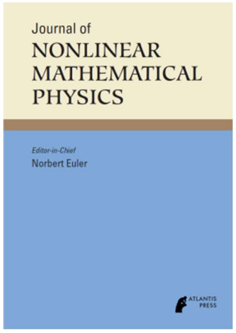

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A NOVEL RICCATI SEQUENCE

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Hierarchies of evolution partial differential equations have become well-established in the literature over the last thirty years. More recently sequences of ordinary differential equations have been introduced. Of these perhaps the most notable is the Riccati Sequence which has beautiful singularity, symmetry and integrability properties. We examine a variation of this sequence and find that there are some remarkable changes in properties consequential upon this variation.

Keywords: Differential sequence; recursion operator; Riccati equation; logarithmic singularity.

Mathematics Subject Classification: 34A05, 34A34

1. Introduction

Some thirty years ago the concept of an hierarchy of evolution partial differential equations was introduced [13]. The hierarchy is generated by means of the action of a recursion operator on a seed equation. A classic example is the Korteweg–de Vries Equation

$$u_t + u_{3x} + 6uu_x = 0,$$

where u_t denotes differentiation with respect to the evolution of variable, t, and u_{nx} denotes differentiation n times with respect to the "space" variable, x, for which a recursion operator is

$$R_{KdV} = D_x^2 + 4u + 2u_x D_x^{-1},$$

where D_x denotes total differentiation with respect to x and D_x^{-1} integration with respect to the same variable. It is a simple matter to show that the second member of the hierarchy is

$$u_t + u_{5x} + 10uu_{3x} + 20u_x u_{3x} + 30u^2 u_x = 0.$$

Further members of the hierarchy are calculated by means of repeated action of the recursion operator on the nonevolution part of the equation.

The elements for the theoretical basis for these recursion operators can be found in a text such as Bluman and Kumei [4].

Following a classification of linearisable evolution partial differential equations by Petersson *et al.* [14] particular attention was paid to the corresponding ordinary differential equations of two of these evolution partial differential equations [8]. The base equations were the Riccati equation [16] and the Ermakov–Pinney equation [7,15]. The analysis of the sequence based upon the Riccati equation indicated a surprising richness of property and this was further investigated by Andriopoulos *et al.* [2]. Subsequently a generalised version of the Riccati Sequence was studied [11].

In this paper we consider a variation of the fundamental definition of the basic equation of the Riccati Differential Sequence and examine the properties of two sequences which can be derived using a recursion operator appropriate to this new definition. Although it is usual to write to the Riccati Differential Sequence in terms of the Riccati equation and recursion operator,

$$y' + y^2 = 0 \quad \text{and} \quad R = D + y, \tag{1.1}$$

respectively, in terms of the standard definition and properties of recursion operators it is more appropriate to commence with the autonomous expression for the potential form of Burgers equation, namely

$$u'' + u'^2 = 0 \quad \text{and} \quad R = D + u', \tag{1.2}$$

where in both definitions the operator D represents total differentiation with respect to the single independent variable, x. Clearly the two equations are related through $y = u'$. Our basic sequence is based upon (1.2), but we also consider the sequence related through $y = u'$. The properties which we wish to consider relate to the existence of Lie point symmetries, singularity properties and integrability.

2. The Two Sequences

We divide the potential form of Burgers equation by u'^2 to obtain

$$\frac{u''}{u'^2} + 1 = 0. \tag{2.1}$$

We calculate a first-order recursion operator for (2.1) of the form $A(u, u', \ldots)D + B(u, u', \ldots)$ according to the standard procedure [4]. It is

$$\begin{aligned} R &= \frac{1}{u'}D - \frac{u''}{u'^2} \\ &= D\left(\frac{1}{u'}\right), \end{aligned} \tag{2.2}$$

which is certainly a compact expression.

To describe the members of the differential sequence generated from (2.1) by means of the recursion operator (2.2) we write the equation[a] as E_n and to the left side of the equation as $\tilde{E}_n$. For ease of computation when it comes to the calculation of symmetries and singularity properties we multiply the equation by a suitable power of u' to remove any fractions and use the notations Q_n and $\tilde{Q}_n$, respectively. When we replace u' with y, the corresponding notation is P_n and $\tilde{P}_n$.

The first few members of the sequences are

$$E_2\colon \frac{u''}{u'^2} + 1 = 0$$

$$E_3\colon -\frac{u''}{u'^2} - \frac{3u''^2}{u'^4} + \frac{u'''}{u'^3} = 0$$

$$E_4\colon \frac{3u''^2}{u'^4} + \frac{15u''^3}{u'^6} - \frac{u'''}{u'^3} - \frac{10u''u'''}{u'^5} + \frac{u''''}{u'^4} = 0$$

$$E_5\colon -\frac{15u''^3}{u'^6} - \frac{105u''^4}{u'^8} + \frac{10u''u'''}{u'^5} + \frac{105u''^2u'''}{u'^7} - \frac{10u'''^2}{u'^6} - \frac{u''''}{u'^4} - \frac{15u''u''''}{u'^6} + \frac{u^{(5)}}{u'^5} = 0$$

$$E_6\colon \frac{105u''^4}{u'^8} + \frac{945u''^5}{u'^{10}} - \frac{105u''^2u'''}{u'^7} - \frac{1260u''^3u'''}{u'^9} + \frac{10u'''^2}{u'^6} + \frac{280u''u'''^2}{u'^8} + \frac{15u''u''''}{u'^6} + \frac{210u''^2u''''}{u'^8} - \frac{35u'''u''''}{u'^7} - \frac{u^{(5)}}{u'^5} - \frac{21u''u^{(5)}}{u'^7} + \frac{u^{(6)}}{u'^6} = 0$$

$$E_7\colon -\frac{945u''^5}{u'^{10}} - \frac{10395u''^6}{u'^{12}} + \frac{1260u''^3u'''}{u'^9} + \frac{17325u''^4u'''}{u'^{11}} - \frac{280u''u'''^2}{u'^8} - \frac{6300u''^2u'''^2}{u'^{10}} + \frac{280u'''^3}{u'^9} - \frac{210u''^2u''''}{u'^8} - \frac{3150u''^3u''''}{u'^{10}} + \frac{35u'''u''''}{u'^7} + \frac{1260u''u'''u''''}{u'^9} - \frac{35u''''^2}{u'^8} + \frac{21u''u^{(5)}}{u'^7} + \frac{378u''^2u^{(5)}}{u'^9} - \frac{56u'''u^{(5)}}{u'^8} - \frac{u^{(6)}}{u'^6} - \frac{28u''u^{(6)}}{u'^8} + \frac{u^{(7)}}{u'^7} = 0,$$

$$Q_2\colon u'' + u'^2 = 0$$

$$Q_3\colon -u'^2u'' - 3u''^2 + u'u''' = 0$$

$$Q_4\colon 3u'^2u''^2 + 15u''^3 - u'^3u''' - 10u'u''u''' + u'^2u'''' = 0$$

$$Q_5\colon -15u'^2u''^3 - 105u''^4 + 10u'^3u''u''' + 105u'u''^2u''' - 10u'^2u'''^2 - u'^4u'''' - 15u'^2u''u'''' + u'^3u^{(5)} = 0$$

$$Q_6\colon 105u'^2u''^4 + 945u''^5 - 105u'^3u''^2u''' - 1260u'u''^3u''' + 10u'^4u'''^2 + 280u'^2u''u'''^2 + 15u'^4u''u'''' + 210u'^2u''^2u'''' - 35u'^3u'''u'''' - u'^5u^{(5)} - 21u'^3u''u^{(5)} + u'^4u^{(6)} = 0$$

[a]The numbering of the equations generated using the operator (2.2) is in terms of the order of the equation rather than the number in the sequence.

$$\begin{aligned}
Q_7\colon & -945u'^2u''^5 - 10395u''^6 + 1260u'^3u''^3u''' + 17325u'u''^4u''' \\
& - 280u'^4u''u'''^2 - 6300u'^2u''^2u'''^2 + 280u'^3u'''^3 - 210u'^4u''^2u'''' \\
& - 3150u'^2u''^3u'''' + 35u'^5u'''u'''' + 1260u'^3u''u'''u'''' - 35u'^4u''''^2 \\
& + 21u'^5u''u^{(5)} + 378u'^3u''^2u^{(5)} - 56u'^4u'''u^{(5)} - u'^6u^{(6)} \\
& - 28u'^4u''u^{(6)} + u'^5u^{(7)} = 0
\end{aligned}$$

and

$$\begin{aligned}
& P_2\colon v' + v^2 = 0 \\
& P_3\colon -v^2v' - 3v'^2 + vv'' = 0 \\
& P_4\colon 3v^2v'^2 + 15v'^3 - v^3v'' - 10vv'v'' + v^2v''' = 0 \\
& P_5\colon -15v^2v'^3 - 105v'^4 + 10v^3v'v'' + 105vv'^2v'' - 10v^2v''^2 - v^4v''' \\
& \quad - 15v^2v'v''' + v^3v'''' = 0 \\
& P_6\colon 105v^2v'^4 + 945v'^5 - 105v^3v'^2v'' - 1260vv'^3v'' + 10v^4v''^2 + 280v^2v'v''^2 \\
& \quad + 15v^4v'v''' + 210v^2v'^2v''' - 35v^3v''v''' - v^5v'''' - 21v^3v'v'''' + v^4v^{(5)} = 0 \\
& P_7\colon -945v^2v'^5 - 10395v'^6 + 1260v^3v'^3v'' + 17325vv'^4v'' - 280v^4v'v''^2 \\
& \quad - 6300v^2v'^2v''^2 + 280v^3v''^3 - 210v^4v'^2v''' - 3150v^2v'^3v''' + 35v^5v''v''' \\
& \quad + 1260v^3v'v''v''' - 35v^4v'''^2 + 21v^5v'v'''' + 378v^3v'^2v'''' \\
& \quad - 56v^4v''v'''' - v^6v^{(5)} - 28v^4v'v^{(5)} + v^5v^{(6)} = 0.
\end{aligned}$$

3. Symmetries of the Q and P Sequences

We calculate the Lie point symmetries of the first few members of the Q and P Sequences using the Mathematica add-on, Sym [3, 5, 6]. The symmetries of the former vary in number with the order in the sequence whereas in the case of the latter all elements of the sequence possess the same symmetries. For the Q sequence we have

$$\begin{array}{ll}
Q_2 & \partial_x,\ \partial_u,\ x\partial_x,\ \mathrm{e}^u\partial_x,\ \mathrm{e}^{-u}\partial_u,\ x\mathrm{e}^{-u}\partial_u \\
Q_3 & \partial_x,\ \partial_u,\ x\partial_x,\ u\partial_x,\ \mathrm{e}^u\partial_x \\
Q_4 & \partial_x,\ \partial_u,\ x\partial_x,\ u\partial_x,\ u^2\partial_x,\ \mathrm{e}^u\partial_x \\
Q_5 & \partial_x,\ \partial_u,\ x\partial_x,\ u\partial_x,\ u^3\partial_x,\ u^2\partial_x,\ \mathrm{e}^u\partial_x \\
Q_6 & \partial_x,\ \partial_u,\ x\partial_x,\ u\partial_x,\ u^4\partial_x,\ u^3\partial_x,\ u^2\partial_x,\ \mathrm{e}^u\partial_x \\
Q_7 & \partial_x,\ \partial_u,\ x\partial_x,\ u\partial_x,\ u^5\partial_x, u^4\partial_x,\ u^3\partial_x,\ u^2\partial_x,\ \mathrm{e}^u\partial_x.
\end{array}$$

and for the P sequence, for the members P_3 to P_7, simply ∂_x and $x\partial_x - v\partial_v$.

In the case of the P sequence the A_2 algebra is of Lie's Type III. The algebra of the symmetries of the Q sequence is $2A_1 \oplus_s (n+1)A_1$, where n refers to the place of the element in the sequence and the two-dimensional subalgebra is of Lie's Type IV.

The algebra of the symmetries of the Q is suggestive. We recall that the number [12] of Lie point symmetries of a linear ordinary differential equation of order $m \geq 3$ is $m+1$, $m+2$ or $m+3$ with algebra $A_1 \oplus_s mA_1$, $2A_1 \oplus_s mA_1$ or $\{A_1 \oplus sl(2,R)\} \oplus_s mA_1$, respectively.

Evidently the algebra for the Q sequence coincides with the middle algebra for the linear equation of the same order. Indeed it is as if the dependent and independent variables have been interchanged. This becomes evident when one performs the hodograph transformation

$$x = U \quad \text{and} \quad u = X. \tag{3.1}$$

The transformed equation has the form

$$U^{(n+1)} - U^{(n)} = 0 \tag{3.2}$$

for Q_n.

4. Singularity Properties of the Q and P Sequences

When we substitute $u = a\chi^p$, where $\chi = x - x_0$ and x_0 is the location of the putative singularity, into, say, $\tilde{Q}_3$, we obtain

$$-a^2p^2x^{-4+2p} + 3a^2p^3x^{-4+2p} - 2a^2p^4x^{-4+2p} + a^3p^3x^{-4+3p} - a^3p^4x^{-4+3p}. \tag{4.1}$$

All terms are dominant only if $p = 0$ which removes (4.1) completely. If the terms with exponent $3p - 4$ are assumed to be dominant, it follows that $p = 1$ which means that the other terms would be dominant were it not for the fact that they vanish identically. If the contrary assumption is made, the first three terms can be dominant with an acceptable value for the exponent, $p = \frac{1}{2}$. The resonances corresponding to this value of the exponent are $s = -1, -\frac{1}{2}, 0$, i.e. the Laurent series is a Left Painlevé Series [10]. This is unfortunate as the nondominant terms require increasing powers of χ for consistency to be possible. We infer that the Q sequence does not possess the Painlevé Property. As the solution of (3.2) is

$$U(X) = \sum_{i=0}^{n-1} k_iX^i + k_n\mathrm{e}^X, \tag{4.2}$$

where k_i, $i = 1, n$, are the constants of integration, one is not surprised that the inversion is not expressible in terms of an analytic function. In the specific case of Q_3 the implicit solution

$$C_0u + C_1\mathrm{e}^u = x - x_0, \tag{4.3}$$

where C_0, C_1 and x_0 are the constants of integration, underlines the general comment.

When we turn to the P sequence, an unusual result is found. For all of the examined cases, P_3 to P_7, the exponent of the leading-order term is $p = -1$, the coefficient of the leading-order term is $a = 1$ and the resonances are $s = -1(n - 1)$ for P_n. The occurrence of multiple resonances at -1 is known, but this seems to be the first instance that **all** resonances are there. We infer that these features persist for all elements of the sequence.

5. Discussion

We have seen that this version of the Riccati sequence has some unusual properties by comparison with the "standard" Riccati Differential Sequence. In its original form this sequence is rich in symmetry and can be transformed into a linear equation of the same order by means of an hodograph transformation whereas the standard sequence is linearised by means of a nonlocal transformation. This is because the standard sequence is really in the P form rather than a Q form. The P form has very disappointing properties in terms of the number of point symmetries possessed while the Q form is a disaster as far as the singularity analysis is concerned. At least the P sequence has the saving grace of having the unusual property that all of its resonances are at -1 and so provides an esoteric example.

A recent development in the study of sequences of ordinary differential equations is the determination of alternate sequences [9], i.e. sequences in which the higher-order elements are replaced by nonhomogeneous equations of order equal to the seed equation. In the case of this differential sequence the construction is particularly simple due to the form of the recursion relation. Commencing with $\tilde{E}_2$ we have

$$\tilde{E}_3 = D\left[\frac{1}{u'}\left(\frac{u''}{u'^2}+1\right)\right] \tag{5.1}$$

$$\tilde{E}_4 = D\left[\frac{1}{u'}\tilde{E}_3\right] \tag{5.2}$$

$$\tilde{E}_5 = D\left[\frac{1}{u'}\tilde{E}_4\right] \tag{5.3}$$

$$\tilde{E}_6 = D\left[\frac{1}{u'}\tilde{E}_5\right] \tag{5.4}$$

$$\vdots$$

so that the construction of the alternate sequence proceeds as follows. From (5.1) we write

$$\tilde{E}_3 = 0 \quad \Rightarrow \quad \frac{u''}{u'^2}+1 = C_0u' \tag{5.5}$$

from which it follows that the alternate to Q_3 is

$$\bar{Q}_3 : \; u'' + u'^2 = C_0u'^3. \tag{5.6}$$

In a similar fashion we arrive at

$$\bar{Q}_4 : \quad u'' + u'^2 = (C_0u + C_1)u'^3$$

$$\bar{Q}_5 : \quad u'' + u'^2 = \left(\frac{1}{2}C_0u^2 + C_1u + C_2\right)u'^3$$

$$\vdots$$

so that the general structure of the alternate sequence is clear.

In [9] a distinction is made between an alternate sequence which is compatible with the original sequence and an alternate sequence which is completely compatible with the original sequence. In the case of the latter the solution of the equation of the alternate sequence is the solution of the corresponding equation of the original sequence. The solution of (5.6) is given implicitly by (4.3). When this solution is substituted into Q_3, the equation is identically satisfied and so these two equations are certainly completely compatible. One could proceed case-by-case to verify the property of complete compatibility, but it is possible to demonstrate the result in general. It is evident that $\bar{Q}_n$ is given by

$$u'' + u'^2 = u'^3 \sum_{i=0}^{n-3} \alpha_i u^i \tag{5.7}$$

for some constants α_i, $i = 0, n-3$. When we divide by u'^2 and make the substitution $u = \log v$, (5.7) becomes

$$\frac{v''}{v'^2} = \frac{v'}{v^2} \sum_{i=0}^{n3} \alpha_i (\log v)^i$$

and this can be directly integrated to give

$$\frac{1}{v'} = C_1 + \frac{1}{v} \sum_{i=0}^{n-3} \alpha_i \sum_{j=0}^{i} \frac{i!}{j!} (\log v)^j$$

which can be rearranged as

$$u' \sum_{i=0}^{n-3} \alpha_i \sum_{j=0}^{i} \frac{i!}{j!} u^j + C_1 u' \mathrm{e}^u = 1.$$

In turn this can be integrated to give the implicit solution

$$\sum_{i=0}^{n-3} \alpha_i \sum_{j=0}^{i} \frac{i!}{(j+1)!} u^{j+1} + C_1 \mathrm{e}^u = x - x_0. \tag{5.8}$$

When we make the hodograph transformation used to obtain (4.2), we see that (5.8) is indeed the solution of Q_n. We conclude that the two sequences are completely compatible.

It is common to consider the complete symmetry group [1] for these sequences. In the case of the Q sequence it is a simple matter as there is a sufficient supply of Lie point symmetries to specify completely the appropriate element of the sequence. The representation of the complete symmetry group is $A_1 \oplus_s (n+1) A_1$ for the nth element of the sequence with the A_1 being $x\partial_x$. From this representation it is an easy matter to construct a representation of the complete symmetry group of the elements of the P sequence as

$$\partial_x,\ x\partial_x - v\partial_v,\ \left(\int v \mathrm{d}x \right)^j \partial_x - jv^2 \left(\int v \mathrm{d}x \right)^{j-1} \partial_v,\ j = 1, m-1,$$

where $m+1$ is the order of the equation.

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