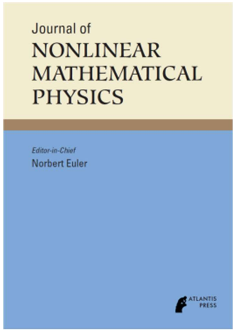

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TWISTED SYMMETRIES OF DIFFERENTIAL EQUATIONS

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We review the basic ideas lying at the foundation of the recently developed theory of twisted symmetries of differential equations, and some of its developments.

Keywords: Symmetry; differential equations.

0. Introduction

The study of nonlinear differential equations was the main motivation to Sophus Lie when he created what is nowadays known as the theory of Lie groups and Lie algebras, which is by now recognized as one of the most effective tools for the study of differential equations — both in geometrical sense and for what concerns the search for explicit solutions [25, 37, 55, 59].

The attention of Lie was mainly focused on what we now call *Lie-point symmetries*, but very soon generalizations of these were investigated, starting with *contact symmetries*. The effectiveness of symmetry methods for differential equations — greatly increased in recent years by the possibility to perform the often involved computations required by these via symbolic manipulation languages, i.e. by computer — led various authors to consider generalizations in several directions [15, 16, 23]; all of these however — at the exception of one, i.e. the subject of this paper — are based on the same scheme: one considers a vector field X acting on the full phase manifold M — the manifold of independent and dependent variables[a] — and then takes into account its action on the jet manifold $J^k M$ of suitable order (i.e. with k the order of the differential equation one wishes to study), which is obtained by the standard *prolongation* procedure. That is, once we know how X acts on the independent variables x^i and on the dependent ones u^a, we also know how it acts on

[a] In most symmetry applications, this is just a linear space $M = B \times U$ with B and U the spaces of independent and dependent variables respectively. However equations can be defined on a manifold, in which case B is not a linear space, or dependent variables take value in a nonlinear manifold U, so that $M = B \times U$ is also not such; or the equation can be complemented with side conditions, in which case M is in general a bundle over B.

the derivatives (of any order) of the u's with respect to the x's. In computational terms, this is readily described by the usual *prolongation formula* [25, 37, 55, 59]; in geometrical terms, this makes use of the *contact structure* which naturally equips the jet manifold J^kM [8, 56]. Thus, the generalizations lie in the class of admitted vector fields X on M, i.e. on what qualifies (if satisfying certain conditions specific to the equation under study) as a "symmetry"; while once this is given, the action on higher order jet bundles J^kM is just the natural prolongation of the one in M; in other words, we have the vector field $X^{(k)}$ — the natural prolongation of X — acting in J^kM.

Here we are interested in the exception mentioned above, i.e. in what we would like to call **twisted symmetries** (the reason for this name will be clear in the following). These were first proposed by M. C. Muriel and J. L. Romero under the name of "$C^\infty(M^{(1)})$-symmetries" or "C^∞-symmetries" for short, or finally of "λ-symmetries", in the framework of ODEs [41, 45–47, 57], and then generalized to PDEs under the name of "μ-symmetries" [22, 26, 29].

It should be stressed that in this case the generalization with respect to standard Lie symmetries lies not in the definition of symmetry,[b] but in the way vector fields are prolonged from the phase bundle M to jet bundles J^kM. In this, they are deeply different from other proposed generalizations of Lie symmetries.

Definition and properties of twisted symmetries will be discussed below, but we like to recall immediately that these allow to integrate — by symmetry methods — equations which are known not to admit any standard Lie symmetry [41, 43, 44]; needless to say, this is a major reason (but not the only one) for the interest they raised. It also turned out that twisted symmetries are interesting from the point of view of the geometry of differential equations. Here we will try to keep some balance between these two points of view, the (mostly analytic) applicative and the (mostly geometric) theoretical one.

1. Standard Prolongations and Symmetries

In this section we will very briefly recall the standard (and well known) notions of prolongation and symmetry, to the aim of fixing notation.

We suppose the reader has some familiarity with symmetry of differential equations and hence do not enter in details; these can be obtained e.g. from [25, 37, 55, 59]. Summation over repeated indices will always be assumed unless otherwise explicitly stated.

1.1. *Prolongations*

As already stated, we will denote the phase bundle as $M = B \times U$, where B is the manifold to which the independent variables belong, equipped with local coordinates x^i $(i = 1, \dots, q)$; and U is the manifold in which the dependent variables (or fields) take values, which is equipped with local coordinates u^a $(a = 1, \dots, p)$.

We consider a (Lie-point) vector field X in M; this will be written in coordinates as

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \varphi^a(x, u)\frac{\partial}{\partial u^a}. \tag{1}$$

[b] I.e. in the class of acceptable vector fields or in the equation-related specific conditions to be satisfied by the prolonged vector field.

As well known, given a section $\sigma_0 = (x, u = f_0(x))$ of the bundle $M = B \times U$, this is transformed under the infinitesimal action $e^{\varepsilon X}$ of X into the section $\sigma_\varepsilon = (x, u = f_\varepsilon(x))$ with

$$f_\varepsilon(x) = f_0(x) + \varepsilon(\varphi - \xi^i u_i^a)_0,$$

where the subscript "0" reminds that the functions φ and ξ — as well as the derivatives u_i^a — should be computed on σ_0, i.e.

$$\varphi^a|_{\sigma_0} = \varphi^a(x, f_0(x)), \quad \xi^i|_{\sigma_0} = \xi^i(x, f_0(x)), \quad u_i^a|_{\sigma_0} = (\partial f_0(x)/\partial x).$$

This is also said by considering the *evolutionary representative* (or vertical representative) X_v of X,

$$X_v = Q^a \frac{\partial}{\partial u^a}, \quad \text{with } Q^a = \varphi^a(x, u) - u_i^a \xi^i(x, u).$$

The jet bundles $J^k M$ associated to M can be seen also as bundles over B, $J^k M = B \times U^{(k)}$ (here $U^{(k)}$ is the manifold of fields and their derivatives of order up to k; we will denote by $U^{[k]}$ the manifold of field derivatives of order exactly k). They can also be seen as bundles over the jet bundles of lower order, $J^k M = J^{k-1} M \times U^{[k]}$, and it is this structure which comes into play when considering the prolongation operation in recursive terms.

The first jet bundle JM is naturally equipped with local coordinates (x^i, u^a, u_i^a). If X is given in coordinates by (1), its first prolongation is

$$X^{(1)} = \xi^i(x, u)\frac{\partial}{\partial x^i} + \varphi^a(x, u)\frac{\partial}{\partial u^a} + \psi_i^a(x, u)\frac{\partial}{\partial u_i^a}, \tag{2}$$

where the coefficients ψ_i^a are given by (see below for the derivation of this formula)

$$\psi_i^a = D_i \varphi^a - u_j^a D_i \xi^j. \tag{3}$$

Similarly, the jet bundles $J^k M$ are equipped with local coordinates (x^i, u^a, u_J^a) with J multi-indices of order up to k. If X is given in coordinates by (1), its kth prolongation is

$$X^{(k)} = \xi^i(x, u)\frac{\partial}{\partial x^i} + \varphi^a(x, u)\frac{\partial}{\partial u^a} + \sum_{|J|=1}^{k} \psi_J^a(x, u)\frac{\partial}{\partial u_J^a}, \tag{4}$$

where the sum is over all multi-indices J of order $1 \leq |J| \leq k$, and the coefficients ψ_J^a are given by the recursive prolongation formula (here one should understand $\psi_0^a \equiv \varphi^a$):

$$\psi_{J,i}^a = D_i \psi_J^a - u_{J,k}^a D_i \xi^k. \tag{5}$$

Note that, as well known, for vector fields in evolutionary (vertical) form, the prolongation formula reduces to

$$\psi_{J,i}^a = D_i \psi_J^a.$$

In the case of ODEs, when we have only one independent variable x, the above notation is a bit too heavy, so it is convenient to also have a simpler notation for this case. We will denote the derivative of order j as

$$u^a_j := \partial^j u^a / \partial x^j,$$

and the kth prolongation of a vector field $X = \xi(x,u)(\partial/\partial x) + \varphi^a(x,u)(\partial/\partial u^a)$ will be written as

$$X^{(k)} = \xi(x,u)\frac{\partial}{\partial x} + \varphi^a(x,u)\frac{\partial}{\partial u^a} + \sum_{j=1}^{k} \psi^a_j(x,u)\frac{\partial}{\partial u^a_j} \tag{6}$$

with the ψ^a_j satisfying

$$\psi^a_{j+1} = D_x \psi^a_j - u^a_{j+1} D_x \xi. \tag{7}$$

Last but not least, we recall that given any two vector fields in M, the commutator of their prolongations is the prolongation of their commutator, i.e.

$$[X^{(k)}, Y^{(k)}] = ([X,Y])^{(k)}. \tag{8}$$

The prolongation formula — analytic derivation

We would like to recall how the prolongation formula is obtained in analytical terms; in a later subsection we will discuss its geometrical meaning and correspondingly a geometrical derivation.

Let us consider a section $\sigma = (x,u)$, corresponding locally to a function $u = f(x)$; let x_1 and $x_2 = x_1 + \delta x^i$ be two nearby points in B, with δx^i an infinitesimal displacement in the direction x^i. The corresponding points on σ will be $p_1 = (x_1, u_1 = f(x))$ and $p_2 = (x_2, u_2 = f(x_2))$; the partial derivatives of u on σ are given by

$$u^a_i(x) = \lim_{\delta x^i \to 0} \frac{u^a_2 - u^a_1}{x_2 - x_1} := \lim_{\delta x^i \to 0} R.$$

Needless to say, $u^a_2 = u^a_1 + u^a_i \delta x^i + o(\delta x^i)$.

If now we act in M by X, i.e. by its infinitesimal action $e^{\varepsilon X}$, the points p_1 and p_2 are mapped to new points $\widehat{p}_1 = (\widehat{x}_1, \widehat{u}_1)$ and $\widehat{p}_2 = (\widehat{x}_2, \widehat{u}_2)$ with

$$\widehat{x}_j = x_j + \varepsilon \xi^i(x_j, u_j), \quad \widehat{u}_j = u_j + \varepsilon \varphi(x_j, u_j).$$

We have now to compute (the limit of)

$$\begin{aligned} \widehat{R} &= \frac{\widehat{u}_2 - \widehat{u}_1}{\widehat{x}_2 - \widehat{x}_1} \\ &= \frac{[u_2 + \varepsilon\varphi(x_2,u_2)] - [u_1 + \varepsilon(x_1,u_1)]}{[x_2 + \varepsilon\xi(x_1,u_1)] - [x_1 + \varepsilon(x_1,u_1)]} \\ &= \frac{(u_2 - u_1) + \varepsilon[\varphi(x_2,u_2) - \varphi(x_1,u_1)]}{(x_2 - x_1) + \varepsilon[\xi(x_2,u_2) - \xi(x_1,u_1)]}. \end{aligned}$$

It results immediately, with all derivatives evaluated in (x_1, y_1),

$$\begin{aligned} \xi^k(x_2,u_2) &= \xi^k(x_1,u_1) + \frac{\partial \xi^k}{\partial x^i}\delta x^i + \frac{\partial \xi^k}{\partial u^b} u^b_i \delta x^i \\ &= \xi(x_1,u_1) + [D_i(\xi)]\delta x^i; \end{aligned}$$

$$\begin{aligned}\varphi^a(x_2,u_2) &= \varphi^a(x_1,u_1) + \frac{\partial \varphi^a}{\partial x^i}\delta x^i + \frac{\partial \varphi^a}{\partial u^b} u^b_i \delta x^i \\ &= \varphi(x_1,u_1) + [D_i(\varphi)]\delta x^i.\end{aligned}$$

Here and below, D_i denotes the total derivative with respect to x^i,

$$D_i = \frac{\partial}{\partial x^i} + u^a_i \frac{\partial}{\partial u^a} + u^a_{ij}\frac{\partial}{\partial u^a_j} + \cdots . \tag{9}$$

We have therefore, recalling $(x_2 - x_1) = \delta x^i$, writing $(u_2 - u_1) = \delta u$ for short, and omitting terms of higher order in ε,

$$\begin{aligned}\widehat{R} &= \frac{\delta u + \varepsilon[(D_i\varphi)\delta x^i]}{(x_2 - x_1) + \varepsilon[(D_i\xi)\delta x^i]} \\ &= \frac{\delta u + \varepsilon[(D_i\varphi)\delta x^i]}{\delta x^i[1 + \varepsilon(D_i\xi)]} \\ &= \frac{[\delta u + \varepsilon[(D_i\varphi)\delta x^i][1 - \varepsilon(D_i\xi)]}{\delta x^i} \\ &= \frac{\delta u}{\delta x^i} + \varepsilon\left[(D_i\varphi) - \frac{\delta u}{\delta x^i}(D_i\xi)\right] \\ &= R + \varepsilon\left[(D_i\varphi) - \frac{\delta u}{\delta x^i}(D_i\xi)\right].\end{aligned}$$

This yields, upon taking the limit $\delta x^i \to 0$, precisely the prolongation formula: if X is given in coordinates by (1), its first prolongation is

$$X^{(1)} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \varphi^a(x,u)\frac{\partial}{\partial u^a} + \psi^a_i(x,u)\frac{\partial}{\partial u^a_i}, \tag{10}$$

where the coefficient ψ^a_i is given by

$$\psi^a_i = D_i\varphi^a - u^a_j D_i \xi^j. \tag{11}$$

The computation for higher order prolongation is exactly the same (with the role of u^a taken by u^a_J).

1.2. *Symmetries*

As well known, a differential equation E of order k defines a submanifold S_E (of codimension one) in J^kM, also called the *solution manifold* for E. The same applies for a system $\Delta = \{E^1, \ldots, E^p\}$ of equations; in this case S_Δ is of codimension p (if the system is non degenerate).

If the vector field X on M is such that its kth prolongation $X^{(k)}$ leaves S_Δ invariant, i.e.

$$X^{(k)} : S_\Delta \to \mathrm{T}S_\Delta, \tag{12}$$

we say that X is a (Lie-point) *symmetry of* Δ. The set of symmetries of Δ will be denoted as $\mathcal{X}_\Delta$.[c]

[c]More precisely, X is a symmetry generator, and $e^{\alpha X}$ is a one-parameter symmetry group for Δ. This slight abuse of notation is commonplace in the literature, and we will keep to it.

It follows immediately from this definition — and from (8) — that the symmetries of a given Δ are a Lie algebra (under the commutator):

$$X \in \mathcal{X}_\Delta, \quad Y \in \mathcal{X}_\Delta \Rightarrow [X,Y] \in \mathcal{X}_\Delta. \tag{13}$$

In order to check if a given vector field X is a symmetry of Δ, one just has to check that

$$[X^{(k)}(E^a)]_{S_\Delta} = 0.$$

1.3. *The geometric meaning of prolongation*

The prolongation operation was defined before as the lifting of the action of a vector field on independent variables and fields, to the derivatives of the latter with respect to the former. In this way, it was defined in an analytic way. But, prolongation has a deep and intrinsic geometric meaning as well, and it is convenient to focus on this in order to consider, later on, twisted prolongations.

The first jet bundle JM is naturally equipped with local coordinates (x^i, u^a, u^a_i); in terms of these, the *contact structure* Θ^1 is described by the *contact forms*

$$\theta^a = \mathrm{d}u^a - u^a_i \mathrm{d}x^i. \tag{14}$$

The prolongation $X^{(1)}$ of X is the unique vector field in J^1M which coincides with X on M and preserves the contact structure Θ^1.

Similarly, the jet bundles J^kM are equipped with a contact structure Θ^k; in terms of the local coordinates (x^i, u^a, u^a_J), the contact structure Θ^k is described by the contact forms

$$\theta^a_J = \mathrm{d}u^a_J - u^a_{J,i}\mathrm{d}x^i, \tag{15}$$

where $0 \le |J| \le k-1$. The prolongation $X^{(k)}$ of X is the unique vector field in J^kM which coincides with X on M and preserves the contact structure Θ^k.

It is maybe worth defining more precisely, for the benefit of the reader less used to this geometric language, what is meant by "preservation of the contact structure".

Consider a set $\{\vartheta_1, \dots, \vartheta_r\}$ of generators for Θ^k, and denote by $\mathcal{E}$ the $C^\infty(J^kM)$ module generated by these same generators.[d] We say that a vector field Y on J^kM preserves Θ^k if and only if

$$\mathcal{L}_Y(\vartheta) \in \mathcal{E} \quad \forall \vartheta \in \mathcal{E}; \tag{16}$$

here and in the following $\mathcal{L}_Y$ is the Lie derivative under Y.

Note, for later reference, that the condition (16) can be expressed equivalently in terms of conditions involving the commutator of Y with the total derivative operators D_i; in particular, it is equivalent to either one of

$$[D_i, Y] \lrcorner \vartheta = 0 \quad \forall \vartheta \in \mathcal{E}, \tag{17}$$

$$[D_i, Y] = h^m_i D_m + V, \tag{18}$$

with $h^m_i \in C^\infty(J^kM)$ and V a vertical vector field in J^kM (seen as a bundle over $J^{k-1}M$, see above).

[d] I.e. the set of one-forms obtained as sum of the ϑ_i with coefficients which are C^∞ functions on J^kM, $\theta = \sum_{i=1}^r \alpha^i(x, u^{(k)})\vartheta^i$.

The prolongation formula — geometric derivation

We have recalled above how the prolongation formula is obtained in analytic terms; here we want to show for comparison how it is derived in geometric terms, i.e. from (16). We will again limit to consider the first prolongation, computations being the same for higher ones.

First of all we note that, from the general properties of the Lie derivative, we have just to show that $\mathcal{L}_Y(\theta) \in \mathcal{E}$ for all θ in the generating set of the contact structure. In facts, for $\vartheta = \beta_a \theta^a$ with $\beta_a \in C^\infty(JM)$, we have

$$\mathcal{L}_Y(\beta_a \theta^a) = [\mathcal{L}_Y(\beta_a)]\theta^a + \beta_a[\mathcal{L}_Y(\theta^a)];$$

the first term in the right-hand side is by definition in $\mathcal{E}$, and the second is always in $\mathcal{E}$ if and only if $\mathcal{L}_Y(\theta^a) \in \mathcal{E}$ for all $a = 1, \dots, p$.

In our case, $\mathrm{d}\theta^a = \mathrm{d}u^a_i \wedge \mathrm{d}x^i$. We will write

$$Y = \xi^i \frac{\partial}{\partial x^i} + \varphi^a \frac{\partial}{\partial u^a} + \psi^a_i \frac{\partial}{\partial u^a_i}, \tag{19}$$

so that $Y \lrcorner \theta^a = \varphi^a - u^a_i \xi^i$ and $\mathrm{d}(Y \lrcorner \theta^a) = \mathrm{d}\varphi^a - u^a_i \mathrm{d}\xi^i - \xi^i \mathrm{d}u^a_i$; moreover $Y \lrcorner \mathrm{d}\theta = \xi^i \mathrm{d}u^a_i - \psi^a_i \mathrm{d}x^i$. Therefore, using the Cartan formula $\mathcal{L}_Y(\theta) = Y \lrcorner \mathrm{d}\theta + \mathrm{d}(Y \lrcorner \theta)$, we get immediately

$$\mathcal{L}_Y(\theta^a) = \mathrm{d}\varphi^a - u^a_j \mathrm{d}\xi^j - \psi^a_i \mathrm{d}x^i.$$

Using $\mathrm{d}u^a = \theta^a + u^a_i \mathrm{d}x^i$, we have $\mathrm{d}\varphi^a = (\partial\varphi^a/\partial u^b)\theta^b + (D_i\varphi^a)\mathrm{d}x^i$ and $\mathrm{d}\xi^j = (\partial\xi^j/\partial u^b)\theta^b + (D_i\xi^j)\mathrm{d}x^i$; thus we can rewrite

$$\mathcal{L}_Y(\theta^a) = [(\partial\varphi^a/\partial u^b) + (\partial\xi^j/\partial u^b)]\theta^b + [(D_i\varphi^a - u^a_j D_i\xi^j) - \psi^a_i]\mathrm{d}x^i.$$

The first term on the right-hand side is surely in $\mathcal{E}$, by definition, while the second is either zero or surely not in $\mathcal{E}$. We conclude that $\mathcal{L}_Y(\theta^a) \in \mathcal{E}$ if and only if

$$\psi^a_i = (D_i\varphi^a) - u^a_j(D_i\xi^j).$$

We have thus obtained again the prolongation formula (2).

2. Lambda-Prolongations and Symmetries

The generalization of classical Lie-point symmetries we wish to consider, i.e. twisted symmetries, modifies the standard prolongation operation. That is, one considers vector fields Y in J^nM which are not the prolongation of some vector field X in M, i.e. $Y \neq X^{(n)}$ for any X, yet are related to such a vector field in a precise manner to be discussed below. Quite surprisingly, when these are symmetries of a differential equation Δ — i.e. when $Y : S_\Delta \to \mathrm{T}S_\Delta$ — they are still effective in obtaining symmetry reductions (for ODEs) or invariant solutions (for PDEs) of differential equations.

We will start by discussing the first class of twisted symmetries to be discovered, i.e. λ-symmetries.

2.1. *The work of Muriel and Romero*

In 2001, Muriel and Romero [41], analyzing the case where Δ is a scalar ODE, noticed a rather puzzling fact.

They substitute the standard prolongation formula (7) with a "lambda-prolongation" formula

$$\Psi_{k+1} = (D_x + \lambda)\Psi_k - u_{k+1}(D_x + \lambda)\xi; \tag{20}$$

here λ is a real C^∞ function defined on J^1M (or on J^kM if one is ready to deal with generalized vector fields). For $\lambda \equiv 0$ one recovers standard prolongations.

We say that X is a "lambda-symmetry" of Δ if its "lambda-prolongation" Y is tangent to the solution manifold $S_\Delta \subset J^nM$.

Then, as mentioned above, it turns out that "lambda-symmetries" are as good as standard symmetries for what concerns symmetry reduction of the differential equation Δ and hence determination of its explicit solutions. As pointed out by Muriel and Romero, it is quite possible to have equations which have no standard symmetries, but possess lambda-symmetries and can therefore be integrated by means of their approach; see their works [41, 43] for examples.

It is quite remarkable that the λ-symmetries approach is able to explain the — rather puzzling — fact that there were equations explicitly integrable by quadratures and not possessing any (standard) symmetry [4, 31, 32, 55].

2.2. *The invariants-by-differentiation property*

The possibility of using λ-symmetries as effectively as standard ones for order reduction of differential equations has its origin in the fact the recursion formula allowing to build higher order differential invariants from lower order ones [55, 56], also called "invariants-by-differentiation", applies for λ-symmetries as well. This point of view was stressed by Muriel and Romero in another paper [47], where other examples are also provided.

If η and ζ are differential invariants for the vector field Y, the invariants-by-differentiation property states that $\rho := (D_x\zeta)/(D_x\eta)$ (with $D_x\eta \neq 0$) will also be a differential invariant. The proof of this property goes as follows: acting with Y on ρ we have

$$Y\left(\frac{D_x\zeta}{D_x\eta}\right) = \frac{[Y(D_x\zeta)](D_x\eta) - (D_x\zeta)[Y(D_x\eta)]}{(D_x\eta)^2}.$$

Thus ρ is a differential invariant if and only if

$$[Y(D_x\zeta)](D_x\eta) = (D_x\zeta)[Y(D_x\eta)];$$

we can rewrite this equation as

$$(D_x\eta)([Y, D_x](\zeta)) - (D_x(Y\zeta))(D_x\eta) = (D_x\zeta)([Y, D_x](\eta)) - (D_x\zeta)(D_x(Y\eta)).$$

We assumed η and ζ are differential invariants for Y, hence $Y(\eta) = Y(\zeta) = 0$, and the above equation reduces to

$$(D_x\eta)([Y, D_x](\zeta)) = (D_x\zeta)([Y, D_x](\eta)). \tag{21}$$

If Y is a λ-prolongation, it satisfies (up to a term which vanishes on the contact distribution)

$$[D_x, Y] = \lambda Y + h D_x \tag{22}$$

with h a smooth function; with this, and recalling again $Y(\eta) = Y(\zeta) = 0$, the above equation reduces to the trivial identity

$$h(D_x \eta)(D_x \zeta) = (D_x \zeta) h(D_x \eta).$$

2.3. *Systems of ODE*

Muriel and Romero also considered how the concept of λ-symmetry would extend to systems of ODEs [45].

They considered a λ-prolongation given by

$$\psi^a_{k+1} = (D_x + \lambda)\psi^a_k - u^a_{k+1}(D_x + \lambda)\xi, \tag{23}$$

i.e. the immediate generalization of (20), and studied a system Δ of first order ODEs (which we will also call a dynamical system),

$$u^a_x = F^a(x, u^1, \ldots, u^r), \quad (a = 1, \ldots, r).$$

In this case, if X is a λ-symmetry of the system Δ, there exists a change of variables $(y, w) = \beta(x, u)$ under which the system reduces to a system of $(r-1)$ first order ODEs,

$$w^a_y = G^a(x, w^1, \ldots, w^{r-1}), \quad (a = 1, \ldots, r-1)$$

and the variable w^r must satisfy an auxiliary ODE

$$H(y, w^r, w^r_y) = 0.$$

2.4. *Lambda symmetries and nonlocal standard symmetries*

In their seminal work, Muriel and Romero also remarked (see [41, Sec. 5]) that there is an intriguing relation between λ-symmetries and exponential symmetries of differential equations.

Exponential symmetries [55] represent a specific form of nonlocal vector fields,

$$X = e^{\int P(x,u)\mathrm{d}x} \left(\xi(x, u) \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right) := e^{\int P(x,u)\mathrm{d}x} X_0. \tag{24}$$

Needless to say, their prolongation X^* must satisfy the usual symmetry condition $X^* : S_\Delta \to \mathrm{T}S_\delta$.

Muriel and Romero proved that if X given by (24) is an exponential symmetry for Δ, then X_0 defined in (24) is a λ-symmetry for δ, with

$$\lambda = P(x, u).$$

As nonlocal symmetries can be used, pretty much as standard symmetries, to reduce and integrate differential equations [7], this shows that all the reduction/integration methods based on nonlocal symmetries (of exponential type) [5–7, 55] can be automatically formulated in terms of λ-symmetries.

In a later work [48], Muriel and Romero went further in their study; in particular they studied the connection between λ-symmetries and the so called "type I hidden symmetries" (see below). The key property of use here, shown in their work, is that if an equation $\mathrm{d}^n u/\mathrm{d}x^n = f(x, u^{(n-1)})$ admits a λ-symmetry X, then when the equation is written in new variables (y, v) as

$$\mathrm{d}^n v/\mathrm{d}y^n = g(y, v^{(n-1)}),$$

the vector field X is a $\widehat{\lambda}$-symmetry for the equation, with

$$\widehat{\lambda} = \frac{\lambda}{D_x y}.$$

Muriel and Romero focused in particular on first order equations; in this case a theorem by Adam and Mahomed [6] gives a criterion which must be satisfied by a nonlocal symmetry of a first-order equation in order that a symmetry-related (computable) transformation maps the latter into an integrable equation; this theorem can be reformulated (in a simpler way) in the language of λ-symmetries [48]. This result was then applied to relevant equations such as Riccati equations and Abel equations of the second kind [48].

The relation between λ-symmetries and nonlocal standard symmetries was also investigated by Catalano-Ferraioli [12] by means of the theory of *coverings* [36,37,61]. In this, one embeds $J^k M$ into a space of higher dimension by adding one or more extra variables w^α with their (first) derivatives, and augments the equation under study with extra equations

$$\mathrm{d}w^\alpha/\mathrm{d}x = h^\alpha(x, u^{(n)}, w);$$

nonlocal symmetries can then be expressed as local symmetries of the augmented system. The connection with λ-symmetries arises (with a single extra variable w) when one chooses $h = \lambda$. We refer to the original paper [12] for details.

2.5. *Lambda symmetries and other types of symmetries*

Lambda symmetries turned out to be relevant, and providing a way to soundly understand, also other kinds of symmetries. Here we will just briefly mention these interrelations, referring the reader to the original papers.

Gandarias, Medina and Muriel [30] discussed the relation between λ-symmetries and the so called *potential symmetries* of differential equations [11]. They also discussed how this relation can be of help in the integration of differential equations not possessing Lie-point symmetries. The same problem was also tackled in a paper by Muriel and Romero [46], where they also discuss the relation with potential symmetries.

Similarly, λ-symmetries are instrumental in integrating equations with non-solvable symmetry algebras; this point was discussed by Muriel and Romero both in general terms [43] and for specific algebras relevant in Physics (and in other applications as well) [42,44].

Lambda symmetries also bear some interesting relation to so called *hidden symmetries* (such as symmetries which got lost in the reduction process) [1]. See [2,3] for a discussion.

Hidden symmetries also have relations [33] with so called *solvable structures* [9,10]; these in turn have been recently studied in connection to λ-symmetries [13]. We refer the reader to the original paper [13] for this matter.

2.6. *Lambda symmetries and integrating factors*

It turns out that λ-symmetries are also relevant for the determination of *integrating factors* for higher order ODEs. As well known, any first order ODE

$$N(x,u)\frac{\mathrm{d}u}{\mathrm{d}x} + M(x,u) = 0$$

admits an integrating factor, i.e. a scalar function $\rho(x,u)$ such that

$$\rho[N(x,u)\mathrm{d}u + M(x,u)\mathrm{d}x] = \mathrm{d}F$$

for some function $F = F(x,u)$.

The integrating factor of an nth order equation

$$N(x,u,u',\ldots,u^{(n-1)})\frac{\mathrm{d}^n u}{\mathrm{d}x^n} + M(x,u,u',\ldots,u^{(n-1)}) = 0$$

is a scalar function ρ such that

$$\mathcal{D}_u(\rho(M + Nu_n)) = 0,$$

where $\mathcal{D}_u$ is the *variational derivative*,

$$\mathcal{D}_u = \frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} + \cdots .$$

The problem of integrating factors for higher order equations[e] was tackled by Muriel and Romero using symmetries and λ-symmetries [49].

In particular, they were able to identify, given an integrating factor, a λ-symmetry associated to it; and conversely, given a λ-symmetry, a corresponding integrating factor. As any ODE admits λ-symmetries, this implies in particular that — similarly to what happens for first order ODEs, as stated by the very classical result by Clairaut — any differential equation of order n,

$$\frac{\mathrm{d}^n u}{\mathrm{d}x^n} = f(x,u^{(n-1)})$$

with f analytic in some open subset of $J^{n-1}M$, admits an integrating factor $\rho(x,u^{(k)})$ with some $k < n$.[f]

2.7. *Lambda symmetries and first integrals*

As mentioned above, λ-symmetries can also be considered for systems of ODEs, and in particular for dynamical systems. In this case, one is specially interested in first integrals. The relation between these and λ-symmetries was considered by Zhang and Li [63].

A dynamical system

$$u_x^a = f^a(x,u); \tag{25}$$

[e] For the relation between ordinary symmetries and integrating factors, see e.g. [38].

[f] After the first version of this paper was submitted, Muriel and Romero published another paper, in which they study in depth the interrelations between λ-symmetries, integrating factors and first integrals of second order equations [50]; this also includes application to Ermakov–Pinney equation. Here we can just urge the reader to read this paper.

this is represented by the dynamical vector field $F = f^a(x,u)(\partial/\partial u^a)$. With $\mathcal{L}$ the Lie derivative, a vector field $Z = \varphi^a(x,u)(\partial/\partial u^a)$ is a λ-Liouville vector field (or simply a Liouville vector field when $\lambda = 0$) if

$$\left(\frac{\partial}{\partial x} + \mathcal{L}_F + \lambda\right) Z + (\operatorname{div} F) Z = 0.$$

If X is a λ-symmetry for (25), then $(\partial_x + \mathcal{L}_F + \lambda)X = 0$. Moreover, let Z be Liouville for (25) and the scalar function $p(x,u)$ satisfy the equation $(\partial p/\partial x) + \operatorname{div}(pF) = 0$; then the vector field $X := (1/p)Z$ is a λ-symmetry for (25) [63].

Now, let γ be a scalar analytic function, solution to

$$(\partial_x + \mathcal{L}_F)\gamma = \lambda\gamma.$$

Under the assumption that γZ is divergence-free, Zhang and Li propose a way, starting from a Liouville vector field Z and γ, and having determined the $n-1$ invariants of the vector field Z (obtained by solving the associate characteristic equation), to build $(n-1)$ first integrals for (25). Note that this associates to a single symmetry not one, but $(n-1)$ first integrals; on the other hand, the requirement of determining a λ-Liouville vector field *and* its characteristic lines is a rather strong one; in particular, solving for the invariants of Z is in general not any easier than integrating (25).

3. The Geometric Meaning of Lambda Symmetries

The results reviewed in the previous section are rather impressive, in particular in that they were able to include in the symmetry theory of differential equations some features which appeared, beforehand, to be definitely out of its reach.

Nevertheless, these results were reached by an essentially analytic approach, and thus shed little light on the geometric meaning of lambda-prolongations and symmetries. Also, they were limited to ODEs; one would hope that understanding the geometry of lambda-prolongation would allow for their extension to PDEs. In this respect, two papers were quite instrumental in bridging the gap between analysis and geometry of λ-prolongations, and between ODEs and PDEs as far as λ-symmetries were concerned.

3.1. *The work of Pucci and Saccomandi*

In 2002, Pucci and Saccomandi [57] devoted further study to lambda-symmetries, and stressed a very interesting geometrical property of lambda-prolongations: that is, lambda-prolonged vector fields in J^nM can be characterized as the *most general* vector fields in J^nM which have the same characteristics as some standardly-prolonged vector field.

We stress that if Y is the lambda-prolongation of a vector field X in M, then the characteristics of Y will not be the same as those of the standard prolongation $X^{(n)}$ of X, but as those of the standard prolongation $\widetilde{X}^{(n)}$ of a generally different (for λ nontrivial) vector field $\widetilde{X}$ in M.

This property can also be understood by recalling (5), (7) and making use of a general property of Lie derivatives: indeed, for α any form on J^nM,

$$\mathcal{L}_{\lambda Y}(\alpha) = \lambda Y \lrcorner \mathrm{d}\alpha + \mathrm{d}(\lambda Y \lrcorner \alpha) = \lambda \mathcal{L}_Y(\alpha) + \mathrm{d}\lambda \wedge (Y \lrcorner \alpha). \tag{26}$$

3.2. *The work of Morando*

It was noted [29, 45] that lambda-prolongations can be given a characterization similar to the one discussed above for standard prolongations; that is, with h_i^m and V as above, (20) is equivalent to either one of[g]

$$[D_x, Y] \lrcorner \vartheta = \lambda(Y \lrcorner \vartheta) \quad \forall \vartheta \in \mathcal{E},$$
$$[D_x, Y] = \lambda Y + h_i^m D_m + V.$$

This, as remarked by Morando, also allows to provide a characterization of lambda-prolonged vector fields in terms of their action on the contact forms, analogously to (16).

In this context, it is natural to focus on the one-form $\mu := \lambda \mathrm{d}x$; note this is horizontal for $J^n M$ seen as a bundle over B, and obviously satisfies $D\mu = 0$, with D the total exterior derivative operator. Then, Y *is a lambda-prolonged vector field if and only if, for all* $\vartheta \in \mathcal{E}$,

$$\mathcal{L}_Y(\vartheta) + (Y \lrcorner \vartheta)\mu \in \mathcal{E}. \tag{27}$$

4. Twisted Symmetries for PDEs: Mu-Prolongations and Mu-Symmetries

The result given above about the geometrical characterization (27) of λ-symmetries, immediately opens the way to extend λ-symmetries to PDEs [29]. As here the main object will be the one-form μ, we prefer to speak of "μ-prolongations" and "μ-symmetries".

4.1. *Mu-prolongations*

Consider a semi-basic one-form

$$\mu := \lambda_i \mathrm{d}x^i \tag{28}$$

on $(J^n M, \pi_n, B)$, satisfying $D\mu = 0$. Then we say that the vector field Y in $J^n M$ *μ-preserves the contact structure* if and only if, for all $\vartheta \in \mathcal{E}$,

$$\mathcal{L}_Y(\vartheta) + (Y \lrcorner \vartheta)\mu \in \mathcal{E}. \tag{29}$$

Note that $D\mu = 0$ means $D_i \lambda_j = D_j \lambda_i$ for all i, j; hence *locally* $\mu = D\Phi$ for some smooth real function Φ.

With standard computations [29], one obtains that (29) implies the **scalar μ-prolongation formula**

$$\Psi_{J,i} = (D_i + \lambda_i)\Psi_J - u_{J,m}(D_i + \lambda_i)\xi^m. \tag{30}$$

Let Y as in (2) be the μ-prolongation of the Lie-point vector field X (1), and write the standard prolongation of the latter as $X^{(n)} = \xi^i \partial_i + \Phi_J \partial_u^J$; note that $\Psi_0 = \Phi_0 = \varphi$. We can obviously always write $\Psi_J = \Phi_J + F_J$, and $F_0 = 0$. Then it can be proved [29] that the difference terms F_J satisfy the recursion relation

$$F_{J,i} = (D_i + \lambda_i)F_j + \lambda_i D_J Q \tag{31}$$

where $Q := \varphi - u_i \xi^i$ is the characteristic [25, 55, 59] of the vector field X.

[g] Note that the second of these relations was already remarked — and used — by Muriel and Romero; see Eq. (22) above.

This shows at once that *the μ-prolongation of X coincides with its standard prolongation on the X-invariant space I_X*; indeed, $I_X \subset J^n M$ is the subspace identified by $D_J Q = 0$ for all J of length $0 \le |J| < n$. It follows that the standard PDE symmetry reduction method [25, 55, 59] works equally well when X is a μ-symmetry of Δ as in the case where X is a standard symmetry of Δ; see e.g. [22, 29] for examples.[h]

4.2. *Mu-symmetries for systems of PDEs*

The developments described in the previous subsection do not include the case of (systems of) PDEs for several dependent variables, i.e. the case with $q > 1$ in our present notation. This was dealt within two works [22, 29], to which we refer for details.

In this case the relevant contact forms are

$$\vartheta^a_J := \mathrm{d}u^a_J - u^a_{J,i}\mathrm{d}x^i, \tag{32}$$

and it is convenient to see them as the components of a vector-valued contact form ϑ_J [60]. We will denote by Θ the module over q-dimensional smooth matrix functions generated by the ϑ_J, i.e. the set of vector-valued forms which can be written as $\eta = (R_J)^a_b \vartheta^b_J$ with $R_J : J^n M \to \mathrm{Mat}(q)$ smooth matrix functions.

Correspondingly, the fundamental form μ will be a horizontal one-form with values in the Lie algebra $g\ell(q)$ (the algebra of the group $GL(q)$, consisting of nonsingular q-dimensional real matrices) [60]. We will thus write

$$\mu = \Lambda_i \mathrm{d}x^i \tag{33}$$

where Λ_i are smooth matrix functions satisfying additional compatibility conditions stated and discussed below.

We will say that the vector field Y in $J^n M$ μ-preserves the vector contact structure Θ if, for all $\vartheta \in \Theta$,

$$\mathcal{L}_Y(\vartheta) + (Y \lrcorner (\Lambda_i)^a_b \vartheta^b)\mathrm{d}x^i \in \Theta; \tag{34}$$

this should be compared to standard preservation of the contract structure in the form (16).

In terms of the coefficients of Y, see (19), this is equivalent to the requirement that the Ψ^a_J obey the **vector μ-prolongation formula**

$$\Psi^a_{J,i} = (\nabla_i)^a_b \Psi^b_J - u^b_{J,m}[(\nabla_i)^a_b \xi^m], \tag{35}$$

where we have introduced the (matrix) differential operators

$$\nabla_i := I D_i + \Lambda_i,$$

with I the $q \times q$ identity matrix.

We note for later reference that for vertical vector fields $X = Q^a(\partial/\partial u^a)$, (35) yields for the coefficients of the first prolongation $Y = X + \psi^a_i(\partial/\partial u^a_i)$, simply

$$\psi^a_i = (\nabla_i)^a_b Q^b = D_i Q^a + (R_i)^a_b Q^b. \tag{36}$$

[h]The concept of μ-symmetries is also generalized to an analogue of standard conditional and partial symmetries [15, 20], i.e. partial (conditional) μ-symmetries [22].

If Y is the μ-prolongation of the vector field X, and $Y : S_\Delta \to \mathrm{T}S_\Delta$, we say that X is a μ-symmetry for Δ.

If again we consider a vector field Y as in (19) which is the μ-prolongation of a Lie-point vector field X, and write the standard prolongation of the latter as $X^{(n)} = \xi^i \partial_i + \Phi^a_J \partial^J_a$ (with $\Psi^a_0 = \Phi^a_0 = \varphi^a$), we can write $\Psi^a_J = \Phi^a_J + F^a_J$, with $F^a_0 = 0$. Then the difference terms F_J satisfy the recursion relation

$$F^a_{J,i} = \delta^a_b [D_i(\Gamma^J)^b_c](D_j Q^c) + (\Lambda_i)^a_b[(\Gamma^J)^b_c(D_J Q^c) + D_J Q^b], \tag{37}$$

where $Q^a := \varphi^a - u^a_i \xi^i$, and Γ^J are certain matrices (see [29] for the explicit expression). This, as for the scalar case, shows that the μ-prolongation of X coincides with its standard prolongation on the X-invariant space I_X; hence, again, the standard PDE symmetry reduction method works equally well for μ-symmetries as for standard ones. See [29] for examples.

4.3. *Compatibility condition, and gauge equivalence*

As mentioned above the form μ, see (33), is not arbitrary: it must satisfy a compatibility condition (this guarantees the Ψ^a_J defined by (35) are uniquely determined), expressed by

$$[\nabla_i, \nabla_k] \equiv D_i \Lambda_k - D_k \Lambda_i + [\Lambda_i, \Lambda_k] = 0. \tag{38}$$

It is quite interesting to remark [22] that this is nothing but the coordinate expression for the horizontal Maurer–Cartan equation[i]

$$D\mu + \frac{1}{2}[\mu, \mu] = 0. \tag{39}$$

Based on this condition, and on classical results of differential geometry [14,24,58] and a theorem by Marvan [39], it follows that locally in any contractible neighborhood $A \subseteq J^n M$, there exists $\gamma_A : A \to GL(q)$ such that (locally in A) μ is the Darboux derivative of γ_A.[j]

In other words, any μ-prolonged vector field is *locally* gauge-equivalent to a standard prolonged vector field [22], the gauge group being $GL(q)$.

It should be mentioned that when $J^n M$ is topologically nontrivial, or μ presents singular points, one can have nontrivial μ-symmetries; this is shown by means of very concrete examples in [22].

Finally, we note that when we consider symmetries of a given equation Δ, the compatibility condition (39) needs to be satisfied only on $S_\Delta \subseteq J^n M$. Indeed when μ is not satisfying everywhere (39), μ-symmetries can happen to be gauge-equivalent to standard *nonlocal symmetries* of exponential form, as noted in [41] and also remarked in [22]; in this respect, see also Subsec. 4.5 below.

[i] This expresses the requirement that the standard Maurer–Cartan equation is satisfied modulo contact forms, i.e. $\mathrm{d}\mu + (1/2)[\mu, \mu] \in \mathcal{E}$.

[j] In a different context, it should also be stressed that the compatibility condition (38) — or equivalently the appearance of the horizontal Maurer–Cartan equation — also hints at a relation between μ-symmetries and *zero curvature representations* for PDEs [39], and hence the theory of integrable systems. This aspect of μ-symmetries has not been studied, and could provide interesting results.

4.4. *The work of Cicogna: Rho-symmetries*

The approach of μ-symmetries is inherently multi-dimensional[k]; one could thus hope that it would provide better results for the analysis of systems of ODEs.

This task was undertaken by Cicogna, who focused on systems of first order ODEs [17]. He considered in particular the possibility of reducing such a system; reduction is achieved by passing to suitable (symmetry-adapted) coordinates.

Consider a system of n equations for $u^a(x)$, $a = 1, \dots, n$, given by

$$F^a(x, u, u_x) = 0 \quad (a = 1, \dots, n) \tag{40}$$

and which admits a μ-symmetry X, with $\mu = \Lambda \mathrm{d}x$; let $(y, w^1, \dots, w^{n-1}, z)$ be the symmetry adapted coordinates, with y the new independent variable and z the invariant dependent variable. Then the system (40), when written in the (y, w, z) variables, depend only on the (y, w) variables and on the n first order differential invariants ζ^a, obtained solving the characteristic equation

$$\mathrm{d}z = \frac{\mathrm{d}w^a_y}{M^{(a)}} = \frac{\mathrm{d}z_y}{M^n}.$$

Here the $M^{(a)}$ are matrix functions given by

$$M^{(a)} = \frac{\partial w^a}{\partial u^b}(\Lambda Q)^b \;\; (a = 1, \dots, n-1), \quad M^{(n)} = \frac{\partial z}{\partial u^b}(\Lambda Q)^b.$$

We have denoted by Q the characteristic of the vector field X: if this is written in the form (1), we have $Q^a = \varphi^a - \xi u^a_x$.

This results gets more clear if instead of general first order systems (40) we consider dynamical systems,

$$u^a_x = f^a(x, u). \tag{41}$$

In this case one would like that the transformation to symmetry adapted coordinates preserves the functional form of the system, i.e. to get

$$w^a_y = g^a(y, w, z) \;\; (a = 1, \dots, n-1), \quad z_y = h(y, w, z).$$

In this case, assuming again (41) admits X as a μ-symmetry with $\mu = \Lambda \mathrm{d}x$, we actually have

$$(\partial g^a / \partial z) = M^{(a)}, \quad (\partial h / \partial z) = M^{(n)}.$$

If $M^{(a)} = 0$ for all a, so that no f^a depends on z, the system splits into an $(n-1)$-dimensional system for the w^a plus a scalar ODE $z_y = h(y, w, z)$ which can be seen as a "reconstruction equation" whose solution allows to pass from solutions to the reduced system to solutions to the full one and hence to the original system (41).

This result is rather similar to what is obtained for standard symmetries, but it should be noted that while in that case the "last equation" reduces to a quadrature,[l] in this case

[k] Actually the one-dimensional case is a very degenerate one, first of all because the compatibility condition is trivially satisfied.

[l] If $\Lambda = 0$, so that we have a standard symmetry, we obviously have $M^{(a)} = M^{(n)} = 0$, and the reconstruction equation is a quadrature. Conversely, one can prove that if $M^{(a)} = M^{(n)} = 0$, then X is a standard symmetry [17].

we have in general to deal with a possibly nontrivial equation. When $\Lambda = \lambda I$, one is reduced to the situation studied by Muriel and Romero [45], see Subsec. 2.3.

As stressed by Cicogna, while in general λ-symmetries lead to a lowering of the order of the equation under study (or of one of the equations in the system), μ-symmetries for dynamical system do not reduce the order of the system, but rather split it into a reduced one and a reconstruction equation. Thus, he suggests that this special kind of twisted symmetry is given the name of ρ-symmetries, where ρ stands for "reducing" (in this context, λ would also mean "lowering") [17].

4.5. *Twisted symmetries and nonlocal Lie-point symmetries for PDEs*

The connection between twisted symmetries and standard nonlocal ones, already discussed in the ODEs case (see above), is also present for PDEs.

In this case one considers nonlocal vector fields of the form

$$X = e^{\int P_i(x,u^{(n)})dx^i} Z$$

with X_0 a local vector field in M. This is a nonlocal exponential symmetry of Δ if Z is a symmetry of Δ and P satisfies, at least on the solution manifold S_Δ, the compatibility condition $D_i P_j = D_j P_i$.

Then, consider the form $\mu = P_i \mathrm{d}x^i$. If Z is a μ-symmetry of Δ, and $D_i P_j = D_j P_i$ on S_Δ, then X is a (standard) nonlocal exponential symmetry for Δ; and conversely, if X as above is a (standard) nonlocal exponential symmetry for Δ, then Z is a μ-symmetry for Δ [22].

4.6. *Twisted symmetries and gauging the exterior derivative*

It was remarked by Morando [40] that μ-prolongations and symmetries can also be described in terms of a deformation of the derivation operations (Lie derivative and exterior derivative).

In fact, one can consider a deformed differential based on a smooth function f and corresponding to a "gauging" by f:

$$\mathrm{d}^{\mathrm{d}f}\beta := e^{-f}\mathrm{d}(e^f\beta) = \mathrm{d}\beta + \mathrm{d}f \wedge \beta. \tag{42}$$

It follows by straightforward computation that this is a first order differential operator, preserving wedge product, and satisfying $\mathrm{d}^{\mathrm{d}f} \circ \mathrm{d}^{\mathrm{d}f} = 0$ (so that one can build complexes and a cohomology on its basis).

Similarly, one can consider a deformed Lie derivative based on a smooth function f, defined on forms as

$$\mathcal{L}_X^{\mathrm{d}f}\beta := e^{-f}\mathcal{L}_{(e^f X)}\beta = \mathcal{L}_X\beta + \mathrm{d}f \wedge (X \lrcorner \beta). \tag{43}$$

This coincides with the ordinary Lie derivative on functions, and in general satisfies

$$\mathcal{L}_X^{\mathrm{d}f}(\beta_1 + \beta_2) = \mathcal{L}_X^{\mathrm{d}f}\beta_1 + \mathcal{L}_X^{\mathrm{d}f}\beta_2;$$
$$\mathcal{L}_X^{\mathrm{d}f}(\beta_1 \wedge \beta_2) = (\mathcal{L}_X^{\mathrm{d}f}\beta_1) \wedge \beta_2 + \beta_1 \wedge (\mathcal{L}_X^{\mathrm{d}f}\beta_2).$$

This deformed Lie derivative will be defined on vector fields as

$$\mathcal{L}_X^{\mathrm{d}f}(Y) := e^{-f}\mathcal{L}_{(e^f X)}Y = \mathcal{L}_X Y - (Y \lrcorner \mathrm{d}f)X. \tag{44}$$

It is immediate to check [40] this entails, for all vector fields X, Y and forms β on M,

$$\mathcal{L}_X^{\mathrm{d}f}(Y \lrcorner \beta) = \mathcal{L}_X^{\mathrm{d}f}(Y) \lrcorner \beta + Y \lrcorner (\mathcal{L}_X^{\mathrm{d}f}\beta).$$

Moreover, Cartan formulas — with the deformed exterior derivative taking the place of the ordinary one — hold for the deformed Lie derivative:

$$\begin{aligned} \mathcal{L}_X^{\mathrm{d}f}(\beta) &= X \lrcorner \mathrm{d}\beta + \mathrm{d}^{\mathrm{d}f}(Y \lrcorner \beta), \\ \mathcal{L}_X^{\mathrm{d}f}(\mathrm{d}\beta) &= \mathrm{d}^{\mathrm{d}f}(\mathcal{L}_X^{\mathrm{d}f}(\beta)). \end{aligned} \tag{45}$$

As stressed by notation, the deformed derivatives actually depend on $\mathrm{d}f$ rather than on f; it is thus quite natural to define them depending on a general closed (but not necessarily exact) one-form $\mu \in \Lambda^1(M)$. This yields

$$\mathrm{d}^\mu \beta := \mathrm{d}\beta + \mu \wedge \beta; \tag{46}$$

$$\mathcal{L}_X^\mu \beta := \mathcal{L}_X \beta + \mu \wedge (X \lrcorner \beta), \tag{47}$$

$$\mathcal{L}_X^\mu Y := \mathcal{L}_X Y - (Y \lrcorner \mu) X. \tag{48}$$

One can check, again by explicit computation [40] that $\mathrm{d}\mu = 0$, $\mathrm{d}^\mu \circ \mathrm{d}^\mu = 0$, and $\mathcal{L}_X^\mu \mathrm{d}\beta = \mathrm{d}^\mu(\mathcal{L}_X^\mu \beta)$ are all equivalent.

Needless to say, these definitions can be extended from M to $J^k M$, i.e. to Jet bundles of any order, in a rather obvious way. In this case denote, as earlier on, by $\mathcal{E}$ the contact ideal (the ideal generated by the contact forms). Then, $\mathrm{d}\mu \in \mathcal{E}$, $\mathrm{d}^\mu(\mathrm{d}^\mu \beta) \in \mathcal{E}$ and $\mathcal{L}_X^\mu(\mathrm{d}\beta) - \mathrm{d}^\mu(\mathcal{L}_X^\mu \beta) \in \mathcal{E}$ are all equivalent.

Consider now a vector field Y on $J^n M$, which leaves M invariant and which reduces to X when restricted to M. It can be proven [40] that Y is the μ-prolongation of X if and only if the deformed Lie derivative preserves $\mathcal{E}$, i.e.

$$\mathcal{L}_Y^\mu(\theta) \in \mathcal{E} \quad \forall \theta \in \mathcal{E}.$$

Moreover, with $\mathcal{D}$ the distribution generated by the operators D_i, Y is the μ-prolongation of X if and only if

$$\mathcal{L}_Y^\mu(Z) \in \mathcal{D} \quad \forall Z \in \mathcal{D}.$$

This formalism is also helpful when one considers variational twisted symmetries; the latter will be discussed in a later section.

5. Twisted Symmetries and Gauged Vector Fields

The discussion so far, in particular for ODEs, mentioned at several points the concept of *gauge transformations*. This is one of the central ideas in 20th century Physics [14, 52, 53], and in a way it is surprising that it has been absent so far in the symmetry theory of differential equations.[m]

As stated above, see Sec. 4, μ-prolonged vector fields are (locally) gauge equivalent to standard-prolonged ones. More precisely, if Y is the μ-prolongation of a vector field X,

[m]It is maybe worth warning the experienced reader that the gauge transformations to be considered here are, in general, of a more general type than standard Yang–Mills ones.

then there are vector fields W and Z, gauge-equivalent via the same gauge transformation (acting respectively in $\mathrm{T}(J^k)M$ and in $\mathrm{T}(M)$) to Y and X, and such that W is the standard prolongation of Z. This is schematically summarized in the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & Z \\ \Big\downarrow{\scriptstyle \mu-\mathrm{prol}} & & \Big\downarrow{\scriptstyle \mathrm{prol}} \\ Y & \xrightarrow{\gamma^{(k)}} & W \end{array}$$

For these considerations, it is convenient to deal with evolutionary representatives of vector fields [55], which we will implicitly do.

We recall that the gauge group Γ (modeled over a Lie group G) acts in the same way on the vector $\{\varphi^1, \ldots, \varphi^q\}$ of the components of the vector field X in M, and on the vectors $\{\psi^1_J, \ldots, \psi^q_J\}$ of components (relative to a given multi-index J, i.e. to partial derivatives with respect to the same array of independent variables) of the vector field Y in J^kM. One also says that Γ acts via a *Jet representation*.

It is thus quite natural to consider a framework in which gauge transformations are also taken into account; that is — similarly to what happens in gauge theories of Theoretical Physics — one introduces, beside dependent and independent variables, also *gauge variables*; these keep track of the reference frame changes [27].[n]

At difference with the approach of gauge theories, however, the tradition in Applied Mathematics (in general, and in the symmetry theory of differential equations in particular) is to deal with standard partial derivatives rather than with *covariant derivatives*. In the specific prolongations framework, this means that the total derivatives appearing in the prolongation formula (5)

$$\psi^a_{J,i} = D_i \psi^a_J$$

(for evolutionary vector field the prolongation formula reduces to this) change when we change frame.

Let us choose a representation for G, and a basis $\{L_1, \ldots, L_r\}$ of (left-invariant) vector fields for the Lie algebra $\mathcal{G}$ of G; then the gauge transformations can be written as

$$S(x) = \exp[\alpha^i(x) L_i]. \tag{49}$$

Denote by $\{\mathbf{e}_1, \ldots, \mathbf{e}_p\}$ the reference frame for the tangent space $\mathrm{T}U$ to the manifold of dependent variables, and by $\{\mathbf{f}_1, \ldots, \mathbf{f}_p\}$ another frame, related to the previous one by

$$\mathbf{f}_a(x) = S_a^{\ b}(x) \mathbf{e}_b(x).$$

A field $\Phi(x)$ will be given by

$$\Phi(x) = u^a(x) \mathbf{f}_a(x)$$

[n]It has to be stressed that here one considers changes of reference frame, and *not* changes of variables.

with u^a the components in the $\mathbf{f}$ frame; in the $\mathbf{e}$ frame, we would have

$$\Phi(x) = u^a(x) S_a^b(x) \mathbf{e}_b(x) := v^a(x) \mathbf{e}_a(x).$$

If now we look at x-derivatives, we get (we omit to indicate the x dependence for the sake of notation)

$$D_i \Phi = u_i^a \mathbf{f}_a$$

in the $\mathbf{f}$ frame; while in the $\mathbf{e}$ frame we get, using the representation (49),

$$D_i \Phi = [\alpha^m (L_m^T)_b^a v_i^b + \alpha_i^m (L_m^T)_b^a v^b]. \tag{50}$$

That is, field derivatives do not change in the same way as the field; as well known this phenomenon can be eliminated by considering *covariant derivatives* instead of standard ones [14, 52, 53].

We will thus consider gauge variables beside standard ones; making use of the basis $\{L_1, \dots, L_r\}$ we can take the α^m as gauge variables. These index the gauge transformation, see (49) above. The function $\alpha : B \to \mathcal{G}$ is identified with a section of a bundle $(\mathcal{A}_\mathcal{G}, \pi_\mathcal{G}, B)$, which is an associated bundle to P_G (the principal bundle of fiber G over B defining the gauge action).

In this way, the phase bundle is augmented from (M, π, B) to $\widehat{M} = (M \oplus \mathcal{A}_\mathcal{G}, \pi \oplus \pi_\mathcal{G}, B)$. Correspondingly one would consider vector fields

$$\widehat{X} = \xi^i(x,u,\alpha) \frac{\partial}{\partial x^i} + \varphi^a(x,u,\alpha) \frac{\partial}{\partial u^a} + A^m(x,u,\alpha) \frac{\partial}{\partial \alpha^m}; \tag{51}$$

we will actually consider their evolutionary representatives,

$$X = Q^a(x,u,\alpha;u_x,\alpha_x) \frac{\partial}{\partial u^a} + P^m(x,u,\alpha;u_x,\alpha_x) \frac{\partial}{\partial \alpha^m} \tag{52}$$

with components

$$Q^a = \varphi^a - u_i^a \xi^i, \quad P^m = A^m - \alpha_i^m \xi^i.$$

Note these imply that the Q^a will not depend on α_x, nor the P^m on u_x.

If X can be expressed as the gauge transformed of a vector field in M, we say it is a *reframed* vector field. It is immediate to see that such vector fields can be characterized as those for which Q can be written in the form

$$Q^a(x,u,\alpha;u_x,\alpha_x) = [K_b^a(\alpha)] Q_0^b(x,u,u_x) \tag{53}$$

with $K(\alpha) \in G$. One can choose $K(0) = I$; in general we will have $K(\alpha) = \exp[\alpha^m L_m]$. We then say that X is gauge equivalent to the vector field $Z = Q_0^a(x,u,u_x)(\partial/\partial u^a)$.

When considering prolongations in $J\widehat{M}$, the prolongation operation will also act on the components in the α^m direction. Applying the (standard) prolongation formula on X we will obtain a vector field

$$Y = \psi_J^a \frac{\partial}{\partial u_J^a} + \chi_J^m \frac{\partial}{\partial \alpha_J^m} = (\widehat{D}_J Q^a) \frac{\partial}{\partial u_J^a} + (\widehat{D}_J P^m) \frac{\partial}{\partial \alpha_J^m} \tag{54}$$

(the sum is also over all multi-indices J of suitable module $|J|$).

Here the notation $\widehat{D}_J$ recalls that total derivatives should be computed by taking into account the gauge variables $\alpha^m(x)$ as well:

$$\widehat{D}_i = \frac{\partial}{\partial x^i} + u^a_{J,i} \frac{\partial}{\partial u^a_J} + \alpha^m_{J,i} \frac{\partial}{\partial \alpha^m_J} := D_i + Z_i. \tag{55}$$

If now we apply (54) on vector fields of the form (53), we get — for coefficients of first derivative variables — in explicit form [27]

$$\psi^a_i = (D_i Q^a) + (R_i)^a_b Q^b \tag{56}$$

where the matrices R_i are defined as

$$R_i = [Z_i(K)]K^{-1}. \tag{57}$$

These matrices satisfy

$$Z_i R_j - Z_j R_i + [R_i, R_j] = 0; \tag{58}$$

this is nothing else than the horizontal Maurer–Cartan equation[o] for the R_i. This also guarantees no compatibility problem will arise when considering higher order prolongations.

It is obvious that (56) will reproduce (36) if and only if $R_i = \Lambda_i$. This relation is, however, by itself meaningless as $R_i = R_i(\alpha, \alpha_x)$ and $\Lambda_i = \Lambda_i(x, u, u_x)$. On the other hand, it makes sense if we consider it *on a section of the gauge bundle* $(J^k\widehat{M}, \rho_k, J^k M)$ with fiber $\rho^{-1}(p) = \mathrm{T}^k \mathcal{G}_p$.

A section σ_γ of the gauge bundle is defined in local coordinates by

$$\gamma^m := \alpha^m - f^m(x, u) = 0;$$

its lift to first derivatives is hence given by

$$\gamma^m_i := \alpha^m_i - \left(\frac{\partial f^m}{\partial x^i} + u^a_i \frac{\partial f^m}{\partial u^a} \right) = \alpha^m_i - (D_i f^m) = 0.$$

Therefore, the restriction of $R_i = \Lambda_i$ to the section σ makes perfect sense, and reads

$$\Lambda_i = [\alpha^m_i]_\sigma L_m = (D_i f^m) L_m. \tag{59}$$

It should be noted that the restriction to a gauge section σ makes sense provided this section is itself invariant under the action of the vector field we are considering; this is the case provided

$$[P^m - (\partial_a f^m) Q^a]_\sigma = 0.$$

This can always be achieved by simply requiring $P^m = (\partial_a f^m) Q^a$; in facts, our discussion constrained Q to the form (53), but did not set any constraint on P. Finally, let us note that a section σ of the gauge bundle $(J^k\widehat{M}, \rho_k, J^k M)$ is by definition isomorphic to $J^k M$;

[o]Note that as the K only depends on α, actually $Z_i R_j = \widehat{D}_i R_j$; hence this equation is equivalently written as $\widehat{D}_i R_j - \widehat{D}_j R_i + [R_i, R_j] = 0$.

thus the vector field $X_\sigma^{(k)}$ (the restriction of X to σ) in σ uniquely defines a vector field in J^kM.

In this way we have established a well-defined relation between reframed (via a gauge transformation K) vector fields on $J^k\widehat{M}$, restricted to a gauge section α, and μ-prolonged vector fields on J^kM. The form μ is given by $\mu = \Lambda_i \mathrm{d}x^i$, and the Λ_i satisfy

$$\Lambda_i = [K^{-1} Z_i(K)]_{\sigma^{(1)}}. \tag{60}$$

The reader is referred to [27] for further details and other results on this line.

6. A Gauge-Theoretic Approach to Twisted Symmetries

The discussion of the previous section (and of [27]) was based on considering the gauge variables as new dependent variables; as seen above, this led to some inconsistency, which could be cured only by restricting on a given section, i.e. *de facto* forcing the gauge variables α^m to depend on (x, u) in a given manner.

This situation is of course not satisfactory, and calls for a fully coherent — and fully gauge-theoretic — treatment. This was proposed in a recent paper [28], and we report here the main lines of the construction proposed there. We also refer to [28] (see the appendix there) for a discussion of the relation of this approach to that of [27], and also to the approach to μ-symmetries based on the formalism of coverings [12].

In the approach discussed in the previous section, gauge variables were considered as new, auxiliary, dependent variables and treated as such in complete parallel to standard dependent variables (fields) up to the point where restriction to a section of the gauge bundle was needed. This approach is quite unnatural to anybody familiar with gauge theories, as gauge variables are different than standard fields (matter fields in usual gauge theories [14, 52, 53]) and should be treated accordingly. Moreover, the gauge variables control a change of reference frame, which is the same for derivatives of any order — in other words, we do not need to consider prolongations of the gauge variables beyond order one.[p]

The key construction is still that of gauge bundles sketched in Sec. 5; by this we mean both the basic gauge bundle $(\rho, \widehat{M}, M)$ with fiber the gauge group Γ (i.e. the set of maps $\gamma : M \to G$), and the higher order gauge bundles $(\rho_k, J^k\widehat{M}, J^kM)$ which also have fiber Γ; this is an important feature.

When one considers that M is also a bundle (π, M, B) and that jet bundles J^kM have several fiber structures, and in particular can be seen as bundles both over B — in which case we write (π_k, J^kM, B) — and over M — in which case we write (σ_k, J^kM, M) — and moreover that the same considerations hold for $J^k\widehat{M}$, the relations among all these structures are embodied in the following "star" diagram [28]:

$$\begin{array}{ccccc} J^k\widehat{M} & & \xrightarrow{\rho_k} & & J^kM \\ & \searrow^{\widehat{\pi}_k} & & \swarrow^{\pi_k} & \\ \downarrow \widehat{\sigma}_k & & B & & \downarrow \sigma_k \\ & \nearrow^{\widehat{\pi}} & & \nwarrow^{\pi} & \\ \widehat{M} & & \xrightarrow{\rho} & & M \end{array} \tag{61}$$

[p]The connection form is related to derivatives of the gauge fields through the well known formula $A = g^{-1}\mathrm{d}g$.

As stressed above, we have (topologically) $J^k\widehat{M} = J^k M \times \Gamma$; that is, the jets only concern standard variables, and not gauge ones. Hence, the prolongation operation does not involve the latter.

Thus, the prolongation operation leading from $\widehat{M}$ to $J^k\widehat{M}$ should be based on the usual total derivative operators D_i, and hence does *not* involve derivation with respect to the gauge variables. On the other hand, a vector field in $\widehat{M}$ will have components both in the M and in the G directions, the prolongation operation should be applied *only* to the M components.

As usual, it will be convenient to consider the vector bundle associated to $J^k\widehat{M}$; in concrete terms, this means passing to consider coordinates α^m in the Lie algebra $\mathcal{G}$ of Γ, similarly to what we made in the previous section.[q] For ease of notation we will keep the same notation $J^k\widehat{M}$ for this.[r]

A vector field in $\widehat{M}$ will then be $\widehat{X} = \xi^i(x,u,\alpha)(\partial/\partial x^i) + \varphi^a(x,u,\alpha)(\partial/\partial u^a) + B^m(x,u,\alpha)(\partial/\partial\alpha^m)$. As usual in considerations involving vector fields on jet bundles, it will be convenient to work with evolutionary representatives [55, 59]; we will consistently use these. The evolutionary representative of $\widehat{X}$ is

$$X \equiv \widehat{X}_v = Q^a \frac{\partial}{\partial u^a} + P^m \frac{\partial}{\partial \alpha^m}, \tag{62}$$

where $Q^a := \varphi^a - u^a_i \xi^i$, $P^m := B^m$.

The coordinate expression of the prolongation $X^{(k)} \in \mathcal{X}(J^k\widehat{M})$ of X is given by the (standard) prolongation formula; we stress once again that no prolongation of α^m components appears. For the evolutionary representative $Y := (X^{(k)})_v = X_v^{(k)}$ we get, with $Q^a_J = D_J Q^a$,

$$Y = Q^a_J \frac{\partial}{\partial u^a_J} + P^m \frac{\partial}{\partial \alpha^m}. \tag{63}$$

We are specially interested in a particular class of vector fields, i.e. those for which

$$Q^a(x,u,g;u_x) = [\Psi(g)]^a_b \Theta^b(x,u;u_x). \tag{64}$$

These will be called, for obvious reasons, **gauged vector fields**.

Using local coordinates (x,u,α), Eq. (64) becomes

$$Q^a(x,u,\alpha;u_x) = [K(\alpha)]^a_b \Theta^b(x,u;u_x). \tag{65}$$

Here $K(\alpha)$ is the representation of the group element $g(\alpha) = \exp(\alpha)$; i.e. if G acts in U via the representation Ψ, we have $K(\alpha) = \Psi[\exp(\alpha)]$. Note that (64) and (65) do not constrain in any way the components P^m of the vector fields along the α^m variables.

[q]We can also think of this operation as a restriction to a neighborhood of a reference section in the gauge bundle, in which we can use Lie algebra coordinates.

[r]A more careful discussion is provided in [28], to which we refer for all missing details.

Keeping in mind that the total derivative operators do not act on gauge variables, we obtain immediately that

$$Q^a_J = D_J Q^a = [K(\alpha)]^a_b D_J \Theta^b. \tag{66}$$

This implies that — writing $\Theta_J = D_J \Theta$ — the prolongation Y of the vector field X, see (63), is given by

$$Y = [K(\alpha)]^a_b \Theta^b_J \frac{\partial}{\partial u^a_J} + P^m \frac{\partial}{\partial \alpha^m}. \tag{67}$$

Now, if $X_0 = \rho_* X$ and $Y_0 = \rho_*^{(k)} Y$ are the projection of the vector fields X and Y to, respectively, M and $J^k M$, we can state formally that Y_0 *is the μ-prolongation of X_0 for a suitable μ.* In fact, we have

$$X_0 = \rho_* X = [K(\alpha)]^a_b \Theta^b \frac{\partial}{\partial u^a}; \quad Y_0 = \rho_*^{(k)} Y = [K(\alpha)]^a_b \Theta^b_J \frac{\partial}{\partial u^a_J}. \tag{68}$$

Thus X_0 and Y_0 are the gauge transformed — via the same gauge transformation — of vector fields $= X_0$ and $= Y_0$ such that $= Y_0$ is the ordinary prolongation of $= X_0$.

Note this statement is only formal, as the α variables have no meaning when we work in M and $J^k M$; in order to make this into a real theorem, we will need to "fix the gauge", as discussed below (and similarly to what has been done in the previous section). Note also the relation between X_0 and Y_0 depends substantially on the assumption that X is a gauged vector field.

Fixing the gauge means selecting a section γ of the gauge bundle. We will denote by $\omega^{(\gamma)}$ the operator of restriction from $\widehat{M}$ to $\widehat{M}_\gamma$, and by $\rho^{(\gamma)}$ the restriction of the projection $\rho : \widehat{M} \to M$ to $\widehat{M}_\gamma$. We also denote by $\omega_k^{(\gamma)} : J^k \widehat{M} \to \widehat{M}_\gamma^{(k)}$ and by $\rho_k^{(\gamma)} : \widehat{M}_\gamma^{(k)} \to J^k M$ the lift of the maps $\omega^{(\gamma)}$ and $\rho^{(\gamma)}$ to maps between corresponding jet spaces of order k.[s]

We will summarize relations and maps between relevant fiber bundles in the following diagram:

$$\begin{array}{ccccc} \widehat{M} & \xrightarrow{\omega^{(\gamma)}} & \widehat{M}_\gamma & \xrightarrow{\rho^{(\gamma)}} & M \\ \big\downarrow {\scriptstyle j^k[\widehat{M}]} & & \big\downarrow {\scriptstyle j^k[\widehat{M}_\gamma]} & & \big\downarrow {\scriptstyle j^k[M]} \\ J^k \widehat{M} & \xrightarrow{\omega_k^{(\gamma)}} & \widehat{M}_\gamma^{(k)} & \xrightarrow{\rho_k^{(\gamma)}} & J^k M \end{array} \tag{69}$$

As anticipated, passing to consider $\widehat{M}_\gamma$ rather than the full $\widehat{M}$, and $\widehat{M}_\gamma^{(k)}$ rather than the full $\widehat{M}^{(k)}$, corresponds in physical terms to a *gauge fixing*.

Needless to say, restriction to M_γ makes sense only if this manifold is invariant under the vector field we are considering; it is easy to check that if X be in the form (62), then the submanifold $\widehat{M}_\gamma \subset \widehat{M}$ identified by $\alpha^m = A^m(x,u)$ is invariant under X_γ if and only if

$$P^m_\gamma = (\partial A^m / \partial u^a) Q^a_\gamma. \tag{70}$$

[s]Note that while ρ is of course not invertible, it follows from $M_\gamma \simeq M$ that $\rho^{(\gamma)}$ is invertible, with $(\rho^{(\gamma)})^{-1} = \gamma$. similarly, $\rho^{(\gamma)}$ is invertible.

As remarked above, P_m was not constrained by our previous considerations, so it can be adjusted to the M_γ, i.e. to the gauge fixing γ, we wish to consider. More precisely, one can show [28] that: given arbitrary smooth functions $Q^a(x,u,\alpha)$, and an arbitrary section γ of the gauge bundle, there is always a vector field X_v of the form $X_v = Q^a\partial_a + P^m\partial_m$ such that X_v leaves $\widehat{M}_\gamma$ invariant.

It should also be noted that X_γ projects in turn to a vertical vector field W on M, $W = Q^a(x,u)(\partial/\partial u^a)$; and conversely any such W lifts to a vertical vector field $W^\gamma = X_\gamma$ on $\widehat{M}_\gamma$, $W^\gamma = Q^a(x,u)[(\partial/\partial u^a) + ((\partial A^m/\partial u^a)(\partial/\partial \alpha^m)]$.

We can then consider gauging and prolongation of vector fields. In particular, in studying twisted symmetries one is led to consider if it is possible to find suitable operators $\widehat{\mathtt{P}}_\gamma^{(k)}$ and $\mathtt{P}_\gamma^{(k)}$ such that, for a given X on M, we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\omega_*^{(\gamma)}} & X_\gamma & \xrightarrow{\rho_*^{(\gamma)}} & W \\ \downarrow{\scriptstyle \mathrm{Pr}\,k[\widehat{M}]} & & \downarrow{\scriptstyle \widehat{\mathtt{P}}_\gamma^{(k)}} & & \downarrow{\scriptstyle \mathtt{P}_\gamma^{(k)}} \\ X^{(k)} & \xrightarrow{(\omega_k^{(\gamma)})_*} & X_\gamma^{(k)} & \xrightarrow{(\rho_k^{(\gamma)})_*} & Y \end{array} \tag{71}$$

We will just give the main results, referring again the reader to [28] for a complete discussion. We will introduce, in order to state our result in a compact form, operators $\tau^{(\gamma)} := \rho^{(\gamma)} \circ \omega^{(\gamma)}$, $\tau^{(\gamma)} : \widehat{M} \to M$; and correspondingly $\tau_k^{(\gamma)} := \rho_k^{(\gamma)} \circ \omega_k^{(\gamma)}$, $\tau_k^{(\gamma)} : \widehat{M}^{(k)} \to M^{(k)}$.

Then we have that: *The twisted prolongation operator* $\mathtt{P}_\gamma^{(k)}$ *is uniquely defined by the requirement that* $(\tau_k^{(\gamma)})_* \circ (\mathrm{Pr}\,k[\widehat{M}]) = \mathtt{P}_\gamma^{(k)} \circ \tau_*^{(\gamma)}$. *Moreover,* $\mathtt{P}_\gamma^{(k)}$ *corresponds to the μ-prolongation operator of order k with* $\mu = [\mathrm{d}\Psi(\gamma)]\Psi(\gamma^{-1})$. *With the local coordinates* (x,u,α), *this corresponds to* $\mu = \Lambda_i \mathrm{d}x^i$ *where* $\Lambda_i = -R_i^{(\gamma)} = -(D_i K_\gamma)K_\gamma^{-1} = K_\gamma(D_i K_\gamma^{-1})$.

A converse of the result above is as follows: *Let Y be the kth μ-prolongation of the evolutionary vector field W on M, with* $\mu \in \Lambda^1(J^1M, \psi(\mathcal{G}))$ *given in coordinates by* $\mu = \Lambda_i(x,u,u_x)\mathrm{d}x^i$. *Then*: (i) *there is a section γ of the gauge bundle such that* $Y = \mathtt{P}_\gamma(W)$. (ii) *there is a vertical vector field X in $\widehat{M}$ such that* (71) *applies.* (iii) *The matrix function* $K_\gamma(x,u,u_x)$ *satisfies* $D_i K_\gamma = -\Lambda_i K_\gamma$.

To conclude this (long) section, let us briefly comment on the geometrical meaning of the main results just given. We have shown that the μ-prolongation operator appears if we are insisting in restricting our analysis to the phase bundle M (or to the sub-bundle $\widehat{M}_\gamma \subset \widehat{M}$ seen as an image of M under the gauge map γ embedding it into $\widehat{M}$) rather than to the full gauge bundle $\widehat{M}$. The fact we are considering projections of vector fields in $\widehat{M}_\gamma \subset \widehat{M}$ and $\widehat{M}_\gamma^{(k)} \subset \widehat{M}^{(k)}$ to vector fields in M and $M^{(k)}$ makes that the relation between basic vector fields and prolonged ones is not the natural one, described by the prolongation operator, but is the "twisted" one described by the μ-prolongation operator. See also the discussion in the Appendix to [28].

7. Twisted Symmetries and Variational Problems

As well known, symmetry analysis — and symmetry reduction — are specially powerful in dealing with *variational problems*; the key result relating standard symmetry and reduction is in this framework the classical Noether theorem [54], see also [35, 55].

It is thus not surprising that the application of twisted symmetries to variational problems is specially fruitful and — in this author's opinion — also specially fascinating.

Unfortunately, this paper is already way too long — which also accounts for the lack of detailed examples in it — so that we will just very sketchily mention the sources of the main results obtained in this direction, leaving to the reader to look directly the original papers.[t]

The first study in this direction was conducted by Muriel, Romero and Olver [51]; they considered variational problems defined by a Lagrangian (of arbitrary order) in one dependent and one independent variable, and studied both how λ-prolongations allow to construct new methods for the reduction of Euler–Lagrange equations and a version of Noether's theorem adapted to λ-symmetries. In this case one focuses on "variational λ symmetries" — which are the twisted symmetries analogue of standard variational symmetries [55] — and obtains partial conservation laws. With L the Lagrangian, X a vector field and Y its λ-prolongation, X is a variational λ-symmetry of L if

$$Y(L) + L(D+\lambda)\xi = (D+\lambda)B \tag{72}$$

for some function $B(x, u, u_x, \ldots)$.

If the nth order Lagrangian L admits a variational λ-symmetry, then there exists a Lagrangian $\widehat{L}$ of order $(n-1)$ such that a $(2n-1)$-parameter family of solutions to the variational problem described by L can be found from the solution to the variational problem described by $\widehat{L}$ by solving an auxiliary first order equation. The result extends to generalized variational λ-symmetries.

Moreover, denote by $E[L]$ the Euler–Lagrange equation corresponding to L. If L admits a variational λ-symmetry X, and Q is the characteristic of X, then there is some $P(x, u, u_x, \ldots)$ such that

$$QE[L] = (D+\lambda)P. \tag{73}$$

It turns out [51] that in this case X is a λ-symmetry for the equation $P[u] = 0$; the reduced equation obtained from $P = 0$ using the symmetry X is, up to multipliers, the reduced equation $E[\widehat{L}]$ for the variational problem (see above).

Finally, if one restricts on the solutions to the Euler–Lagrange equation $E[L] = 0$, then $(1/P)X$ is a variational symmetry of the variational problem.[u] This allows to associate a partial conservation law to such symmetries. We refer again the reader to [51] for details.

It should be stressed again that the Muriel–Romero–Olver approach is not restricted to first order Lagrangians, and they consider explicitly some higher order example. On the other hand, as implied by consideration of λ-symmetries, they consider Lagrangian problems with only one independent variable.

In her work on deformation of the Lie derivative [40], Morando considered variational λ- and μ-symmetries. She started from a geometrical characterization — in terms of the action of the deformed Lie derivative (see Sec. 5 above) on the Poincaré–Cartan form Θ —

[t]We hope to be able at a later time to also review developments in this direction.

[u]When this happens, i.e. we need the restriction to solutions of the Euler–Lagrange equation to have a variational symmetry, one speaks of "pseudo-variational symmetries" [55].

of variational λ-symmetries, which reads

$$\mathcal{L}_Y^\mu(\Theta) \in \mathcal{E}; \tag{74}$$

here Y is the λ-prolongation of the vector field X on M, and $\mathcal{E}$ is the contact ideal (again, see above).

This characterization is immediately generalized, and written exactly in the same form, to μ-symmetries and hence field theory.

To a divergence variational λ-symmetry X of a first order regular Lagrangian L is associated the "λ-conservation law"

$$D_x(X \lrcorner \Theta - R) + \lambda(X \lrcorner \Theta - R), \tag{75}$$

where R is a suitable smooth function.

In the case of field theory, it is convenient to introduce a form

$$\rho = R^i \Omega_i + \sigma,$$

where $\sigma \in \mathcal{E}$ and $\Omega_i = \partial_i \lrcorner \Omega$, with $\Omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p$ the volume form on the base manifold B. In this case one has a "μ-conservation law"; this reads

$$D_i \lrcorner \mathrm{d}^\mu (X \lrcorner \Theta - \rho) = 0 \tag{76}$$

in compact geometrical notation; in coordinates, and using freely the standard notation introduced earlier on, it yields

$$(D_i + \Lambda_i) \left(\frac{\partial L}{\partial u_i^a} (\varphi^a - u_k^a \xi^k) + \xi^i L - R^i \right) = 0. \tag{77}$$

In a different work [21], Cicogna and Gaeta also looked at μ-symmetries for variational problems; they also obtained that to a μ-symmetry of a Lagrangian is associated a deformed conservation law as above.

On the other hand, they also looked at twisted symmetries of variational problems from the point of view of gauged vector fields. In this framework, it is natural to wonder how the Euler–Lagrange equation themselves are transformed by a change of reference frame. One thus obtains "μ-Euler–Lagrange equations", and this are exactly invariant under (the μ-prolongation of) a vector field which is a μ-symmetry for the underlying Lagrangian; correspondingly, a μ-symmetry for the Lagrangian yields an exact conservation law for the associated μ-Euler–Lagrange equations [21].

All these works were conducted in the Lagrangian framework; in a very recent study, Cicogna considered the Hamiltonian counterpart of these results. We will not discuss at all this aspect, referring the reader to his paper [18].

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