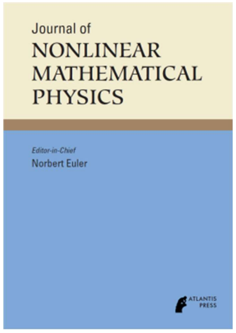

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COMPLETE SPECIFICATION OF SOME PARTIAL DIFFERENTIAL EQUATIONS THAT ARISE IN FINANCIAL MATHEMATICS

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We consider some well-known partial differential equations that arise in Financial Mathematics, namely the Black–Scholes–Merton, Longstaff, Vasicek, Cox–Ingersoll–Ross and Heath equations. Our central aim is to discover any underlying connections taking into account the Lie remarkability property of the heat equation. For a few of these equations there is a known connection with the heat equation through a coordinate transformation. We investigate further that connection with the help of modern group analysis. This is realized by obtaining the Lie point symmetries of these equations and comparing their algebras with that of the heat equation. For those with an algebra identical to that of the heat equation a systematic way is shown to obtain the coordinate transformation that links them: the Lie remarkability property is a direct consequence. For the rest this is achieved only in certain subcases.

Keywords: Modern group analysis; Financial Mathematics; Lie remarkability; Black–Scholes equation; Cox–Ingersoll–Ross equation.

1. Introduction

Over the last forty-odd years there has been a considerable development in the mathematical analysis of partial differential equations which arise in Financial Mathematics. One of the techniques, which has only been seriously engaged in the last decade, is the use of symmetry analysis. For those to whom symmetry analysis is an integral component of their repertoire of mathematical skills and tools this late development does seem to be more than a little strange. When one reads various papers devoted to the resolution of evolution partial differential equations which arise as the, more or less, final stage of the mathematical modeling of some financial process, one can only marvel at the ingenuity displayed in the making of *Ansätze* to bring the equation under consideration under the control of the master. One of the advantages in the employment of symmetry analysis of differential equations is to provide an algorithmic procedure which essentially enables one to convert an *Ansatz* into a *Satz*.

Every differential equation possesses a kind of symmetry. First-order ordinary differential equations possess an infinite number of Lie point symmetries whereas higher-order equations exhibit only a restricted number of such symmetries. It is a commonplace that, if one loosens the restrictions upon the dependence of the coefficient functions of the generator of symmetry, one ends up with an infinite number of symmetries. This in itself is rather artificial. Infinite algebras can even appear in the case of point symmetries. Most of the time they just denote the linearity of the equation at hand or some gauge transformation and therefore are regarded as trivial as they are of no use for the reduction of the equation or for the construction of solutions. Our belief is that those infinite-parameter generators of symmetry are not completely useless. Apart from the well-known fact that they can help to linearize a nonlinear equation, they seem to play a very important part on the classification of Differential Equations [3]. It is needless to mention their attributes with respect to the analysis of boundary and initial value problems.

There are obviously those other rare instances when one encounters differential equations with a sufficient number of Lie point symmetries for anyone's purposes. It is remarkable how often one claims to be so fortunate in the partial differential equations that arise in Financial Mathematics.[a] However, on a closer look this is by far not accidental: it is shown that these equations, in their vast majority, are linked via a coordinate transformation to the heat equation in $1+1$ dimensions. The latter is

$$cu_{xx} - u_t = 0 \tag{1}$$

and possesses the six Lie point symmetries

$$\begin{aligned}
&\Gamma_1 = \partial_x && \Gamma_4 = 2t\partial_t + x\partial_x \\
&\Gamma_2 = \partial_t && \Gamma_5 = 2t\partial_x - \frac{1}{c}xu\partial_u \\
&\Gamma_3 = u\partial_u && \Gamma_6 = 4t^2\partial_t + 4tx\partial_x - \left(\frac{1}{c}x^2 + 2t\right)u\partial_u
\end{aligned} \tag{2}$$

plus the additional infinite-dimensional subalgebra $\Gamma_\infty = \eta_1(x,\, t)\partial_u$, where $\eta_1(x,\, t)$ is any solution of (1) and reflects its linearity. The associated Lie algebra of the above six-parameter Lie group of infinitesimal operators is $sl(2,R) \oplus_s W_3$, where W_3 is the three-dimensional Heisenberg–Weyl algebra, implied by the Lie Brackets, see Table 1.

Bluman and Cole [6] proved that the heat equation is the only polynomial partial differential equation of the second order in two independent variables invariant under the finite group of the heat equation itself. In fact we included the infinite-dimensional subalgebra and extended this result in the following sense: if we further consider any one

[a]This is true for most differential equations that arise from natural applications. This is due to the fact that Nature or Real-World Phenomena are endowed with such structure that is inherent within symmetry notions.

Table 1. Lie Brackets for the heat equation.

$[.,.]_{\rm LB}$	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	0	0	Γ_1	$-\frac{1}{c}\Gamma_3$	$2\Gamma_5$
Γ_2	0	0	0	$2\Gamma_2$	$2\Gamma_1$	$-2\Gamma_3+4\Gamma_4$
Γ_3	0	0	0	0	0	0
Γ_4	$-\Gamma_1$	$-2\Gamma_2$	0	0	Γ_5	$2\Gamma_6$
Γ_5	$\frac{1}{c}\Gamma_3$	$-2\Gamma_1$	0	$-\Gamma_5$	0	0
Γ_6	$-2\Gamma_5$	$2\Gamma_3-4\Gamma_4$	0	$-2\Gamma_6$	0	0

symmetry of the form $\Gamma^\star_\infty = \eta^\star_1(x,\,t)\partial_u$, then Eq. (1) is the only equation of the form $u_{xx} = F(t, x, u, u_t, u_x, u_{tx}, u_{tt})$ invariant under this specific group, i.e. this Lie group completely characterizes the heat equation [3]. Since this group consists solely of point symmetries, the heat equation may be termed Lie remarkable [23, 2].

In this paper we investigate the equations

- **Black–Scholes–Merton:**

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0 \tag{3}$$

- **Longstaff:**

$$u_t - \left(\frac{1}{4}\sigma^2 - \kappa\sqrt{x} - 2\lambda x\right)u_x - \frac{1}{2}\sigma^2 x u_{xx} + xu = 0 \tag{4}$$

- **Vasicek:**

$$u_t + \frac{1}{2}\sigma^2 u_{xx} - (\kappa - \lambda x)u_x - xu = 0 \tag{5}$$

- **Cox–Ingersoll–Ross:**

$$u_t + \frac{1}{2}\sigma^2 x u_{xx} - (\kappa - \lambda x)u_x - xu = 0 \tag{6}$$

- **Heath *et al.*** (Hamilton–Jacobi–Bellman type):

$$u_t + au_x + \frac{1}{2}b^2 u_{xx} - \frac{1}{2}u_x^2 + \nu(x) = 0. \tag{7}$$

We examine these equations for Lie symmetries using the package *SYM* [10, 11, 12] developed for the *Mathematica* environment. Equations (3)–(5) have the exact same Lie algebra as the heat equation, $sl(2,R) \oplus_s W_3$. Equation (6), on the other hand, does not have the same algebra as the heat equation. In spite of that our analysis shows that for two special subcases ($\sigma^2 = -4k, \sigma^2 = -\frac{4}{3}k$) we get the same Lie algebra $sl(2,R) \oplus_s W_3$. Furthermore, a connection with the Ermakov–Pinney equation is established. Finally for (7) it is obvious from the existence of the term $\nu(x)$ that in general the dimension of the Lie algebra is insufficient for such a connection. Yet the existence of an infinite-dimensional symmetry makes possible the linearization of the equation! Furthermore from the analysis of that linear equation using *SYM* a couple of interesting subcases arise.

For the equations with the same Lie algebra as the heat equation we show, via an appropriate isomorphism upon their algebras, that there exists a coordinate transformation that connects them with the heat equation. Being granted the knowledge that the heat equation is uniquely characterized by its Lie group one anticipates the same result for the other equations. In addition, the Lie remarkability of the heat equation ensures that the inversible transformation found in each case always points to the heat equation.

All the above are not necessarily something new; only the approach differs. In [7], [Chapter 6] Bluman and Kumei provide the general framework for the existence and construction of a transformation between two differential equations. The algorithmic procedures outlined in their chapter are based on the symmetries and determining equations of the equation under consideration. Our approach exploits the algebraic properties of the equation under consideration to identify both the transformation and the resulting equation. In Sec. 7 the discussion follows.

2. The Black–Scholes–Merton Equation

The Black–Scholes–Merton equation [4, 5, 24],

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0, \tag{8}$$

is the best known evolution partial differential equation of the mathematics of finance. Merton [24], in his effort to value corporate liabilities, based his model on the option-pricing method introduced by Black and Scholes [4, 5] some years before him. Although the assumptions are regarded as surprisingly strict and unrealistic, this model remains extremely popular and has received many extensions and improvements to loosen some of the restrictions and fit to the current market needs. It is therefore not at all unreasonable that many studies still consider the Black–Scholes–Merton formula. An analytic solution for this equation was provided in [4] and the connection to the heat equation was also given [5]. The first consideration in terms of Lie analysis was presented in [14] and since there are various studies revealing different aspects of this equation [18, 19, 1, 20].

The Lie point symmetries of (8) are

$$\begin{aligned}
\Gamma_1 &= \partial_t \\
\Gamma_2 &= x\partial_x \\
\Gamma_3 &= u\partial_u \\
\Gamma_4 &= 4t\partial_t + 2x\log x\partial_x + \frac{1}{\kappa+\lambda}(\lambda^2 t + 2\kappa\log x)u\partial_u \\
\Gamma_5 &= tx\partial_x + \frac{1}{\kappa+\lambda}(2\log x + \kappa t)u\partial_u \\
\Gamma_6 &= 4t^2\partial_t + 4tx\log x\partial_x + \frac{1}{\kappa+\lambda}[(2\log x + \kappa t)^2 + \lambda t(\lambda t - 2) - \kappa t(\kappa t + 2)]u\partial_u \\
\Gamma_\infty &= f(t,x)\partial_u,
\end{aligned}$$

where $f(t, x)$ is any solution of (8) and $(\kappa, \lambda) = (\sigma^2 - 2r, \sigma^2 + 2r)$.

Table 2. Lie Brackets for the Black–Scholes–Merton equation.

$[.,.]_{\mathrm{LB}}$	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	0	0	$4\Gamma_1 + \dfrac{\lambda^2}{\kappa+\lambda}\Gamma_3$	$\Gamma_2 + \dfrac{\kappa}{\kappa+\lambda}\Gamma_3$	$-2\Gamma_3 + 2\Gamma_4$
Γ_2	0	0	0	$2\Gamma_2 + \dfrac{2\kappa}{\kappa+\lambda}\Gamma_3$	$\dfrac{2}{\kappa+\lambda}\Gamma_3$	$4\Gamma_5$
Γ_3	0	0	0	0	0	0
Γ_4	$-4\Gamma_1 - \dfrac{\lambda^2}{\kappa+\lambda}\Gamma_3$	$-2\Gamma_2 - \dfrac{2\kappa}{\kappa+\lambda}\Gamma_3$	0	0	$2\Gamma_5$	$4\Gamma_6$
Γ_5	$-\Gamma_2 - \dfrac{\kappa}{\kappa+\lambda}\Gamma_3$	$-\dfrac{2}{\kappa+\lambda}\Gamma_3$	0	$-2\Gamma_5$	0	0
Γ_6	$2\Gamma_3 - 2\Gamma_4$	$-4\Gamma_5$	0	$-4\Gamma_6$	0	0

The associated Lie algebra of the above six-parameter Lie group of infinitesimal operators is $sl(2,R) \oplus_s W_3$, implied by the Lie Brackets, Table 2.

The Lie Brackets for the heat equation, Table 1, and the Black–Scholes–Merton equation, Table 2, indicate certain structural similarities. In fact we encounter two different realizations of the same Lie Algebra, $sl(2,R) \oplus_s W_3$: We find a transformation that connects the two equations. The first step in that direction is to choose a better representation for the algebra, a representation which renders the Lie Brackets identical. This is achieved by making the following changes to the base elements of the Lie algebra,

$$\begin{aligned} \Gamma_1' &= 2\Gamma_2 + \frac{2\kappa}{\kappa+\lambda}\Gamma_3 \qquad & \Gamma_4' &= \frac{1}{2}\Gamma_4 \\ \Gamma_2' &= 4\Gamma_1 + \frac{\lambda^2}{\kappa+\lambda}\Gamma_3 \qquad & \Gamma_6' &= \frac{1}{4}\Gamma_6. \end{aligned} \tag{9}$$

In addition we obtain the value of the parameter within the heat equation, $c = -\frac{\kappa+\lambda}{4}$.

The next step involves the determination (if it exists) of a coordinate transformation, $X = X(x,t,u)$, $T = T(x,t,u), U = U(x,t,u)$, which connects the two different representations of the Lie algebra. Indeed Eq. (8) can be transformed via the transformation,

$$\begin{aligned} T(x,t,u) &= \frac{1}{4}t \\ X(x,t,u) &= \frac{1}{2}\log x \\ U(x,t,u) &= \exp\left\{-\frac{\lambda^2 t}{4(\kappa+\lambda)}\right\} x^{-\frac{\kappa}{\kappa+\lambda}} u, \end{aligned}$$

to the heat equation,

$$U_T + \frac{\kappa+\lambda}{4} U_{XX} = 0.$$

The existence of such a transformation is indicated already in [5].

3. The Longstaff Equation

Longstaff [22] constructed a double square root model which seems (using empirical data) to model better the pricing of discount bonds than that of the square root model introduced by Cox, Ingersoll and Ross [9]. Furthermore, in that same paper, the closed-form solution to that equation is given, found by the use of the separation of variables approach.

The Longstaff equation [22] is

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 x \frac{\partial^2 u}{\partial x^2} - \left(\frac{1}{4}(\mu^2 - 2\lambda^2) - \kappa\sqrt{x} - 2\lambda x\right)\frac{\partial u}{\partial x} + xu = 0. \tag{10}$$

Equation (10) possesses the Lie point symmetries [28],

$$\begin{aligned}
\Gamma_1 &= \partial_t \\
\Gamma_2 &= u\partial_u \\
\Gamma_{3\pm} &= \mathrm{e}^{\pm\frac{\mu t}{\sqrt{2}}}\left\{\sqrt{x}\partial_x \mp \left[\frac{(\kappa(\sqrt{2}\lambda \mp \mu) + \mu\sqrt{x}(\sqrt{2}\mu \mp 2\lambda))}{\mu(\mu^2 - 2\lambda^2)}\right] u\partial_u\right\} \\
\Gamma_{5\pm} &= \frac{1}{\mu}e^{\mp\sqrt{2}\mu t}\left\{\mp 2\sqrt{2}\partial_t + \frac{4}{\mu}(\kappa\lambda\sqrt{x} + \mu^2 x)\partial_x \pm \frac{u}{\mu^2(\mu^2 - 2\lambda^2)}\left(2\sqrt{2}\kappa^2\lambda^2\right.\right. \\
&\quad \pm 4\kappa\lambda\left(\kappa + 2\lambda\sqrt{x}\right)\mu + \sqrt{2}\left(\kappa^2 + 8\kappa\lambda\sqrt{x} + 2\lambda^3\right)\mu^2 \\
&\quad \left.\left.\pm 2\ \left(2\kappa\sqrt{x} + 4\lambda x + \lambda^2\right)\mu^3 + \sqrt{2}(4x - \lambda)\mu^4 \mp \mu^5\right)\partial_u\right\} \\
\Gamma_\infty &= f(t,x)\partial_u,
\end{aligned}$$

where $f(t,x)$ is any solution of (10).

The associated Lie algebra of the above six-parameter Lie group of infinitesimal operators is $sl(2,R) \oplus_s W_3$ (see also Table 3), where $\Lambda = 8\sqrt{2}(\lambda\mu - \kappa^2)/\mu^3$.

Table 3. Lie Brackets for the Longstaff equation.

$[\cdot,\cdot]_{\mathrm{LB}}$	Γ_1	Γ_2	Γ_{3+}	Γ_{3-}	Γ_{5+}	Γ_{5-}
Γ_1	0	0	$\frac{\mu}{\sqrt{2}}\Gamma_{3+}$	$-\frac{\mu}{\sqrt{2}}\Gamma_{3-}$	$-\sqrt{2}\mu\Gamma_{5+}$	$\sqrt{2}\mu\Gamma_{5-}$
Γ_2	0	0	0	0	0	0
Γ_{3+}	$-\frac{\mu}{\sqrt{2}}\Gamma_3$	0	0	$\frac{\sqrt{2}\mu}{\mu^2 - 2\lambda^2}\Gamma_2$	$4\Gamma_{3-}$	0
Γ_{3-}	$\frac{\mu}{\sqrt{2}}\Gamma_{3-}$	0	$-\frac{\sqrt{2}\mu}{\mu^2 - 2\lambda^2}\Gamma_2$	0	0	$4\Gamma_{3+}$
Γ_{5+}	$\sqrt{2}\mu\Gamma_{5+}$	0	$-4\Gamma_{3-}$	0	0	$-\frac{16\sqrt{2}}{\mu}\Gamma_1 - \Lambda\Gamma_2$
Γ_{5-}	$-\sqrt{2}\mu\Gamma_{5-}$	0	0	$-4\Gamma_{3+}$	$\frac{16\sqrt{2}}{\mu}\Gamma_1 + \Lambda\Gamma_2$	0

Again by suitable reordering of the base elements of the Lie algebra and using its centre,

$$\begin{aligned}
&\Gamma_1' = \frac{1}{2}\Gamma_{3+} \qquad && \Gamma_4' = -\frac{\sqrt{2}}{\mu}\Gamma_1 + \left(\frac{1}{2} - \frac{1}{\sqrt{2}}\frac{\lambda\mu^2-\kappa^2}{\mu^3}\right)\Gamma_2 \\
&\Gamma_2' = \frac{1}{4}\Gamma_{5-} && \Gamma_5' = -\Gamma_{3-} \\
&\Gamma_3' = \Gamma_2 && \Gamma_6' = -\Gamma_{5+},
\end{aligned} \tag{11}$$

we succeed in making the two representations identical. In addition $c = \sqrt{2}(-2\lambda^2+\mu^2)/\mu$.

Equation (10) can be transformed via the change of variables,

$$\begin{aligned}
T(x,t,u) &= -e^{-\sqrt{2}\mu t} \\
X(x,t,u) &= \frac{4e^{-\frac{\mu t}{\sqrt{2}}}(\kappa\lambda + \mu^2\sqrt{x})}{\mu^2} \\
vU(x,t,u) &= \exp\left\{\frac{1}{4}\left(-2\lambda + \frac{2\kappa^2}{\mu^2} + \sqrt{2}\mu\right)t + \frac{-2\kappa\sqrt{x}+\sqrt{2}\mu x}{\mu(\sqrt{2}\lambda+\mu)}\right\}u,
\end{aligned}$$

to the heat equation,

$$U_T + \frac{\sqrt{2}(\mu^2-2\lambda^2)}{\mu}U_{XX} = 0.$$

For the special case $\mu = 0$ the point symmetries of (10) are

$$\begin{aligned}
\Gamma_1 &= \partial_t \\
\Gamma_2 &= u\partial_u \\
\Gamma_3 &= 2\sqrt{x}\partial_x + \frac{\kappa t - 2\sqrt{x}}{\lambda}u\partial_u \\
\Gamma_4 &= 4t\sqrt{x}\partial_x + \frac{\sqrt{x}(4-4\lambda t) + \kappa t(-2+\lambda t)}{\lambda^2}u\partial_u \\
\Gamma_5 &= (8x + 6t^2\kappa\lambda\sqrt{x})\partial_x + 8t\partial_t + \\
&\quad \frac{-8\lambda x - 2\kappa\sqrt{x}(2-6\lambda t+3\lambda^2t^2) + t(4\lambda^3+\kappa^2(2-\lambda 3t+\lambda^2t^2))}{\lambda^2}u\partial_u \\
\Gamma_6 &= 8(4tx + \kappa\lambda\sqrt{x}t^3)\partial_x + 16t^2\partial_t \\
&\quad + \frac{x(16-32\lambda t) - 8\kappa t\sqrt{x}(2-3\lambda t+\lambda^2t^2) + t(-4\kappa^2\lambda t^2 - 8\lambda^2 + \kappa^2\lambda^2t^3 + 4t(\kappa^2+2\lambda^3))}{\lambda^2}u\partial_u \\
\Gamma_\infty &= f(t,x)\partial_u,
\end{aligned}$$

where $f(t,x)$ is any solution of

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 x\frac{\partial^2 u}{\partial x^2} + \left(\frac{1}{2}\lambda^2 + \kappa\sqrt{x} + 2\lambda x\right)\frac{\partial u}{\partial x} + xu = 0. \tag{12}$$

The associated Lie algebra of the above six-parameter Lie group of infinitesimal operators is $sl(2,R) \oplus_s W_3$ (see also Table 4), where $\Omega = 2(\kappa^2 + 2\lambda^3)/\lambda^2$.

Table 4. Lie Brackets for the Longstaff equation when $\mu = 0$.

$[.,.]_{\text{LB}}$	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	0	$\frac{\kappa}{\lambda}\Gamma_2$	$-\frac{2\kappa}{\lambda^2}\Gamma_2 + 2\Gamma_3$	$8\Gamma_1 + \Omega\Gamma_2 + 3\kappa\lambda\Gamma_4$	$-8\Gamma_2 + 4\Gamma_5$
Γ_2	0	0	0	0	0	0
Γ_3	$-\frac{\kappa}{\lambda}\Gamma_2$	0	0	$\frac{4}{\lambda^2}\Gamma_2$	$-\frac{4\kappa}{\lambda^2}\Gamma_2 + 2\Gamma_3$	$8\Gamma_4$
Γ_4	$\frac{2\kappa}{\lambda^2}\Gamma_2 - 2\Gamma_3$	0	$-\frac{4}{\lambda^2}\Gamma_2$	0	$-4\Gamma_4$	0
Γ_5	$-8\Gamma_1 - \Omega\Gamma_2 - 3\kappa\lambda\Gamma_4$	0	$\frac{4\kappa}{\lambda^2}\Gamma_2 - 2\Gamma_3$	$4\Gamma_4$	0	$8\Gamma_6$
Γ_6	$8\Gamma_2 - 4\Gamma_5$	0	$-8\Gamma_4$	0	$-8\Gamma_6$	0

In this case the reordering of the basis elements is

$$\begin{aligned} \Gamma_1' &= \Gamma_3 + \frac{\kappa}{\lambda^2}\Gamma_2 & \Gamma_4' &= \frac{1}{4}\Gamma_5 \\ \Gamma_2' &= \Gamma_1 + \frac{\kappa^2 + 2\lambda^3}{4\lambda^2}\Gamma_2 + \frac{\kappa\lambda}{4}\Gamma_4 & \Gamma_5' &= \Gamma_4 \\ \Gamma_3' &= \Gamma_2 & \Gamma_6' &= \frac{1}{4}\Gamma_6, \end{aligned} \tag{13}$$

and, after we use the centre of the algebra, we find that the Lie Brackets of (12) and of the heat equation are the same when we make the identification $c = -\frac{\lambda^2}{4}$.

Under the change of variables

$$\begin{aligned} T(x,t,u) &= t \\ X(x,t,u) &= \sqrt{x} - \frac{\kappa\lambda}{4}t^2 \\ U(x,t,u) &= \exp\left\{\frac{12\lambda x - 12\kappa(-1+\lambda t)\sqrt{x} + (-6\lambda^3 + \kappa^2(-3+\lambda^2 t^2))t}{12\lambda^2}\right\} u, \end{aligned}$$

(12) is transformed to the heat equation given by

$$U_T + \frac{\lambda^2}{4}U_{XX} = 0.$$

4. The Vasicek Equation

Cox, Ingersoll and Ross [9] proposed a model for the pricing of zero-coupons, bonds or swaps using the spot (interest) rate as the single factor and its volatility being sensitive to the riskless rate (see next section). However, similar zero-coupon bond-pricing equations include the model by Vasicek [30] and other considerations [13, 8]. Equations of this form have been studied in [8, 16, 26].

In the case of the Vasicek equation [30],

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} - (\kappa - \lambda x)\frac{\partial u}{\partial x} - xu = 0, \tag{14}$$

the Lie point symmetries are

$$\begin{aligned}
\Gamma_1 &= \partial_t \\
\Gamma_2 &= u\partial_u \\
\Gamma_3 &= e^{\lambda t}\left\{\partial_x + \frac{u}{\lambda}\partial_u\right\} \\
\Gamma_4 &= e^{-\lambda t}\left\{\partial_x + \frac{2\lambda^2 x - 2\kappa\lambda + \sigma^2}{\lambda\sigma^2}u\partial_u\right\} \\
\Gamma_5 &= \frac{e^{2\lambda t}}{\lambda}\left\{2\frac{\lambda^2 x - \kappa\lambda + \sigma^2}{\lambda}\partial_x + 2\partial_t + \frac{2\lambda^2 x + \sigma^2}{\lambda^2}u\partial_u\right\} \\
\Gamma_6 &= \frac{e^{-2\lambda t}}{\lambda}\left\{2\frac{\lambda^2 x - \kappa\lambda + \sigma^2}{\lambda}\partial_x - 2\partial_t\right. \\
&\quad \left. - \frac{4\lambda^2(\kappa - \lambda x)^2 + 2\lambda(-2\kappa + \lambda(3x + \lambda))\sigma^2 + \sigma^4}{\lambda^2\sigma^2}u\partial_u\right\} \\
\Gamma_\infty &= f(t,x)\partial_u,
\end{aligned}$$

where $f(t,x)$ is any solution of (14).

The Lie Brackets of the six-dimensional subalgebra of the symmetries are given in Table 5 and it is evident that the Lie algebra is $sl(2,R) \oplus_s W_3$, where $\Xi = 8(2\kappa\lambda + \lambda^3 - \sigma^2)/\lambda^3$.

The procedure is the same. Firstly we perform a suitable reordering of the base elements of the Lie algebra and using its centre,

$$\begin{aligned}
\Gamma_1' &= \Gamma_3 & \Gamma_4' &= -\frac{1}{\lambda}\Gamma_1 + \left(\frac{1}{2} - \frac{1}{16}\Xi\right)\Gamma_2 \\
\Gamma_2' &= -\frac{1}{2}\Gamma_5 & \Gamma_5' &= \Gamma_4 \\
\Gamma_3' &= \Gamma_2 & \Gamma_6' &= \frac{1}{2}\Gamma_6,
\end{aligned} \tag{15}$$

we succeed in making the Lie Brackets for Eq. (14) and the heat equation identical. In addition we have that $c = \frac{\sigma^2}{2\lambda}$.

Table 5. Lie Brackets for the Vasicek equation.

$[.,.]_{LB}$	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	0	$\lambda\Gamma_3$	$-\lambda\Gamma_4$	$2\lambda\Gamma_5$	$-2\lambda\Gamma_6$
Γ_2	0	0	0	0	0	0
Γ_3	$-\lambda\Gamma_3$	0	0	$-\frac{2\lambda}{\sigma^2}\Gamma_2$	0	$4\Gamma_4$
Γ_4	$\lambda\Gamma_4$	0	$\frac{2\lambda}{\sigma^2}\Gamma_2$	0	$4\Gamma_3$	0
Γ_5	$-2\lambda\Gamma_5$	0	0	$-4\Gamma_3$	0	$\frac{16}{\lambda}\Gamma_1 + \Xi\Gamma_2$
Γ_6	$2\lambda\Gamma_6$	0	$-4\Gamma_4$	0	$-\frac{16}{\lambda}\Gamma_1 - \Xi\Gamma_2$	0

For completion Eq. (14) can be transformed to the heat equation,

$$U_T - \frac{\sigma^2}{2\lambda}U_{XX} = 0,$$

via the change of variables,

$$\begin{aligned} T(x,t,u) &= \frac{1}{2}e^{-2\lambda t} \\ X(x,t,u) &= \frac{e^{-t\lambda}(\lambda^2 x - \kappa\lambda + \sigma^2)}{\lambda^2} \\ U(x,t,u) &= \exp\left\{\frac{-2(x+\kappa t)\lambda + \sigma^2 t}{2\lambda^2}\right\} u. \end{aligned}$$

For the special case $\lambda = 0$ the point symmetries of (14) are

$$\begin{aligned} \Gamma_1 &= \partial_t \\ \Gamma_2 &= u\partial_u \\ \Gamma_3 &= \partial_x + tu\partial_u \\ \Gamma_4 &= 2t\partial_x + \frac{2x + t(2\kappa + \sigma^2 t)}{\sigma^2}u\partial_u \\ \Gamma_5 &= (2x + 3\sigma^2 t^2)\partial_x + 4t\partial_t + \frac{2x(\kappa + 3\sigma^2 t) + t(2\kappa^2 + 3\sigma^2\kappa t + \sigma^4 t^2)}{\sigma^2}u\partial_u \\ \Gamma_6 &= (8tx + 4\sigma^2 t^3)\partial_x + 8t^2\partial_t - \frac{4x^2 + 4tx(2\kappa + 3\sigma^2 t) + t((4\kappa^2 + \sigma^4 t^2)t + 4\sigma^2(\kappa t^2 - 1))}{\sigma^2}u\partial_u \\ \Gamma_\infty &= f(t,x)\partial_u, \end{aligned}$$

where $f(t,x)$ is any solution of

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} - \kappa\frac{\partial u}{\partial x} - xu = 0. \tag{16}$$

Again the associated Lie algebra of the above six-parameter Lie group of infinitesimal operators is $sl(2,R) \oplus_s W_3$ (see also Table 6). This is indicative of a link to the heat equation.

When we reorder the basis elements of the Lie algebra according to

$$\begin{aligned} \Gamma_1' &= \Gamma_3 + \frac{\kappa}{\sigma^2}\Gamma_2 & \Gamma_4' &= \frac{1}{2}\Gamma_5 \\ \Gamma_2' &= \Gamma_1 + \frac{\kappa^2}{2\sigma^2}\Gamma_2 + \frac{\sigma^2}{2}\Gamma_4 & \Gamma_5' &= \Gamma_4 \\ \Gamma_3' &= \Gamma_2 & \Gamma_6' &= \frac{1}{2}\Gamma_6 \end{aligned} \tag{17}$$

and use its centre, we bring the tables of Lie Brackets for (16) and the heat equation into coincidence. By a comparison of coefficients it follows that $c = -\frac{1}{2}\sigma^2$. The change of

Table 6. Lie Brackets for the Vasicek equation when $\lambda = 0$.

$[.,.]_{\mathrm{LB}}$	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	0	Γ_2	$\frac{2\kappa}{\sigma^2}\Gamma_2 + 2\Gamma_3$	$4\Gamma_1 + \frac{2\kappa^2}{\sigma^2}\Gamma_2 + 3\sigma^2\Gamma_4$	$-4\Gamma_2 + 4\Gamma_5$
Γ_2	0	0	0	0	0	0
Γ_3	$-\Gamma_2$	0	0	$\frac{2}{\sigma^2}\Gamma_2$	$\frac{2\kappa}{\sigma^2}\Gamma_2 + 2\Gamma_3$	$4\Gamma_4$
Γ_4	$-\frac{2\kappa}{\sigma^2}\Gamma_2 - 2\Gamma_3$	0	$-\frac{2}{\sigma^2}\Gamma_2$	0	$-2\Gamma_4$	0
Γ_5	$-4\Gamma_1 - \frac{2\kappa^2}{\sigma^2}\Gamma_2 - 3\sigma^2\Gamma_4$	0	$-\frac{2\kappa}{\sigma^2}\Gamma_2 - 2\Gamma_3$	$2\Gamma_4$	0	$4\Gamma_6$
Γ_6	$4\Gamma_2 - 4\Gamma_5$	0	$-4\Gamma_4$	0	$-4\Gamma_6$	0

variables to achieve that is

$$
\begin{aligned}
T(x,t,u) &= t \\
X(x,t,u) &= x - \frac{\sigma^2}{2}t^2 \\
U(x,t,u) &= \exp\left\{\frac{1}{6}\left(-6tx - \frac{3\kappa(2x+\kappa t)}{\sigma^2} + \sigma^2 t^3\right)\right\} u.
\end{aligned}
$$

5. The Cox–Ingersoll–Ross Equation

The Cox–Ingersoll–Ross (CIR) equation [9] is

$$
\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x \frac{\partial^2 u}{\partial x^2} - (\kappa - \lambda x)\frac{\partial u}{\partial x} - xu = 0. \tag{18}
$$

Equation (18) possesses the symmetries,

$$
\begin{aligned}
\Gamma_1 &= \partial_t \\
\Gamma_2 &= u\partial_u \\
\Gamma_{3\pm} &= \mathrm{e}^{\pm\beta t}\left\{\pm\frac{1}{\beta}\partial_t + x\partial_x - \frac{1}{\sigma^2\beta}(-\beta \pm \lambda)(\kappa \pm \beta x)u\partial_u\right\} \\
\Gamma_\infty &= f(t,x)\partial_u,
\end{aligned}
$$

where $f(t,x)$ is any solution of (18) and $\beta = \sqrt{2\sigma^2 + \lambda^2}$. In this case the finite algebra is $sl(2,R) \oplus A_1$.

It is apparent that in general there is no transformation that connects the CIR equation to the heat equation. However, our analysis using *SYM* indicates that for two special subcases we get the same Lie algebra, $sl(2,R) \oplus_s W_3$. These special cases are $\sigma^2 = -4\kappa$ and $\sigma^2 = -4\kappa/3$. Naturally, if κ is negative, these cases are of mathematical interest only and we only treat them briefly at the end of this section. For the general range of values of the parameters in Eq. (18) the algebra is evidently $\{sl(2,R) \oplus A_1\} \oplus_s \infty A_1$ and this is insufficient for the equation to be Lie remarkable. This algebra has been identified in the context of the Schrödinger equation corresponding to the classical Ermakov–Pinney equation [21]. The corresponding equation in the present context is

$$c_2 u_{xx} + x^{-2}u - u_t = 0. \tag{19}$$

Equation (19) possesses the symmetries

$$\begin{aligned} &\Gamma_1 = \partial_t && \Gamma_3 = 2t\partial_t + x\partial_x \\ &\Gamma_2 = u\partial_u && \Gamma_4 = 4t^2\partial_t + 4tx\partial_x - (2t + c_2^{-1}x^2)u\partial_u \end{aligned} \tag{20}$$

plus the infinite-dimensional subalgebra $\Gamma_\infty = \zeta(x,\, t)\partial_u$, where $\zeta(x,\, t)$ is any solution of

$$\zeta + x^2\zeta_t - c_2 x^2 \zeta_{xx} = 0.$$

The transformation that connects the CIR equation, (18), to (19) is

$$\begin{aligned} T(x,t,u) &= -\exp\{\beta t\} \\ X(x,t,u) &= c_2 \exp\left\{\frac{1}{2}(\beta t + \log x)\right\} \\ U(x,t,u) &= cux^{-\frac{1}{4}-\frac{\kappa}{\sigma^2}} \exp\left\{\left(\frac{\kappa\lambda}{\sigma^2} - \frac{1}{4}\beta\right)t + \frac{x}{\sigma^2}(\lambda+\beta)\right\}, \end{aligned}$$

where $c_2^2 = 32\beta\sigma^2/(4\kappa + 3\sigma^2)(4\kappa + \sigma^2)$.

For the special case $\sigma^2 = -4\kappa$ the point symmetries of (18) are

$$\begin{aligned} \Gamma_1 &= \partial_t \\ \Gamma_2 &= u\partial_u \\ \Gamma_{3,4} &= 4e^{\pm\frac{\beta}{2}t}\left\{\sqrt{x}\partial_x + \frac{2\sqrt{x}}{\lambda \pm \beta}u\partial_u\right\} \\ \Gamma_{5,6} &= 4e^{\pm\beta t}\left\{x\partial_x \pm \frac{1}{\beta}\partial_t + 2\frac{\pm\beta x + \kappa}{\beta(\beta \pm \lambda)}u\partial_u\right\} \\ \Gamma_\infty &= f(t,x)\partial_u, \end{aligned}$$

where $f(t,x)$ is any solution of

$$\frac{\partial u}{\partial t} - 2\kappa x\frac{\partial^2 u}{\partial x^2} - (\kappa - \lambda x)\frac{\partial u}{\partial x} - xu = 0 \tag{21}$$

and $\beta = \sqrt{\lambda^2 - 8\kappa}$.

Table 7 represents the associated Lie Brackets which become identical to the one for the heat equation (for $c = \frac{1}{2}\kappa/\beta$) by the reordering,

$$\begin{aligned} &\Gamma_1' = \frac{1}{2}\Gamma_4 && \Gamma_4' = \frac{2}{\beta}\Gamma_1 + \frac{1}{2}(2 + 2\lambda/\beta)\Gamma_2 \\ &\Gamma_2' = -\frac{1}{4}\Gamma_6 && \Gamma_5' = \Gamma_3 \\ &\Gamma_3' = \Gamma_2 && \Gamma_6' = \Gamma_5. \end{aligned} \tag{22}$$

Table 7. Lie Brackets for the CIR equation when $\sigma^2 = -4\kappa$.

$[.,.]_{\text{LB}}$	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	0	$\frac{\beta}{2}\Gamma_3$	$-\frac{\beta}{2}\Gamma_4$	$\beta\Gamma_5$	$-\beta\Gamma_6$
Γ_2	0	0	0	0	0	0
Γ_3	$-\frac{\beta}{2}\Gamma_3$	0	0	$\frac{4\beta}{\kappa}\Gamma_2$	0	$4\Gamma_4$
Γ_4	$\frac{\beta}{2}\Gamma_4$	0	$-\frac{4\beta}{\kappa}\Gamma_2$	0	$4\Gamma_3$	0
Γ_5	$-\beta\Gamma_5$	0	0	$-4\Gamma_3$	0	$\frac{32}{\beta}\Gamma_1 + \frac{8\lambda}{\beta}\Gamma_2$
Γ_6	$\beta\Gamma_6$	0	$-4\Gamma_4$	0	$-\frac{32}{\beta}\Gamma_1 - \frac{8\lambda}{\beta}\Gamma_2$	0

Under the change of variables,

$$\begin{aligned} T(x,t,u) &= e^{\beta t} \\ X(x,t,u) &= \exp\left\{\frac{1}{2}\beta t\right\}\sqrt{x} \\ U(x,t,u) &= \exp\left\{\frac{2(x+\kappa t)}{\beta-\lambda}\right\}u, \end{aligned}$$

(21) is transformed to the heat equation

$$U_T - \frac{\kappa}{2\beta}U_{XX} = 0.$$

In the particular case that $\sigma^2 = -4\kappa/3$ the point symmetries of (18) are

$$\begin{aligned} \Gamma_1 &= \partial_t \\ \Gamma_2 &= u\partial_u \\ \Gamma_{3,4} &= e^{\pm\frac{\beta}{2\sqrt{3}}t}\left\{4\sqrt{x}\partial_x + \frac{-2\kappa + (3\lambda \mp \sqrt{3}\beta)x}{\kappa\sqrt{x}}u\partial_u\right\} \\ \Gamma_{5,6} &= 4e^{\pm\frac{\beta}{\sqrt{3}}t}\left\{x\partial_x \pm \frac{\sqrt{3}}{\beta}\partial_t - \frac{(\sqrt{3}\beta \mp 3\lambda)(\sqrt{3}\kappa \pm \beta x)}{4\kappa\beta}u\partial_u\right\} \\ \Gamma_\infty &= f(t,x)\partial_u, \end{aligned}$$

where now $f(t,x)$ is a solution of

$$\frac{\partial u}{\partial t} - \frac{2}{3}\kappa x\frac{\partial^2 u}{\partial x^2} - (\kappa - \lambda x)\frac{\partial u}{\partial x} - xu = 0 \qquad (23)$$

and $\beta = \sqrt{3\lambda^2 - 8\kappa}$.

The algebra given by the Lie Brackets of the six-dimensional subalgebra is still $sl(2,R)\oplus_s W_3$ and the details of the brackets are given in Table 8.

Table 8. Lie Brackets for the CIR equation when $\sigma^2 = -\frac{4}{3}\kappa$.

$[\cdot,\cdot]_{\mathrm{LB}}$	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	0	$\frac{\beta}{2\sqrt{3}}\Gamma_3$	$-\frac{\beta}{2\sqrt{3}}\Gamma_4$	$\frac{\beta}{\sqrt{3}}\Gamma_5$	$-\frac{\beta}{\sqrt{3}}\Gamma_6$
Γ_2	0	0	0	0	0	0
Γ_3	$-\frac{\beta}{2\sqrt{3}}\Gamma_3$	0	0	$\frac{4\sqrt{3}\beta}{\kappa}\Gamma_2$	0	$4\Gamma_4$
Γ_4	$\frac{\beta}{2\sqrt{3}}\Gamma_4$	0	$-\frac{4\sqrt{3}\beta}{\kappa}\Gamma_2$	0	$4\Gamma_3$	0
Γ_5	$-\frac{\beta}{\sqrt{3}}\Gamma_5$	0	0	$-4\Gamma_3$	0	$\frac{32\sqrt{3}}{\beta}\Gamma_1 + \frac{24\sqrt{3}\lambda}{\beta}\Gamma_2$
Γ_6	$\frac{\beta}{\sqrt{3}}\Gamma_6$	0	$-4\Gamma_4$	0	$-\frac{32\sqrt{3}}{\beta}\Gamma_1 - \frac{24\sqrt{3}\lambda}{\beta}\Gamma_2$	0

The suitable reordering of the base elements of the Lie algebra is,

$$\begin{aligned} \Gamma_1' &= \frac{1}{2}\Gamma_4 & \Gamma_4' &= \frac{2\sqrt{3}}{\beta}\Gamma_1 + \frac{3\sqrt{3}\lambda+\beta}{2\beta}\Gamma_2 \\ \Gamma_2' &= -\frac{1}{4}\Gamma_6 & \Gamma_5' &= \Gamma_3 \\ \Gamma_3' &= \Gamma_2 & \Gamma_6' &= \Gamma_5, \end{aligned} \tag{24}$$

and the Lie Brackets for Eq. (23) and the heat equation are identical, while $c = \kappa/(2\sqrt{3}\beta)$. The transformation that links Eq. (23) to the heat equation is

$$\begin{aligned} T(x,t,u) &= \exp\left\{\frac{\beta}{\sqrt{3}}t\right\} \\ X(x,t,u) &= \exp\left\{\frac{\beta}{2\sqrt{3}}t\right\}\sqrt{x} \\ U(x,t,u) &= \exp\left\{-\frac{3x(3\lambda+\sqrt{3}\beta)+(9\lambda+\sqrt{3}\beta)\kappa t}{12\kappa}\right\}\sqrt{x}u. \end{aligned}$$

6. The Heath Equation

Heath [17] presented an equation for mean-variance hedging which is an equation of Hamilton–Jacobi–Bellman-type not related to the equations described above in an obvious way [25]. However, as is clearly seen in what follows there are cases between the parameters of the equation that in fact exists a transformation linking it to the heat equation.

The Heath equation [17] is

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \nu(x) = 0. \tag{25}$$

Equation (25) possesses the Lie point symmetries [25],

$$\Gamma_1 = \partial_t$$

$$\Gamma_2 = \partial_u$$

$$\Gamma_\infty = b^2 e^{\frac{u}{b^2}} f(t,x)\partial_u,$$

where $f(t,x)$ is any solution of

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial x^2} + \nu(x)u = 0. \tag{26}$$

Due to the nature of Eq. (25) and the presence of an arbitrary function of x, we should not have expected more than the rather obvious symmetries. Paradoxically the infinite one redeems the situation. Its existence indicates that Eq. (25) can be linearized to (26)! The coordinate transformation achieving this feature can be found by seeking for a function $U = U(x,t,u)$ which transforms the representation of the symmetry Γ_∞ to $\tilde{f}(x,t)\partial_u$. This transformation is $U = -\exp\{-u/b^2\}$. Note that the transformation is of the same form as the Cole–Hopf transformation well-known from its linearizing effect upon the Burgers equation.

Through the above transformation we arrive at the linear Eq. (26) as expected. This concludes all that can be said for Eq. (25) for arbitrary function $\nu(x)$. However, from the symmetry analysis of Eq. (26) using *SYM* two special cases arise naturally: $\nu(x) = a_1 + a_2x + a_3x^2$ and $\nu(x) = a_1 + a_2x + a_3x^2 + a_4/(a_2 + 2a_3x)^2$. The former has the same Lie algebra as the heat equation and the coordinate transformation linking them is shown below. The latter has fewer point symmetries and is analyzed at the end of the section. Note that these special cases encompass all the special cases examined in [25].

For $\nu(x) = a_1 + a_2x + a_3x^2$ the point symmetries of (26) are

$$\Gamma_1 = \partial_t$$

$$\Gamma_2 = u\partial_u$$

$$\Gamma_3 = 2\sin(\sqrt{a_3}bt)\partial_x + \frac{b(a_2+2a_3x)\cos(\sqrt{a_3}bt) - 2a\sqrt{a_3}\sin(\sqrt{a_3}bt)}{\sqrt{a_3}b^2}u\partial_u$$

$$\Gamma_4 = 2\cos(\sqrt{a_3}bt)\partial_x - \frac{2a\sqrt{a_3}\cos(\sqrt{a_3}bt) + b(a_2+2a_3x)\sin(\sqrt{a_3}bt)}{\sqrt{a_3}b^2}u\partial_u$$

$$\begin{aligned}\Gamma_5 = {} & 4\frac{(a_2+2a_3x)\sin(2\sqrt{a_3}bt)}{a_3}\partial_x - \frac{8\cos(2\sqrt{a_3}bt)}{\sqrt{a_3}b}\partial_t \\ & + \frac{1}{\sqrt{a_3^3}b^3}\left\{(-4a^2a_3 + b^2(a_2^2 + 8a_2a_3x + 4a_3(a_1 + 2a_3x^2)))\cos(2\sqrt{a_3}bt)\right. \\ & \left. - 4\sqrt{a_3}b(a_3b^2 + a(a_2+2a_3x))\sin(2\sqrt{a_3}bt)\right\} u\partial_u\end{aligned}$$

$$\begin{aligned}
\Gamma_6 &= 4\frac{(a_2+2a_3x)\cos(2\sqrt{a_3}bt)}{a_3}\partial_x + \frac{8\sin(2\sqrt{a_3}bt)}{\sqrt{a_3}b}\partial_t \\
&\quad + \frac{1}{\sqrt{a_3^3}b^3}\left\{(4a^2a_3 - b^2(a_2^2+8a_2a_3x+4a_3(a_1+2a_3x^2)))\sin(2\sqrt{a_3}bt)\right. \\
&\quad \left. - 4\sqrt{a_3}b(a_3b^2+a(a_2+2a_3x))\cos(2\sqrt{a_3}bt)\right\} u\partial_u \\
\Gamma_\infty &= f(t,x)\partial_u,
\end{aligned}$$

where $f(t,x)$ is any solution of (26) with $\nu(x) = a_1 + a_2x + a_3x^2$.

The associated Lie algebra of the above six-parameter Lie group of infinitesimal operators is $sl(2,R) \oplus_s W_3$ (see also Table 9), where $A = 16(-4a^2a_3 - a_2^2b^2 + 4a_1a_3b^2)/(\sqrt{a_3^3}\beta^3)$.

The appropriate reordering of the base elements of the Lie algebra (using its centre) is

$$\begin{aligned}
\Gamma_1' &= \frac{1}{2}\Gamma_4 & \Gamma_2' &= -\frac{1}{4\sqrt{a_3}b}\Gamma_1 + \frac{1}{512}A\Gamma_2 + \frac{1}{32}\Gamma_5 \\
\Gamma_3' &= \Gamma_2 & \Gamma_4' &= \frac{1}{2}\Gamma_2 + \frac{1}{8}\Gamma_6 \\
\Gamma_5' &= -4\Gamma_3 & \Gamma_6' &= -\frac{4}{\sqrt{a_3}b}\Gamma_1 + \frac{1}{32}A\Gamma_2 - \frac{1}{2}\Gamma_5.
\end{aligned}$$

Thence, for $c = b/(16\sqrt{a_3})$, the Lie Brackets for Eq. (26) and the heat equation are identical.

Equation (26) can be transformed via the change of variables,

$$\begin{aligned}
T(x,t,u) &= -2\tan(\sqrt{a_3}bt) \\
X(x,t,u) &= \frac{(a_2+2a_3x)\sec(\sqrt{a_3}bt)}{4a_3} \\
U(x,t,u) &= \exp\left\{\frac{1}{8}\left(4a_1t - \frac{a_2^2t}{a_3} - \frac{4a(at-2x)}{b^2} + \frac{(a_2+2a_3x)^2\tan(\sqrt{a_3}bt)}{\sqrt{a_3^3}b}\right)\right\} \\
&\quad \times\sqrt{\cos(\sqrt{a_3}bt)}u,
\end{aligned}$$

Table 9. Lie Brackets for the Heath equation when $\nu(x) = a_1 + a_2x + a_3x^2$.

$[.,.]_{LB}$	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	0	$\sqrt{a_3}b\Gamma_4$	$-\sqrt{a_3}b\Gamma_3$	$2\sqrt{a_3}b\Gamma_6$	$-2\sqrt{a_3}b\Gamma_5$
Γ_2	0	0	0	0	0	0
Γ_3	$-\sqrt{a_3}b\Gamma_4$	0	0	$-\frac{4\sqrt{a_3}}{b}\Gamma_2$	$8\Gamma_4$	$-8\Gamma_3$
Γ_4	$\sqrt{a_3}b\Gamma_3$	0	$\frac{4\sqrt{a_3}}{b}\Gamma_2$	0	$8\Gamma_3$	$8\Gamma_4$
Γ_5	$-2\sqrt{a_3}b\Gamma_6$	0	$-8\Gamma_4$	$-8\Gamma_3$	0	$-\frac{128}{\sqrt{a_3}\beta}\Gamma_1 + A\Gamma_2$
Γ_6	$2\sqrt{a_3}b\Gamma_5$	0	$8\Gamma_3$	$-8\Gamma_4$	$\frac{128}{\sqrt{a_3}\beta}\Gamma_1 - A\Gamma_2$	0

to the heat equation,

$$U_T - \frac{b}{16\sqrt{a_3}}U_{XX} = 0.$$

In the case of $\nu(x) = a_1 + a_2x + a_3x^2 + a_4/(a_2 + 2a_3x)^2$, the algebra is $\{sl(2,R) \oplus A_1\} \oplus_s \infty A_1$, which, as was previously mentioned, is characteristic of evolution equations derived from the Ermakov–Pinney equation, (19) [21].

The transformation that connects this special case to (19) is

$$\begin{aligned} T(x,t,u) &= \frac{1}{2}\sqrt{a_3}b\tan(\sqrt{a_3}bt), \\ X(x,t,u) &= b(a_2 + 2a_3x)c_2\sec(\sqrt{a_3}bt), \\ U(x,t,u) &= c\exp\left\{\frac{1}{8}\left(4a_1t - \frac{a_2^2t}{a_3} - \frac{4a(at-2x)}{b^2} + \frac{(a_2+2a_3x)^2\tan(\sqrt{a_3}bt)}{a_3^{3/2}b}\right)\right\} \\ &\quad \times\sqrt{\cos(\sqrt{a_3}bt)u} \end{aligned}$$

where $c_2^2 = \frac{a_3}{a_4}$.

7. Conclusion

We have considered five $(1+1)$ evolution equations which arise in Financial Mathematics in the context of the pricing of options and bonds, the term structure of interest rates and the value of commodities and assets in general from the point of view of their complete specification through their Lie point symmetries. Although the application of the Lie theory of infinitesimal groups to the resolution of differential equations has become a commonplace in the traditionally exact sciences since the resurgence of interest in the theory in the 1950s, its application in the more qualitative sciences is really only just beginning. Even in the case of Financial Mathematics for which the mathematical modeling of the problems mentioned above in terms of evolution equations has been with us for some forty years there has been very little analysis of these equations in terms of their Lie point symmetries until the last few years with the notable exception of [14] and it should be remembered that the problem solved in that paper was scarcely relevant to real finance. The idea of the complete specification of a differential equation in terms of its symmetries is not new, but its flourishing is definitely a matter of the current century. Since Bluman and Cole [6] had already presented, albeit not precisely explicitly, the idea in their well-known text, this can only be described as curious.

The general principle of the complete specification of differential equations in terms of their symmetries has had two lines of evolution. In one of these lines the type of Lie symmetry is restricted to point [3, 23, 2] and is strongly associated with the property of Lie remarkability, i.e. the equation is completely specified by the totality of the point symmetries of the equation. In the second line there is no restriction upon the type of Lie symmetry used to specify the equation. Evidently this approach is more appropriate for equations which do not have a sufficient supply of point symmetries for complete specification. Here we have inclined to the former approach since it does have a tendency to unify the models from the somewhat disparate underlying problems. Nevertheless we have to accept the latter approach when the former fails to produce the goods.

In our investigations of these five equations which model various phenomena in Financial Mathematics we were guided by the principle of Lie remarkability and this led us to a consideration of the commonality of the Black–Scholes, Longstaff and Vasicek equations with the standard heat equation for all values of the parameters contained in those three equations. This similarity within the algebraic structures means that the various effects introduced into the three models do not destroy the basic balance between evolution in time and diffusion in the other independent variable. There has been no symmetry breaking. In the case of the Cox–Ingersoll–Ross equation this commonality could only be achieved provided two of the parameters in the equation were constrained (which led to two cases economically not acceptable). Otherwise the algebra was of lower dimension which rendered it impossible for the Cox–Ingersoll–Ross equation in general to be Lie remarkable. Nevertheless the algebra was sufficient to be able to identify an elementary form of an heat equation to which it could be transformed. Curiously this form was the classical Ermakov–Pinney equation. The very existence of a general function, $\nu(x)$, in the Heath equation precluded a transformation to the heat equation, at least by means of point transformation. Yet the existence of the infinite-dimensional subalgebra implied the existence of a linearizing transformation. Further analysis showed that for functions $\nu(x)$ of specific form the Heath equation could either belong to the class related to the classical heat equation by means of point transformation or, once again, to the Ermakov–Pinney equation. One should note that the heat equation and the Ermakov–Pinney equation cannot be linked even with a noninvertible (point or contact) transformation; the symmetries in both cases are exactly the same and the two additional ones for the former overimpose restrictions upon the form of the transformations.

In the case of modeling of commodities [15, 27] we showed how the introduction of various factors into the different models could leave untouched the algebraic integrity of the equation [29]. This was achieved by group theoretical methods. For the CIR and Heath equations we have seen the introduction of terms which break that symmetry contrary to what happened in the first three equations. Eventually with the most general form for the Heath equation there has been an almost complete loss of symmetry in terms of the finite subalgebra. Although it is not the purpose of this paper to discuss the process of modeling in Financial Mathematics, it does seem that there is ample scope for research into the effects which may be considered in the modeling process in terms of their algebraic implications. If the finite-dimensional subalgebra can be maintained to a reasonable number of dimensions and certainly the four of the general Cox–Ingersoll–Ross equation has been shown to be sufficient, then symmetry methods can be used to solve the standard terminal-condition problem.

The purpose of this work was two-fold. Firstly we have been proven able to connect these equations to the well-known and widely-studied heat equation and therefore we have shown how one can use the results for the latter to obtain solutions of any of the former.[b] This in itself is very important. Since similar equations are even nowadays widely used by financial institutions, practitioners have a direct method to obtain solutions important to their needs.

[b]As was noted already in the Introduction, this link, for some cases of the equations presented here, is not new but it is a commonplace in the economical and mathematical literature.

Secondly we have completely characterized all these equations by simply highlighting this connection to a Lie remarkable partial differential equation. This is a natural consequence of papers devoted to complete symmetry groups and the notion of Lie remarkability [3, 23, 2].

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