Statistical Properties and Different Estimation Procedures of Poisson–Lindley Distribution

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ABSTRACT

In this paper, we propose a new class of distributions by compounding Lindley distributed random variates with the number of variates being zero-truncated Poisson distribution. This model is called a compound zero-truncated Poisson–Lindley distribution with two parameters. Different statistical properties of the proposed model are discussed. We describe different methods of estimation for the unknown parameters involved in the model. These methods include maximum likelihood, least squares, weighted least squares, Cramer–von Mises, maximum product of spacings, Anderson–Darling and right-tail Anderson–Darling methods. Numerical simulation experiments are conducted to assess the performance of the so obtained estimators developed from these methods. Finally, the potentiality of the model is studied using one real data set representing the monthly highest snowfall during February 2018, for a subset of stations in the Global Historical Climatological Network of USA.

1. INTRODUCTION

In recent years, many researches are interested in obtaining several new continuous distributions by compounding an absolutely continuous distribution with a discrete distribution. This method is used widely in engineering applications including risk measurement, floods reliability and survical analysis. For example, Adamidis and Loukas [1] proposed a two-parameter lifetime distribution by compounding exponential and geometric distributions. The exponential Poisson (EP) and exponential logarithmic distributions were introduced by Kus [2] and Tahmasbi and Rezaei [3], respectively. Marshall and Olkin [4] developed some new extensions based on random minimum and maximum. Barreto-Souza and Cribari-Neto [5] introduced the exponentiated exponential Poisson (EEP).

Cancho *et al.* [6] introduced the Poisson-exponential (PE) distribution by compounding the exponential and zero-truncated Poisson distributions. Chahkandi and Ganjali [7] introduced a class of distributions, namely exponential power series (EPS) distributions by compounding exponential and power series distributions. Also in the same way, Mahmoudi and Jafari [8] introduced the generalized exponential power series (GEPS) distribution by compounding the generalized exponential (GE) distribution with the power series distribution. The performances of the estimators using intensive simulation experiments have received considerable attention in the literature by several authors. Among them, Gupta and Kundu [9], Kundu and Raqab [10], Alkasabeh and Raqab [11], Asgharzadeh *et al.* [12], Dey *et al.* [13] and Rodrigues *et al.* [14].

The main aim of the present study is two-fold. The first main aim is to introduce a new model which is flexible in fitting a wide range of data sets by compounding Lindley and zero-truncated Poisson distributions. The basic idea can be described as follows. Consider a random variable X having the Lindley distribution with probability density function (PDF)

$$f_X(x; \lambda) = \frac{\lambda^2}{\lambda + 1}(1 + x)e^{-\lambda x}, \quad x > 0, \ \lambda > 0,$$ (1)

and cumulative distribution function (CDF)

$$F_X(x; \lambda) = 1 - \left(1 + \frac{\lambda}{\lambda + 1}x\right)e^{-\lambda x}, \ x > 0, \ \lambda > 0.$$ (2)

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Given $M = m$, let $X_1, X_2, ..., X_m$ be independent and identically distributed (iid) random variables from Lindley distribution. The random variable M follows zero-truncated Poisson distribution with PDF

$$P(M = m) = \frac{\theta^m}{m!\,(e^\theta - 1)}, \quad m = 1, 2, 3, ..., \quad \theta > 0. \tag{3}$$

Here, we introduce a new class of distributions based on the maximal random variate $Y = \max\{X_1, X_2 ..., X_M\}$. Now we have

$$
\begin{aligned}
P(Y \le y) &= P(\max\{X_1, X_2 ..., X_M\} \le y) \\
&= \sum_{m=1}^\infty P\left(\max\{X_1, X_2 ..., X_M\} \le y | M = m\right) P(M = m) \\
&= \sum_{m=1}^\infty [F(y; \lambda, \theta)]^m \, P(M = m) \\
&= \sum_{m=1}^\infty \frac{[\theta\, F_X(y; \lambda, \theta)]^m}{m!} \frac{1}{e^\theta - 1} \\
&= \frac{e^{\theta\, F_X(y;\lambda,\theta)} - 1}{e^\theta - 1},
\end{aligned}
\tag{4}
$$

where $F_X(x; \lambda)$ is the CDF of the Lindley distribution defined in (2). The so obtained model is called the compound zero-truncated Poisson Lindley (ZTPL) distribution with two parameters. It has an absolute continuous distribution function. Moreover, the Lindley distribution can be obtained as a special case of the compound ZTPL.

The second aim is to present various estimation methods for estimating the two parameters of the compound ZTPL model. The estimators to be considered are maximum likelihood estimators (MLEs), least square estimators (LSEs), weighted least square estimators (WLSEs), Cramer–von Mises type minimum distance estimators (CMDEs), maximum product of spacings, Anderson–Darling estimators (ADEs) and right-tail Anderson–Darling estimation (RTADE). An intensive simulation study is performed for comparing the effectiveness of the so developed of estimators.

This paper is organized as follows: In Section 2, the ZTPL model is described and its distributional properties are discussed. Also, different methods for estimating the parameters of the ZTPL model are developed in Section 3. Numerical simulation results are presented in Section 4. The analysis of monthly highest snowfall data during the month of February 2018, for a subset of stations in the Global Historical Climatological Network of USA is performed for validation purposes in Section 5. Some concluding remarks are presented in Section 6.

2. ZTPL DISTRIBUTION

A random variable Y is said to have a compound ZTPL distribution if its CDF is given by

$$F_Y(y; \lambda, \theta) = \frac{e^{\theta\left[1 - \left(1 + \frac{\lambda y}{1 + \lambda}\right)e^{-\lambda y}\right]} - 1}{e^\theta - 1}. \tag{5}$$
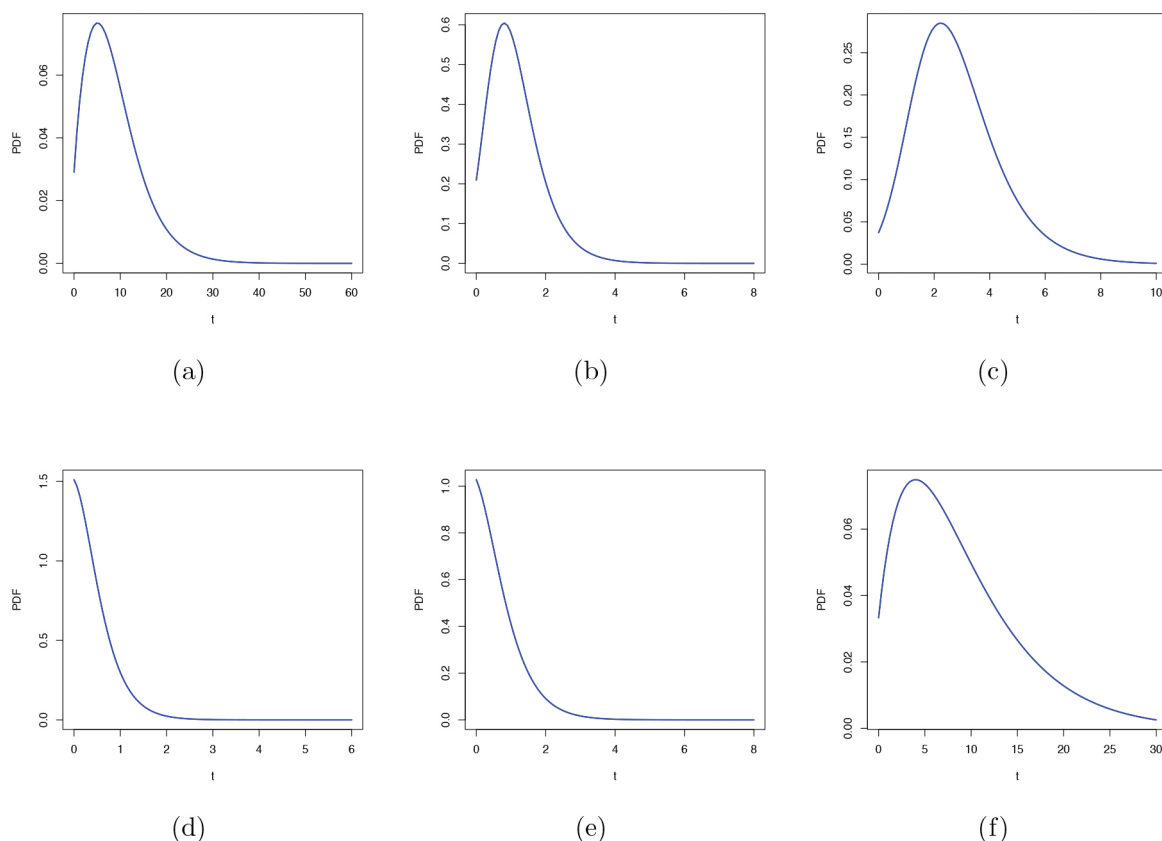
The corresponding PDF of Y can be obtained to be

$$f_Y(y; \lambda, \theta) = \frac{\theta\lambda^2(1 + y)e^{-\lambda y}}{(e^\theta - 1)(\lambda + 1)} e^{\theta\left[1 - \left(1 + \frac{\lambda y}{1 + \lambda}\right)e^{-\lambda y}\right]}, \quad y > 0; \ \theta, \lambda > 0. \tag{6}$$

From (6), it is easily seen that the Lindley distribution is a special case of compound ZTPL when $\theta \to 0$. Hence the parameter θ can be interpreted as a concentration parameter. Figure 1 provides plots of the compound ZTPL distribution for some selected choices of λ and θ. It is observed that the compound ZTPL distribution can be decreasing and unimodal.

The joint PDF of Y and M is given as

$$f_{Y,M}(y, m) = \frac{\theta\lambda^2}{(e^\theta - 1)(m - 1)!\,(\lambda + 1)}(1 + y)e^{-\lambda y}\left[\theta\left(1 - \left(1 + \frac{\lambda y}{1 + \lambda}\right)e^{-\lambda y}\right)\right]^{m-1}. \tag{7}$$

Figure 1 | Probability density function (PDF) plots of zero-truncated Poisson–Lindley (ZTPL) distribution for different parameter values: (a) (0.25, 1) (b) (2, 3) (c) (1, 4) (d) (3, 0.75) (e) (2, 0.5) (f) (0.2, 0.0001).

Further, from (6) and (7), it can be shown that the PDF of the conditional distribution of M given $Y = y$ is just Poisson random variate with mean $\theta \left(1 - \left(1 + \dfrac{\lambda y}{1 + \lambda}\right) e^{-\lambda y}\right)$. That is,

$$P(M = m | Y = y) = \frac{\left[\theta \left(1 - \left(1 + \dfrac{\lambda y}{1 + \lambda}\right) e^{-\lambda y}\right)\right]^{m-1}}{(m-1)! \; e^{\theta \left[1 - \left(1 + \dfrac{\lambda y}{1 + \lambda}\right) e^{-\lambda y}\right]}}, \; m = 1, 2, \ldots.$$

The survival function and hazard rate (HR) of the $ZTPL(\lambda, \theta)$ distribution, are given respectively by

$$S(y) = \frac{e^{\theta}}{e^{\theta} - 1} \left\{1 - e^{\left[1 - \left(1 + \dfrac{\lambda y}{1 + \lambda}\right) e^{-\lambda y}\right]}\right\},$$
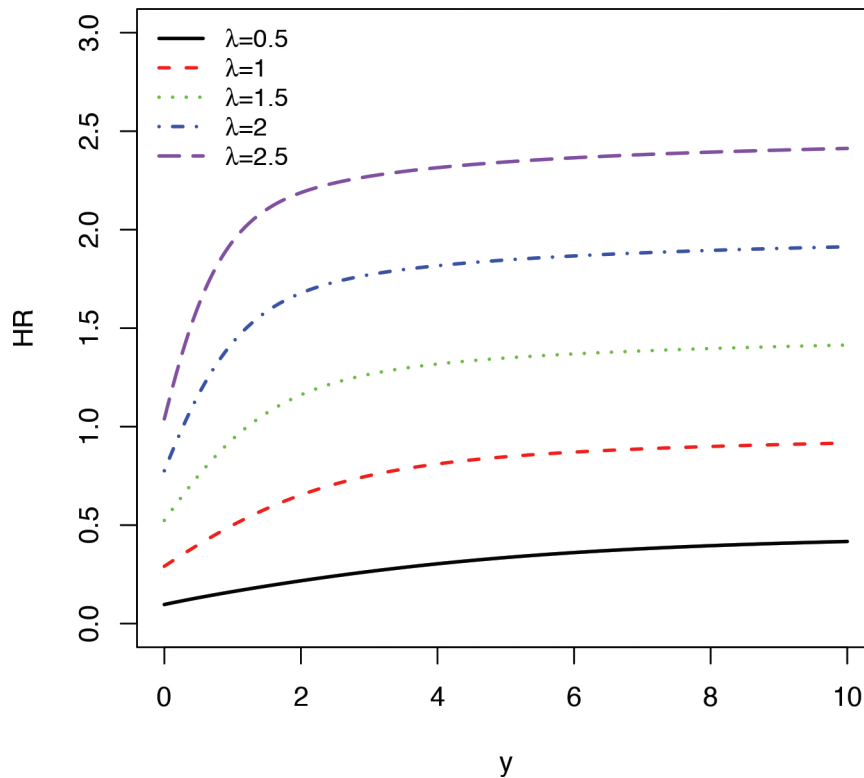
and

$$h(y) = \frac{\theta \lambda^2 (1 + y) e^{-\lambda y}}{(\lambda + 1) e^{\theta}} \frac{e^{\theta \left[1 - \left(1 + \dfrac{\lambda y}{1 + \lambda}\right) e^{-\lambda y}\right]}}{1 - e^{\left[1 - \left(1 + \dfrac{\lambda y}{1 + \lambda}\right) e^{-\lambda y}\right]}}.$$

Figure 2 presents different shapes of HR for the compound $ZTPL(\lambda, \theta)$ distribution considering different values of λ and θ. It is observed from Figure 2 that the HR function is increasing for all $\lambda > 0$ and $\theta > 0$.

The following expression for the r-th moment of Y can be obtained as

$$E(Y^r) = \frac{\theta \lambda^2}{(e^{\theta} - 1)(\lambda + 1)} C_{k,i,j} \left\{ \frac{\Gamma(r + j + 1)}{[\lambda(i + 1)]^{r+j+1}} + \frac{\Gamma(r + j + 2)}{[\lambda(i + 1)]^{r+j+2}} \right\},$$

Figure 2 | HR Plots for $\lambda = 0.5, 1, 1.5, 2, 2.5$ and fxed $\theta = 1$.

where

$$C_{k,i,j} = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\theta^k}{k!} \binom{k}{i} (-1)^i \binom{i}{j} \left(\frac{\lambda}{\lambda + 1} \right)^j.$$

Therefore, the mean and variance of Y are given by

$$E(Y) = \frac{\theta \lambda^2}{(e^\theta - 1)(\lambda + 1)} C_{k,i,j} \left(\frac{\Gamma(j + 2)}{[\lambda(i + 1)]^{j+2}} + \frac{\Gamma(j + 3)}{[\lambda(i + 1)]^{j+3}} \right),$$

and

$$Var(Y) = \frac{\theta \lambda^2}{(e^\theta - 1)(\lambda + 1)} C_{k,i,j} \left\{ \frac{\Gamma(j + 3)}{[\lambda(i + 1)]^{j+3}} + \frac{\Gamma(j + 4)}{[\lambda(i + 1)]^{j+4}} \right. $$
$$\left. - \frac{\theta \lambda^2}{(e^\theta - 1)(\lambda + 1)} C_{k,i,j} \left(\frac{\Gamma(j + 2)}{[\lambda(i + 1)]^{j+2}} + \frac{\Gamma(j + 3)}{[\lambda(i + 1)]^{j+3}} \right)^2 \right\}.$$

The skewness measure of Y is given by

$$\gamma_3 = \left(\frac{\lambda(\lambda + 1)(e^\theta - 1)}{\theta} \right)^{1/2} C_{k,i,j}^{-1/2} \left\{ \frac{\Gamma(j + 3)}{[\lambda(i + 1)]^{j+3}} + \frac{\Gamma(j + 4)}{[\lambda(i + 1)]^{j+4}} \right.$$
$$\left. - \frac{\theta \lambda^2}{(e^\theta - 1)(\lambda + 1)} C_{k,i,j} \left(\frac{\Gamma(j + 2)}{[\lambda(i + 1)]^{j+2}} + \frac{\Gamma(j + 3)}{[\lambda(i + 1)]^{j+3}} \right)^2 \right\}^{-3/2}$$
$$\times \left\{ \frac{\Gamma(j + 4)}{[\lambda(i + 1)]^{j+4}} + \frac{\Gamma(j + 5)}{[\lambda(i + 1)]^{j+5}} \right\}.$$

The moment generating function (MGF) of Y, $M_Y(t)$, is

$$M_Y(t) = \frac{\theta \lambda^2}{(e^\theta - 1)(\lambda + 1)} C_{k,i,j} \left\{ \frac{\Gamma(j + 1)}{[\lambda(i + 1) - t]^{j+1}} + \frac{\Gamma(j + 2)}{[\lambda(i + 1) - t]^{j+2}} \right\}.$$

3. METHODS OF ESTIMATION

In this section, we present different estimation methods for obtaining the estimators of the parameters λ and θ of the compound ZTPL distribution. These methods are quite useful in obtaining the estimators of λ and θ and other related inferences.

3.1. Maximum Likelihood Estimation and its Asymptotics

Let $\{y_1, ..., y_n\}$ be a random sample of size n from $ZTPL(\lambda, \theta)$. Then the log-likelihood function is given as

$$l(\lambda, \theta) = n\log(\theta) + 2n\log(\lambda) + \sum_{i=1}^{n}\log(1 + y_i) - \lambda\sum_{i=1}^{n}y_i + \theta\sum_{i=1}^{n}\left[1 - \left(1 + \frac{\lambda y_i}{1 + \lambda}\right)e^{-\lambda y_i}\right]$$
$$-n\log(e^{\theta} - 1) - n\log(1 + \lambda). \tag{8}$$

The MLEs $\hat{\lambda}_{MLE}$ and $\hat{\theta}_{MLE}$ of λ and θ, are obtained respectively by solving the two nonlinear equations:

$$\frac{\partial l(\lambda, \theta)}{\partial \lambda} = \frac{2n}{\lambda} - \frac{n}{1 + \lambda} - \sum_{i=1}^{n}y_i + \frac{\theta}{(\lambda + 1)^2}\sum_{i=1}^{n}\lambda y_i e^{-\lambda y_i}(\lambda y_i + y_i + \lambda + 2) = 0,$$

and

$$\frac{\partial l(\lambda, \theta)}{\partial \theta} = \frac{n}{\theta} - \frac{n}{e^{\theta} - 1} + \sum_{i=1}^{n}\left[1 - \left(1 + \frac{\lambda y_i}{1 + \lambda}\right)e^{-\lambda y_i}\right] = 0.$$

Another aspect of estimation is to construct confidence intervals (CIs) of the parameters by making use of the asymptotic distribution theory of MLE. By denoting the parameter vector $\boldsymbol{\vartheta} = (\lambda, \theta)$, the asymptotic distribution of $\boldsymbol{\vartheta}$ is $(\hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}) \xrightarrow{D} N_2(0, \boldsymbol{\Gamma}^{-1})$, where $\hat{\boldsymbol{\vartheta}}$ is the MLE of $\boldsymbol{\vartheta}$ and $\boldsymbol{\Gamma}^{-1}(\boldsymbol{\vartheta})$ is the inverse of the observed information matrix of $\boldsymbol{\vartheta} = (\lambda, \theta)$, which can be approximated by $\boldsymbol{\Gamma}^{-1}(\hat{\boldsymbol{\vartheta}})$, where

$$I(\hat{\boldsymbol{\vartheta}}) = \left(-\frac{\partial^2 l}{\partial \vartheta_i \, \partial \vartheta_j}\right)|_{\vartheta = \hat{\vartheta}},$$

with $\vartheta_1 = \lambda$ and $\vartheta_2 = \theta$.

The elements of matrix $I(\boldsymbol{\vartheta})$ are derived based on (8) as follows:

$$I_{11} = \frac{2n}{\lambda^2} - \frac{n}{(1 + \lambda)^2} + \frac{\theta}{(\lambda + 1)^3}\sum_{i=1}^{n}y_i e^{-\lambda y_i}(\lambda^3 y_i^2 + \lambda^3 y_i + 2\lambda^2 y_i^2 + 3\lambda^2 y_i + \lambda y_i^2 + \lambda y_i - y_i - 2),$$

$$I_{12} = I_{21} = -\frac{1}{(\lambda + 1)^2}\sum_{i=1}^{n}\lambda y_i e^{-\lambda y_i}(\lambda y_i + y_i + \lambda + 2),$$

and

$$I_{22} = \frac{n}{\theta^2} - \frac{ne^{\theta}}{(e^{\theta} - 1)^2}.$$

Therefore, the lower confidence limit (LCL) and upper confidence limit (UCL) of $(1 - \alpha)\%$ CI of ϑ_j are

$$LCB = \hat{\vartheta}_j - z_{\alpha/2}\sqrt{I^{-1}(\hat{\vartheta}_j)}, j = 1, 2,$$

and

$$UCB = \hat{\vartheta}_j + z_{\alpha/2}\sqrt{I^{-1}(\hat{\vartheta}_j)}, j = 1, 2,$$

where $z_{\alpha/2}$ is upper $\alpha/2$ quantile of the standard normal distribution, $N(0, 1)$. However, the drawback of this method is that the lower limit of the CI may be negative, which is inadmissible. For this reason, one may use the delta method and logarithmic transformation can avoid this problem. The asymptotic distribution of $\ln \hat{\vartheta}_j$ is

$$\left(\ln \hat{\vartheta}_j - \ln \vartheta_j\right) \xrightarrow{D} N(0, var(\ln \hat{\vartheta}_j)),$$

where

$$var(\ln \hat{\vartheta}_j) = \frac{var(\hat{\vartheta}_j)}{\hat{\vartheta}_j^2} = \frac{[I^{-1}(\hat{\vartheta})]_j}{\hat{\vartheta}_j^2}.$$

Then the $(1 - \alpha)\%$ CI of ϑ_j can be written as

$$\left(\frac{\hat{\vartheta}_j}{Exp\left(z_{\alpha/2}\sqrt{var(\ln \hat{\vartheta}_j)}\right)}, \ \hat{\vartheta}_j \ Exp\left(z_{\alpha/2}\sqrt{var(\ln \hat{\vartheta}_j)}\right) \right).$$

3.2. Least Square and Weighted LSEs

Swain *et al.* [15] proposed an alternative method to compute the estimation of unknown parameters, which is called the LSEs or WLSEs. The basic idea can be defined as follows. Let $y_1, ..., y_n$ is a random sample of size n from ZTPL distribution and $y_{(1)} < ... < y_{(n)}$ denote the order statistics of the random sample. The LSEs of λ and θ (say, $\hat{\lambda}_{LSE}$ and $\hat{\theta}_{LSE}$) can be obtained by minimizing

$$\sum_{i=1}^{n}\left[F(y_{(i)}|\lambda, \theta) - \frac{i}{n+1}\right]^2,$$

with respect to λ and θ, where $F(y|\lambda, \theta)$ is given by (5). Equivalently, they can be obtained by solving the following nonlinear equations:

$$\sum_{i=1}^{n}\left[F(y_{(i)}|\lambda, \theta) - \frac{i}{n+1}\right]\eta_1(y_{(i)}|\lambda, \theta) = 0,$$

$$\sum_{i=1}^{n}\left[F(y_{(i)}|\lambda, \theta) - \frac{i}{n+1}\right]\eta_2(y_{(i)}|\lambda, \theta) = 0,$$

where

$$\eta_1(y_{(i)}|\lambda, \theta) = \frac{\theta\lambda y_{(i)}e^{-\lambda y_{(i)}}}{(1+\lambda)^2(e^\theta - 1)} \exp\left\{\theta\left[1 - \left(1 + \frac{\lambda y_{(i)}}{1+\lambda}\right)e^{-\lambda y_{(i)}}\right]\right\}$$
$$\times \left(\lambda(1 + y_{(i)}) + y_{(i)} + 2\right), \tag{9}$$

$$\eta_2(y_{(i)}|\lambda, \theta) = \frac{1}{(e^\theta - 1)^2}\left\{(e^\theta - 1)\left[1 - \left(1 + \frac{\lambda y_{(i)}}{1+\lambda}\right)e^{-\lambda y_{(i)}}\right]\exp\left(\theta\left[1 - \left(1 + \frac{\lambda y_{(i)}}{1+\lambda}\right)e^{-\lambda y_{(i)}}\right]\right)\right.$$
$$\left. - \exp\left(\theta^2\left[1 - \left(1 + \frac{\lambda y_{(i)}}{1+\lambda}\right)e^{-\lambda y_{(i)}}\right]\right) + e^\theta\right\}. \tag{10}$$

The WLSEs of λ and θ, say $\hat{\lambda}_{WLSE}$ and $\hat{\theta}_{WLSE}$, respectively, can be found by minimizing

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(y_{(i)}|\lambda, \theta) - \frac{i}{n+1}\right]^2.$$

Hence, the estimates $\hat{\lambda}_{WLSE}$ and $\hat{\theta}_{WLSE}$, respectively, can be obtained by solving the following nonlinear equations:

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(y_{(i)}|\lambda, \theta) - \frac{i}{n+1}\right]\eta_1(y_{(i)}|\lambda, \theta) = 0,$$

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(y_{(i)}|\lambda, \theta) - \frac{i}{n+1}\right]\eta_2(y_{(i)}|\lambda, \theta) = 0.$$

3.3. Maximum Product of Spacings

Cheng and Amin [16,17] introduced an elaborate technique to compute the estimation of unknown parameters of continuous univariate distributions, namely the maximum product spacing (MPS) method. It was developed by Ranneby [18] independently as an approximation to the Kullback–Leibler measure of information. The simple idea can be described as follows. Let

$$D_i(\lambda, \theta) = F(y_{(i)}|\lambda, \theta) - F(y_{(i-1)}|\lambda, \theta), \qquad i = 1, ..., n+1,$$

where

$$F(y_{(0)}|\lambda, \theta) = 0, \text{ and } F(y_{(n+1)}|\lambda, \theta) = 1.$$

Clearly $\sum_{i=1}^{n+1} D_i(\lambda, \theta) = 1$.

The MPS estimators (MPSEs) of λ and θ, say $\hat{\lambda}_{MPS}$ and $\hat{\theta}_{MPS}$, respectively, can be obtained by maximizing the geometric mean of the spacings

$$G(\lambda, \theta) = \left[\prod_{i=1}^{n+1} D_i(\lambda, \theta) \right]^{1/(n+1)}. \tag{11}$$

Equivalently, they can be obtained by maximizing the logarithm of the geometric mean of sample spacings:

$$H(\lambda, \theta) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log D_i(\lambda, \theta). \tag{12}$$

The estimates $\hat{\lambda}_{MPS}$ and $\hat{\theta}_{MPS}$ can be obtained by solving the two nonlinear equations:

$$\frac{\partial H(\lambda, \theta)}{\partial \lambda} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\lambda, \theta)} [\eta_1(y_{(i)}|\lambda, \theta) - \eta_1(y_{(i-1)}|\lambda, \theta)] = 0,$$

and

$$\frac{\partial H(\lambda, \theta)}{\partial \theta} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\lambda, \theta)} [\eta_2(y_{(i)}|\lambda, \theta) - \eta_2(y_{(i-1)}|\lambda, \theta)] = 0,$$

where $\eta_1(.|\lambda, \theta)$ and $\eta_2(.|\lambda, \theta)$ are given in (9) and (10), respectively.

3.4. Cramer–von Mises Minimum Distance Estimators

The CMDEs can be obtained as the difference between the estimate of the CDF and its respective empirical distribution function. MacDonald [19] provided empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. The Cramer–von Mises estimates $\hat{\lambda}_{CME}$ and $\hat{\theta}_{CME}$ are obtained by minimizing

$$C(\lambda, \theta) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(y_{(i)}|\lambda, \theta) - \frac{2i-1}{2n} \right]^2. \tag{13}$$

These estimates can also be obtained by solving the nonlinear equations:

$$\sum_{i=1}^{n} \left[F(y_{(i)}|\lambda, \theta) - \frac{2i-1}{2n} \right] \eta_1(y_{(i)}|\lambda, \theta) = 0,$$

$$\sum_{i=1}^{n} \left[F(y_{(i)}|\lambda, \theta) - \frac{2i-1}{2n} \right] \eta_2(y_{(i)}|\lambda, \theta) = 0,$$

where $\eta_1(.|\lambda, \theta)$ and $\eta_2(.|\lambda, \theta)$ are given in (9) and (10), respectively.

3.5. Anderson–Darling and Right-Tail Anderson–Darling

The ADE is another type of minimum distance estimator and it was introduced by Anderson and Darling [20]. The ADEs $\hat{\lambda}_{ADE}$ and $\hat{\theta}_{ADE}$ of λ and θ are, respectively, obtained by minimizing

$$A(\lambda, \theta) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i - 1)\{\log F(y_{(i)}|\lambda, \theta) + \log S(y_{(n+1)}|\lambda, \theta)\}.$$

These estimates can also be obtained by solving the following equations:

$$\sum_{i=1}^{n} (2i - 1) \left[\frac{\eta_1(y_{(i)}|\lambda, \theta)}{F(y_{(i)}|\lambda, \theta)} - \frac{\eta_1(y_{(n+1)}|\lambda, \theta)}{S(y_{(n+1)}|\lambda, \theta)} \right] = 0,$$

$$\sum_{i=1}^{n} (2i - 1) \left[\frac{\eta_2(y_{(i)}|\lambda, \theta)}{F(y_{(i)}|\lambda, \theta)} - \frac{\eta_2(y_{(n+1)}|\lambda, \theta)}{S(y_{(n+1)}|\lambda, \theta)} \right] = 0,$$

where $\eta_1(.|\lambda, \theta)$ and $\eta_2(.|\lambda, \theta)$ are given in (9) and (10), respectively.

Similarly, the RTADEs, $\hat{\lambda}_{RTADE}$ and $\hat{\theta}_{RTADE}$ are obtained by minimizing

$$R(\lambda, \theta) = \frac{n}{2} - 2\sum_{i=1}^{n} \log F(y_{(i)}|\lambda, \theta) - \frac{1}{n} \sum_{i=1}^{n} (2i - 1) \log S(y_{(n+1)}|\lambda, \theta).$$

These estimates can be obtained by solving the the following equations:

$$-2\sum_{i=1}^{n} \frac{\eta_1(y_{(i)}|\lambda, \theta)}{F(y_{(i)}|\lambda, \theta)} + \frac{1}{n} \sum_{i=1}^{n} (2i - 1)\frac{\eta_1(y_{(n+1)}|\lambda, \theta)}{S(y_{(n+1)}|\lambda, \theta)} = 0,$$

$$-2\sum_{i=1}^{n} \frac{\eta_2(y_{(i)}|\lambda, \theta)}{F(y_{(i)}|\lambda, \theta)} + \frac{1}{n} \sum_{i=1}^{n} (2i - 1)\frac{\eta_2(y_{(n+1)}|\lambda, \theta)}{S(y_{(n+1)}|\lambda, \theta)} = 0,$$

where $\eta_1(.|\lambda, \theta)$ and $\eta_2(.|\lambda, \theta)$ are given in (9) and (10), respectively.

4. NUMERICAL EXPERIMENTS AND DISCUSSIONS

In this section, we present some results of Monte Carlo simulation study to compare the efficiency of the different estimation procedures proposed in the previous sections. For a given set of parameter values for λ and θ and for a given sample size, we first generate a random sample of size n from the compound ZTPL model. Secondly, we compute the average estimates (AEs) and the associated mean squared errors (MSEs) over on 1000 replications. The results are recorded in Tables 1 and 2. The simulation study was computed using the software R. The performance of different methods of estimation are evaluated in terms of MSEs.

Some of the points are quite clear from Tables 1 and 2. As the sample size increases, the AEs based on all estimation methods tend to the true parameter values and the MSEs decrease. This indicates that all estimators are consistent and asymptotically unbiased. Furthermore, it is observed that as λ and θ increase, the MSEs increase for all the estimates. Based on the MSE as an optimality criterion, the MPSEs have superior performance than the MLEs and other types of estimation for the compound ZTPL distribution. These results are shown with other studies, see, for example, Ramos and Louzada [21] and Sharma *et al.* [22]. However, the RTADE has smaller MSE than the ADE. Figures 3 and 4 confirm that these concluding remarks.

5. APPLICATION TO MONTHLY MAXIMUM SNOWFALL DATA

In this section, we discuss the analysis of real-life data representing the monthly highest snowfall during the month of February 2018, obtained from a subset of stations in the United States and it is measured in inches (in). This data set was reported in: the National Centers for Environmental Information (NCEI) (https://www.ncdc.noaa.gov/cdoweb/datatools/records). Here, we only wish to demonstrate the use of the estimation procedures based on samples from compound ZTPL model. Table 3 summarizes some basic statistics of the monthly maximum snowfall data set.

The following distributions are used in the literature as fitting models for the data set. For example, Poisson Lomax PL (α, β, λ) distribution is introduced by Bander and Hanaa [23], exponentiated Weibull–Poisson (EWP) $(\alpha, \beta, \gamma, \theta)$ distribution is considered by Mahmoudi and

Table 1 | The AE and the associated MSEs for the estimates of λ and θ considering different sample sizes.

Par	n	Est.	MLE	LSE	WLSE	CME	MPS	ADE	RTADE
$\lambda = 0.5$	25	AE	0.5673	0.4897	0.4929	0.5263	0.4816	0.5102	0.4686
		MSE	0.0200	0.0316	0.0315	0.0313	0.0041	0.0257	0.0218
	50	AE	0.5437	0.4829	0.4802	0.5040	0.4983	0.4975	0.4902
		MSE	0.0089	0.0170	0.0146	0.0158	0.0024	0.0175	0.0113
	100	AE	0.5227	0.4873	0.5003	0.4997	0.4992	0.4914	0.5027
		MSE	0.0042	0.0096	0.0067	0.0084	0.0011	0.0046	0.0064
	300	AE	0.5098	0.4891	0.4990	0.4986	0.5004	0.4942	0.5050
		MSE	0.0015	0.0036	0.0020	0.0028	0.0009	0.0020	0.0023
	500	AE	0.5026	0.5049	0.4968	0.5007	0.5090	0.5008	0.5022
		MSE	0.0010	0.0014	0.0013	0.0016	0.0003	0.0013	0.0010
$\theta = 0.5$	25	AE	1.1585	0.2207	0.1742	0.6038	0.4670	0.4126	0.1417
		MSE	1.6850	2.6828	3.2857	3.2989	0.0076	2.7555	3.8645
	50	AE	0.8743	0.2743	0.2389	0.4794	0.4896	0.4011	0.3443
		MSE	0.8223	2.4417	1.7143	1.6978	0.0010	1.9692	1.4611
	100	AE	0.5960	0.3460	0.4395	0.4713	0.4963	0.4337	0.4771
		MSE	0.3872	0.9877	0.7283	0.9177	0.0008	0.5503	0.8661
	300	AE	0.3400	0.4181	0.4931	0.4756	0.5049	0.4486	0.5061
		MSE	0.1264	0.3812	0.2301	0.2816	0.0007	0.2131	0.2165
	500	AE	0.5292	0.5073	0.4734	0.5011	0.5097	0.4915	0.5216
		MSE	0.0960	0.1335	0.1338	0.1657	0.0003	0.1373	0.0969

AE, average estimate; MSE, mean squared error; MLE, maximum likelihood estimator; LSE, least square estimator; WLSE, weighted least square estimator; MPS, maximum product spacing; ADE, Anderson–Darling estimator; RTADE, right-tail Anderson–Darling estimator.

Table 2 | The AE and the associated MSEs for the estimates of λ and θ considering different sample sizes.

Par	n	Est.	MLE	LSE	WLSE	CME	MPS	ADE	RTADE
$\lambda = 1$	25	AE	1.0759	0.9318	0.9798	1.0336	0.9598	0.9866	1.0077
		MSE	0.0508	0.1168	0.0937	0.0938	0.0202	0.0864	0.0954
	50	AE	1.0092	0.9861	0.9562	1.0146	0.9808	0.9430	1.0216
		MSE	0.0207	0.0601	0.0437	0.0565	0.0084	0.0446	0.0477
	100	AE	1.0223	0.9873	1.0170	1.0215	0.9912	0.9958	1.0153
		MSE	0.0143	0.0351	0.0220	0.0262	0.0030	0.0244	0.0248
	300	AE	1.0015	0.9865	0.9780	1.0080	1.0056	1.0096	0.9913
		MSE	0.0048	0.0098	0.0082	0.0086	0.0026	0.0085	0.0070
	500	AE	1.0034	1.0043	1.0053	1.0014	1.0176	1.0025	1.0078
		MSE	0.0037	0.0049	0.0044	0.0047	0.0020	0.0044	0.0034
$\theta = 0.75$	25	AE	1.0774	0.2428	0.5457	0.8378	0.7257	0.4698	0.7177
		MSE	1.2320	3.6427	2.4906	2.2156	0.0071	2.8561	3.0602
	50	AE	0.8486	0.6438	0.5907	0.7168	0.7453	0.4715	0.8298
		MSE	0.5147	1.7936	1.2246	1.1642	0.0012	1.3342	1.3694
	100	AE	0.8745	0.6277	0.8020	0.8503	0.7480	0.6720	0.8176
		MSE	0.4271	0.8848	0.5889	0.5837	0.0008	0.6024	0.6815
	300	AE	0.7589	0.6678	0.6496	0.7904	0.7520	0.7519	0.7044
		MSE	0.1057	0.2395	0.1890	0.1721	0.0006	0.1865	0.1870
	500	AE	0.7682	0.7524	0.7613	0.7523	0.7583	0.7775	0.8085
		MSE	0.1000	0.1024	0.1045	0.1041	0.0005	0.1090	0.0978

AE, average estimate; MSE, mean squared error; MLE, maximum likelihood estimator; LSE, least square estimator; WLSE, weighted least square estimator; MPS, maximum product spacing; ADE, Anderson–Darling estimator; RTADE, right-tail Anderson–Darling estimator.

Table 3 | Basic statistics of monthly highest snowfall data set.

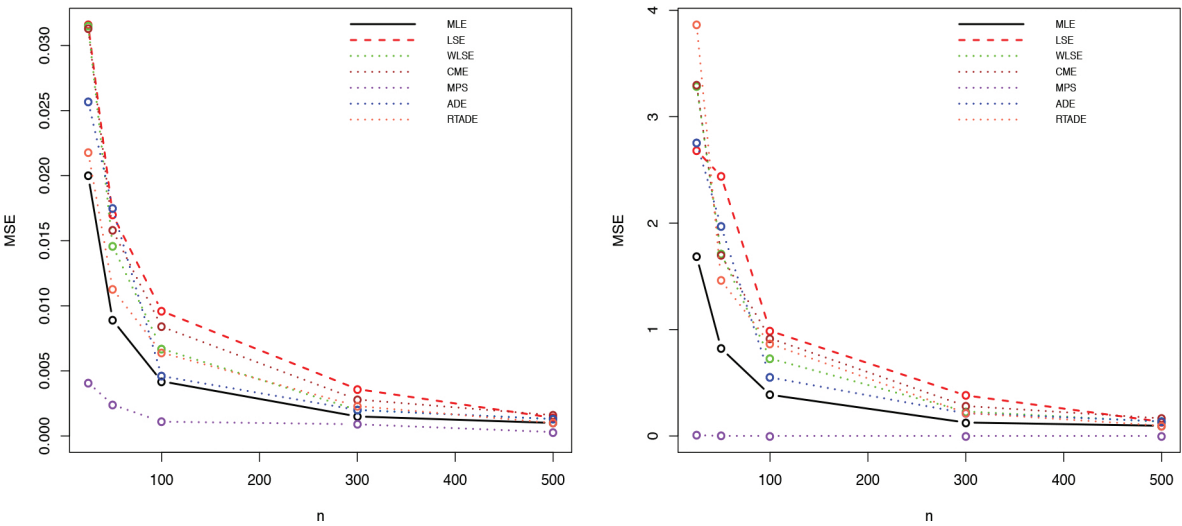| Mean	Median	Std.Dev.	Q_1	Q_3
8.25	7.99	3.50	5.98	10.09

Figure 3 | MSE of $\hat{\lambda}$ for $\lambda = 0.5$ (left) and MSE of $\hat{\theta}$ for $\theta = 0.5$ (right).
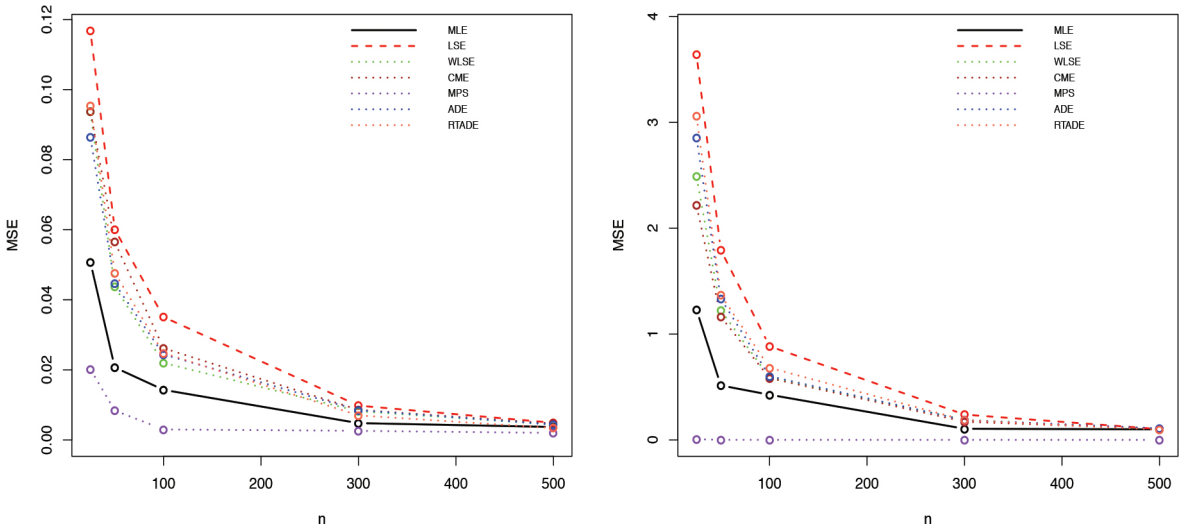

Figure 4 | MSE of $\hat{\lambda}$ for $\lambda = 1$ (left) and MSE of $\hat{\theta}$ for $\theta = 0.75$ (right).

Table 4 | The MLEs and the corresponding log-likelihood (ll) values for different fitting distributions.

Distribution	Parameters	ll
Lindley	$\hat{\lambda}$=0.2205	−156.4475
ZTPL	$\hat{\lambda}$=0.4259 $\hat{\theta}$=5.8033	−143.1997
PL	$\hat{\alpha}$=0.0107 $\hat{\beta}$=6.7197 $\hat{\lambda}$=29.0468	−145.3201
EWP	$\hat{\alpha} = 0.1814$ $\hat{\beta} = 4.2158$ $\hat{\gamma}$=0.4335 $\hat{\theta}$=319.3034	−152.0237

MLE, maximum likelihood estimator; ZTPL, zero-truncated Poisson–Lindley; EWP, exponentiated Weibull–Poisson.

Sepahdar [24] and Lindley (θ) distribution is also used by Ghitany *et al.* [25]. Here, we show that the compound ZTPL distribution is also a correct fitting distribution as an alternative to the PL, EWP and LI distributions. We fit the compound ZTPL distribution to the monthly highest snowfall data set. The MLEs and the corresponding log-likelihood (ll) values for each distribution are computed. The results are reported in Table 4. The MLEs of λ and θ are computed numerically using Newton–Raphson (NR) method to be $\hat{\lambda} = 0.4259$ and $\hat{\theta} = 5.8033$. The Kolmogorov–Smirnov (K-S) distance between the fitted and the empirical distribution functions is 0.1109, and the corresponding p values is 0.5197, respectively. Therefore, these values indicate that the two-parameter compound ZTPL distribution fits the data set well.

For further checking model validity and comparisons, the values of K-S and other criteria and their corresponding p values for other distributions including Lindley, PL and EWP distributions are computed. The additional considered criteria are Akaike information criterion (AIC), Akaike information criterion correction (AICc), Hannan–Quinn information criterion (HQIC), Bayesian information criterion (BIC). Table 5 presents the values of these statistics. Note that the smaller the value of the considered criterion, the better the fit to the data.

Table 5 | The goodness of fit tests for monthly highest snowfall data set.

| Distribution	AIC	AICc	HQIC	BIC	K-S	*p*- value
Lindley	314.8949	314.9719	315.6620	316.8839	0.2398	0.004
ZTPL	290.3994	290.6347	291.9335	294.3774	0.1109	0.5197
PL	296.6401	297.1201	298.9414	302.6071	0.1267	0.3512
EWP	312.0474	312.8637	315.1156	320.0033	0.1587	0.1317

MLE, maximum likelihood estimator; ZTPL, zero-truncated Poisson–Lindley; EWP, exponentiated Weibull–Poisson; AIC, Akaike information criterion; AICc, Akaike information criterion correction (AICc); HQIC, Hannan–Quinn information criterion; BIC, Bayesian information criterion; K-S, Kolmogorov–Smirnov.
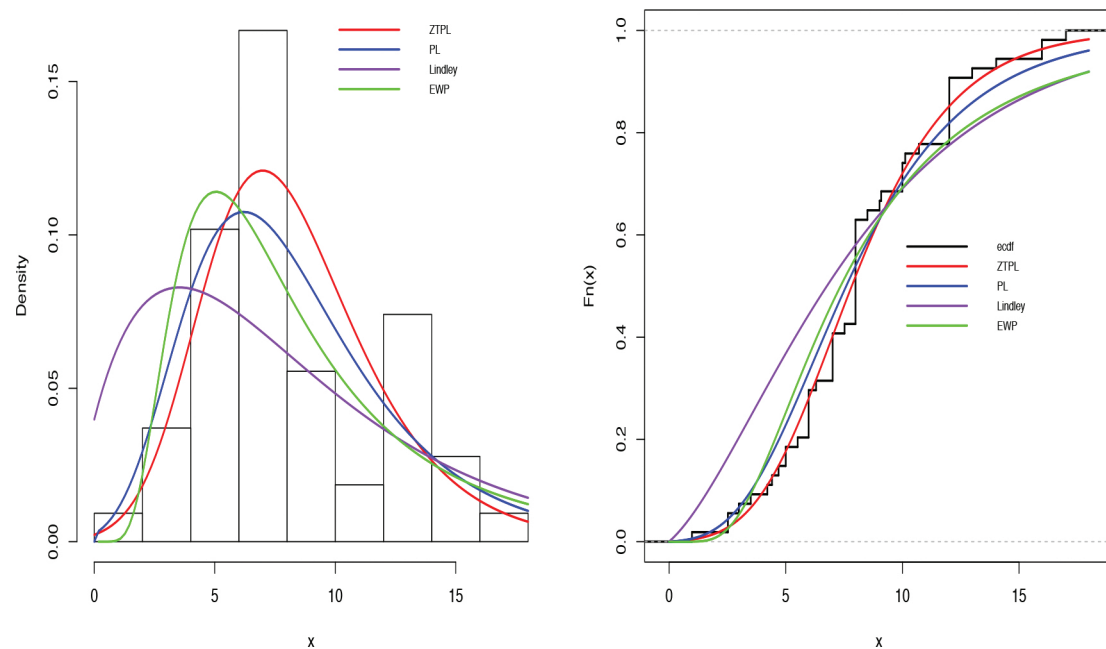
Figure 5 | Fitted and empirical densities and cumulative distribution functions (CDFs) of zero-truncated Poisson–Lindley (ZTPL), PL, Lindley and exponentiated Weibull–-Poisson (EWP) distributions for the monthly highest snowfall data set.

Table 6 | Estimates of λ and θ and the respective 95% CIs under various methods and goodness of fit statistics.

Method	$\hat{\lambda}$			$\hat{\theta}$			ll	K-S
	Est.	LCB	UCB	Est.	LCB	UCB		
MLE	0.4258	0.3471	0.5078	5.8033	4.1696	7.4369	−143.1997	0.1109
LSE	0.4309	0.3477	0.5140	6.1238	4.4186	7.8289	−143.2309	0.1173
WLSE	0.4239	0.3421	0.5056	5.8177	4.1809	7.4544	−143.2049	0.1165
CME	0.4423	0.3569	0.5276	6.6075	4.7906	8.4243	−143.3629	0.1162
MPS	0.4341	0.3503	0.5178	5.8224	4.1845	7.4602	−143.1989	0.1104
ADE	0.4278	0.3453	0.5102	5.9683	4.2981	7.6384	−143.2100	0.1160
RTADE	0.4383	0.3537	0.5228	6.4321	4.6562	8.2079	−143.3023	0.1165

CI, confidence interval; MSE, mean squared error; MLE, maximum likelihood estimator; LSE, least square estimator; WLSE, weighted least square estimator; MPS, maximum product spacing; ADE, Anderson–Darling estimator; RTADE, right-tail Anderson–Darling estimator.

Clearly, the compound ZTPL distribution is a good alternative model comparing with other fitted models. Figure 5 shows the plots of the fitted PDFs and CDFs with their corresponding empirical values. In addition, the empirical survival function (ESF) and fitted survival function are presented in Figure 6. All these plots confirm the same conclusion. Now, we obtain the estimators of the unknown parameters for the compound ZTPL model using different methods of estimation discussed in Section 3. The results for estimates as well as LCL and UCL for 95% CIs of the parameters are displayed in Table 6. Based on K-S distance and log-likelihood criteria, it can be checked that the MPS method competes the other methods but their values are close.
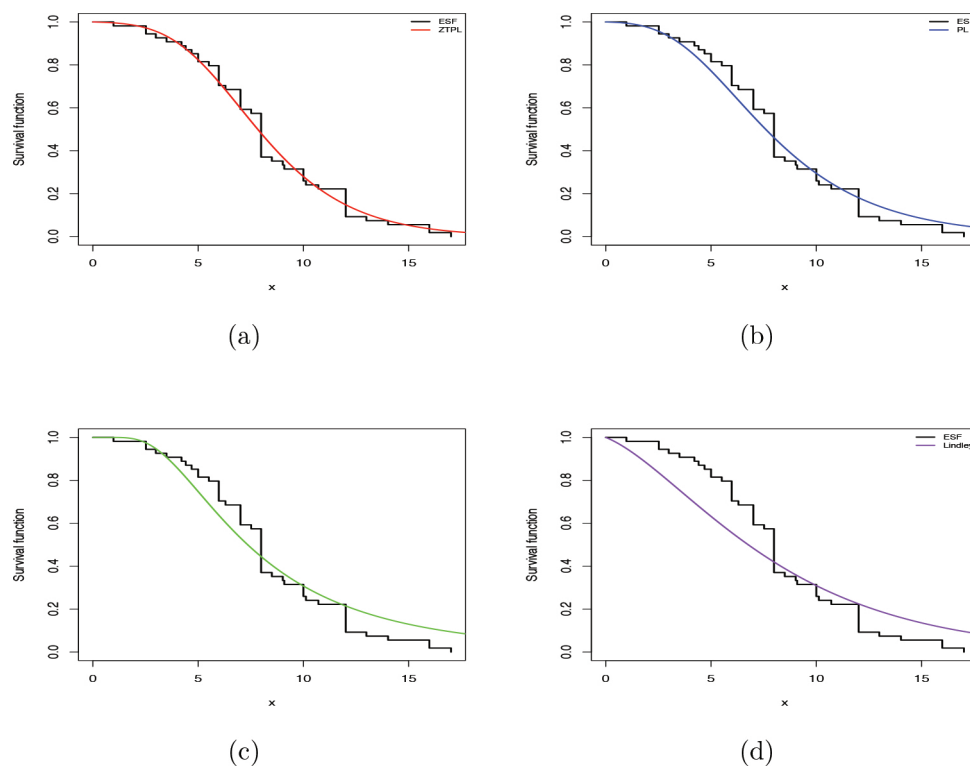
Figure 6 | The empirical survival function (ESF) and fitted survival functions for different distributions: (a) zero-truncated Poisson–Lindley (ZTPL), (b) PL, (c) exponentiated Weibull–-Poisson (EWP) and (d) Lindley.

6. CONCLUSION

In this paper, a new family of distributions is proposed based on a maxima of Poisson number of Lindely random variates. It is called a compound ZTPL model. Some distributional properties of this model are discussed and different methods of estimation are derived for the unknown parameters, namely, maximum likelihood, least squares, weighted least squares, Cramer–von Mises, maximum product of spacing, Anderson–Darling and right tailed Anderson–Darling. It is observed that the estimators obtained by maximum product of spacing method outperform all other estimators when the mean square error is considered as an optimality criterion. For fitting the maximal values of random number observations, it is evident that the compound ZTPL model provides a consistently better fit than Lindley, Poisson Lomax and Exponentiated Weibul–Poisson distributions.

CONFLICTS OF INTEREST

The authors have no conflicts of interests to declare.

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