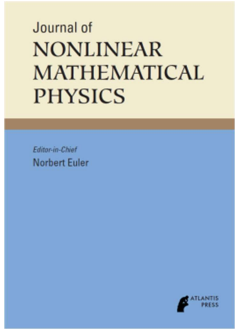

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A *SYMMETRY INVARIANCE* ANALYSIS OF THE MULTIPLIERS & CONSERVATION LAWS OF THE JAULENT–MIODEK AND SOME FAMILIES OF SYSTEMS OF KdV TYPE EQUATIONS

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In this paper, we study and classify the conservation laws of the Jaulent–Miodek equations and other systems of KdV type equations which arises in, inter alia, shallow water equations. The main focus of the paper is the construction of the conservation laws as a consequence of the interplay between symmetry generators and "multipliers" (integration factors), particularly, the higher-order ones and the significance of the Euler operators for systems of equations.

Keywords: Symmetry invariance; Jaulent–Miodek equations; multipliers.

1. Introduction and Background

The evolutionary Jaulent–Miodek system of equations

$$\begin{aligned} &u_t + u_{xxx} + \frac{3}{2}vv_{xxx} + \frac{9}{2}v_xv_{xx} - 6uu_x - 6uvv_x - \frac{3}{2}v^2u_x = 0, \\ &v_t + v_{xxx} - 6vu_x - 6uv_x - \frac{15}{2}v^2v_x = 0 \end{aligned} \tag{1.1}$$

arises from the study of the "Inverse Scattering Transform" on a class of nonlinear equations associated with the inverse problem for the one-dimensional Schrödinger equation [1]. It has been analyzed via a Hamiltonian formulation [2], Bäcklund transformation [3], He's iteration and Adomian Decomposition [4] and Homotopy Analysis Methods, inter alia. It has been a system of great interest both, numerically and analytically. It is clear that finding the conserved quantities of (1.1) is not a trivial task. Determining a Lagrangian, if it exists, would in itself be a nontrivial undertaking.

The use of symmetry properties of a given system of partial differential equations (pdes) to construct or generate new conservation laws from known conservation laws has been investigated [5, 6]. Furthermore, the role of "multipliers" has been been shown to play a

significant role in the construction of conserved densities and flux [7] and in determining hierarchies. In short, a knowledge of a multiplier, by formula, leads to a conserved flow. Here, we apply the notion of **symmetry invariance properties of multipliers** to construct a large class of conserved flows as opposed to using variational techniques or some cumbersome standard methods, particularly, in finding the higher-order multipliers.

Consider an rth-order system of partial differential equations of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$

$$G^\mu(x, u, u_{(1)}, \dots, u_{(r)}) = 0, \quad \mu = 1, \dots, \tilde{m}, \tag{1.2}$$

where $u_{(1)}, u_{(2)}, \dots, u_{(r)}$ denote the collections of all first, second, $\dots$, rth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \dots$ respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n, \tag{1.3}$$

where the summation convention is used whenever appropriate.

A current $\Phi = (\Phi^1, \dots, \Phi^n)$ is conserved if it satisfies

$$D_i \Phi^i = 0 \tag{1.4}$$

along the solutions of (1.2).

It can be shown that every admitted conservation law arises from *multipliers* $Q_\mu(x, u, u_{(1)}, \dots)$ such that

$$Q_\mu G^\mu = D_i \Phi^i \tag{1.5}$$

holds identically (i.e., off the solution space) for some current Φ^i "modulo a curl". When the PDE system is variational, multipliers are variational symmetries. There is a determining system for finding multipliers (and hence conservation laws) for any given PDE system. Then, the conserved density is determined by, amongst other formulae, a homotopy formula.

Let $\hat{X} = P^\mu(x, u, u_{(1)}, \dots)\partial_{u^\mu}$ be a symmetry in characteristic form. So $pr\hat{X}G^\mu = 0$ on the solution space. Then off the solution space $pr\hat{X}G^\mu = \hat{R}(G^\mu)$ holds for some linear differential operator $\hat{R}$.

In what follows, we will be making wide use of the following result.

Proposition 1.1. *If Φ^i is a conserved current with multiplier Q_μ then $\Phi^i_X := pr\hat{X}\Phi^i$ is also a conserved current and has multiplier $Q^X_\mu := Q'_\mu(P) + \hat{R}^*(Q_\mu)$ where $\hat{R}^*$ is the adjoint of the operator $\hat{R}$. In the case of a point symmetry, this becomes $\Phi^i_X = prX\Phi^i + 2\Phi^{[i}D_j\xi^{j]}$ modulo curls and $Q^X_\mu = prXQ_\mu + Q_\mu D_i\xi^i + R^*(Q_\mu)$ where $R = \hat{R} + \xi^i D_i$, i.e., $prXG^\mu = R(G^\mu)$.*

The above proposition or versions of it are well known, for e.g., see [5] wherein the proof is done using *differential forms*. However, the action of X on Q is not — the proof amounts to applying $pr\hat{X}$ to the multiplier equation.

In addition to the Jaulent–Miodek equations, we discuss a number of scalar and vector examples whose conservation laws are interesting and previously not determined. These could serve as illustrative examples.

2. Applications — Scalar Examples

In this section, we consider a variety of pdes of interest. This section will not only serve to illustrate the method but also to construct previously unknown conserved flows of the pdes in question.

2.1. *Example 1*

For weakly nonlinear waves (with weak nonlinearity), the progressive is described by the KdV equation ([8])

$$u_t - \alpha u_x - \beta u u_x - \gamma u_{xxx} = 0 \tag{2.1}$$

where α, β and γ are constants which take on certain forms for specific fluids like two-layer fluids (and may be functions of u in the case of *strongly* nonlinear waves). In addition to translations in x and t, it has two point symmetries

$$X = \left(\alpha t - \frac{1}{2}x\right)\partial_x - \frac{3}{2}\partial_t + u\partial_u, \quad Y = -\beta t\partial_x + \partial_u.$$

The conserved vector $(\Phi^1, \Phi^2) = (\Phi^x, \Phi^t)$, as described above, satisfies (1.4) along the solutions of (2.1). Moreover, we have

$$\frac{\delta}{\delta u}(Q(u_t - \alpha u_x - \beta u u_x - \gamma u_{xxx})) = 0. \tag{2.2}$$

If we suppose $Q = f(x, t, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$, the tedious calculations after expansion and separation by monimials reveal that f is of order zero or two in derivatives with respect to u. We list these as

$$\begin{aligned} f_1 &= 1 \\ f_2 &= u \\ f_3 &= x + \alpha t + \beta t u \\ f_4 &= \beta u^2 + 2\gamma u_{xx} \end{aligned} \tag{2.3}$$

and another two second order ones given by

$$f_5 = \frac{1}{216\gamma}\left(\gamma(144uu_{xx} - 36\beta u_x^2 + 216u_{xt}) + 108\beta u^2\left(\alpha + \frac{10}{9}\beta u\right)\right) \tag{2.4}$$

and

$$\begin{aligned} f_6 = \frac{1}{216\gamma}\Big\{&108\beta\gamma^2 u_{xx}^2 + [\beta^2(132u^2u_{xx} - 12uu_x^2) + \beta\gamma(-72u_xu_t + 72uu_{xt} \\ &+ 216\alpha uu_{xx} + 216u_{tt}] + 108\beta u^2\left(\frac{35}{108}\beta^2u^2 + 10\beta u + \alpha^2\right)\Big\}. \end{aligned} \tag{2.5}$$

A symmetry study of the multipliers in (2.3) is done using X and Y, i.e.,

$$Xf_1 = 0, \quad Xf_2 = f_2, \quad Xf_3 = -\frac{1}{2}f_3, \quad Xf_4 = 2f_4 \tag{2.6}$$

so that f_1 is strictly invariant under X but $f_2, \ldots, f_4$ are ray invariant under X. Also,

$$Yf_1 = 0, \quad Yf_2 = f_1, \quad Yf_3 = 0, \quad Yf_4 = 2\beta f_2. \tag{2.7}$$

Thus f_1 and f_3 are strictly invariant under Y and f_2 and f_4 are not invariant under Y but rather **the action of Y on these leads to another multiplier**.

Notes. Invariance of a multiplier under a symmetry implies association of the symmetry with the corresponding conservation law. One may generate new multipliers by the action of a symmetry on a known multiplier.

The corresponding conserved flows for those in (2.3) are

$$\begin{array}{ll}
\Phi^t, & \Phi^x \\
u, & -\gamma u_{xx} - \alpha u - \frac{1}{2}\beta u^2 \\
\frac{1}{2}u^2, & -\gamma u u_{xx} + \frac{1}{2}\gamma u_x^2 - \frac{1}{2}\alpha u^2 - \frac{1}{3}\beta u^3 \\
\alpha t u + \frac{1}{2}\beta t u^2 + xu, & -\alpha\gamma t u x x - \beta\gamma t u u_{xx} - \gamma x u_{xx} + \frac{1}{2}\beta\gamma t u_x^2 \\
 & \quad + \gamma u_x - \alpha^2 t u - \alpha\beta t u^2 - \alpha x u - \frac{1}{3}\beta^2 t u^3 - \frac{1}{2}\beta x u^2 \\
\gamma u_x^2 - \frac{1}{3}\beta u^3, & -2\gamma u_x u_t + \gamma^2 u_{xx}^2 + \beta\gamma u^2 u_{xx} + \gamma\alpha u_x^2 + \frac{1}{3}\alpha\beta u^3 + \frac{1}{4}\beta^2 u^4.
\end{array} \tag{2.8}$$

2.2. *Example 2*

A particular case of a strongly nonlinear wave with exact long wave velocity $au^{\alpha-\beta}u_x$ with the instantaneous value of *layer depths* ([8]) is given by

$$u_t + au^{\alpha-\beta}u_x + b((3\alpha-\beta)u^{3\alpha-\beta-1}u_x u_{xx} + u^{3\alpha-\beta}u_{xxx}) = 0. \tag{2.9}$$

A similar calculation as above reveals that there are no multipliers of order one or two and the zero order multipliers are $Q_1 = 1$ and $Q_2 = u^{\beta-3\alpha+1}$; the first being the obvious one with flow given by $(u, \frac{a}{\alpha-\beta+1}u^{\alpha-\beta+1} + bu^{3\alpha-\beta}u_{xx})$ — the flow following Q_2 may be computed in a way described above.

3. Systems of pdes

Our main example is the Jaulent–Miodek system of equations wherein we emphasize the role of the Euler operator in two variables and its complexity as the order function on which it operates increases (we choose possible second order multipliers). A symmetry analysis of the multipliers based on Proposition 1.1 ia carried out. As a closing example, we also discuss the vector KdV equations for which, it will be shown, we obtain first but no second order multipliers.

3.1. *Main example — Jaulent–Miodek equations*

The J–M equation with energy dependent Schrödinger potential [1]

$$\begin{aligned} eqt1 &= u_t + u_{xxx} + \frac{3}{2}vv_{xxx} + \frac{9}{2}v_x v_{xx} - 6uu_x - 6uvv_x - \frac{3}{2}v^2 u_x = 0, \\ eqt2 &= v_t + v_{xxx} - 6vu_x - 6uv_x - \frac{15}{2}v^2 v_x = 0. \end{aligned} \tag{3.1}$$

The Lie point symmetry generators are

$$X = \partial_t, \quad Y = \partial_x, \quad Z = -x\partial_x - 3t\partial_t + 2u\partial_u + v\partial_v.$$

It can be shown that a dependence of the multipliers Q and P on second-order derivatives leads to

$$Q = f(x, t, u, v, v_x, u_{xx}, v_{xx})$$

and

$$P = g(x, t, u, v, v_x, u_{xx}, v_{xx}).$$

The "joint" Euler operator annihilates the total divergence to give

$$\frac{\delta}{\delta(u, v)}(f * eqt1 + g * eqt2) = 0. \tag{3.2}$$

The expansion of the left-hand side of (3.2) is extensive and requires software to enumerate, particularly in the separation of the monomials and solving the overdetermined system of pdes. In summarized form, the multipliers are given by

$$\begin{aligned} f_1 &= x + \frac{9}{2}tv^2 + 6tu, & g_1 &= \frac{45}{12}tv^3 + \frac{432}{48}tuv - \frac{3}{2}v_{xx}t + \frac{1}{2}xv, \\ f_2 &= v_{xx} - \frac{5}{2}v^3 - 6uv, & g_2 &= -\frac{35}{16}v^4 - \frac{15}{2}uv^2 + \frac{30}{12}v_{xx}v + \frac{1}{48}(60v_x^2 - 144u^2 + 48u_{xx}), \\ f_3 &= \frac{3}{4}v^2 + u, & g_3 &= \frac{15}{24}v^3 + \frac{3}{2}uv - \frac{1}{4}v_{xx}, \\ f_4 &= 1, & g_4 &= \frac{1}{2}v, \\ f_5 &= F(t)v, & g_5 &= \frac{1}{6}xF' + \left(\frac{3}{4}v^2 + u\right)F, \\ f_6 &= 0, & g_6 &= H(t). \end{aligned} \tag{3.3}$$

We now do a symmetry analysis of the multipliers and relate the results to Proposition 1.1. The most interesting relationship follows from the generator Z so that

$$\begin{array}{lll} Zf_1 = -f_1, & Zg_1 = 0, & \textit{strict invariance} \\ Zf_2 = 3f_2, & Zg_2 = 4g_2, & \textit{ray invariance} \\ Zf_3 = 2f_3, & Zg_3 = 3g_3, & \textit{ray invariance} \\ Zf_4 = 0, & Zg_4 = g_4, & \textit{strict invariance} \\ Zf_5 = f_5, & Zg_5 = 2g_5 \quad (F = 4), & \textit{ray invariance.} \end{array} \tag{3.4}$$

Regarding X, we get $Xf_1 = f_4$, $Xg_1 = g_4$ and $Xf_5 = \tilde{f}_5$, $Xg_5 = \tilde{g}_5$ and the remainder are strictly invariant. With respect to Y, we get $Yf_1 = f_4$, $Yg_1 = g_4$ and the remainder are strictly invariant.

If the *symmetry invariance condition on the multipliers* route is adopted from the begining (which is the main emphasis here), then via the symmetry Z, for e.g., and Proposition 1.1, we get the conditions

$$ZQ = (\lambda - 1)Q, \quad ZP = \mu P$$

for which we need to solve for Q and P. If Q and P are assumed to be of second-order and independent of t derivatives, then, for e.g.,

$$\begin{aligned} Q &= v^{\lambda-1} f(\alpha, \beta, \kappa, \gamma, \delta, \xi, \eta), \\ P &= v^{\mu} g(\alpha, \beta, \kappa, \gamma, \delta, \xi, \eta), \end{aligned}$$

where $\alpha = xv$, $\beta = tv^3$, $\kappa = v^2/u$, $\gamma = v^3/u_x$, $\delta = v^4/u_{xx}$, $\eta = v^3/v_{xx}$ and $\xi = v^2/v_{xx}$ are invariants of the operator Z. If we put these into (3.2), we obtain form of f and g equivalent to the first five cases in (3.3).

(1) If we assume the result on all λs and μs, the calculations give $f = 0$ and so that $Q = 0$ and $P = t^{-\frac{1}{3}\mu}$ which corresponds to the sixth case in (3.3).

(2) If we assume the multipliers to be derivative independent, i.e.,

$$Q = v^{\lambda-1} f(\alpha, \beta, \kappa), \quad P = v^{\mu} g(\alpha, \beta, \kappa),$$

the calculations reveal that

(a) For $\lambda = \mu$,

$$Q = t^{\frac{1}{3}(2-\mu)} v, \quad P = u t^{\frac{1}{3}(2-\mu)} - \frac{\mu - 2}{18} x t^{\frac{1}{3}(1+\mu)} + \frac{3}{4} v^2 t^{\frac{1}{3}(2-\mu)}$$

so that for $\mu = 2$, we get (3.3)c and for $\mu = -1$, we obtain (3.3)e with $F = t$.

(b) For $\lambda = 1$, we get

$$Q = 1, \quad P = \frac{1}{\mu} v$$

which is generalization of (3.3)d.

(c) For $\lambda = \mu - 1$, we get

$$Q = t^{\frac{1}{3}(2-\mu)}, \quad P = -\frac{\mu - 2}{18} x v^{-1} t^{-\frac{1}{3}(1+\mu)} + \frac{1}{4} t^{\frac{1}{3}(2-\mu)}$$

so that with $\mu = 2$, we obtain (3.3)d and with $\mu = -1$, we get

$$Q = t, \quad P = \frac{1}{6} x v^{-1} + \frac{1}{4} t v,$$

which is not in the list (3.3).

(3) If we include derivative dependence, we obtain the following three sets of f and g equivalent to the first three of multipliers in (3.3)

$$\begin{aligned} f_1 &= \frac{1}{12}(2\alpha + 9\beta) + \frac{\beta}{\kappa}, & g_1 &= -\frac{\beta}{4\eta} + \frac{1}{12}\alpha + \frac{3}{2}\frac{\beta}{\kappa} + \frac{5}{8}\beta \\ f_2 &= 1 + \frac{12}{5\kappa} - \frac{2}{5\eta}, & g_2 &= -\frac{1}{2\xi^2} - \frac{1}{\eta} - \frac{2}{5\delta} + \frac{3}{\kappa} + \frac{6}{5\kappa^2} + \frac{7}{8} \\ f_3 &= \frac{3}{4} + \frac{1}{\kappa}, & g_3 &= -\frac{1}{\kappa\eta} + \frac{3}{2\kappa} + \frac{5}{8}. \end{aligned} \tag{3.5}$$

Finally, having the multipliers, we then determine the conserved flows (Φ^t, Φ^x). Corresponding to the first five of the list in (3.3) with $F = 4$ in case 5, the conserved densities are

$$\begin{aligned} \Phi_1^t &= 3tv_x^2 + 12tu^2 + 18tuv^2 + \frac{15}{4}tv^4 + xv^2 \\ \Phi_2^t &= 16vu_{xx} - 20vv_x^2 - 48u^2v - 40uv^3 - 7v^5 \\ \Phi_3^t &= v_x^2 + 4u^2 + 6uv^2 + \frac{5}{4}v^4 \\ \Phi_4^t &= 2u + \frac{1}{2}v^2 \\ \Phi_5^t &= 4uv + v^3 \quad \text{(e.g., } F = 4\text{)}. \end{aligned} \tag{3.6}$$

3.2. *KdV vector equations*

There are a number of versions of this system dependent on the choice of the *norm*. We consider the form

$$\begin{aligned} v_t &= v_{xxx} + \frac{3}{2}v_x v^4 + 3v_x v^2 w^2 + \frac{3}{2}v_x w^4, \\ w_t &= w_{xxx} + \frac{3}{2}w_x v^4 + 3w_x v^2 w^2 + \frac{3}{2}w_x w^4. \end{aligned} \tag{3.7}$$

The procedure of the previous example shows that there are zero and first order multipliers $(Q, P) = (v, w)$ and $(Q, P) = (w_x, -v_x)$ with conserved flows (Φ^t, Φ^x),

$$\left(\frac{1}{2}(v^2 + w^2), -vv_{xx} + \frac{1}{2}v_x^2 - ww_{xx} + \frac{1}{2}w_x^2 - \frac{1}{4}v^6 - \frac{3}{4}v^4w^2 - \frac{3}{4}v^2w^4 - \frac{1}{4}w^6\right)$$

and

$$(-v_x w, v_t w - v_{xx}w_x + v_x w_{xx})$$

where, in the second case, interestingly, a number of necessary terms are required after a direct calculation of the divergence $D_t\Phi^t + D_x\Phi^x$.

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References

[1] M. Jaulent and I. Miodek, Nonlinear evolution equations associated with "energy dependent Schrödinger potentials", *Letters in Mathematical Physics* **1** (1976) 243–250.

[2] F. G. Guererro and L. M. Alonso, Hamiltonian formulation for the Jaulent–Miodek family of nonlinear evolution equations, *Lettere al Nuovo Cimento* **27**(1) (1980) 28–31.

[3] C. Laddomada and T. Gui-Zhang, Bäcklund transformations of the Jaulent–Miodek equations, *Letters in Mathematical Physics* **6** (1982) 453–462.

[4] D. D. Ganji, M. Jannatabadi and E. Mohseni, Application of Hes variational iteration method to nonlinear Jaulent–Miodek equations and comparing it with ADM, *J. Computational Applied Mathematics* **207** (2007) 35–45.

[5] A. H. Kara and F. M. Mahomed, Relationaship between symmetries and conservation laws, *Int. J. Theoretical Physics* **39**(1) (2000) 23–40.

[6] A. H. Kara and F. M. Mahomed, A basis of conservation laws for partial differential equations, *J. Nonlinear Math. Phys.* **9** (2002) 60–72.

[7] N. Euler and M. Euler, On nonlocal symmetries and nonlocal conservation laws and nonlocal transformations of evolution equations: Two linearisable hierarchies, *J. Nonlinear Mathematical Physics*, arXiv: 0811.1105.

[8] L. A. Ostrovsky and J. Grue, Evolution equations for strongly nonlinear internal waves, *Physics of Fluids* **15**(10) (2003) 2934–2948.