8)$$

$$\rho\eta^1\mathcal{P}'' + [\rho(\xi_t^1 + 2u\xi_x^1 - \xi_x^2 + \eta_\alpha^1 - \eta_u^2) - 2\xi_u^1R]\mathcal{P}' = 0, \quad (4.9)$$

$$\rho(u\xi_t^1 + u^2\xi_x^1 - \xi_t^2 - u\xi_x^2 + \eta^2 - \alpha\eta_\alpha^2) + (\alpha\xi_x^1 + \eta_u^1)\mathcal{P}' - (u\xi_u^1 - \xi_u^2 + \alpha\xi_\alpha^1)R = 0, \quad (4.10)$$

where a prime denotes differentiation with respect to α and subscripts denote partial differentiation.

From Eqs. (4.3), (4.6) and (4.10) we obtain

$$\alpha(\rho\eta_\alpha^2 + \xi_\alpha^1R) - \eta_u^1\mathcal{P}' = 0. \quad (4.11)$$

Since $\rho > 0$ and we require $R > 0$, Eq. (4.11) holds provided $\xi_\alpha^1 = 0$, $\eta_\alpha^2 = 0$ and $\eta_u^1 = 0$ or $\mathcal{P}'(\alpha) = 0$.

We proceed with the analysis of the above determining equations by considering the cases: $\mathcal{P}'(\alpha) \neq 0$ and $\mathcal{P}'(\alpha) = 0$.

Case I. $\mathcal{P}'(\alpha) \neq 0$.

From Eq. (4.11) we obtain

$$\xi^1 = \xi^1(t, x, u), \quad \eta^1 = \eta^1(t, x, \alpha), \quad \eta^2 = \eta^2(t, x, u). \quad (4.12)$$

According to the first equation in (4.12), Eqs. (4.3) and (4.7) imply that

$$\xi^1 = \xi^1(t, x), \quad \xi^2 = \xi^2(t, x). \quad (4.13)$$

Taking into account the last two equations in (4.12) and Eq. (4.13), the determining equations are

$$\rho(\eta_t^1 + u\eta_x^1 + \alpha\eta_x^2) + \alpha\xi_x^1R = 0, \quad (4.14)$$

$$\alpha\eta_u^2 + \eta^1 - \alpha\eta_\alpha^1 - \alpha\xi_x^2 + 2\alpha u\xi_x^1 + \alpha\xi_t^1 = 0, \quad (4.15)$$

$$\rho(u\xi_t^1 + u^2\xi_x^1 - \xi_t^2 - u\xi_x^2 + \eta^2) + \alpha\xi_x^1\mathcal{P}' = 0, \quad (4.16)$$

$$\rho(\eta_t^2 + u\eta_x^2) + \eta_x^1 \mathcal{P}' + \eta^1 R_\alpha + \eta^2 R_u + (\xi_t^1 + u\xi_x^1 - \eta_u^2)R = 0, \quad (4.17)$$

$$\eta^1 \mathcal{P}'' + (\xi_t^1 + 2u\xi_x^1 - \xi_x^2 + \eta_\alpha^1 - \eta_u^2) \mathcal{P}' = 0. \quad (4.18)$$

Using the similar reasoning as above, Eq. (4.14) implies that $\eta_t^1 + u\eta_x^1 + \alpha\eta_x^2 = 0$ and $\xi_x^1 = 0$. Thus, the remaining determining equations (4.14)–(4.18) are

$$\eta_t^1 + u\eta_x^1 + \alpha\eta_x^2 = 0, \quad (4.19)$$

$$\alpha\eta_u^2 + \eta^1 - \alpha\eta_\alpha^1 - \alpha\xi_x^2 + \alpha\xi_t^1 = 0, \quad (4.20)$$

$$u\xi_t^1 - \xi_t^2 - u\xi_x^2 + \eta^2 = 0, \quad (4.21)$$

$$\rho(\eta_t^2 + u\eta_x^2) + \eta_x^1 \mathcal{P}' + \eta^1 R_\alpha + \eta^2 R_u + (\xi_t^1 - \eta_u^2)R = 0, \quad (4.22)$$

$$\eta^1 \mathcal{P}'' + (\xi_t^1 - \xi_x^2 + \eta_\alpha^1 - \eta_u^2) \mathcal{P}' = 0. \quad (4.23)$$

Solving Eqs. (4.45) and (4.46) we obtain

$$\eta^1 = C(t, x)\alpha, \quad \eta^2 = (\xi_x^2 - \xi_t^1)u + \xi_t^2 \quad (4.24)$$

where C is an arbitrary function. Therefore, summarizing we have

$$\xi^1 = \xi^1(t), \quad \xi^2 = \xi^2(t, x), \quad \eta^1 = C(t, x)\alpha, \quad \eta^2 = (\xi_x^2 - \xi_t^1)u + \xi_t^2 \quad (4.25)$$

and the determining equations are

$$C_t + u(C_x + \xi_{xx}^2) + \xi_{tx}^2 = 0, \quad (4.26)$$

$$C\alpha\mathcal{P}'' + [C + 2(\xi_t^1 - \xi_x^2)]\mathcal{P}' = 0, \quad (4.27)$$

$$\begin{aligned} &\rho[(2\xi_{xt}^2 - \xi_{tt}^1)u + \xi_{tt}^2 + u^2\xi_{xx}^2] + C_x\alpha\mathcal{P}' + C\alpha R_\alpha \\ &+ [(\xi_x^2 - \xi_t^1)u + \xi_t^2]R_u + (2\xi_t^1 - \xi_x^2)R = 0. \end{aligned} \quad (4.28)$$

Therefore, the symmetry generator (4.1) takes the form

$$X = \xi^1(t)\frac{\partial}{\partial t} + \xi^2(t, x)\frac{\partial}{\partial x} + C(t, x)\alpha\frac{\partial}{\partial \alpha} + [(\xi_x^2 - \xi_t^1)u + \xi_t^2]\frac{\partial}{\partial u}. \quad (4.29)$$

In order to satisfy the remaining determining equations (4.26)–(4.28), since R depends on more than one variable, it becomes difficult to employ the procedure discussed in [14, 26]. However, this procedure can be used to solve for $\mathcal{P}(\alpha)$ in Eq. (4.27) and hence, remaining with only $R(\alpha, u)$ to solve from Eq. (4.28). Since the direct Lie symmetry classification is difficult to implement, instead we make use of classification with respect to three- and four-dimensional real Lie algebras by Patera and Winternitz [27] to solve Eqs. (4.26)–(4.28).

4.1. Classification with respect to 3D Lie algebras

We consider the three-dimensional Lie algebras having the basis (e_1, e_2, e_3) . The two operators are those that span the principal Lie algebra and a third operator (which is to be

obtained) is in form of the symmetry generator (4.29). Thus, in order to obtain the third operator we consider the following possible realizations:

$$\begin{array}{lll} \text{(a)} & e_1 = \partial_t, & e_2 = \partial_x, & e_3 = X_{tn} \\ \text{(b)} & e_1 = \partial_t, & e_2 = X_{tn}, & e_3 = \partial_x \\ \text{(c)} & e_1 = \partial_x, & e_2 = \partial_t, & e_3 = X_{tn} \\ \text{(d)} & e_1 = \partial_x, & e_2 = X_{tn}, & e_3 = \partial_t \\ \text{(e)} & e_1 = X_{tn}, & e_2 = \partial_t, & e_3 = \partial_x \\ \text{(f)} & e_1 = X_{tn}, & e_2 = \partial_x, & e_3 = \partial_t \end{array}$$

where X_{tn} is the extension of the symmetry Lie algebra of the form

$$X_{tn} = A(t)\partial_t + B(t, x)\partial_x + C(t, x)\alpha\partial_\alpha + [(B_x - A_t)u + B_t]\partial_u, \quad (4.30)$$

$\partial_{\mathbf{x}}$ denotes $\partial/\partial\mathbf{x}$ for $\mathbf{x} \equiv (t, x, \alpha, u)$.

Our goal is to find the functions A, B, C for which X_{tn} is a symmetry generator of the underlying system and thus, obtaining the corresponding forms of $\mathcal{P}$ and R .

We illustrate the calculations involved by considering the possibilities (a)–(f) for the algebras $3A_1$ and $A_1 \oplus A_2$.

At first we consider $3A_1$ with the commutation relations: $[e_1, e_2] = 0, [e_1, e_3] = 0, [e_2, e_3] = 0$.

(a) The first commutation relation is identically satisfied. The second commutation relation implies that

$$e_3 = c_1\partial_t + B(x)\partial_x + C(x)\alpha\partial_\alpha + B_x u\partial_u,$$

for c_1 arbitrary. The last commutation relation yields

$$e_3 = c_1\partial_t + c_2\partial_x + c_3\alpha\partial_\alpha,$$

where c_2 and c_3 are arbitrary constants. Therefore, the determining equations (4.26)–(4.28) reduce to

$$c_3\alpha R_\alpha = 0, \quad c_3(\alpha\mathcal{P}'' + \mathcal{P}') = 0. \quad (4.31)$$

We take $e_3 = \alpha\partial_\alpha$ and hence, the corresponding forms of the arbitrary functions from the classifying equations (4.31) are

$$\mathcal{P} = k_1 \ln \alpha + k_2, \quad R = f(u), \quad (4.32)$$

where $k_1 \neq 0, k_2$ are arbitrary constants and f is an arbitrary function of u .

We reach the same conclusion for the remaining possibilities (b)–(f).

Secondly consider $A_1 \oplus A_2$ with the commutation relations: $[e_1, e_2] = e_2, [e_1, e_3] = 0, [e_2, e_3] = 0$.

These commutation relations imply that (b), (d), (e) and (f) are the only possible realizations.

(b) $[e_1, e_2] = e_2$ implies that

$$e_2 = e^t[\bar{c}_1\partial_t + b(x)\partial_x + c(x)\alpha\partial_\alpha + [(b_x - \bar{c}_1)u + b]\partial_u].$$

The second commutation relation is identically satisfied. The third commutation yields

$$e_2 = e^t[\bar{c}_1\partial_t + \bar{c}_2\partial_x + \bar{c}_3\alpha\partial_\alpha + (\bar{c}_2 - \bar{c}_1 u)\partial_u],$$

where $\bar{c}_1$, $\bar{c}_2$ and $\bar{c}_3$ are arbitrary constants. The determining equations (4.26)–(4.28) reduce to

$$\bar{c}_3 e^t \alpha = 0, \quad (4.33)$$

$$e^t [\bar{c}_3 \alpha R_\alpha + (\bar{c}_2 - \bar{c}_1 u) R_u + 2\bar{c}_1 R + \rho(\bar{c}_2 - \bar{c}_1 u)] = 0, \quad (4.34)$$

$$e^t [\bar{c}_3 \alpha \mathcal{P}'' + (2\bar{c}_1 + \bar{c}_3) \mathcal{P}'] = 0. \quad (4.35)$$

With $\bar{c}_3 = 0$, Eq. (4.35) implies that $\bar{c}_1 = 0$ and $\mathcal{P}(\alpha)$ is arbitrary since we require that $\mathcal{P}'(\alpha) \neq 0$. Thus, $e_2 = e^t(\partial_x + \partial_u)$ and solving Eq. (4.34) we obtain

$$R = g(\alpha) - \rho u, \quad (4.36)$$

where g is an arbitrary function of α .

(e) $[e_1, e_2] = e_2$ and $[e_1, e_3] = 0$ imply that $e_1 = -t\partial_t + \tilde{c}_3 \alpha \partial_\alpha + u\partial_u$, for arbitrary $\tilde{c}_3$. The determining equations (4.26)–(4.28) become

$$\tilde{c}_3 \alpha R_\alpha + u R_u - 2R = 0, \quad \tilde{c}_3 \alpha \mathcal{P}'' + (\tilde{c}_3 - 2) \mathcal{P}' = 0. \quad (4.37)$$

Solving (4.37) for nonzero $\tilde{c}_3$ we obtain

$$R = \alpha^{2/\tilde{c}_3} \tilde{f}(u \alpha^{-1/\tilde{c}_3}), \quad \mathcal{P} = \frac{1}{2} \tilde{k}_1 \tilde{c}_3 \alpha^{2/\tilde{c}_3} + \tilde{k}_2, \quad (4.38)$$

where $\tilde{k}_1 \neq 0$, $\tilde{k}_2$ are arbitrary constants and $\tilde{f}$ is an arbitrary function of its argument.

The complete results are tabulated below, excluding the cases which yield $R = 0$.

In addition to the possible realizations (a)–(f), we also consider the 3D Lie algebras containing the two-dimensional subalgebras $2A_1$ and A_2 (c.f. Table I [27]). The 2D subalgebra, $2A_1$ yields the same results as those presented in Table 1.

We show some calculations for the 3D Lie algebra $A_1 \oplus A_2$ containing A_2 with the basis $(e_1 + p e_3; e_2)$, where $p \neq 0$. The case $p = 0$ leads to the cases which are already treated.

Let $\hat{e}_1 = e_1 + p e_3$, $\hat{e}_2 = e_2$, $\hat{e}_3 = e_1$ or e_3 , where e_1, e_3 can be either ∂_t or ∂_x and $\hat{e}_2$ is of the form (4.30). Likewise, we can have $\hat{e}_1 = e_1$ or e_3 , $\hat{e}_2 = e_2$, $\hat{e}_3 = e_1 + p e_3$.

Consider $\hat{e}_1 = \partial_t + p \partial_x$, $\hat{e}_2 = e_2$, $\hat{e}_3 = \partial_t$. The commutation relations $[\hat{e}_1, \hat{e}_2] = \hat{e}_2$ and $[\hat{e}_2, \hat{e}_3] = 0$ yield $e_2 = e^{x/p} [k_2 \partial_x + k_3 \alpha \partial_\alpha + (k_2 u/p) \partial_u]$ for arbitrary constants k_2 and k_3 . Eventually e_2 takes the form $e_2 = e^{x/p} (p \partial_x - \alpha \partial_\alpha + u \partial_u)$ and the corresponding arbitrary elements are

$$R = \frac{\hat{k}_1 - \rho \alpha^2 u^2 + p \alpha \Gamma(\alpha u)}{p \alpha^2}, \quad \mathcal{P} = \hat{k}_2 - \frac{\hat{k}_1}{2 \alpha^2},$$

where $\hat{k}_1, \hat{k}_2$ are arbitrary constants and $\Gamma(\alpha u)$ is an arbitrary function.

The complete results are given in the following table.

4.2. Classification with respect to 4D Lie algebras

Since each 4D Lie algebra contains a 3D subalgebra, equations admitting 4D symmetry Lie algebras will stem from specifications of arbitrary functions in equations admitting 3D symmetry Lie algebra. Thus, we have to examine all 4D Lie algebras admitting the previously obtained 3D symmetry Lie algebras (given in Table 1) as subalgebras.

Table 1. Classification with respect to three-dimensional Lie algebras for $\mathcal{P}'(\alpha) \neq 0$ where Γ_i s are arbitrary functions and $0 < |\alpha| < 1$.

Algebra	$\mathcal{P}$	R	Condition on consts.	Extra operator
$3A_1$	$k_1 \ln \alpha + k_2$	$\Gamma_1(u)$	$k_1 \neq 0$	$\alpha \partial_\alpha$
$A_1 \oplus A_2$	Arbitrary	$\Gamma_2(\alpha) - \rho u$		$e^t(\partial_x + \partial_u)$
	$\bar{k}_2 - \frac{\bar{k}_1}{2\alpha^2}$	$\frac{\Gamma_3(\alpha u)}{\alpha} + \frac{\bar{k}_1}{\alpha^2} - \rho u^2$	$\bar{k}_1 \neq 0$	$e^x(\partial_x - \alpha \partial_\alpha + u \partial_u)$
	$\frac{1}{2}\bar{k}_3 k \alpha^{2/k} + \bar{k}_4$	$\alpha^{2/k} \Gamma_4(u \alpha^{-1/k})$	$\bar{k}_3, k \neq 0$	$t \partial_t - k \alpha \partial_\alpha - u \partial_u$
	$\bar{k}_6 - \frac{1}{2}\bar{k}_5 \lambda \alpha^{-2/\lambda}$	$\alpha^{-1/\lambda} \Gamma_5(u \alpha^{1/\lambda})$	$\bar{k}_5, \lambda \neq 0$	$x \partial_x - \lambda \alpha \partial_\alpha + u \partial_u$
$A_{3,1}$	$k_3 \ln \alpha + k_4$	$\Gamma_6\left(u - \frac{\ln \alpha}{h}\right)$	$k_3, h \neq 0$	$t \partial_x + h \alpha \partial_\alpha + \partial_u$
	Arbitrary	$\Gamma_7(\alpha)$	$h = 0$	$t \partial_x + \partial_u$
$A_{3,2}$	$k_5 \ln \alpha + k_6$	$\alpha^{-1/\kappa} \Gamma_8\left(u - \frac{\ln \alpha}{\kappa}\right)$	$k_5, \kappa \neq 0$	$t \partial_t + (t+x) \partial_x + \kappa \alpha \partial_\alpha + \partial_u$
	Arbitrary	$e^{-u} \Gamma_9(\alpha)$	$\kappa = 0$	$t \partial_t + (t+x) \partial_x + \partial_u$
$A_{3,3}$	$k_7 \ln \alpha + k_8$	$\alpha^{-1/\ell} \Gamma_{10}(u)$	$k_7, \ell \neq 0$	$t \partial_t + x \partial_x + \ell \alpha \partial_\alpha$
$A_{3,4}$	$\tilde{k}_2 - \frac{1}{4}\tilde{k}_1 m \alpha^{-4/m}$	$\alpha^{-3/m} \Gamma_{11}(u \alpha^{2/m})$	$\tilde{k}_1, m \neq 0$	$t \partial_t - x \partial_x + m \alpha \partial_\alpha - 2u \partial_u$
$A_{3,5}^a$	$\frac{\hat{k}_1 n}{2(a-1)} \alpha^{\frac{2(a-1)}{n}} + \hat{k}_2$	$\alpha^{\frac{a-2}{n}} \Gamma_{12}(u \alpha^{\frac{1-a}{n}})$	$\hat{k}_1, n \neq 0$	$t \partial_t + a x \partial_x + n \alpha \partial_\alpha + (a-1)u \partial_u$
	$\frac{\hat{k}_3 \bar{n}}{2(1-a)} \alpha^{\frac{2(1-a)}{\bar{n}}} + \hat{k}_4$	$\alpha^{\frac{1-2a}{\bar{n}}} \Gamma_{13}(u \alpha^{\frac{a-1}{\bar{n}}})$	$\hat{k}_5, \bar{n} \neq 0$	$a t \partial_t + x \partial_x + \bar{n} \alpha \partial_\alpha + (1-a)u \partial_u$

We follow the same procedure used in the classification with respect to 3D Lie algebra and the cases which lead to $R = 0$ are discarded. Below we show some calculations for the 4D Lie algebra containing 3D Lie algebra $A_1 \oplus A_2$ as subalgebra.

The algebra $2A_2$ with the nonzero commutation relations $[e_1, e_2] = e_2$ and $[e_3, e_4] = e_4$ is the only 4D Lie algebra containing the 3D Lie algebra $A_1 \oplus A_2$.

Consider the possible realization $(\partial_x, e_2, -t \partial_t + k \alpha \partial_\alpha + u \partial_u, \partial_t)$: the commutation relations $[e_1, e_2] = e_2$ and $[e_2, e_4] = 0$ of the algebra $2A_2$ give $e_2 = e^x(c_2 \partial_x + c_3 \alpha \partial_\alpha + c_2 u \partial_u)$, for arbitrary constants c_2 and c_3 . The determining equations (4.26)–(4.28) become

$$e^x \alpha u (c_2 + c_3) = 0, \quad (4.39)$$

$$\bar{k}_3 e^x \alpha^{2/k} (k c_2 - c_3) = 0, \quad (4.40)$$

$$\begin{aligned} e^x [k(c_2 \rho u^2 + c_3 \bar{k}_3 \alpha^{2/k}) - (k c_2 - 2 c_3) \alpha^{2/k} \Gamma_4(u \alpha^{-1/k}) \\ + (k c_2 - c_3) u \alpha^{1/k} \Gamma_4'(u \alpha^{-1/k})] = 0. \end{aligned} \quad (4.41)$$

The first two equations (4.39) and (4.40) imply that $c_3 = -c_2$ and thus, $k = -1$. Therefore, $e_2 = e^x(\partial_x - \alpha \partial_\alpha + u \partial_u)$, $e_3 = -t \partial_t - \alpha \partial_\alpha + u \partial_u$ and $\Gamma_4 = \bar{k}_3 - \rho \alpha^2 u^2$, i.e., $R = \bar{k}_3 \alpha^{-2} - \rho u^2$.

The complete results are presented in tabular form below.

It can be shown by direct calculations that there are no realizations of the 4D Lie algebras containing the 3D Lie algebras from Table A.1.

Case II. $\mathcal{P}'(\alpha) = 0$, i.e., $\mathcal{P} = \mathcal{P}_0 \equiv \text{constant}$.

According to Eq (4.11), the symmetry generator (4.1) is given by

$$X = \xi^1(t, x, u)\partial_t + \xi^2(t, x, u)\partial_x + \eta^1(t, x, \alpha, u)\partial_\alpha + \eta^2(t, x, u)\partial_u. \quad (4.42)$$

The corresponding determining equations are

$$u\xi_u^1 - \xi_u^2 = 0, \quad (4.43)$$

$$\rho(\eta_t^1 + u\eta_x^1 + \alpha\eta_x^2) + (\alpha\xi_x^1 - \eta_u^1)R = 0, \quad (4.44)$$

$$\alpha\eta_u^2 + \eta^1 - \alpha\eta_\alpha^1 - \alpha\xi_x^2 + 2\alpha u\xi_x^1 + \alpha\xi_t^1 = 0, \quad (4.45)$$

$$u\xi_t^1 + u^2\xi_x^1 - \xi_t^2 - u\xi_x^2 + \eta^2 = 0, \quad (4.46)$$

$$\rho^2(\eta_t^2 + u\eta_x^2) + \rho[\eta^1 R_\alpha + \eta^2 R_u + (\xi_t^1 + u\xi_x^1 - \eta_u^2)R] - \xi_u^1 R^2 = 0, \quad (4.47)$$

$$\rho(u\xi_t^1 + u^2\xi_x^1 - \xi_t^2 - u\xi_x^2 + \eta^2) - (u\xi_u^1 - \xi_u^2)R = 0. \quad (4.48)$$

Proceeding as in the previous case, our aim is to find an extra operator in the form of (4.42). Hence, obtain the corresponding form of R whenever the extra operator exists.

We give an illustration of some calculations for the 3D Lie algebra $A_1 \oplus A_2$ and the 4D Lie algebra containing this algebra.

Table 2. Classification with respect to four-dimensional Lie algebras for $\mathcal{P}'(\alpha) \neq 0$ where ℓ_i s are arbitrary nonzero constants and $0 < |b| < 1$.

$\mathcal{P}$	R	Condition on consts.	Operators
$k_3 \ln \alpha + k_4$	$\ell_1 \alpha^{-1/\bar{\ell}} \exp[hu/\bar{\ell}]$	$\bar{\ell}, h \neq 0$	$\partial_t, \partial_x, t\partial_x + h\alpha\partial_\alpha + \partial_u, t\partial_t + x\partial_x + \bar{\ell}\alpha\partial_\alpha$
$k_7 \ln \alpha + k_8$	$\ell_2 \alpha^{-1/\ell}$	$k_7, \ell \neq 0$	$\partial_t, \partial_x, t\partial_x + \partial_u, t\partial_t + x\partial_x + \ell\alpha\partial_\alpha$
$\bar{k}_4 - \frac{\bar{k}_3}{2\alpha^2}$	$\rho u^2 - \frac{\bar{k}_3}{\alpha^2}$	$\bar{k}_3 \neq 0$	$\partial_t, \partial_x, e^x(\partial_x - \alpha\partial_\alpha + u\partial_u), t\partial_t + \alpha\partial_\alpha - u\partial_u$
$\bar{k}_8 - \frac{1}{2}\bar{k}_7\lambda\alpha^{-2/\lambda}$	$\ell_3 \alpha^{-1/\lambda} - \rho u$	$\bar{k}_7, \lambda \neq 0$	$\partial_t, \partial_x, e^t(\partial_x + \partial_u), x\partial_x - \lambda\alpha\partial_\alpha + u\partial_u$
$\lambda_2 - \frac{1}{2}\lambda_1\gamma\alpha^{-2/\gamma}$	$\ell_4 \alpha^{-2/\gamma}$	$\lambda_1, \gamma \neq 0$	$\partial_t, \partial_x, t\partial_x + \partial_u, t\partial_t + \gamma\alpha\partial_\alpha - u\partial_u$
$\frac{1}{2b}\lambda_3\nu\alpha^{\frac{2b}{\nu}} + \lambda_4$	$\ell_5 \alpha^{\frac{b-1}{\nu}}$	$\lambda_3, \nu \neq 0$	$\partial_t, \partial_x, t\partial_x + \partial_u, t\partial_t + (1+b)x\partial_x + \nu\alpha\partial_\alpha + bu\partial_u$
$\frac{1}{2}\lambda_5\sigma\alpha^{2/\sigma} + \lambda_6$	ℓ_6	$\lambda_5, \sigma \neq 0$	$\partial_t, \partial_x, t\partial_x + \partial_u, t\partial_t + 2x\partial_x + \sigma\alpha\partial_\alpha + u\partial_u$
$\frac{\hat{k}_1 n}{2(a-1)}\alpha^{\frac{2(a-1)}{n}} + \hat{k}_2$	$\ell_7 \alpha^{\frac{a-2}{n}}$	$\hat{k}_1, n \neq 0$	$\partial_t, \partial_x, t\partial_x + \partial_u, t\partial_t + ax\partial_x + n\alpha\partial_\alpha + (a-1)u\partial_u$
$\frac{\hat{k}_3 \bar{n}}{2(1-a)}\alpha^{\frac{2(1-a)}{\bar{n}}} + \hat{k}_4$	$\ell_8 \alpha^{\frac{1-2a}{\bar{n}}}$	$\hat{k}_3, \bar{n} \neq 0$	$\partial_t, \partial_x, t\partial_x + \partial_u, at\partial_t + x\partial_x + \bar{n}\alpha\partial_\alpha + (1-a)u\partial_u$

Consider first the 3D Lie algebra having the realization $(e_1, \partial_x, \partial_t)$, the commutation relations $[e_1, e_2] = e_2$ and $[e_1, e_3] = 0$ yield

$$e_1 = a(u)\partial_t + (-x + b(u))\partial_x + C(\alpha, u)\partial_\alpha - d(u)\partial_u. \quad (4.49)$$

Substituting the forms of ξ s and η s from (4.49) into the determining equations (4.43)–(4.48) we have

$$e_1 = a(u)\partial_t + \left(-x + a(u)u - \int a(u) du + C_2\right)\partial_x + C_3\alpha\partial_\alpha - u\partial_u, \quad (4.50)$$

where C_2 and C_3 are arbitrary constants.

From (4.50) we investigate the cases: $a'(u) \neq 0$ and $a'(u) = 0$. Firstly for $a'(u) \neq 0$, $C_3 \neq 0$ we take

$$e_1 = u\partial_t + \left(-x + \frac{u^2}{2}\right)\partial_x + C_3\alpha\partial_\alpha - u\partial_u \quad (4.51)$$

and thus, $R = -\rho/(\alpha^{1/C_3} \exp[\rho\Phi(u\alpha^{1/C_3})] - 1)$, where Φ is an arbitrary function of its argument.

For $C_3 = 0$, $R = -\rho u/(\exp[\rho\bar{\Phi}(\alpha)] - u)$, where $\bar{\Phi}$ is an arbitrary function of α .

Secondly for $a'(u) = 0$, $C_3 \neq 0$ the operator (4.50) takes the form

$$e_1 = -x\partial_x + C_3\alpha\partial_\alpha - u\partial_u \quad (4.52)$$

and thus, $R = \alpha^{-1/C_3}\Lambda(u\alpha^{1/C_3})$, where Λ is an arbitrary function of its argument.

For $C_3 = 0$, $R = u\bar{\Lambda}(\alpha)$ and $\tilde{e}_1 = -x\partial_x - u\partial_u$, where $\bar{\Lambda}$ is an arbitrary function of α .

We now turn to the 4D Lie algebra containing $A_1 \oplus A_2$ for the extended operator (4.52) and show calculations for the possible realizations $(\partial_t, \bar{e}_2, -x\partial_x + C_3\alpha\partial_\alpha - u\partial_u, \partial_x)$ and $(-x\partial_x + C_3\alpha\partial_\alpha - u\partial_u, \partial_x, \bar{e}_3, \partial_t)$.

Taking the first realization, the commutation relations $[\bar{e}_1, \bar{e}_2] = \bar{e}_2$, $[\bar{e}_2, \bar{e}_3] = 0$ and $[\bar{e}_2, \bar{e}_4] = 0$ yield

$$\bar{e}_2 = e^t[C_1\partial_t + C_2u\partial_x + \bar{C}(u\alpha^{1/C_3})\alpha\partial_\alpha + C_4u\partial_u],$$

where C_1, C_2, C_4 are arbitrary constants and $\bar{C}$ is an arbitrary constant function. Substituting the forms of ξ s and η s in $\bar{e}_2$ into the determining equations (4.43)–(4.48) we obtain

$$\bar{e}_2 = e^t(\partial_t - u\partial_u) \quad \text{and} \quad G = \rho u\alpha^{1/C_3} + K_1 u^2 \alpha^{2/C_3},$$

for arbitrary constant K_1 . Therefore, $R = u(\rho + K_1 u\alpha^{1/C_3})$, provided $C_3 \neq 0$.

For the second realization we have

$$\bar{e}_3 = -t\partial_t + \bar{C}_3\alpha\partial_\alpha + u\partial_u \quad \text{and} \quad R = K_2 u(C_3 + \bar{C}_3)[(C_3 + \bar{C}_3)u\alpha^{1/C_3}]^{\frac{C_3}{\bar{C}_3 + C_3}},$$

where $C_3 \neq -\bar{C}_3$ and $K_2 \neq 0$ are arbitrary constants.

Considering $C_3 = 0$, the first realization gives $\bar{e}_2 = e^t(\partial_t - u\partial_u)$, $R = \rho u$ and the second realization yields $\bar{e}_3 = t\partial_t - \bar{C}_3\alpha\partial_\alpha - u\partial_u$, $R = \bar{K}_2 u\alpha^{1/\bar{C}_3}$, where $\bar{C}_3$ and $\bar{K}_2$ are nonzero arbitrary constants.

The complete results for both the 3D Lie algebras and 4D Lie algebras are given in the Appendix.

Also, presented in the Appendix are the results involving the 2D subalgebra A_2 (Table A.2.).

5. Conclusion

We obtained the forms of the arbitrary elements that appear in the one-dimensional collapsible tube model via the classification with respect to subalgebras of real three- and four-dimensional Lie algebras.

The procedure employed in finding extra operators and hence, the arbitrary elements does not guarantee the full symmetry Lie algebra, it only provides a subalgebra of the symmetry Lie algebra. That is, there might be extra operators for the equations admitting 3D and 4D Lie algebras. The guarantee for obtaining the maximal symmetry Lie algebra would be reached only after performing symmetry analysis of individual cases from Tables 1 and 3 for 3D Lie algebras and Tables 2 and 4 for 4D Lie algebras. This will be reported in future work.

Acknowledgments

M. Molati acknowledges with gratitude financial support from the National University of Lesotho through Research and Conference Committee (RCC) which initially funded this project.

Appendix A. Classification Results for the 3D and 4D Lie Algebras

Table A.1. Classification with respect to 3D Lie algebra $A_1 \oplus A_2$ for $\mathcal{P}'(\alpha) \neq 0$. The arbitrary functions Γ s are in general different and $p \neq 0$.

$\mathcal{P}$	R	Condition on consts.	Operators
$\hat{k}_2 - \frac{\hat{k}_1}{2\alpha^2}$	$\frac{\hat{k}_1 - \rho\alpha^2 u^2 + p\alpha\Gamma(\alpha u)}{p\alpha^2}$	$\hat{k}_1 \neq 0$	$\partial_t, \partial_t + p\partial_x,$ $e^{x/p} (p\partial_x - \alpha\partial_\alpha + u\partial_u)$
Arbitrary	$\Gamma(\alpha) - \frac{\rho u}{p}$		$\partial_x, p\partial_t + \partial_x, e^{t/p} (p\partial_x + \partial_u)$
$\hat{k}_4 - \frac{\hat{k}_3}{2\alpha^2}$	$\frac{\rho\alpha^2(u^2 - 2pu + p^2) - \hat{k}_3 + p\alpha\Gamma(\alpha u - p\alpha)}{p\alpha^2}$	$\hat{k}_3 \neq 0$	$\partial_t, \partial_t + p\partial_x,$ $e^{t-x/p} [p\partial_x + \alpha\partial_\alpha + (p-u)\partial_u]$
$\hat{k}_6 - \frac{\hat{k}_5}{2\alpha^2}$	$\frac{\hat{k}_5 + \rho\alpha^2(2pu - u^2 - p^2) + \alpha\Gamma(\alpha u - p\alpha)}{\alpha^2}$	$\hat{k}_5 \neq 0$	$\partial_x, \partial_t + p\partial_x,$ $e^{x-pt} [\partial_x - \alpha\partial_\alpha + (u-p)\partial_u]$
$\hat{k}_8 - \frac{\hat{k}_7}{2\alpha^2}$	$\frac{\rho\alpha^2(1 - 2pu + p^2 u^2) - \hat{k}_7 p^2 + p\alpha\Gamma(\alpha u - \alpha/p)}{p\alpha^2}$	$\hat{k}_7 \neq 0$	$\partial_t, p\partial_t + \partial_x,$ $e^{t-px} [\partial_x + p\alpha\partial_\alpha + (1-pu)\partial_u]$
$\hat{k}_{10} - \frac{\hat{k}_9}{2\alpha^2}$	$\frac{\hat{k}_9 p^2 + \rho\alpha^2(2pu - 1 - p^2 u^2) + p^2 \alpha\Gamma(\alpha u - \alpha/p)}{p^2 \alpha^2}$	$\hat{k}_9 \neq 0$	$\partial_x, p\partial_t + \partial_x,$ $e^{x-t/p} [p\partial_x - p\alpha\partial_\alpha + (pu-1)\partial_u]$

Table A.2. Classification with respect to 3D Lie algebra containing 2D subalgebra A_2 for $\mathcal{P}(\alpha) = \mathcal{P}_0$. The arbitrary functions $\bar{G}(\alpha)$ are in general different and $0 \leq \phi < \pi$.

R	Condition on consts.	Operators
$\rho u^2/p$	$p \neq 0, 1$	$\partial_t, \partial_t + p\partial_x, e^{x/p}u\alpha\partial_\alpha$
$\frac{\rho}{p}\left(\frac{u^2}{2} - \bar{k}\right)$	$p \neq 0, 1$	$\partial_t, \partial_t + p\partial_x,$ $e^{x/p}\left[p u\partial_t + p\left(\frac{u^2}{2} + \bar{k}\right)\partial_x + u\left(\bar{k} - \frac{u^2}{2}\right)\partial_u\right]$
$\rho u(u-r)/p$	$p \neq 0, 1$	$\partial_t, \partial_t + p\partial_x,$ $e^{x/p}[p\partial_t + pr\partial_x + (r-u)u\partial_u]$
$\rho u/p$	$p \neq 0, 1$	$\partial_x, p\partial_t + \partial_x, e^{t/p}u\alpha\partial_\alpha,$ $e^{t/p}(2pu\partial_t + pu^2\partial_x + 2pu\alpha\partial_\alpha - u^2\partial_u)$
$\rho\left(\frac{u}{p} + i\sqrt{\frac{2\bar{k}}{p}}\right)$	$\bar{k} \neq 0$	$\partial_x, p\partial_t + \partial_x,$
$\times \tan\left[2i \arctan\left[\frac{u}{\sqrt{2\bar{k}}}\right] + \bar{G}(\alpha)\right]$	$p \neq 0, 1$	$e^{t/p}\left[p u\partial_t + p\left(\frac{u^2}{2} + \bar{k}\right)\partial_x + \left(\bar{k} - \frac{u^2}{2}\right)\partial_u\right]$
$\frac{\rho}{p}(u-s) + (u-s)^2\bar{G}(\alpha)$	$p \neq 0, 1$	$\partial_x, p\partial_t + \partial_x,$ $e^{t/p}[p\partial_t + ps\partial_x + (s-u)\partial_u]$
$\rho u(p-u)/p$	$p \neq 0$	$\partial_t, \partial_t + p\partial_x, e^{t-x/p}u\alpha\partial_\alpha$
$\frac{\rho}{2pu}(p-u)(u^2 - 2\bar{k})$	$p \neq 0$	$\partial_t, \partial_t + p\partial_x,$ $e^{t-x/p}\left[\partial_t + \left(\frac{u^2}{2} + \bar{k}\right)\partial_x - \frac{p-u}{2p}(u^2 - 2\bar{k})\partial_u\right]$
$\rho(p-u)(u-y)/p$	$p \neq 0$	$\partial_t, \partial_t + p\partial_x,$ $e^{t-x/p}[p\partial_t + py\partial_x - (p-u)(u-y)\partial_u]$
$\rho u(u-p)$	$p \neq 0$	$\partial_x, \partial_t + p\partial_x, e^{x-pt}u\alpha\partial_\alpha$
$\frac{\rho}{2u}(u-p)(u^2 - 2\bar{k})$	$p \neq 0$	$\partial_x, \partial_t + p\partial_x,$ $e^{x-pt}\left[u\partial_t + \left(\frac{u^2}{2} + \bar{k}\right)\partial_x + (p-u)\left(\frac{u^2}{2} - \bar{k}\right)\partial_u\right]$
$\rho(p-u)(z-u)$	$p \neq 0$	$\partial_x, \partial_t + p\partial_x,$ $e^{x-pt}[\partial_t + z\partial_x + (u-z)(p-u)\partial_u]$

Table A.2. (Continued).

R	Condition on consts.	Operators
$\rho u(1 - pu)$	$p \neq 0$	$\partial_t, p\partial_t + \partial_x, e^{t-px} u \alpha \partial_\alpha$
$\frac{\rho}{2u}(1 - pu)(u^2 - 2\bar{k})$	$p \neq 0$	$\partial_t, p\partial_t + \partial_x,$ $e^{t-px} \left[u\partial_t + \left(\frac{u^2}{2} + \bar{k} \right) \partial_x + (pu - 1) \left(\frac{u^2}{2} - \bar{k} \right) \partial_u \right]$
$\rho(pu - 1)(\tau - u)$	$p \neq 0$	$\partial_t, p\partial_t + \partial_x,$ $e^{t-px} [\partial_t + \tau\partial_x + (\tau u - 1)(u - \tau)\partial_u]$
$\rho u(pu - 1)/p$	$p \neq 0$	$\partial_x, p\partial_t + \partial_x, e^{x-t/p} u \alpha \partial_\alpha$
$\frac{\rho}{2pu}(pu - 1)(u^2 - 2\bar{k})$	$p \neq 0$	$\partial_x, p\partial_t + \partial_x,$ $e^{x-t/p} \left[u\partial_t + \left(\frac{u^2}{2} + \bar{k} \right) \partial_x - \frac{pu - 1}{2p} (u^2 - 2\bar{k}) \partial_u \right]$
$\rho(pu - 1)(u - \omega)/p$	$p \neq 0$	$\partial_x, p\partial_t + \partial_x,$ $e^{x-t/p} [p\partial_t + p\omega\partial_x + (pu - 1)(u - \omega)\partial_u]$
$-\rho u^2$	$\phi \neq \pi/2$	$\partial_x, e^{-x} u \alpha \partial_\alpha, \cos \phi e^{-x} u \alpha \partial_\alpha$
$\rho \left(\bar{k} - \frac{u^2}{2} \right)$	$\phi \neq \pi/2$	$\partial_x, \cos \phi e^{-x} \left[u\partial_t + \left(\frac{u^2}{2} + \bar{k} \right) \partial_x + u \left(\frac{u^2}{2} - \bar{k} \right) \partial_u \right],$ $e^{-x} \left[u\partial_t + \left(\frac{u^2}{2} + \bar{k} \right) \partial_x + u \left(\frac{u^2}{2} - \bar{k} \right) \partial_u \right]$
$\rho u(\mu - u)$	$\phi \neq \pi/2$	$\partial_x, \cos \phi e^{-x} [\partial_t + \mu\partial_x + (u - \mu)u\partial_u],$ $e^{-x} [\partial_t + \mu\partial_x + (u - \mu)u\partial_u]$
$(u - \sigma)^2 \bar{G}(\alpha) - \rho(u - \sigma)$	$\phi \neq \pi/2$	$\partial_t, \cos \phi e^{-t} [\partial_t + \sigma\partial_x + (u - \sigma)\partial_u],$ $e^{-t} [\partial_t + \sigma\partial_x + (u - \sigma)\partial_u]$

Table A.3. Classification with respect to three-dimensional Lie algebras for $\mathcal{P}'(\alpha) = 0$ where G s are (different) arbitrary functions of their arguments.

Algebra	R	Condition on consts.	Extra operator
$3A_1$	$-\frac{\rho\bar{c}}{\ln \alpha + \rho\bar{c}G(u)}$ $G(u)$	$\bar{c} \neq 0$	$u\partial_t + \frac{u^2}{2}\partial_x + \bar{c}\alpha\partial_\alpha$ $\alpha\partial_\alpha$
$A_1 \oplus A_2$	ρu		$e^t u \alpha \partial_\alpha$
$\rho(u + i\sqrt{2c_1})$ $\times \tan \left[2i \arctan \left[\frac{u}{\sqrt{2c_1}} \right] + G(\alpha) \right]$		$c_1 \neq 0$	$e^t \left[u\partial_t + \left(\frac{u^2}{2} + c_1 \right) \partial_x + \left(c_1 - \frac{u^2}{2} \right) \partial_u \right]$
$\frac{\rho u (2u + \exp[\rho G(\alpha)])}{2(u + \exp[\rho G(\alpha)])}$		$c_1 = 0$	$e^t \left(u\partial_t + \frac{u^2}{2}\partial_x - \frac{u^2}{2}\partial_u \right)$
$\rho(u - c_2) + (u - c_2)^2 G(\alpha)$ $G(\alpha) - \rho u$			$e^t [\partial_t + c_2 \partial_x + (c_2 - u) \partial_u]$ $e^t (\partial_x + \partial_u)$
$\rho \left(\frac{u^2}{2} - c_3 \right)$ ρu^2 $\rho u(u - c_4)$ $\alpha^{-1} G(\alpha u) - \rho u^2$			$e^x \left[u\partial_t + \left(\frac{u^2}{2} + c_3 \right) \partial_x + \left(c_3 - \frac{u^2}{2} \right) u\partial_u \right]$ $e^x u \alpha \partial_\alpha$ $e^x [\partial_t + c_4 \partial_x + (c_4 - u) u \partial_u]$ $e^x (\partial_x - \alpha \partial_\alpha + u \partial_u)$
$\frac{2\rho\alpha^{2/c_5} \exp[2\rho G(u\alpha^{-1/c_5})]}{1 - \alpha^{2/c_5} \exp[2\rho G(u\alpha^{-1/c_5})]}$		$c_5 \neq 0$	$(u - t)\partial_t + \frac{u^2}{2}\partial_x + c_5 \alpha \partial_\alpha + u\partial_u$
$\frac{2\rho u^2 \exp[2\rho G(\alpha)]}{1 - u^2 \exp[2\rho G(\alpha)]}$		$c_5 = 0$	$(u - t)\partial_t + \frac{u^2}{2}\partial_x + u\partial_u$
$\alpha^{2/c_6} G(u\alpha^{-1/c_6})$		$c_6 \neq 0$	$t\partial_t - c_6 \alpha \partial_\alpha - u\partial_u$
$u^2 G(\alpha)$		$c_6 = 0$	$t\partial_t - u\partial_u$
$\frac{\rho}{1 - \alpha^{1/c_7} \exp[\rho G(u\alpha^{1/c_7})]}$		$c_7 \neq 0$	$u\partial_t + \left(\frac{u^2}{2} - x \right) \partial_x + c_7 \alpha \partial_\alpha - u\partial_u$
$\frac{\rho u}{u - \exp[\rho G(\alpha)]}$		$c_7 = 0$	$u\partial_t + \left(\frac{u^2}{2} - x \right) \partial_x - u\partial_u$
$\alpha^{-1/c_8} G(u\alpha^{1/c_8})$		$c_8 \neq 0$	$x\partial_x - c_8 \alpha \partial_\alpha + u\partial_u$
$uG(\alpha)$		$c_8 = 0$	$x\partial_x + u\partial_u$

Table A.3. (Continued)

Algebra	R	Condition on consts.	Extra operator
$A_{3,1}$	$\frac{2\rho u}{1 + 2u\alpha^3\rho G(u\alpha)}$		$(x + u)\partial_t + \frac{u^2}{2}\partial_x + u\alpha\partial_\alpha - u^2\partial_u$
	$\alpha^{-3}G(\alpha u)$		$x\partial_t + u\alpha\partial_\alpha - u^2\partial_u$
	$\frac{\rho\alpha^{1/c_9}\exp[\rho G(u\alpha^{-1/c_9})]}{1 - \alpha^{1/c_9}\exp[\rho G(u\alpha^{-1/c_9})]}$	$c_9 \neq 0$	$u\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x + c_9\alpha\partial_\alpha + u\partial_u$
	$\frac{\rho u \exp[\rho G(\alpha)]}{1 - u \exp[\rho G(\alpha)]}$	$c_9 = 0$	$u\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x + u\partial_u$
	$\alpha^{1/c_{10}}G(u\alpha^{-1/c_{10}})$	$c_{10} \neq 0$	$x\partial_x + c_{10}\alpha\partial_\alpha + u\partial_u$
	$uG(\alpha)$	$c_{10} = 0$	$x\partial_x + u\partial_u$
$A_{3,2}$	$\frac{2\rho u^3}{u^2 - u + [2\rho c_{11}^3 G(\ln[u\alpha] - \frac{c_{11}}{u}) - {}^a\text{Ei}[-\frac{1}{u}]]e^{1/u}}$	$c_{11} \neq 0$	$(t + x + u)\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x + (u + c_{11})\alpha\partial_\alpha - u^2\partial_u$
	$27u^3 e^{-1/u} G\left(\ln[u\alpha] - \frac{1}{3u}\right)$		$(t + x)\partial_t + x\partial_x + \left(u + \frac{1}{3}\right)\alpha\partial_\alpha - u^2\partial_u$
	$\alpha^{-3}e^{-1/3u}G(u\alpha)$		$(t + x)\partial_t + x\partial_x + u\alpha\partial_\alpha - u^2\partial_u$
	$\frac{\rho}{1 - \alpha^{1/c_{12}}\exp[\rho G(u - \frac{\ln \alpha}{c_{12}})]}$	$c_{12} \neq 0$	$(t + u)\partial_t + \left(t + x + \frac{u^2}{2}\right)\partial_x + c_{12}\alpha\partial_\alpha + \partial_u$
	$\frac{\rho}{1 - \exp[u + \rho G(\alpha)]}$	$c_{12} = 0$	$(t + u)\partial_t + \left(t + x + \frac{u^2}{2}\right)\partial_x + \partial_u$
	$\alpha^{-1/c_{13}}G\left(u - \frac{\ln \alpha}{c_{13}}\right)$	$c_{13} \neq 0$	$t\partial_t + (t + x)\partial_x + c_{13}\alpha\partial_\alpha + \partial_u$
	$e^{-u}G(\alpha)$	$c_{13} = 0$	$t\partial_t + (t + x)\partial_x + \partial_u$
$A_{3,3}$	$\frac{\rho}{1 - \alpha^{1/c_{14}}\exp[\rho G(u)]}$	$c_{14} \neq 0$	$(t + u)\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x + c_{14}\alpha\partial_\alpha$
	ρ	$c_{14} = 0$	$(t + u)\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x$
	$\alpha^{-1/c_{15}}G(u)$	$c_{15} \neq 0$	$t\partial_t + x\partial_x + c_{15}\alpha\partial_\alpha$
$A_{3,4}$	$\frac{3\rho}{1 - \alpha^{3/c_{16}}\exp[3\rho G(u\alpha^{2/c_{16}})]}$	$c_{16} \neq 0$	$(t + u)\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x + c_{16}\alpha\partial_\alpha - 2u\partial_u$
	$\frac{3\rho u^{3/2}}{u^{3/2} - \exp[3\rho G(\alpha)]}$	$c_{16} = 0$	$(t + u)\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x - 2u\partial_u$
	$\alpha^{-3/c_{17}}G(u\alpha^{2/c_{17}})$	$c_{17} \neq 0$	$t\partial_t - x\partial_x + c_{17}\alpha\partial_\alpha - 2u\partial_u$
	$u^{3/2}G(\alpha)$	$c_{17} = 0$	$t\partial_t - x\partial_x - 2u\partial_u$

Table A.3. (*Continued*)

Algebra	R	Condition on consts.	Extra operator
$A_{3,5}^a$	$\frac{\rho(2-a)}{1 - \alpha^{\frac{2-a}{c_{18}}} \exp[\rho(2-a)G(u\alpha^{\frac{1-a}{c_{18}}})]}$	$c_{18} \neq 0$	$(t+u)\partial_t + \left(ax + \frac{u^2}{2}\right)\partial_x + c_{18}\alpha\partial_\alpha$ $+ (a-1)u\partial_u$
	${}^b\text{Inv} \left[\frac{\ln[\frac{\#1}{\rho(a-2)+\#1}]}{\rho(a-2)} \& \left[\frac{\ln u}{\rho(a-1)} + G(\alpha) \right] \right.$	$c_{18} = 0$	$(t+u)\partial_t + \left(ax + \frac{u^2}{2}\right)\partial_x + (a-1)u\partial_u$
	$\alpha^{\frac{a-2}{c_{19}}} G(u\alpha^{\frac{1-a}{c_{19}}})$	$c_{19} \neq 0$	$t\partial_t + ax\partial_x + c_{19}\alpha\partial_\alpha + (a-1)u\partial_u$
	$u^{\frac{a-2}{a-1}} G(\alpha)$	$c_{19} = 0$	$t\partial_t + ax\partial_x + (a-1)u\partial_u$
	$\frac{\rho(2a-1)}{1 - \alpha^{\frac{2a-1}{c_{20}}} \exp[\rho(2a-1)G(u\alpha^{\frac{a-1}{c_{20}}})]}$	$c_{20} \neq 0$	$(at+u)\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x + c_{20}\alpha\partial_\alpha$ $+ (1-a)u\partial_u$
	${}^b\text{Inv} \left[\frac{\ln[\frac{\#1}{\rho(1-2a)+\#1}]}{\rho(2a-1)} \& \left[\frac{\ln u}{\rho(a-1)} + G(\alpha) \right] \right.$	$c_{20} = 0$	$(at+u)\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x + (1-a)u\partial_u$
	$\alpha^{\frac{1-2a}{c_{21}}} G(u\alpha^{\frac{a-1}{c_{21}}})$	$c_{21} \neq 0$	$at\partial_t + x\partial_x + c_{21}\alpha\partial_\alpha + (1-a)u\partial_u$
	$[(1-a)u]^{\frac{1-2a}{1-a}} G(\alpha)$	$c_{21} = 0$	$at\partial_t + x\partial_x + (1-a)u\partial_u$
	$\frac{2\rho(1+u^2)^{3/2}}{u\sqrt{1+u^2} + \text{arc sinh } u}$		$(x+u)\partial_t + \left(\frac{u^2}{2} - t\right)\partial_x + (u+c_{22})\alpha\partial_\alpha$ $- (1+u^2)\partial_u$
	$\frac{2\rho G(c_{22} \arctan u + \ln[\alpha\sqrt{1+u^2}])}{(1+u^2)^{3/2} G(c_{23} \arctan u + \ln[\alpha\sqrt{1+u^2}])}$		$x\partial_t - t\partial_x + (u+c_{23})\alpha\partial_\alpha - (1+u^2)\partial_u$
$A_{3,7}^a$	$(1-au+u^2)^{3/2} \exp \left[\frac{a \arctan[\frac{2u-a}{\sqrt{4-a^2}}]}{\sqrt{4-a^2}} \right]$		$(at+x)\partial_t + (ax-t)\partial_x + (u+c_{24})\alpha\partial_\alpha$ $+ (au-u^2-1)\partial_u$
	$\times G \left(-\frac{(a+2c_{24}) \arctan[\frac{2u-a}{\sqrt{4-a^2}}]}{\sqrt{4-a^2}} \right.$		
	$\left. - \ln[\alpha\sqrt{1-au+u^2}] \right)$		
	$\frac{(1+u^2)^{3/2}}{\exp[a \arctan u]} G(\ln[\alpha\sqrt{1+u^2}] - c_{25} \arctan u)$		$(at-x)\partial_t + (t+ax)\partial_x + (c_{25}-u)\alpha\partial_\alpha$ $+ (1+u^2)\partial_u$

Note: ${}^a\text{Ei}$ is the ExpIntegralEi $[-\frac{1}{u}]$.

${}^b\text{Inv}$ is the InverseFunction.

Table A.4. Classification with respect to four-dimensional Lie algebras for $\mathcal{P}'(\alpha) = 0$ where K s are arbitrary nonzero constants which in general are different. The symmetry Lie algebra for each case include the operators ∂_t and ∂_x , this is the same for the Tables that follow.

R	Condition on consts.	Operators
ρu		$\alpha\partial_\alpha, e^t u \alpha\partial_\alpha, (x+u)\alpha\partial_\alpha, x\partial_x + u\partial_u, e^t(\partial_t - u\partial_u)$
$\rho(u + i\sqrt{2\bar{c}_1})$ $\times \tan\left[2i\arctan\left[\frac{u}{\sqrt{2\bar{c}_1}}\right] + K\right]$	$\bar{c}_1 \neq 0$	$\alpha\partial_\alpha, e^t\left[u\partial_t + \left(\bar{c}_1 + \frac{u^2}{2}\right)\partial_x + \left(\bar{c}_1 - \frac{u^2}{2}\right)\partial_u\right]$
$\frac{\rho u(2u + \exp[\rho K])}{2(u + \exp[\rho K])}$	$\bar{c}_1 = 0$	$\alpha\partial_\alpha, e^t\left(u\partial_t + \frac{u^2}{2}\partial_x - \frac{u^2}{2}\partial_u\right)$
$\rho(u - \bar{c}_2) + K(u - \bar{c}_2)^2$		$\alpha\partial_\alpha, e^t[\partial_t + \bar{c}_2\partial_x + (\bar{c}_2 - u)\partial_u]$
$K - \rho u$		$\alpha\partial_\alpha, e^t(\partial_x + \partial_u)$
ρu^2		$\alpha\partial_\alpha, e^x u \alpha\partial_\alpha, e^x(\partial_x - \alpha\partial_\alpha + u\partial_u), e^x(\partial_t - u^2\partial_u),$ $e^{t-\frac{1}{u}}\alpha\partial_\alpha, t\partial_t + \partial_x - u\partial_u$
$Ku - \rho u^2$		$\alpha\partial_\alpha, e^x(\partial_x - \alpha\partial_\alpha + u\partial_u)$
$\frac{2\rho u^2 \exp[2\rho K]}{1 - u^2 \exp[2\rho K]}$		$\alpha\partial_\alpha, (u - t)\partial_t + \frac{u^2}{2}\partial_x + u\partial_u$
Ku^2		$\alpha\partial_\alpha, t\partial_t - u\partial_u$
$\frac{\rho u}{u - \exp[\rho K]}$		$\alpha\partial_\alpha, u\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x - u\partial_u$
Ku		$\alpha\partial_\alpha, x\partial_x + u\partial_u$
$\frac{2\rho u^3}{u^2 + 2\rho K}$		$\alpha\partial_\alpha, (x+u)\partial_t + \frac{u^2}{2}\partial_x + u\alpha\partial_\alpha - u^2\partial_u$
Ku^3		$\alpha\partial_\alpha, x\partial_t + u\alpha\partial_\alpha - u^2\partial_u, 2t\partial_t + x\partial_x - u\partial_u$
$-\frac{\rho}{u + \rho K}$		$\alpha\partial_\alpha, u\partial_t + \left(t + \frac{u^2}{2}\right)\partial_x + \partial_u$
K		$\alpha\partial_\alpha, t\partial_x + \partial_u$
ρ		$\alpha\partial_\alpha, (t+u)\alpha\partial_\alpha, (t+u)\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x,$ $\left(x + \frac{3}{2}u^2\right)\partial_t + u^3\partial_x + u\alpha\partial_\alpha - u^2\partial_u,$ $\left(x + 1 + \frac{3}{2}u^2\right)\partial_t + (u^3 - t)\partial_x + u\alpha\partial_\alpha - (1+u^2)\partial_u$
$\frac{2\rho(1+u^2)^{3/2}}{u\sqrt{1+u^2} + \arcsinh u + 2\rho K}$		$\alpha\partial_\alpha, (x+u)\partial_t + \left(\frac{u^2}{2} - t\right)\partial_x + u\alpha\partial_\alpha - (1+u^2)\partial_u$
$K(1+u^2)^{3/2}$		$\alpha\partial_\alpha, x\partial_t - t\partial_x + u\alpha\partial_\alpha - (1+u^2)\partial_u$
$\frac{2\rho(1+u^2)^{3/2}}{(u-a)\sqrt{1+u^2} + \frac{2\rho K}{\exp[a \arctan u]}}$ $+ 2(a-i)\exp[i \arctan u]^c H$		$\alpha\partial_\alpha, (at+x+u)\partial_t + (ax-t+\frac{u^2}{2})\partial_x + u\alpha\partial_\alpha - (1+u^2)\partial_u$

Note: $^c H$ is the Hypergeometric2F1 $[\frac{1-ia}{2}, 1, \frac{3-ia}{2}, -\exp[2i \arctan u]]$.

Table A.4. (*Continued*)

R	Condition on consts.	Operators
$K(1+u^2)^{3/2} \exp[a \arctan u]$		$\alpha \partial_\alpha, (at+x)\partial_t + (ax-t)\partial_x + u\alpha \partial_\alpha - (1+u^2)\partial_u$
$\rho(1-au)$		$\alpha \partial_\alpha, \left(at+x + \frac{(1+a^2) \left[\frac{1}{1-au} - \ln[au-1] \right] - 2au}{a^2} \right) \partial_t$ $+ \left(\frac{a+u}{1-au} \right) u^2 \partial_x + (ax-t)\alpha \partial_\alpha - (a+u)u \partial_u$
$\frac{\rho u [(u-1) \exp[2\rho K] - u\alpha]}{(u-2) \exp[2\rho K] - u\alpha}$	$c_1 = 0$	$e^t \left(u \partial_t + \frac{u^2}{2} \partial_x - \frac{u^2}{2} \partial_u \right),$ $e^{(x+u)} [\partial_t + (u-1)\partial_x + \alpha \partial_\alpha - u \partial_u]$
$\frac{1}{2} \rho u \left[2 - \frac{1}{1+u\alpha^{1/m_1} \exp[-\rho K]} \right]$	$c_1 = 0$ $m_1 \neq 0$	$e^t \left(u \partial_t + \frac{u^2}{2} \partial_x - \frac{u^2}{2} \partial_u \right), \partial_t - \partial_x + m_1 \alpha \partial_\alpha - u \partial_u$
$\frac{\rho(c_2-u) [1+u(\alpha \exp[\rho K] - 1)]}{c_2(1-\alpha \exp[\rho K]) - 1}$		$e^t [\partial_t + c_2 \partial_x + (c_2-u)\partial_u],$ $e^x (c_2-u)^{c_2} \left[\frac{c_2}{c_2-u} \partial_t + \frac{c_2+u(c_2-1)}{c_2-u} \partial_x - \alpha \partial_\alpha + u \partial_u \right]$
$\rho u \left(\frac{u}{c_2} - 1 \right)$	$c_2 \neq 0$	$e^t [\partial_t + c_2 \partial_x + (c_2-u)\partial_u], e^x (c_2-u)^{c_2} \alpha \partial_\alpha,$ $\frac{c_2}{u-c_2} \partial_t - \left[x + c_2 \left(\ln[u-c_2] - \frac{c_2}{u-c_2} \right) \right] \partial_x - u \partial_u$
$\rho \left(\frac{1}{1-\alpha \exp[\rho K]} - u \right)$		$e^t (\partial_x + \partial_u), e^{(x-u)} [\partial_t + (u+1)\partial_x - \alpha \partial_\alpha + u \partial_u]$
$\frac{\rho(c_2-u) [u\alpha^{1/m_2} - \exp[\rho K]]}{\exp[\rho K] - c_2 \alpha^{1/m_2}}$	$m_2 \neq 0$	$e^t [\partial_t + c_2 \partial_x + (c_2-u)\partial_u],$ $\frac{c_2}{u-c_2} \partial_t - \left[x + c_2 \left(\ln[u-c_2] - \frac{c_2}{u-c_2} \right) \right] \partial_x + m_2 \alpha \partial_\alpha - u \partial_u$
$\rho(u-c_2)$	$m_2 = 0$	$e^t [\partial_t + c_2 \partial_x + (c_2-u)\partial_u],$ $\frac{c_2}{u-c_2} \partial_t - \left[x + c_2 \left(\ln[u-c_2] - \frac{c_2}{u-c_2} \right) \right] \partial_x - u \partial_u$
$K\alpha^{-1/m_3} - \rho u$	$m_3 \neq 0$	$e^t (\partial_x + \partial_u), \partial_t + x \partial_x - m_3 \alpha \partial_\alpha + u \partial_u$
$-\rho u$	$m_3 = 0$	$e^t (\partial_x + \partial_u), \partial_t + x \partial_x + u \partial_u$
$\rho \left(\frac{u^2}{2} - c_3 \right)$		$e^x \left[u \partial_t + \left(\frac{u^2}{2} + c_3 \right) \partial_x + \left(c_3 - \frac{u^2}{2} \right) u \partial_u \right],$ $\exp \left[t - \frac{\sqrt{2} \operatorname{arctanh}[\frac{u}{\sqrt{2c_3}}]}{\sqrt{c_3}} \right] \alpha \partial_\alpha$
$\rho u^2/2$	$c_3 = 0$	$e^x \left(u \partial_t + \frac{u^2}{2} \partial_x - \frac{u^3}{2} \partial_u \right), e^{(t-2/u)} \alpha \partial_\alpha$
$\rho u(u-c_4)$	$c_4 \neq 0$	$e^x [\partial_t + c_4 \partial_x + (c_4-u)\partial_u], e^t \left(\frac{u-c_4}{u} \right)^{\frac{1}{c_4}} \alpha \partial_\alpha,$

Table A.4. (Continued)

R	Condition on consts.	Operators
$u[K(1+m_4)[(1+m_4)u\alpha]^{\frac{1}{1+m_4}} - \rho u]$	$m_4 \neq -1$	$e^x(\partial_x - \alpha\partial_\alpha + u\partial_u), t\partial_t - m_4\alpha\partial_\alpha - u\partial_u$
$-\rho u^2$	$m_4 = -1$	$e^x(\partial_x - \alpha\partial_\alpha + u\partial_u), t\partial_t + \alpha\partial_\alpha - u\partial_u$
$\rho \left[\frac{4}{2+u^2-K(1+c_5)^2u^2} - 2 \right]$	$c_5 \neq 0, -1$	$(u-t)\partial_t + \frac{u^2}{2}\partial_x + c_5\alpha\partial_\alpha + u\partial_u,$ $e^{(x-\frac{u^2}{2})} \left[\frac{u}{2}\partial_t + \left(1 + \frac{u^2}{2}\right)\partial_x - \alpha\partial_\alpha + u\partial_u \right]$
$\times [(1+c_5)u\alpha^{-1/c_5}]^{-\frac{c_5}{1+c_5}}$		
$-\frac{2\rho u^2}{u^2+2}$	$c_5 = -1$	$(u-t)\partial_t + \frac{u^2}{2}\partial_x - \alpha\partial_\alpha + u\partial_u,$ $e^{(x-\frac{u^2}{2})} \left[\frac{u}{2}\partial_t + \left(1 + \frac{u^2}{2}\right)\partial_x - \alpha\partial_\alpha + u\partial_u \right]$
$\rho \left[\frac{4}{2+u^2(1+\alpha\exp[4\rho K])} - 2 \right]$	$c_5 = 0$	$(u-t)\partial_t + \frac{u^2}{2}\partial_x + u\partial_u,$ $e^{(x-\frac{u^2}{2})} \left[\frac{u}{2}\partial_t + \left(1 + \frac{u^2}{2}\right)\partial_x - \alpha\partial_\alpha + u\partial_u \right]$
$2\rho \left[\frac{1}{1-\alpha^{2/c_5}\exp[2\rho K]} - 1 \right]$	$c_5 \neq 0, -m_5$	$(u-t)\partial_t + \frac{u^2}{2}\partial_x + c_5\alpha\partial_\alpha + u\partial_u,$ $\frac{u}{2}\partial_t + \left(x + \frac{u^2}{4}\right)\partial_x - m_5\alpha\partial_\alpha + u\partial_u$
$\times [2\rho(c_5+m_5)u\alpha^{-1/c_5}]^{\frac{c_5+2m_5}{c_5+m_5}}$		
$2 \left[\frac{1}{1-u^2\alpha^{1/m_5}\exp[2\rho K]} - 1 \right]$	$c_5 = 0$	$(u-t)\partial_t + \frac{u^2}{2}\partial_x + u\partial_u,$ $\frac{u}{2}\partial_t + \left(x + \frac{u^2}{4}\right)\partial_x - m_5\alpha\partial_\alpha + u\partial_u$
$K\alpha^{2/c_6}[(1+c_6)u\alpha^{-1/c_6}]^{\frac{c_4+2}{c_4+1}} - \rho u^2$	$c_6 \neq 0, -1$	$t\partial_t - c_6\alpha\partial_\alpha - u\partial_u, e^x(\partial_x - \alpha\partial_\alpha + u\partial_u)$
$(\alpha K - \rho)u^2$	$c_6 = 0$	$t\partial_t - u\partial_u, e^x(\partial_x - \alpha\partial_\alpha + u\partial_u)$
$K(c_6+m_6)^2u^2[(c_6+m_6)u\alpha^{-1/c_6}]^{-\frac{c_6}{c_6+m_6}}$	$c_6 \neq 0, -m_6$	$t\partial_t - c_6\alpha\partial_\alpha - u\partial_u, x\partial_x - m_6\alpha\partial_\alpha + u\partial_u$
$Ku^2\alpha^{1/m_6}$	$c_6, m_6 \neq 0$	$t\partial_t - u\partial_u, x\partial_x - m_6\alpha\partial_\alpha + u\partial_u$
$\rho \left[1 + \frac{1}{u(u\alpha^{1/c_7}\exp[\rho K] - 1)} - 1 \right]$	$c_7 \neq 0$	$u\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x + c_7\alpha\partial_\alpha - u\partial_u,$ $e^{(t+u)}[(u+1)\partial_t + u^2\partial_x - u\partial_u]$
$\frac{\rho u}{u+1}$	$c_7 = 0$	$u\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x - u\partial_u,$ $e^{(t+u)}[(u+1)\partial_t + u^2\partial_x - u\partial_u]$

Table A.4. (*Continued*)

R	Condition on consts.	Operators
$\frac{\rho}{1 - \alpha^{1/c_7} [\rho(c_7 + m_7)u\alpha^{1/c_7}]^{-\frac{2c_7+m_7}{c_7+m_7}} \exp[\rho K]}$	$c_7 \neq 0, -m_7$	$u\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x + c_7\alpha\partial_\alpha - u\partial_u,$ $(t + 2u)\partial_t + u^2\partial_x - m_7\alpha\partial_\alpha - u\partial_u$
$\frac{\rho u}{u - \alpha^{-1/m_7} \exp[\rho K]}$	$c_7 = 0$	$u\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x - u\partial_u,$
$u(\rho + Ku\alpha^{1/c_8})$	$m_7 \neq 0$	$(t + 2u)\partial_t + u^2\partial_x - m_7\alpha\partial_\alpha - u\partial_u$
$K(c_8 + m_8)u[(c_8 + m_8)u\alpha^{1/c_8}]^{\frac{c_8}{c_8+m_8}}$	$c_8 \neq 0$	$x\partial_x - c_8\alpha\partial_\alpha + u\partial_u, e^t(\partial_t - u\partial_u)$
$Ku\alpha^{1/m_8}$	$c_8 \neq 0, -m_8$	$x\partial_x - c_8\alpha\partial_\alpha + u\partial_u, t\partial_t - m_8\alpha\partial_\alpha - u\partial_u$
$\frac{2\rho u^3}{2\rho K + u^2}$	$c_8 = 0, m_8 \neq 0$	$x\partial_x + u\partial_u, t\partial_t - m_8\alpha\partial_\alpha - u\partial_u$
$\frac{\rho p_1 u^3}{\rho p_1 K - \ln[u\alpha]}$	$p_1 \neq 0$	$(x + u)\partial_t + \frac{u^2}{2}\partial_x + u\alpha\partial_\alpha - u^2\partial_u, \alpha\partial_\alpha$
$\frac{2\rho p_2 u^3}{p_2(2\rho K + u^2) - 2\ln[u\alpha]}$	$p_2 \neq 0$	$x\partial_t + u\alpha\partial_\alpha - u^2\partial_u, \frac{1}{2u^2}\partial_t + \frac{1}{u}\partial_x - p_1\alpha\partial_\alpha$
$\frac{\rho p_3 u}{c_9(K - \ln[u\alpha^{-1/c_9}]) + p_3 u}$	$p_2 \neq 0$	$(x + u)\partial_t + \frac{u^2}{2}\partial_x + u\alpha\partial_\alpha - u^2\partial_u,$ $\frac{1}{2u^2}\partial_t + \frac{1}{u}\partial_x - p_2\alpha\partial_\alpha$
$\frac{\rho p_3 u}{\ln \alpha - p_3(\rho K + u)}$	$c_9, p_3 \neq 0$	$u\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x + c_9\alpha\partial_\alpha + u\partial_u,$ $\ln u\partial_t + u\partial_x - p_3\alpha\partial_\alpha$
$\frac{2\rho u^3}{1 + 2\rho K u^3 \alpha^3 [(1 - p_4)u\alpha]^{\frac{1-3p_4}{p_4-1}}}$	$c_9 = 0$	$u\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x + u\partial_u,$
$2\rho u^3$	$p_3 \neq 0$	$\ln u\partial_t + u\partial_x - p_3\alpha\partial_\alpha$
$K\alpha^{-3}[(1 + p_5)u\alpha]^{\frac{1+3p_5}{1+p_5}}$	$p_4 \neq 1$	$x\partial_t + u\alpha\partial_\alpha - u^2\partial_u,$ $\frac{1}{2u^2}\partial_t + \left(\frac{1}{u} - x\right)\partial_x + p_4\alpha\partial_\alpha - u\partial_u$
$\frac{2\rho u}{1 + 2\rho K u \alpha^3 [(1 + p_6)u\alpha]^{-\frac{1+3p_6}{1+p_6}}}$	$p_4 = 1$	$x\partial_t + u\alpha\partial_\alpha - u^2\partial_u,$ $\frac{1}{2u^2}\partial_t + \left(\frac{1}{u} - x\right)\partial_x + \alpha\partial_\alpha - u\partial_u$
$\frac{\rho u}{p_7 - u + (u\alpha^{-1/c_9})^{-\frac{2c_9}{1+c_9}} \exp[\frac{2Kc_9}{1+c_9}]}$	$p_5 \neq -1$	$x\partial_t + u\alpha\partial_\alpha - u^2\partial_u, x\partial_x + p_5\alpha\partial_\alpha + u\partial_u$
	$p_6 \neq -1$	$(x + u)\partial_t + \frac{u^2}{2}\partial_x + u\alpha\partial_\alpha - u^2\partial_u,$ $x\partial_x + p_6\alpha\partial_\alpha + u\partial_u$
	$c_9 \neq 0, -1$	$u\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x + c_9\alpha\partial_\alpha + u\partial_u,$

Table A.4. (Continued)

R	Condition on consts.	Operators
$\frac{\rho u}{p_7 - u}$	$c_9 = -1$	$\left(x + 2p_7u - \frac{3}{2}u^2\right)\partial_t + (p_7u^2 - u^3)\partial_x$ $+ u\alpha\partial_\alpha - u^2\partial_u$ $u\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x - \alpha\partial_\alpha + u\partial_u,$
$\frac{\rho u}{p_7 - u + \alpha^2 \exp[\rho K]}$	$c_9 = 0$	$\left(x + 2p_7u - \frac{3}{2}u^2\right)\partial_t + (p_7u^2 - u^3)\partial_x$ $+ u\alpha\partial_\alpha - u^2\partial_u$ $u\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x + u\partial_u,$ $\left(x + 2p_7u - \frac{3}{2}u^2\right)\partial_t + (p_7u^2 - u^3)\partial_x$ $+ u\alpha\partial_\alpha - u^2\partial_u$
$\frac{\rho u(1 + c_{10})}{p_8(1 + c_{10}) + \rho K[(1 + c_{10})u\alpha^{-1/c_{10}}]^{-\frac{2c_{10}}{1+c_{10}}}}$	$c_{10} \neq 0, -1$	$x\partial_x + c_{10}\alpha\partial_\alpha + u\partial_u,$
$\rho u/p_8$	$c_{10} = -1$	$(x + 2p_8u)\partial_t + p_8u^2\partial_x + u\alpha\partial_\alpha - u^2\partial_u$ $x\partial_x - \alpha\partial_\alpha + u\partial_u,$
$\frac{\rho}{p_8 - \alpha^2 \exp[\rho K]}$	$p_8 \neq 0$	$(x + 2p_8u)\partial_t + p_8u^2\partial_x + u\alpha\partial_\alpha - u^2\partial_u$
	$c_{10} = 0$	$x\partial_x + u\partial_u,$
$K\alpha^{-3}[(b - p_9)u\alpha]^{\frac{1+2b-3p_9}{p_4-b}}$	$p_9 \neq b$	$(x + 2p_8u)\partial_t + p_8u^2\partial_x + u\alpha\partial_\alpha - u^2\partial_u$ $x\partial_t + u\alpha\partial_\alpha - u^2\partial_u,$ $(b + 1)t\partial_t + x\partial_x + p_9\alpha\partial_\alpha - bu\partial_u$
$\frac{2\rho u}{1 + 2\rho Ku\alpha^3[(b - p_{10})u\alpha]^{-\frac{1+2b-3p_{10}}{b-p_{10}}}}$	$p_{10} \neq b$	$(x + u)\partial_t + \frac{u^2}{2}\partial_x + u\alpha\partial_\alpha - u^2\partial_u,$ $(b + 1)\left(t + \frac{1}{2}\ln u\right)\partial_t + \left[x + (b + 1)\frac{u}{2}\right]\partial_x$ $+ p_{10}\alpha\partial_\alpha - bu\partial_u$
$2\rho u$	$p_{10} = b$	$(x + u)\partial_t + \frac{u^2}{2}\partial_x + u\alpha\partial_\alpha - u^2\partial_u,$ $(b + 1)\left(t + \frac{1}{2}\ln u\right)\partial_t + \left[x + (b + 1)\frac{u}{2}\right]\partial_x$ $+ b\alpha\partial_\alpha - bu\partial_u,$ $\left(t + \frac{\ln u}{2}\right)\partial_t + \left(x + \frac{u}{2}\right)\partial_x, x\partial_x - \alpha\partial_\alpha + u\partial_u$
$Ku^{\frac{3p_{11}-1}{p_{11}}}\alpha^{-1/p_{11}}$	$p_{11} \neq 0$	$x\partial_t + u\alpha\partial_\alpha - u^2\partial_u, t\partial_t + x\partial_x + p_{11}\alpha\partial_\alpha$
$\frac{2\rho u^3}{u^2 + 2\rho K(u\alpha)^{1/p_{12}}}$	$p_{12} \neq 0$	$(x + u)\partial_t + \frac{u^2}{2}\partial_x + u\alpha\partial_\alpha - u^2\partial_u,$ $\left(t + \frac{\ln u}{2}\right)\partial_t + \left(x + \frac{u}{2}\right)\partial_x + p_{12}\alpha\partial_\alpha$

Table A.4. (*Continued*)

R	Condition on consts.	Operators
$\frac{2\rho q_1 u^3}{e^{1/u}[2(\ln[u\alpha] - \frac{c_{11}}{u} + \rho K c_{11}^3 q_1) - {}^1\text{Ei} q_1] + q_1(u-1)u}$	$c_{11}, q_1 \neq 0$	$(t+x+u)\partial_t + \left(x + \frac{u^2}{2}\right)x\partial_x$ $+ (c_{11}+u)\alpha\partial_\alpha - u^2\partial_u,$ $e^{\frac{1}{u}}\left(\frac{1}{u}-1\right)\partial_t + e^{\frac{1}{u}}\partial_x + q_1\alpha\partial_\alpha$
$\frac{2\rho u^3}{u^2 - u + e^{1/u}(2\rho K c_{11}^3 - {}^1\text{Ei})}$		$(t+x+u)\partial_t + \left(x + \frac{u^2}{2}\right)x\partial_x$ $+ (c_{11}+u)\alpha\partial_\alpha - u^2\partial_u, \alpha\partial_\alpha$
$\frac{2\rho u^3}{u^2 - u - {}^1\text{Ei}e^{1/u}}$	$c_{11} = 0$	$(t+x+u)\partial_t + \left(x + \frac{u^2}{2}\right)x\partial_x$ $+ u\alpha\partial_\alpha - u^2\partial_u, \alpha\partial_\alpha$
$\frac{81\rho q_2 u^4 e^{-1/u}}{1 + (81\ln[u\alpha] - 3\rho q_2 K)u}$	$q_2 \neq 0$	$(t+x)\partial_t + x\partial_x + \left(u + \frac{1}{3}\right)\alpha\partial_\alpha - u^2\partial_u,$ $e^{\frac{1}{u}}\left(\frac{1}{u}-1\right)\partial_t + e^{\frac{1}{u}}\partial_x + q_2\alpha\partial_\alpha$
$27Ku^3 e^{-1/u}$		$(t+x)\partial_t + x\partial_x + \left(u + \frac{1}{3}\right)\alpha\partial_\alpha - u^2\partial_u, \alpha\partial_\alpha$
$\frac{\rho q_3 u^3 e^{-1/u}}{\ln[u\alpha] + \rho q_3 K}$	$q_3 \neq 0$	$(t+x)\partial_t + x\partial_x + u\alpha\partial_\alpha - u^2\partial_u,$ $e^{\frac{1}{u}}\left(\frac{1}{u}-1\right)\partial_t + e^{\frac{1}{u}}\partial_x + q_3\alpha\partial_\alpha$
$Ku^3 e^{-1/u}$		$(t+x)\partial_t + x\partial_x + u\alpha\partial_\alpha - u^2\partial_u, \alpha\partial_\alpha$
$\frac{\rho q_4}{q_4 - e^u(K + \ln \alpha - c_{12}u)}$	$c_{12}, q_4 \neq 0$	$(t+u)\partial_t + \left(t+x + \frac{u^2}{2}\right)x\partial_x + c_{12}\alpha\partial_\alpha + \partial_u,$ $e^{\frac{1}{u}}\left(\frac{1}{u}-1\right)\partial_t + e^{\frac{1}{u}}\partial_x + q_4\alpha\partial_\alpha$
$\frac{\rho q_4}{q_4 - e^u(\ln \alpha + \rho q_4 K)}$	$c_{12} = 0$	$(t+u)\partial_t + \left(t+x + \frac{u^2}{2}\right)x\partial_x + \partial_u,$
	$q_4 \neq 0$	$e^u\partial_t + e^u(u-1)\partial_x + q_4\alpha\partial_\alpha$
$\frac{\rho}{1 - \exp[u + \rho K]}$	$c_{12} = 0$	$(t+u)\partial_t + \left(t+x + \frac{u^2}{2}\right)x\partial_x + \partial_u, \alpha\partial_\alpha$
$\frac{\rho q_5 e^{-u}}{c_{13}u - \ln \alpha + \rho q_5 K}$	$c_{13}, q_5 \neq 0$	$t\partial_t + (t+x)\partial_x + c_{13}\alpha\partial_\alpha + \partial_u,$ $e^u\partial_t + e^u(u-1)\partial_x + q_5\alpha\partial_\alpha$
$-\frac{\rho q_5 e^{-u}}{\rho q_5 K + \ln \alpha}$	$c_{13} = 0$	$t\partial_t + (t+x)\partial_x + \partial_u, e^u\partial_t + e^u(u-1)\partial_x + q_5\alpha\partial_\alpha$
Ke^{-u}	$c_{13} = 0$	$t\partial_t + (t+x)\partial_x + \partial_u, \alpha\partial_\alpha$

Table A.4. (Continued)

R	Condition on consts.	Operators
$\frac{\rho}{1 - u^{\frac{1-3c_{14}}{c_{14}}} \alpha^{1/c_{14}} \exp[\rho K - \frac{r_1}{c_{14}u}]}$	$c_{14} \neq 0$	$(t+u)\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x + c_{14}\alpha\partial_\alpha,$ $\left(x + \frac{3}{2}u^2\right)\partial_t + u^3\partial_x + (r_1+u)\alpha\partial_\alpha - u^2\partial_u$
$\frac{\rho}{1 - (1+u^2)^{\frac{1-3c_{14}}{2c_{14}}} \alpha^{1/c_{14}} \times \exp[\frac{r_2 \arctan u}{c_{14}} + \rho K]}$	$c_{14} \neq 0$	$(t+u)\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x + c_{14}\alpha\partial_\alpha,$ $\left(x + 1 + \frac{3}{2}u^2\right)\partial_t + (u^3 - t)\partial_x$ $+ (r_2+u)\alpha\partial_\alpha - (1+u^2)\partial_u$
$Ku^{\frac{3c_{15}-1}{c_{15}}} \alpha^{-1/c_{15}} \exp[\frac{r_3}{c_{15}u}]$	$c_{15} \neq 0$	$t\partial_t + x\partial_x + c_{15}\alpha\partial_\alpha, x\partial_t + (r_3+u)\alpha\partial_\alpha - u^2\partial_u$
$K\alpha^{-1/c_{15}} \exp[\frac{r_4 u}{c_{15}}]$	$c_{15} \neq 0$	$t\partial_t + x\partial_x + c_{15}\alpha\partial_\alpha, t\partial_x + r_4\alpha\partial_\alpha + \partial_u$
$\frac{K(1+u^2)^{\frac{3c_{15}-1}{2c_{15}}}}{\alpha^{-1/c_{15}} \exp[\frac{r_5 \arctan u}{c_{15}}]}$	$c_{15} \neq 0$	$t\partial_t + x\partial_x + c_{15}\alpha\partial_\alpha,$ $x\partial_t - t\partial_x + (r_5+u)\alpha\partial_\alpha - (1+u^2)\partial_u$
$\frac{\rho u(2 + \ln u)}{t+u}$	$c_{16} = 0$	$(t+u)\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x - 2u\partial_u, e^{(t+\frac{u}{2})} u^{t/2} \alpha\partial_\alpha$
$\frac{3\rho u^{3/2}}{u^{3/2} + 3s_1 - K[2(c_{16} + 2s_2)u\alpha^{2/c_{16}}]^{-\frac{c_{16}}{2c_{16}+4s_2}}}$	$c_{16} \neq 0, -2s_2$	$(t+u)\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x + c_{16}\alpha\partial_\alpha - 2u\partial_u,$ $\left(\frac{s_1}{\sqrt{u}} - \frac{2}{3}u - t\right)\partial_t - \left(\frac{1}{3}u^2 + s_1\sqrt{u}\right)\partial_x$ $+ s_2\alpha\partial_\alpha + u\partial_u$
$\frac{3\rho u^{3/2}}{u^{3/2} - 3s_1 + \alpha^{-1/2s_2} \exp[3\rho K]}$	$c_{16} = 0$	$(t+u)\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x - 2u\partial_u,$
	$s_2 \neq 0$	$\left(\frac{s_1}{\sqrt{u}} - \frac{2}{3}u - t\right)\partial_t - \left(\frac{1}{3}u^2 + s_1\sqrt{u}\right)\partial_x$ $+ s_2\alpha\partial_\alpha + u\partial_u$
$\frac{3\rho u^{3/2}}{u^{3/2} - 3s_1}$	$c_{16}, s_2 = 0$	$(t+u)\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x - 2u\partial_u,$ $\left(\frac{s_1}{\sqrt{u}} - \frac{2}{3}u - t\right)\partial_t - \left(\frac{1}{3}u^2 + s_1\sqrt{u}\right)\partial_x + u\partial_u$
$\frac{12\rho u^{3/2}}{4s_3 u^{3/2} - 3(K - c_{16} \ln[u\alpha^{2/c_{16}}])}$	$c_{16} \neq 0$	$(t+u)\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x + c_{16}\alpha\partial_\alpha - 2u\partial_u,$ $\frac{1}{\sqrt{u}}\partial_t - \sqrt{u}\partial_x + s_3\alpha\partial_\alpha$
$\frac{6\rho s_3 u^{3/2}}{2s_3(u^{3/2} - 3\rho K) - \ln \alpha}$	$c_{16} = 0$	$(t+u)\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x - 2u\partial_u,$
	$s_3 \neq 0$	$\frac{1}{\sqrt{u}}\partial_t - \sqrt{u}\partial_x + s_3\alpha\partial_\alpha$

Table A.4. (*Continued*)

R	Condition on consts.	Operators
$\frac{\rho\sqrt{u}\alpha^{-2/c_{17}}[(c_{17}+2s_5)u\alpha^{2/c_{17}}]^{\frac{2c_{17}+3s_5}{c_{17}+2s_5}}}{\rho K\sqrt{u}\alpha^{1/c_{17}}+s_4(c_{17}+2s_5)} \times [(c_{17}+2s_5)u\alpha^{2/c_{17}}]^{\frac{c_{17}+s_5}{c_{17}+2s_5}}$	$c_{17} \neq 0, -2s_5$	$t\partial_t - x\partial_x + c_{17}\alpha\partial_\alpha - 2u\partial_u,$ $\left(\frac{s_4}{\sqrt{u}} - t\right)\partial_t - s_4\sqrt{u}\partial_x + s_5\alpha\partial_\alpha + u\partial_u$
$\frac{\rho u^{3/2}\alpha^{1/2s_5}}{s_4\alpha^{1/2s_4} - \exp[\rho K]}$	$c_{17} = 0$	$t\partial_t - x\partial_x - 2u\partial_u,$
	$s_5 \neq 0$	$\left(\frac{s_4}{\sqrt{u}} - t\right)\partial_t - s_4\sqrt{u}\partial_x + s_5\alpha\partial_\alpha + u\partial_u$
$\frac{\rho}{s_4}u^{3/2}$	$c_{17}, s_5 = 0, s_4 \neq 0$	$t\partial_t - x\partial_x - 2u\partial_u, \left(\frac{s_4}{\sqrt{u}} - t\right)\partial_t$ $- s_4\sqrt{u}\partial_x + u\partial_u$
$\frac{4\rho s_6 u^{3/2}}{c_{17}\ln[u\alpha^{2/c_{17}}] + 4K\rho s_6}$	$c_{17}, s_6 \neq 0$	$t\partial_t - x\partial_x + c_{17}\alpha\partial_\alpha - 2u\partial_u, \frac{1}{\sqrt{u}}\partial_t$ $- \sqrt{u}\partial_x + s_6\alpha\partial_\alpha$
$\frac{2\rho s_6 u^{3/2}}{\ln \alpha - 2\rho s_6 K}$	$c_{17} = 0, s_6 \neq 0$	$t\partial_t - x\partial_x - 2u\partial_u, \frac{1}{\sqrt{u}}\partial_t - \sqrt{u}\partial_x + s_6\alpha\partial_\alpha$
$Ku^{3/2}$	$c_{17} = 0$	$t\partial_t - x\partial_x - 2u\partial_u, \alpha\partial_\alpha$
$\frac{\rho(a-2)}{u^{\frac{2-a}{a-1}}\exp[\rho(2-a)K]} - 1$	$c_{18} \neq 0$	$(t+u)\partial_t + \left(ax + \frac{u^2}{2}\right)\partial_x$ $+ c_{18}\alpha\partial_\alpha + (a-1)u\partial_u, \alpha\partial_\alpha$
$\frac{\rho y_1(a-1)^2 u^{\frac{a-2}{a-1}}}{\rho y_1 K(a-1)^2 + ac_{19}\ln[u\alpha^{\frac{1-a}{c_{19}}}]}$	$c_{19}, y_1 \neq 0$	$t\partial_t + ax\partial_x + c_{19}\alpha\partial_\alpha + (a-1)u\partial_u,$ $au^{\frac{1}{a-1}}\partial_t + u^{\frac{a}{a-1}}\partial_x + y_1\alpha\partial_\alpha$
$-\frac{\rho y_1(a-1)u^{\frac{a-2}{a-1}}}{\rho y_1 K(a-1) + a\ln[\rho y_1(1-a)\alpha]}$	$c_{19} = 0, y_1 \neq 0$	$t\partial_t + ax\partial_x + (a-1)u\partial_u, au^{\frac{1}{a-1}}\partial_t$ $+ u^{\frac{a}{a-1}}\partial_x + y_1\alpha\partial_\alpha$
$Ku^{\frac{a-2}{a-1}}$	$c_{19} = 0$	$t\partial_t + ax\partial_x + (a-1)u\partial_u, \alpha\partial_\alpha$
$\frac{\rho(a-2)}{y_2}u^{\frac{a-2}{a-1}}$	$c_{19} = 0,$	$t\partial_t + ax\partial_x + (a-1)u\partial_u,$
	$y_2 \neq 0$	$\left(\frac{y_2}{2-a}\right)u^{\frac{2-a}{a-1}}\partial_t + (t+y_2u^{\frac{1}{a-1}})\partial_x + \partial_u$
$\frac{\rho(2a-1)}{1 - [\rho(1-a)u]^{\frac{2a-1}{1-a}}\exp[\rho(2a-1)K]}$	$c_{20} \neq 0$	$(at+u)\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x + c_{20}\alpha\partial_\alpha$ $+ (1-a)u\partial_u, \alpha\partial_\alpha$
$K[(1-a)u]^{\frac{2a-1}{a-1}}$	$c_{21} = 0$	$at\partial_t + x\partial_x + (1-a)u\partial_u, \alpha\partial_\alpha$
$\frac{2\rho z_1(1+u^2)^{3/2}}{z_1(2\rho K + \operatorname{arcsinh} u + u\sqrt{1+u^2}) - 2c_{22}\operatorname{arctan} u - \ln[\alpha^2(1+u^2)]}$	$c_{22}, z_1 \neq 0$	$(x+u)\partial_t + \left(\frac{u^2}{2} - t\right)\partial_x + (c_{22}+u)\alpha\partial_\alpha$

Table A.4. (Continued)

R	Condition on consts.	Operators
		$-(1+u^2)\partial_u, \frac{u}{\sqrt{1+u^2}}\partial_t - \frac{1}{\sqrt{1+u^2}}\partial_x + z_1\alpha\partial_\alpha$
$-\frac{2\rho z_2(1+u^2)^{3/2}}{2c_{23}\arctan u + \ln[\alpha^2(1+u^2)] + 2\rho z_2 K}$	$c_{23} \neq 0$	$x\partial_t - t\partial_x + (c_{23}+u)\alpha\partial_\alpha - (1+u^2)\partial_u,$ $\frac{u}{\sqrt{1+u^2}}\partial_t - \frac{1}{\sqrt{1+u^2}}\partial_x + z_2\alpha\partial_\alpha$
$\frac{\rho(1+u^2)^{3/2}}{z_3 - (\alpha\sqrt{1+u^2})^{1/z_4} \exp[\rho K + \frac{c_{23}\arctan u}{z_4}]}$	$c_{23} \neq 0$	$x\partial_t - t\partial_x + (c_{23}+u)\alpha\partial_\alpha - (1+u^2)\partial_u,$ $\left(t + \frac{z_3 u}{\sqrt{1+u^2}}\right)\partial_t + \left(x - \frac{z_3}{\sqrt{1+u^2}}\right)\partial_x + z_4\alpha\partial_\alpha$
$\frac{\rho}{z_3}(1+u^2)^{3/2}$	$z_3 \neq 0$	$x\partial_t - t\partial_x + u\alpha\partial_\alpha - (1+u^2)\partial_u,$
	$z_4 = 0$	$\left(t + \frac{z_3 u}{\sqrt{1+u^2}}\right)\partial_t + \left(x - \frac{z_3}{\sqrt{1+u^2}}\right)\partial_x$
$K(1-au+u^2)^{3/2} \exp\left[\frac{a\arctan[\frac{2u-a}{\sqrt{4-a^2}}]}{\sqrt{4-a^2}}\right]$		$(at+x)\partial_t + (ax-t)\partial_x + u\alpha\partial_\alpha + (au-u^2-1)\partial_u,$ $\alpha\partial_\alpha$

References

- [1] T. Aittollio, M. Gyllenberg and O. Polo, A model of a snorer's upper airway, *Math. Biosci.* **170** (2001) 79–90.
- [2] W. F. Ames, *Non-Linear Partial Differential Equations in Engineering* (Academic, New York, Vol. I-1965; Vol. II-1972).
- [3] P. Basarab–Horwath, V. Lahno and R. Zhdanov, The structure of Lie algebras and the classification problem for partial differential equations, *Acta Applicandae Mathematicae* **69** (2001) 43–94.
- [4] C. D. Bertram, Flow-induced oscillation of collapsed tubes and airway structures, *Respiratory Physiology & Neurobiology* **163** (2008) 256–265.
- [5] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations* (Springer, New York, 1989).
- [6] B. S. Brook and T. J. Pedley, A model for time-dependent flow in (giraffe jugular) veins: uniform tube properties, *J. Biomech.* **35** (2002) 95–107.
- [7] P. Carbonaro, Group analysis for the equations describing the one-dimensional motion of an ideal gas in the hodograph plane, *Int. J. Non-Linear Mech.* **32** (1997) 455–464.
- [8] J. M. Díaz, “Short guide to **YaLie**: Yet another Lie *Mathematica* package for Lie symmetries”, <http://library.wolfram.com/infocenter/MathSource/4231/YaLie.ps>
- [9] J.-M. Fullana *et al.*, Filling a collapsible tube, *J. Fluid Mech.* **494** (2003) 285–296.
- [10] J. B. Grotberg and O. E. Jensen, Biofluid mechanics in flexible tubes, *Annu. Rev. Fluid Mech.* **36** (2004) 121–147.
- [11] N. H. Ibragimov, M. Torissi and A. Valenti, Preliminary group classification of equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$, *J. Math. Phys.* **32** (1991) 2988–2995.
- [12] N. H. Ibragimov and M. Torissi, A method for group analysis and its application to a model of detonation, *J. Math. Phys.* **33** (1992) 3931–3937.

- [13] N. H. Ibragimov and M. Torissi, Equivalence groups for balance equations, *J. Math. Analysis Appl.* **184** (1994) 441–452.
- [14] N. H. Ibragimov (Ed.), *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. 2 (CRC Press, Boca Raton, 1995).
- [15] N. H. Ibragimov and N. Säfström, The equivalence group and invariant solutions of a tumour growth model, *Comm. Nonlin. Sci. Num. Simul.* **9** (2004) 61–68.
- [16] O. E. Jensen, Flow through deformable airways, Centre for Mathematical Medicine, School of Mathematical Sciences, University of Nottingham (2002).
- [17] J. E. A. John, *Gas Dynamics* (Pearson Prentice Hall, New Jersey, 2006).
- [18] X. Y. Luo and T. J. Pedley, Multiple solutions and flow limitation in collapsible channel flows, *J. Fluid Mech.* **420** (2000) 301–324.
- [19] F. M. Mahomed and P. G. L. Leach, Lie algebras associated with second-order ordinary differential equations, *J. Math. Phys.* **30** (1989) 2770–2777.
- [20] A. Marzo and X. Y. Luo, Numerical simulation of three dimensional flows through a collapsible tube, Summer Bioengineering Conference, Soneta Beach Resort in Key Biscayne, Florida (2003).
- [21] S. V. Meleshko, Group classification of two-dimensional stable viscous gas dynamics equations with arbitrary state equations, *J. Phys. A: Math. Gen.* **35** (2002) 3515–3533.
- [22] M. Molati and C. Wafo Soh, Similarity reduction of energy-transport models for semiconductors, *Math. Comput. Appl.* **10** (2005) 221–230.
- [23] P. Morgan and K. H. Parker, A mathematical model of flow through a collapsible tube-I. Model and steady flow results, *J. Biomechanics* **22** (1989) 1263–1270.
- [24] S. Murata, Non-classical symmetry and Riemann invariants, *Int. J. Non-Linear Mech.* **41** (2006) 242–246.
- [25] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York, 1986).
- [26] L. V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic, New York, 1982).
- [27] J. Patera and P. Winternitz, Subalgebras of real three- and four-dimensional Lie algebras, *J. Math. Phys.* **18** (1977) 1449–1455.
- [28] A. H. Shapiro, Steady flow in collapsible tubes, *ASME J. Biomech. Engineering* **99** (1977) 126–147.
- [29] J. J. Stoker, *Water Waves: Mathematical Theory with Applications* (Interscience, New York, 1957).
- [30] C. Wafo Soh, Incompressible laminar 2D steady thermal boundary layers with temperature-dependent kinematic viscosity and thermal diffusivity, *Int. J. Non-Linear Mech.* **38** (2003) 991–997.