Research Article

Generalized Direct Product of Complex Intuitionistic Fuzzy Subrings

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ABSTRACT

The objective of this article is to present the notion of direct product of two complex intuitionistic fuzzy subrings. We show that the direct product of two complex intuitionistic fuzzy subrings is also complex intuitionistic fuzzy subring and discuss its various algebraic aspects. We also define the level subsets of the direct product of two complex intuitionistic fuzzy subsets and prove that the level subset of the direct product of two complex intuitionistic fuzzy subring is subring and prove some fundamental result of this phenomena. We show that the homomorphic image (preimage) of the direct product of complex intuitionistic fuzzy subrings is a complex intuitionistic fuzzy subring by using the notion of classical homomorphism.

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1. INTRODUCTION

Obscurity is a pervasive part of human life. This world is not pertaining to accurate calculations or hypothesis. This error in assessment is really problematic for human intelligence. A variety of mathematical notions have been formulated as handy approaches to tackle this difficulty wherein fuzzy sets and complex fuzzy sets are also included. The complex fuzzy logic have been formed on a system of group having ambiguous knowledge. Owing to elastic quality of complex intuitionistic fuzzy sets to handle unreliability, this event is regarded wonderfully great for humanistic logic underlying inaccurate reality and limitless knowledge. This doctrine is doubtlessly a core point of classical complex fuzzy sets as it provides further opportunity to put forth incorrect information, leading to more suitable solution for numerous challenges. These certain sets developed favorable models in situation wherein we are to deal with highly limited choices like yes or no. The other significant property of this knowledge as it empowers man to analyze negative and positive aspects of inaccurate concepts. The branch of mathematics related to fuzzy set theory is known as fuzzy mathematics. In 1965, it was innovated after the seminal paper of Zadeh [1] on

fuzzy sets, who is the founder of this theory. Rosenfeld [2] commenced the fuzzification algebraic structure in 1971. He initiated the concept of fuzzy subgroups. Liu [3] developed a link between ring theory and fuzzy sets and presented the theory of fuzzy subring. Atanassov [4] presented the idea of intuitionistic fuzzy sets. He also defined some properties of intuitionistic fuzzy sets in [5]. Dixit *et al.* [6] depicted the notion of level subgroup in 1990. Kumbhokar and Bapat [7] studied correspondence theorem for fuzzy ideals. Gupta and Kantroo [8] defined the relation between intrinsic product and fuzzy subring. Atanassov [9] presented many interesting new operations about intuitionistic fuzzy sets in 1994. The more development about the application of intuitionistic fuzzy sets in different algebraic structure may be viewed in [10]. Biswas [11] started the conception of intuitionistic fuzzy subgroups in 1997. Gang and Jun [12] describe fuzzy factor ring in 1998. A new concept of complex fuzzy sets was presented by Ramot *et al.* [13]. The extension of fuzzy sets to complex fuzzy sets is comparable to the extension of real numbers to complex numbers. The more development of complex fuzzy sets can be viewed in [14]. Banerjee and Basnet [15] studied intuitionistic fuzzy subrings in 2003. Yetkin and Olgun [16] studied the direct product of fuzzy subgroup and fuzzy subring in 2011. Thirunavukarasu *et al.* [17] illustrated possible application including complex fuzzy representation of solar

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activity, time series, forecasting problems, signal processing application and compare the two national economies by using the concept of complex fuzzy relation. Azam *et al.* [18] defined the some properties of anti-fuzzy ideal in 2013. In addition, the ambiguity and uncertainty which exist in the data arises from everyday life, with the phase shift of data. Thus, it is theoretically insufficient to take this information into account, therefore information is lost in process. To overcome this, Alkouri and Salleh [19] gave the idea of complex intuitionistic fuzzy subsets and enlarge the basic properties of this phenomena. This concept became more effective and useful in scientific field because it deals with degree of membership and nonmembership in complex plane. They also initiated the concept of complex intuitionistic fuzzy relation and developed fundamental operation of complex intuitionistic fuzzy sets in [20,21]. Al-Husban and Salleh [22] introduced the concept of complex fuzzy subgroups in 2016. Ali and Tamir [23] innovated the notion complex intuitionistic fuzzy classes in 2016. Salih and Ahmed [24] delineated the fundamental attribution of α -Q-Fuzzy subgroup in 2017. Alsarahead and Ahmed [25,26] proposed the new ideas of complex fuzzy subgroup and complex fuzzy subring in 2017. These concept are different from fuzzy subgroups [2] and fuzzy ideal [3]. They also introduced a new concept of complex intuitionistic fuzzy subrings in [27] and depicted the elementary properties of this fact. Ameri *et al.* [28] studied some fundamental results about strongly transitive fuzzy geometric spaces. Alolaiyan and Abbas [29] established a stability of complete fuzzy hypergraphs and used it to find a safe and scientific way in computerized tomography (CT) scans. Gulstan *et al.* [30] expounded the notion (α, β) complex fuzzy hyperideal. The algebraic structure of weakly and almost T-ABSO fuzzy modules was studied in [31]. Many interesting results about complex intuitionistic fuzzy graph and cellular network provider companies were presented in [32]. Garg and Rani [33–38] made remarkable effort to generalize the notion of complex intuitionistic fuzzy sets in decision-making problems. Aloaiyan *et al.* [39] described a novel frame work of t -intuitionistic fuzzification of Lagranges theorem. Nguyen [40] defined a some powerful operation of intuitionistic fuzzy sets in decision-making problems. We organized this paper as follows: Section 2 contains the introductory definition of complex intuitionistic fuzzy subrings and related result which plays a key role for our further discussion. In Section 3 we define of the direct product two complex fuzzy subsets. We prove that the product of two complex intuitionistic fuzzy subrings is complex intuitionistic fuzzy subring and also the fundamental properties of the direct product of complex intuitionistic fuzzy subrings are discussed deeply in this section. In Section 4, we explicate the level subset of the product of two complex intuitionistic fuzzy subsets. We prove that the level subset of the direct product of two complex intuitionistic fuzzy subrings is subring and also investigate the algebraic properties of this phenomena. Section 5 deal the notion of direct product of complex intuitionistic fuzzy subring under ring homomorphism.

2. PRELIMINARIES

We recall first the elementary notion of complex intuitionistic fuzzy sets and complex intuitionistic fuzzy subrings which play a key role for our further analysis.

Definition 2.1. [1] A fuzzy set λ of a nonempty set P is a mapping, $\lambda : P \rightarrow [0, 1]$.

Definition 2.2. [4] An intuitionistic fuzzy set A of a universe of discourse P is a triplet of the form $A = \{(m, \lambda_A(m), \hat{\lambda}_A(m)) : m \in P\}$, where the function $\lambda_A(m) : P \rightarrow [0, 1]$ and $\hat{\lambda}_A(m) : P \rightarrow [0, 1]$ are represents the membership and nonmembership of an element m of P , respectively. These function must fulfill the condition $0 \leq \lambda_A(m) + \hat{\lambda}_A(m) \leq 1$.

Definition 2.3. [9] Let A and B be any two intuitionistic fuzzy sets of the universe of discourse P . Then the following operation are defined as:

1. $(A \cap B)(m) = \{(m, \lambda_{A \cap B}(m), \hat{\lambda}_{A \cap B}(m)) | m \in P\}$, where $\lambda_{A \cap B}(m) = \min\{\lambda_A(m), \lambda_B(m)\}$ and $\hat{\lambda}_{A \cap B}(m) = \max\{\hat{\lambda}_A(m), \hat{\lambda}_B(m)\}$,
2. $(A \cup B)(m) = \{(m, \lambda_{A \cup B}(m), \hat{\lambda}_{A \cup B}(m)) | m \in P\}$, where $\lambda_{A \cup B}(m) = \max\{\lambda_A(m), \lambda_B(m)\}$ and $\hat{\lambda}_{A \cup B}(m) = \min\{\hat{\lambda}_A(m), \hat{\lambda}_B(m)\}$,
3. $(A + B)(m) = \{(m, \lambda_A(m) + \lambda_B(m) - \lambda_A(m) \cdot \lambda_B(m), \hat{\lambda}_A(m) \cdot \hat{\lambda}_B(m)) | m \in P\}$,
4. $(A \cdot B)(m) = \{(m, \lambda_A(m) \cdot \lambda_B(m), \hat{\lambda}_A(m) + \hat{\lambda}_B(m) - \hat{\lambda}_A(m) \cdot \hat{\lambda}_B(m)) | m \in P\}$.

Definition 2.4. [15] An intuitionistic fuzzy set A of a ring S is called a intuitionistic fuzzy subring S , if the following condition hold:

- $\lambda_A(m - n) \geq \min\{\lambda_A(m), \lambda_A(n)\}$,
- $\lambda_A(mn) \geq \min\{\lambda_A(m), \lambda_A(n)\}$,
- $\hat{\lambda}_A(m - n) \leq \max\{\hat{\lambda}_A(m), \hat{\lambda}_A(n)\}$,
- $\hat{\lambda}_A(mn) \leq \max\{\hat{\lambda}_A(m), \hat{\lambda}_A(n)\}$, for all $m, n \in S$.

Definition 2.5. [13] A complex fuzzy set A of universe of discourse P is identify with the membership function $\theta_A(m) = \eta_A(m)e^{i\varphi_A(m)}$ and is defined as $\theta_A : P \rightarrow \{z \in \mathbb{C} : |z| \leq 1\}$, where $\mathbb{C}$ is set of complex numbers. This membership function receive all membership value from the unit disc on complex plane, where $i = \sqrt{-1}$, where both $\eta_A(m)$ and $\varphi_A(m)$ are the real valued such that $\eta_A(m) \in [0, 1]$ and $\varphi_A(m) \in [0, 2\pi]$. In this paper, we use CFS for complex fuzzy subset.

Remark 2.6. By setting $\varphi_A(m) = 0$ in above definition, we return back to classical fuzzy set.

Definition 2.7. [19] A complex intuitionistic fuzzy set A of crisp nonempty set P is an object of the form $A = \{(m, \theta_A(m), \hat{\theta}_A(m)) : m \in P\}$ where the membership function $\theta_A(m) = \eta_A(m)e^{i\varphi_A(m)}$ and nonmembership function $\hat{\theta}_A(m) = \hat{\eta}_A(m)e^{i\hat{\varphi}_A(m)}$ is defined as $\theta_A : P \rightarrow \{z \in \mathbb{C} : |z| \leq 1\}$ and $\hat{\theta}_A : P \rightarrow \{z \in \mathbb{C} : |z| \leq 1\}$, where $\mathbb{C}$ is set of complex numbers. These membership and nonmembership function receive all degree of membership and nonmembership from the unit disc on complex plane respectively such that the sum of membership and nonmembership value is also lies within unit disc of complex plane, where $i = \sqrt{-1}$, $\eta_A(m)$, $\hat{\eta}_A(m)$, $\varphi_A(m)$ and $\hat{\varphi}_A(m)$ are real-valued function such that $0 \leq \eta_A(m) + \hat{\eta}_A(m) \leq 1$ and $0 \leq \varphi_A(m) + \hat{\varphi}_A(m) \leq 2\pi$. In this paper we use the complex intuitionistic fuzzy set (CIFS) for the complex fuzzy subset. For the sake of simplicity we shall use $\theta_A(m) = \eta_A(m)e^{i\varphi_A(m)}$, $\theta_B(m) = \eta_B(m)e^{i\varphi_B(m)}$ as the membership function and $\hat{\theta}_A(m) = \hat{\eta}_A(m)e^{i\hat{\varphi}_A(m)}$, $\hat{\theta}_B(m) = \hat{\eta}_B(m)e^{i\hat{\varphi}_B(m)}$ as the nonmembership function of CIFS A and B , respectively.

Remark 2.8. It is important to note that one can obtain the traditional intuitionistic fuzzy set by choosing the value $\varphi_A(m) = \hat{\varphi}_A(m) = 0$ in Definition [2.7].

Definition 2.9. [27]

1. A complex intuitionistic fuzzy set (CIFS) A is homogeneous CIFS, if for all $m, x \in S$, we have
 - (a) $\eta_A(m) \leq \eta_A(x)$ if and only if $\varphi_A(m) \leq \varphi_A(x)$,
 - (b) $\hat{\eta}_A(m) \leq \hat{\eta}_A(x)$ if and only if $\hat{\varphi}_A(m) \leq \hat{\varphi}_A(x)$.
2. A CIFS A is homogeneous CIFS with B , if for all $m, x \in S$, we have
 - (a) $\eta_A(m) \leq \eta_B(x)$ if and only if $\varphi_A(m) \leq \varphi_B(x)$,
 - (b) $\hat{\eta}_A(m) \leq \hat{\eta}_B(x)$ if and only if $\hat{\varphi}_A(m) \leq \hat{\varphi}_B(x)$.

Throughout this paper we use CIFS as homogeneous CIFS.

Definition 2.10. [27] Let $A = \{(m, \psi_A(m), \hat{\psi}_A(m))\}$ be a intuitionistic fuzzy subset of S . Then the π -intuitionistic fuzzy subset A_π is defined as

$$A_\pi = \{(m, \psi_{A_\pi}(m), \hat{\psi}_{A_\pi}(m)) : m \in S\},$$

where $\psi_A(m) = 2\pi\psi_A(m)$ and $\hat{\psi}_A(m) = 2\pi\hat{\psi}_A(m)$ be membership function and non-membership function, respectively for all $m \in S$. For convenience, we shall write π -IFS for π -intuitionistic fuzzy subset.

Definition 2.11. [27] A π -IFS A_π of ring S is called π -fuzzy subring (π -FSR) of S , for all $m, n \in S$, if

1. $\psi_{A_\pi}(m - n) \geq \min\{\psi_{A_\pi}(m), \psi_{A_\pi}(n)\}$,
2. $\psi_{A_\pi}(mn) \geq \min\{\psi_{A_\pi}(m), \psi_{A_\pi}(n)\}$,
3. $\hat{\psi}_{A_\pi}(m - n) \leq \max\{\hat{\psi}_{A_\pi}(m), \hat{\psi}_{A_\pi}(n)\}$,
4. $\hat{\psi}_{A_\pi}(mn) \leq \max\{\hat{\psi}_{A_\pi}(m), \hat{\psi}_{A_\pi}(n)\}$.

In this paper, we write π -intuitionistic fuzzy subring as π -IFSR.

Theorem 2.12. [27] A π -IFS A_π of ring S is a π -IFSR if and only if A is Intuitionistic fuzzy subrings (IFSR).

Definition 2.13. [27] A CIFS $A = (\theta_A(m), \hat{\theta}_A(m))$ of ring S is called a complex intuitionistic fuzzy subring, for all $m, n \in S$, if

1. $\theta_A(m - n) \geq \min\{\theta_A(m), \theta_A(n)\}$,
2. $\theta_A(mn) \geq \min\{\theta_A(m), \theta_A(n)\}$,
3. $\hat{\theta}_A(m - n) \leq \max\{\hat{\theta}_A(m), \hat{\theta}_A(n)\}$,
4. $\hat{\theta}_A(mn) \leq \max\{\hat{\theta}_A(m), \hat{\theta}_A(n)\}$.

For convenience, we shall use complex intuitionistic fuzzy subrings (CIFSR) for complex intuitionistic fuzzy subring.

Definition 2.14. [27] Let A be a CIFS of S . For $r \in [0, 1]$ and $t \in [0, 2\pi]$ the level subset of CIFS is defined by

$$A_{(r,t)}^{\hat{r},\hat{t}}(m) = \{m \in P : \eta_A(m) \geq r, \varphi_A(m) \geq t, \hat{\eta}_A(m) \leq \hat{r}, \hat{\varphi}_A(m) \leq \hat{t}\}.$$

For $t = 0$, we obtain the level subset $A_r^{\hat{r}} = \{m \in P : \eta_A(m) \geq r, \hat{\eta}_A(m) \leq \hat{r}\}$ and for $r = 0$, we obtain the level subset $A_t^{\hat{t}} = \{m \in P : \varphi_A(m) \geq t, \hat{\varphi}_A(m) \leq \hat{t}\}$.

Definition 2.15. [27] Let $f : S \rightarrow R$ be a ring homomorphism. Let A and B be two CIFSR of ring S and R , respectively, for all $x \in S, m \in R$. Then the sets $E = \{(m, f(\eta_A)(m), f(\hat{\eta}_A)(m))\}$ and $F = \{(x, f^{-1}(\theta_B)(x), f^{-1}(\hat{\theta}_B)(x))\}$ are called image of A and preimage of B , respectively, where

$$f(\theta_A)(m) = \begin{cases} \sup\{\theta_A(x), \text{ if } f(x) = m\}, & f^{-1} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$f(\hat{\theta}_A)(m) = \begin{cases} \inf\{\hat{\theta}_A(x), \text{ if } f(x) = m\}, & f^{-1} \neq \emptyset \\ 1, & \text{otherwise} \end{cases}$$

$$f^{-1}(\theta_B)(x) = \theta_B(f(x)),$$

$$f^{-1}(\hat{\theta}_B)(x) = \hat{\theta}_B(f(x)).$$

3. PROPERTIES OF THE DIRECT PRODUCT OF COMPLEX INTUITIONISTIC FUZZY SUBRINGS

In this section, we use the concept of CIFS to define direct product of CIFS. We prove that direct product of two CIFSR is CIFSR and investigate their properties.

Definition 3.1. Let A and B be any two π -intuitionistic fuzzy sets of sets S , for all $m \in S$. Then the following operation are defined as:

1. $(A_\pi \cap B_\pi)(m) = \{(m, \psi_{A_\pi \cap B_\pi}(m), \hat{\psi}_{A_\pi \cap B_\pi}(m))\}$, where $\psi_{A_\pi \cap B_\pi}(m) = \min\{\psi_{A_\pi}(m), \psi_{B_\pi}(m)\}$ and $\hat{\psi}_{A_\pi \cap B_\pi}(m) = \max\{\hat{\psi}_{A_\pi}(m), \hat{\psi}_{B_\pi}(m)\}$.
2. $(A_\pi \cup B_\pi)(m) = \{(m, \psi_{A_\pi \cup B_\pi}(m), \hat{\psi}_{A_\pi \cup B_\pi}(m))\}$, where $\psi_{A_\pi \cup B_\pi}(m) = \max\{\psi_{A_\pi}(m), \psi_{B_\pi}(m)\}$ and $\hat{\psi}_{A_\pi \cup B_\pi}(m) = \min\{\hat{\psi}_{A_\pi}(m), \hat{\psi}_{B_\pi}(m)\}$.

Definition 3.2. Let A and B be two π -intuitionistic fuzzy sets of sets S_1 and S_1 , respectively. The Cartesian product of π -intuitionistic fuzzy sets A and B is defined as

$$(A_\pi \times B_\pi)(m, n) = \{(m \times n, \psi_{A_\pi \times B_\pi}(m, n), \hat{\psi}_{A_\pi \times B_\pi}(m, n))\}$$

for all $m \in S_1, n \in S_2$.

Remark 3.3. Let A and B be two π -IFSR of sets S_1 and S_2 , respectively. Then $A_\pi \times B_\pi$ is intuitionistic π -fuzzy subring of $S_1 \times S_2$.

Definition 3.4. [20] Let A and B be two CIFS of sets S_1 and S_1 . The Cartesian product of CIFS A and B is defined by a function

$$\begin{aligned} A \times B &= \{(m, n, \theta_{A \times B}(m, n), \hat{\theta}_{A \times B}(m, n))\}, \\ \theta_{A \times B}(m, n) &= \eta_{A \times B}(m, n) e^{i\varphi_{A \times B}(m, n)} \\ &= \min\{\eta_A(m), \eta_B(n)\} e^{i \min\{\varphi_A(m), \varphi_B(n)\}}, \\ \hat{\theta}_{A \times B}(m, n) &= \hat{\eta}_{A \times B}(m, n) e^{i\hat{\varphi}_{A \times B}(m, n)} \\ &= \max\{\hat{\eta}_A(m), \hat{\eta}_B(n)\} e^{i \max\{\hat{\varphi}_A(m), \hat{\varphi}_B(n)\}}. \end{aligned}$$

For sake of simplicity, throughout this paper we shall use $\theta_{A \times B}(m, n) = \eta_{A \times B}(m, n)e^{i\varphi_{A \times B}(m, n)}$ and $\theta_{C \times D}(m, n) = \eta_{C \times D}(m, n)e^{i\varphi_{C \times D}(m, n)}$ for the membership function of Cartesian product of CIFS $A \times B$ and $C \times D$, respectively, and $\hat{\theta}_{A \times B}(m, n) = \hat{\eta}_{A \times B}(m, n)e^{i\hat{\varphi}_{A \times B}(m, n)}$ and $\hat{\theta}_{C \times D}(m, n) = \hat{\eta}_{C \times D}(m, n)e^{i\hat{\varphi}_{C \times D}(m, n)}$, for the nonmembership function of Cartesian product of CIFS $A \times B$ and $C \times D$, respectively.

Theorem 3.5. Let A and B be to CIFSR of S_1 and S_2 , respectively. Then $A \times B$ is CIFSR of $S_1 \times S_2$.

Proof. Let $m, x \in S_1$ and $n, y \in S_2$ be an elements. Then $(m, n), (x, y) \in S_1 \times S_2$. Consider

$$\begin{aligned}\theta_{A \times B}((m, n) - (x, y)) &= \theta_{A \times B}(m - x, n - y) \\ &= \eta_{A \times B}(m - x, n - y) e^{i\phi_{A \times B}(m - x, n - y)} \\ &= \min\{\eta_A(m - x), \eta_B(n - y)\} \\ &\quad e^{i\min\{\phi_A(m - x), \phi_B(n - y)\}} \\ &= \min\{\eta_A(m - x) e^{i\phi_A(m - x)}, \\ &\quad \eta_B(n - y) e^{i\phi_B(n - y)}\} \\ &= \min\{\theta_A(m - x), \theta_B(n - y)\} \\ &\geq \min\{\min\{\theta_A(m), \theta_A(x)\}, \\ &\quad \min\{\theta_B(n), \theta_B(y)\}\} \\ &= \min\{\min\{\theta_A(m), \theta_B(n)\}, \\ &\quad \min\{\theta_A(x), \theta_B(y)\}\} \\ \theta_{A \times B}((m, n) - (x, y)) &\geq \min\{\theta_{A \times B}(m, n), \theta_{A \times B}(x, y)\}.\end{aligned}$$

$$\begin{aligned}\text{Further, } \theta_{A \times B}((m, n)(x, y)) &= \theta_{A \times B}(mx, ny) \\ &= \eta_{A \times B}(mx, ny) e^{i\phi_{A \times B}(mx, ny)} \\ &= \min\{\eta_A(mx), \eta_B(ny)\} \\ &\quad e^{i\min\{\phi_A(mx), \phi_B(ny)\}} \\ &= \min\{\eta_A(mx) e^{i\phi_A(mx)}, \\ &\quad \eta_B(ny) e^{i\phi_B(ny)}\} \\ &= \min\{\theta_A(mx), \theta_B(ny)\} \\ &\geq \min\{\min\{\theta_A(m), \theta_A(x)\}, \\ &\quad \min\{\theta_B(n), \theta_B(y)\}\}\end{aligned}$$

Consequently, $\theta_{A \times B}((m, n)(x, y)) \geq \min\{\theta_{A \times B}(m, n), \theta_{A \times B}(x, y)\}$.

$$\begin{aligned}\text{Moreover, } \hat{\theta}_{A \times B}((m, n) - (x, y)) &= \hat{\theta}_{A \times B}(m - x, n - y) \\ &= \hat{\eta}_{A \times B}(m - x, n - y) \\ &\quad e^{i\hat{\phi}_{A \times B}(m - x, n - y)} \\ &= \max\{\hat{\eta}_A(m - x), \hat{\eta}_B(n - y)\} \\ &\quad e^{i\max\{\hat{\phi}_A(m - x), \hat{\phi}_B(n - y)\}} \\ &= \max\{\hat{\eta}_A(m - x) e^{i\hat{\phi}_A(m - x)}, \\ &\quad \hat{\eta}_B(n - y) e^{i\hat{\phi}_B(n - y)}\} \\ &= \max\{\hat{\theta}_A(m - x), \hat{\theta}_B(n - y)\} \\ &\leq \max\{\max\{\hat{\theta}_A(m), \hat{\theta}_A(x)\}, \\ &\quad \max\{\hat{\theta}_B(n), \hat{\theta}_B(y)\}\} \\ &= \max\{\max\{\hat{\theta}_A(m), \hat{\theta}_B(n)\}, \\ &\quad \max\{\hat{\theta}_A(x), \hat{\theta}_B(y)\}\}.\end{aligned}$$

As a result, $\hat{\theta}_{A \times B}((m, n) - (x, y)) \leq \max\{\hat{\theta}_{A \times B}(m, n), \hat{\theta}_{A \times B}(x, y)\}$.

$$\begin{aligned}\text{Further, } \hat{\theta}_{A \times B}((m, n)(x, y)) &= \hat{\theta}_{A \times B}(mx, ny) \\ &= \hat{\eta}_{A \times B}(mx, ny) e^{i\hat{\phi}_{A \times B}(mx, ny)} \\ &= \max\{\hat{\eta}_A(mx), \hat{\eta}_B(ny)\} e^{i\max\{\hat{\phi}_A(mx), \hat{\phi}_B(ny)\}} \\ &= \max\{\hat{\eta}_A(mx) e^{i\hat{\phi}_A(mx)}, \hat{\eta}_B(ny) e^{i\hat{\phi}_B(ny)}\} \\ &= \max\{\hat{\theta}_A(mx), \hat{\theta}_B(ny)\} \\ &\leq \max\{\max\{\hat{\theta}_A(m), \hat{\theta}_A(x)\}, \max\{\hat{\theta}_B(n), \hat{\theta}_B(y)\}\} \\ &= \max\{\max\{\hat{\theta}_A(m), \hat{\theta}_B(n), \max\{\hat{\theta}_A(x), \hat{\theta}_B(y)\}\}\} \\ \hat{\theta}_{A \times B}((m, n)(x, y)) &\leq \max\{\hat{\theta}_{A \times B}(m, n), \hat{\theta}_{A \times B}(x, y)\}.\end{aligned}$$

Thus concluded the proof. $\square$

Corollary 3.6. Let $A_1, A_2, \dots, A_n$ be CIFSR of $S_1, S_2, \dots, S_n$, respectively. Then $A_1 \times A_2 \times \dots \times A_n$ is CIFSR of $S_1 \times S_2 \times \dots \times S_n$.

Remark 3.7. Let A and B be two CIFS of S_1 and S_2 , respectively and $A \times B$ be CIFSR of $S_1 \times S_2$. Then, it not compulsory both A and B should be CIFSR of S_1 and S_2 , respectively.

Example 3.8. Let $\bar{Z}_2 = \{0, 1\}$ and $S = \{e, a, b, c\}$ be two rings. Here S is ring of 2×2 matrices over $\bar{Z}_2$ with second row has zero entries, where $e = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $c = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

$\bar{Z}_2 \times S = \{(0, e), (0, a), (0, b), (0, c), (1, e), (1, a), (1, b), (1, c)\}$. Then two CIFS A and B is defined by

$$\begin{aligned}\theta_A &= \{(0, 0.2e^{i\pi/12}), (1, 0.1e^{i\pi/15})\}, \\ \hat{\theta}_A &= \{(0, 0.55e^{i\pi/6}), (1, 0.7e^{i\pi/3})\}, \\ \theta_B &= \{(e, 0.3e^{i\pi/3}), (a, 0.45e^{i\pi/2}), (b, 0.33e^{i\pi/3}), (c, 0.4e^{i\pi})\}, \\ \hat{\theta}_B &= \{(e, 0.24e^{i\pi/8}), (a, 0.2e^{i\pi/10}), (b, 0.1e^{i\pi/9}), (c, 0.23e^{i\pi/6})\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\theta_{A \times B}(x) &= \begin{cases} 0.2e^{i\pi/12}, & \text{for all } m \in \{(0, e), (0, a), (0, b), (0, c)\} \\ 0.1e^{i\pi/15}, & \text{for all } m \in \{(1, e), (1, a), (1, b), (1, c)\} \end{cases} \\ \hat{\theta}_{A \times B}(x) &= \begin{cases} 0.55e^{i\pi/6}, & \text{for all } x \in \{(0, e), (0, a), (0, b), (0, c)\} \\ 0.7e^{i\pi/3}, & \text{for all } x \in \{(1, e), (1, a), (1, b), (1, c)\} \end{cases}\end{aligned}$$

Here, $A \times B$ is CIFSR of $S_1 \times S_2$ and A is CIFSR of B . But B is not a CIFSR of S_2 because $(B)^{\left(\frac{23}{6}, \frac{\pi}{6}\right)} = \{a, c\}$ is not a subring.

Remark 3.9. Let $A \times B$ be two CIFSR of ring $S_1 \times S_2$. Then $\eta_{A \times B}(0, 0') \geq \eta_{A \times B}(m, n)$, $\varphi_{A \times B}(0, 0') \geq \varphi_{A \times B}(m, n)$ and $\hat{\eta}_{A \times B}(0, 0') \leq \hat{\eta}_{A \times B}(m, n)$, $\hat{\varphi}_{A \times B}(0, 0') \leq \hat{\varphi}_{A \times B}(m, n)$, for all $m \in S_1$, for all, $n \in S_2$. Where 0 and $0'$ are identities of S_1 and S_2 , respectively.

Theorem 3.10. Let A and B be two CIFS of rings S_1 and S_2 , respectively. If $A \times B$ is a CIFSR of $S_1 \times S_2$, then at least one of the following assertions must be hold.

1. $\eta_A(0) \geq \eta_B(n)$, $\varphi_A(0) \geq \varphi_B(n)$, $\hat{\eta}_A(0) \leq \hat{\eta}_B(n)$ and $\hat{\varphi}_A(0) \leq \hat{\varphi}_B(n)$, for all $n \in S_2$,
2. $\eta_B(0') \geq \eta_A(m)$, $\varphi_B(0') \geq \varphi_A(m)$, $\hat{\eta}_B(0') \leq \hat{\eta}_A(m)$ and $\hat{\varphi}_B(0') \leq \hat{\varphi}_A(m)$, for all $m \in S_1$.

where 0 and $0'$ are identities of S_1 and S_2 , respectively.

Proof. Let $A \times B$ be a CIFSR of $S_1 \times S_2$. On contrary, suppose that the assertions (1) and (2) do not hold. Then there exist $m \in S_1$ and $n \in S_2$ such that

1. $\eta_A(0) \leq \eta_B(n)$, $\varphi_A(0) \leq \varphi_B(n)$, $\hat{\eta}_A(0) \geq \hat{\eta}_B(n)$ and $\hat{\varphi}_A(0) \geq \hat{\varphi}_B(n)$, for all $n \in S_2$
2. $\eta_B(0') \leq \eta_A(m)$, $\varphi_B(0') \leq \varphi_A(m)$, $\hat{\eta}_B(0') \geq \hat{\eta}_A(m)$ and $\hat{\varphi}_B(0') \geq \hat{\varphi}_A(m)$, for all $m \in S_1$

$$\text{Consider, } \theta_{A \times B}(m, n) = \min\{\eta_A(m), \eta_B(n)\} e^{i \min\{\varphi_A(m), \varphi_B(n)\}} \\ \geq \min\{\eta_A(0), \eta_B(0')\} e^{i \min\{\varphi_A(0), \varphi_B(0')\}}$$

$$\theta_{A \times B}(m, n) \geq \theta_{A \times B}(0, 0')$$

$$\text{And, } \hat{\theta}_{A \times B}(m, n) = \max\{\eta_A(m), \hat{\eta}_B(n)\} e^{i \max\{\varphi_A(m), \varphi_B(n)\}} \\ \leq \max\{\hat{\eta}_A(0), \hat{\eta}_B(0')\} e^{i \max\{\hat{\varphi}_A(0), \hat{\varphi}_B(0')\}}$$

$$\hat{\theta}_{A \times B}(m, n) \leq \hat{\theta}_{A \times B}(0, 0').$$

But $A \times B$ is CIFSR. Hence, at least one of the following assertions must be hold:

1. $\eta_A(0) \geq \eta_B(n)$, $\varphi_A(0) \geq \varphi_B(n)$, $\hat{\eta}_A(0) \leq \hat{\eta}_B(n)$ and $\hat{\varphi}_A(0) \leq \hat{\varphi}_B(n)$, for all $n \in S_2$
2. $\eta_B(0') \geq \eta_A(m)$, $\varphi_B(0') \geq \varphi_A(m)$, $\hat{\eta}_B(0') \leq \hat{\eta}_A(m)$ and $\hat{\varphi}_B(0') \leq \hat{\varphi}_A(m)$, for all $m \in S_1$

Theorem 3.11. Let A and B CIFS of S_1 and S_2 such that $\eta_B(0') \geq \eta_A(m)$, $\varphi_B(0') \geq \varphi_A(m)$, $\hat{\eta}_B(0') \leq \hat{\eta}_A(m)$ and $\hat{\varphi}_B(0') \leq \hat{\varphi}_A(m)$ for all $m \in S_1$ and $0'$ is identity of S_2 . If $A \times B$ is CIFSR of $S_1 \times S_2$, then A is CIFSR of S_1 .

Proof. Let $(m, 0')$, $(x, 0')$ be elements of $S_1 \times S_2$. By given condition $\eta_B(0') \geq \eta_A(m)$ and $\varphi_B(0') \geq \varphi_A(m)$, $\hat{\eta}_B(0') \leq \hat{\eta}_A(m)$ and $\hat{\varphi}_B(0') \leq \hat{\varphi}_A(m)$ for all $m, x \in S_1$ and $0' \in S_2$.

Consider $\theta_A(m - x)$

$$\begin{aligned} &= \eta_A(m - x) e^{i \varphi_A(m - x)} \\ &= \min\{\eta_A(m - x) e^{i \varphi_A(m - x)}, \eta_B(0' - 0') e^{i \varphi_B(0' - 0')}\} \\ &= \{\eta_{A \times B}((m, 0') - (x, 0'))\} e^{i \varphi_{A \times B}((m, 0') - (x, 0'))} \\ &\geq \min\{\eta_{A \times B}(m, 0'), \eta_{A \times B}(x, 0')\} \\ &\quad e^{i \min\{\varphi_{A \times B}(m, 0'), \varphi_{A \times B}(x, 0')\}} \\ &= \min\{\min\{\eta_A(m), \eta_B(0')\}, \min\{\eta_A(x), \eta_B(0')\}\} \\ &\quad \times e^{i \min\{\min\{\varphi_A(m), \varphi_B(0')\}, \min\{\varphi_A(x), \varphi_B(0')\}\}} \\ &= \min\{\min\{\eta_A(m), \eta_A(x)\}, \min\{\eta_A(x), \eta_A(x)\}\} \\ &\quad \times e^{i \min\{\min\{\eta_A(m), \eta_A(x)\}, \min\{\eta_A(x), \eta_A(x)\}\}} \\ &= \min\{\eta_A(m), \eta_A(x)\} e^{i \min\{\varphi_A(m), \varphi_A(x)\}} \\ &= \min\{\theta_A(m), \theta_A(x)\}. \end{aligned}$$

Thus, $\theta_A(m - x) \geq \min\{\theta_A(m), \theta_A(x)\}$.

$$\begin{aligned} \text{Moreover, } \theta_A(mx) &= \eta_A(mx) e^{i \varphi_A(mx)} \\ &= \min\{\eta_A(mx) e^{i \varphi_A(mx)}, \eta_B(0'0') e^{i \varphi_B(0'0')}\} \\ &= \{\eta_{A \times B}((m, 0')(x, 0'))\} e^{i \varphi_{A \times B}((m, 0')(x, 0'))} \\ &\geq \min\{\eta_{A \times B}(m, 0'), \eta_{A \times B}(x, 0')\} \\ &\quad e^{i \min\{\varphi_{A \times B}(m, 0'), \varphi_{A \times B}(x, 0')\}} \\ &= \min\{\min\{\eta_A(m), \eta_B(0')\}, \min\{\eta_A(x), \eta_B(0')\}\} \\ &\quad \times e^{i \min\{\min\{\varphi_A(m), \varphi_B(0')\}, \min\{\varphi_A(x), \varphi_B(0')\}\}} \\ &= \min\{\min\{\eta_A(m), \eta_A(x)\}, \min\{\eta_A(x), \eta_A(x)\}\} \\ &\quad \times e^{i \min\{\min\{\eta_A(m), \eta_A(x)\}, \min\{\eta_A(x), \eta_A(x)\}\}} \\ &= \min\{\eta_A(m), \eta_A(x)\} e^{i \min\{\varphi_A(m), \varphi_A(x)\}} \\ &= \min\{\theta_A(m), \theta_A(x)\}. \end{aligned}$$

Thus, $\theta_A(mx) \geq \min\{\theta_A(m), \theta_A(x)\}$

On the other hand, we have

$$\begin{aligned} \hat{\theta}_A(m - x) &= \hat{\eta}_A(m - x) e^{i \hat{\varphi}_A(m - x)} \\ &= \max\{\hat{\eta}_A(m - x) e^{i \hat{\varphi}_A(m - x)}, \hat{\eta}_B(0' - 0') e^{i \hat{\varphi}_B(0' - 0')}\} \\ &= \{\hat{\eta}_{A \times B}((m, 0') - (x, 0'))\} e^{i \hat{\varphi}_{A \times B}((m, 0') - (x, 0'))} \\ &\leq \max\{\hat{\eta}_{A \times B}(m, 0'), \hat{\eta}_{A \times B}(x, 0')\} \\ &\quad e^{i \max\{\hat{\varphi}_{A \times B}(m, 0'), \hat{\varphi}_{A \times B}(x, 0')\}} \\ &= \max\{\max\{\hat{\eta}_A(m), \hat{\eta}_B(0')\}, \max\{\hat{\eta}_A(x), \hat{\eta}_B(0')\}\} \\ &\quad \times e^{i \max\{\max\{\hat{\varphi}_A(m), \hat{\varphi}_B(0')\}, \max\{\hat{\varphi}_A(x), \hat{\varphi}_B(0')\}\}} \\ &= \max\{\max\{\hat{\eta}_A(m), \hat{\eta}_A(x)\}, \max\{\hat{\eta}_A(x), \hat{\eta}_A(x)\}\} \\ &\quad \times e^{i \max\{\max\{\hat{\eta}_A(m), \hat{\eta}_A(x)\}, \max\{\hat{\eta}_A(x), \hat{\eta}_A(x)\}\}} \\ &= \max\{\hat{\eta}_A(m), \hat{\eta}_A(x)\} e^{i \max\{\hat{\varphi}_A(m), \hat{\varphi}_A(x)\}} \\ &= \max\{\hat{\theta}_A(m), \hat{\theta}_A(x)\}. \end{aligned}$$

As a result, $\hat{\theta}_A(m - x) \leq \max\{\hat{\theta}_A(m), \hat{\theta}_A(x)\}$

$$\begin{aligned} \hat{\theta}_A(mx) &= \hat{\eta}_A(mx) e^{i \hat{\varphi}_A(mx)} \\ &= \max\{\hat{\eta}_A(mx) e^{i \hat{\varphi}_A(mx)}, \hat{\eta}_B(0'0') e^{i \hat{\varphi}_B(0'0')}\} \\ &= \{\hat{\eta}_{A \times B}((m, 0')(x, 0'))\} e^{i \hat{\varphi}_{A \times B}((m, 0')(x, 0'))} \\ &\leq \max\{\hat{\eta}_{A \times B}(m, 0'), \hat{\eta}_{A \times B}(x, 0')\} \\ &\quad e^{i \max\{\hat{\varphi}_{A \times B}(m, 0'), \hat{\varphi}_{A \times B}(x, 0')\}} \\ &= \max\{\max\{\hat{\eta}_A(m), \hat{\eta}_B(0')\}, \max\{\hat{\eta}_A(x), \hat{\eta}_B(0')\}\} \\ &\quad \times e^{i \max\{\max\{\hat{\varphi}_A(m), \hat{\varphi}_B(0')\}, \max\{\hat{\varphi}_A(x), \hat{\varphi}_B(0')\}\}} \\ &= \max\{\max\{\hat{\eta}_A(m), \hat{\eta}_A(x)\}, \max\{\hat{\eta}_A(x), \hat{\eta}_A(x)\}\} \\ &\quad \times e^{i \max\{\max\{\hat{\eta}_A(m), \hat{\eta}_A(x)\}, \max\{\hat{\eta}_A(x), \hat{\eta}_A(x)\}\}} \\ &= \max\{\hat{\eta}_A(m), \hat{\eta}_A(x)\} e^{i \max\{\hat{\varphi}_A(m), \hat{\varphi}_A(x)\}} \\ &= \max\{\hat{\theta}_A(m), \hat{\theta}_A(x)\}. \end{aligned}$$

Consequently, $\hat{\theta}_A(mx) \leq \max\{\hat{\theta}_A(m), \hat{\theta}_A(x)\}$. Hence, proved our claim. $\square$

Theorem 3.12. Let A and B two CIFS of S_1 and S_2 such that $\eta_A(0) \geq \eta_B(n)$, $\varphi_A(0) \geq \varphi_B(n)$, $\hat{\eta}_A(0) \leq \hat{\eta}_B(n)$ and $\hat{\varphi}_A(0) \leq \hat{\varphi}_B(n)$ for all $n \in S_2$ and e is identity of S_1 . If $A \times B$ is CIFS of $S_1 \times S_2$, then B is CIFS of S_2 .

Proof. Similar as Theorem 3.11. $\square$

Corollary 3.13. Let A and B two CIFS of S_1 and S_2 , respectively. If $A \times B$ is CIFS of $S_1 \times S_2$, then A is a CIFS of S_1 or B is a CIFS of S_2 .

Theorem 3.14. Let $A \times B$ be a CIFS of ring $S_1 \times S_2$. Then $A \times B$ is a CIFS of $S_1 \times S_2$ if and only if:

1. The fuzzy set $\overline{A \times B} = \{(m, n), \eta_{A \times B}(m, n), \hat{\eta}_{A \times B}(m, n) : (m, n) \in S_1 \times S_2, \eta_{A \times B}(m, n), \hat{\eta}_{A \times B}(m, n) \in [0, 1]\}$ is an IFSR.
2. The π -fuzzy set $\underline{A \times B} = \{(m, n), \varphi_{A \times B}(m, n), \hat{\varphi}_{A \times B}(m, n) : (m, n) \in S_1 \times S_2, \varphi_{A \times B}(m, n), \hat{\varphi}_{A \times B}(m, n) \in [0, 2\pi]\}$ is a π -IFSR.

Proof. Suppose that $A \times B$ is a CIFS and $(m, n), (x, y) \in S_1 \times S_2$. Then we have

$$\begin{aligned} \eta_{A \times B}((m, n) - (x, y)) e^{i\varphi_{A \times B}((m, n) - (x, y))} \\ = \theta_{A \times B}((m, n) - (x, y)) \\ \geq \min\{\theta_{A \times B}(m, n), \theta_{A \times B}(x, y)\} \\ = \min\{\eta_{A \times B}(m, n) e^{i\varphi_{A \times B}(m, n)}, \eta_{A \times B}(x, y) e^{i\varphi_{A \times B}(x, y)}\} \\ = \min\{\eta_{A \times B}(m, n), \eta_{A \times B}(x, y)\} e^{i \min\{\varphi_{A \times B}(m, n), \varphi_{A \times B}(x, y)\}} \end{aligned}$$

(As $A \times B$ is homogeneous)

$$\eta_{A \times B}((m, n) - (x, y)) \geq \min\{\eta_{A \times B}(m, n), \eta_{A \times B}(x, y)\},$$

$$\varphi_{A \times B}((m, n) - (x, y)) \geq \min\{\varphi_{A \times B}(m, n), \varphi_{A \times B}(x, y)\}$$

$$\text{and, } \eta_{A \times B}((m, n)(x, y)) e^{i\varphi_{A \times B}((m, n)(x, y))}$$

$$\begin{aligned} = \theta_{A \times B}((m, n)(x, y)) \\ \geq \min\{\theta_{A \times B}(m, n), \theta_{A \times B}(x, y)\} \\ = \min\{\eta_{A \times B}(m, n) e^{i\varphi_{A \times B}(m, n)}, \eta_{A \times B}(x, y) e^{i\varphi_{A \times B}(x, y)}\} \\ = \min\{\eta_{A \times B}(m, n), \eta_{A \times B}(x, y)\} e^{i \min\{\varphi_{A \times B}(m, n), \varphi_{A \times B}(x, y)\}} \end{aligned}$$

(By using the homogeneous property)

$$\eta_{A \times B}((m, n)(x, y)) \geq \min\{\eta_{A \times B}(m, n), \eta_{A \times B}(x, y)\},$$

$$\text{and, } \varphi_{A \times B}((m, n)(x, y)) \geq \min\{\varphi_{A \times B}(m, n), \varphi_{A \times B}(x, y)\}$$

$$\text{Further, } \hat{\eta}_{A \times B}((m, n) - (x, y)) e^{i\hat{\varphi}_{A \times B}((m, n) - (x, y))}$$

$$\begin{aligned} = \hat{\theta}_{A \times B}((m, n) - (x, y)) \\ \leq \max\{\hat{\theta}_{A \times B}(m, n), \hat{\theta}_{A \times B}(x, y)\} \\ = \max\{\hat{\eta}_{A \times B}(m, n) e^{i\hat{\varphi}_{A \times B}(m, n)}, \hat{\eta}_{A \times B}(x, y) e^{i\hat{\varphi}_{A \times B}(x, y)}\} \\ = \max\{\hat{\eta}_{A \times B}(m, n), \hat{\eta}_{A \times B}(x, y)\} e^{i \max\{\hat{\varphi}_{A \times B}(m, n), \hat{\varphi}_{A \times B}(x, y)\}} \end{aligned}$$

(As $A \times B$ is homogeneous)

$$\hat{\eta}_{A \times B}((m, n) - (x, y)) \leq \max\{\hat{\eta}_{A \times B}(m, n), \hat{\eta}_{A \times B}(x, y)\}$$

$$\hat{\varphi}_{A \times B}((m, n) - (x, y)) \leq \max\{\hat{\varphi}_{A \times B}(m, n), \hat{\varphi}_{A \times B}(x, y)\}$$

$$\text{Moreover, } \hat{\eta}_{A \times B}((m, n)(x, y)) e^{i\hat{\varphi}_{A \times B}((m, n)(x, y))}$$

$$= \hat{\theta}_{A \times B}((m, n)(x, y))$$

$$\leq \max\{\hat{\theta}_{A \times B}(m, n), \hat{\theta}_{A \times B}(x, y)\}$$

$$= \max\{\hat{\eta}_{A \times B}(m, n) e^{i\hat{\varphi}_{A \times B}(m, n)}, \hat{\eta}_{A \times B}(x, y) e^{i\hat{\varphi}_{A \times B}(x, y)}\}$$

$$= \max\{\hat{\eta}_{A \times B}(m, n), \hat{\eta}_{A \times B}(x, y)\} e^{i \max\{\hat{\varphi}_{A \times B}(m, n), \hat{\varphi}_{A \times B}(x, y)\}}$$

(As $A \times B$ is homogeneous)

$$\hat{\eta}_{A \times B}((m, n) - (x, y))$$

$$\leq \max\{\hat{\eta}_{A \times B}(m, n), \hat{\eta}_{A \times B}(x, y)\} \hat{\varphi}_{A \times B}((m, n)(x, y))$$

$$\leq \max\{\hat{\varphi}_{A \times B}(m, n), \hat{\varphi}_{A \times B}(x, y)\}$$

Consequently, $\overline{A \times B}$ is IFSR and $\underline{A \times B}$ is π -IFSR.

Conversely, suppose that $\overline{A \times B}$ and $\underline{A \times B}$ is IFSR and π -IFSR, respectively. Then we have

$$\begin{aligned} \eta_{A \times B}((m, n) - (x, y)) &\geq \min\{\eta_{A \times B}(m, n), \eta_{A \times B}(x, y)\} \\ \eta_{A \times B}((m, n)(x, y)) &\geq \min\{\eta_{A \times B}(m, n), \eta_{A \times B}(x, y)\} \\ \hat{\eta}_{A \times B}((m, n) - (x, y)) &\leq \max\{\hat{\eta}_{A \times B}(m, n), \hat{\eta}_{A \times B}(x, y)\} \\ \hat{\eta}_{A \times B}((m, n)(x, y)) &\leq \max\{\hat{\eta}_{A \times B}(m, n), \hat{\eta}_{A \times B}(x, y)\} \\ \varphi_{A \times B}((m, n) - (x, y)) &\geq \min\{\varphi_{A \times B}(m, n), \varphi_{A \times B}(x, y)\} \\ \varphi_{A \times B}((m, n)(x, y)) &\geq \min\{\varphi_{A \times B}(m, n), \varphi_{A \times B}(x, y)\} \\ \hat{\varphi}_{A \times B}((m, n) - (x, y)) &\leq \max\{\hat{\varphi}_{A \times B}(m, n), \hat{\varphi}_{A \times B}(x, y)\} \\ \hat{\varphi}_{A \times B}((m, n)(x, y)) &\leq \max\{\hat{\varphi}_{A \times B}(m, n), \hat{\varphi}_{A \times B}(x, y)\} \end{aligned}$$

Consider $\theta_{A \times B}$

$$\begin{aligned} ((m, n) - (x, y)) &= \eta_{A \times B}((m, n) - (x, y)) e^{i\varphi_{A \times B}((m, n) - (x, y))} \\ &\geq \min\{\eta_{A \times B}(m, n), \eta_{A \times B}(x, y)\} \\ &\quad e^{i \min\{\varphi_{A \times B}(m, n), \varphi_{A \times B}(x, y)\}} \end{aligned}$$

$$= \min\{\eta_{A \times B}(m, n) e^{i\varphi_{A \times B}(m, n)},$$

$$\eta_{A \times B}(x, y) e^{i\varphi_{A \times B}(x, y)}\}$$

$$= \min\{\theta_{A \times B}(m, n), \theta_{A \times B}(x, y)\}$$

$$\text{Also, } \theta_{A \times B}((m, n)(x, y)) = \eta_{A \times B}((m, n)(x, y)) e^{i\varphi_{A \times B}((m, n)(x, y))}$$

$$\geq \min\{\eta_{A \times B}(m, n), \eta_{A \times B}(x, y)\} \\ e^{i \min\{\varphi_{A \times B}(m, n), \varphi_{A \times B}(x, y)\}}$$

$$= \min\{\eta_{A \times B}(m, n) e^{i\varphi_{A \times B}(m, n)},$$

$$\eta_{A \times B}(x, y) e^{i\varphi_{A \times B}(x, y)}\}$$

$$= \min\{\theta_{A \times B}(m, n), \theta_{A \times B}(x, y)\}$$

Moreover, $\hat{\theta}_{A \times B}$

$$\begin{aligned} ((m, n) - (x, y)) &= \hat{\eta}_{A \times B}((m, n) - (x, y)) e^{i\hat{\varphi}_{A \times B}((m, n) - (x, y))} \\ &\leq \max\{\hat{\eta}_{A \times B}(m, n), \hat{\eta}_{A \times B}(x, y)\} \\ &\quad e^{i \max\{\hat{\varphi}_{A \times B}(m, n), \hat{\varphi}_{A \times B}(x, y)\}} \end{aligned}$$

$$= \max\{\hat{\eta}_{A \times B}(m, n) e^{i\hat{\varphi}_{A \times B}(m, n)},$$

$$\hat{\eta}_{A \times B}(x, y) e^{i\hat{\varphi}_{A \times B}(x, y)}\}$$

$$= \max\{\hat{\theta}_{A \times B}(m, n), \hat{\theta}_{A \times B}(x, y)\}$$

$$\text{Also, } \hat{\theta}_{A \times B}((m, n)(x, y)) = \hat{\eta}_{A \times B}((m, n)(x, y)) e^{i\hat{\varphi}_{A \times B}((m, n)(x, y))}$$

$$\leq \max\{\hat{\eta}_{A \times B}(m, n), \hat{\eta}_{A \times B}(x, y)\} \\ e^{i \max\{\hat{\varphi}_{A \times B}(m, n), \hat{\varphi}_{A \times B}(x, y)\}}$$

$$= \max\{\hat{\eta}_{A \times B}(m, n) e^{i\hat{\varphi}_{A \times B}(m, n)},$$

$$\hat{\eta}_{A \times B}(x, y) e^{i\hat{\varphi}_{A \times B}(x, y)}\}$$

$$= \max\{\hat{\theta}_{A \times B}(m, n), \hat{\theta}_{A \times B}(x, y)\}.$$

Hence, $A \times B$ is CIFS. Thus concluded the proof. $\square$

4. PROPERTIES OF LEVEL SUBSETS OF DIRECT PRODUCT OF COMPLEX INTUITIONISTIC FUZZY SUBRING

This section is devoted to study the properties of level subrings of direct product of CIFSR. We initiate the concept level subset of the direct product of CIFSR and prove various impactful results of this phenomena.

Definition 4.1. Let $A \times B$ be Cartesian product of CIFS A and B . Then for $r \in [0, 1]$, and $t \in [0, 2\pi]$ the level subset of CIFS $A \times B$ of $S_1 \times S_2$ is defined by

$$(A \times B)_{(r,t)}^{(\hat{r},\hat{t})}(m,n) = \{(m,n) \in S_1 \times S_2 : \eta_A(m,n) \geq r, \varphi_A(m,n) \geq t, \hat{\eta}_A(m,n) \leq \hat{r}, \hat{\varphi}_A(m,n) \leq \hat{t}\}$$

For $t = \hat{t} = 0$, we obtain the level subset $(A \times B)_r^{\hat{r}}(m,n) = \{(m,n) \in P : \eta_A(m,n) \geq r, \hat{\eta}_A(m,n) \leq \hat{r}\}$

and for $r = \hat{r} = 0$, then we obtain the level subset $(A \times B)_t^{\hat{t}}(m,n) = \{(m,n) \in S_1 \times S_2 : \varphi_A(m,n) \geq t, \hat{\varphi}_A(m,n) \leq \hat{t}\}$.

Theorem 4.2. Let A and B be two CIFS of rings S_1 and S_2 . Then $(A \times B)_{(r,t)}^{(\hat{r},\hat{t})} = A_{(r,t)}^{(\hat{r},\hat{t})} \times B_{(r,t)}^{(\hat{r},\hat{t})}$.

Proof. Consider $(m,n) \in A_{(r,t)}^{(\hat{r},\hat{t})} \times B_{(r,t)}^{(\hat{r},\hat{t})}$

$$\begin{aligned} &\Leftrightarrow m \in A_{(r,t)}^{(\hat{r},\hat{t})} \text{ and } n \in B_{(r,t)}^{(\hat{r},\hat{t})} \\ &\Leftrightarrow \eta_A(m) \geq r, \varphi_A(m) \geq t, \hat{\eta}_A(m) \leq \hat{r}, \hat{\varphi}_A(m) \leq \hat{t}, \text{ and} \\ &\quad \eta_B(n) \geq r, \varphi_B(n) \geq t, \hat{\eta}_B(n) \leq \hat{r}, \hat{\varphi}_B(n) \leq \hat{t} \\ &\Leftrightarrow \min\{\eta_A(m), \eta_B(n)\} \geq r, \max\{\hat{\eta}_A(m), \hat{\eta}_B(n)\} \leq \hat{r} \\ &\quad \min\{\varphi_A(m), \varphi_B(n)\} \geq t, \max\{\hat{\varphi}_A(m), \hat{\varphi}_B(n)\} \leq \hat{t} \\ &\Leftrightarrow \eta_{A \times B}(m,n) \geq r, \varphi_{A \times B}(m,n) \geq t, \text{ and} \\ &\quad \hat{\eta}_{A \times B}(m,n) \leq \hat{r}, \hat{\varphi}_{A \times B}(m,n) \leq \hat{t} \\ &\Leftrightarrow (m,n) \in (A \times B)_{(r,t)}^{(\hat{r},\hat{t})} \end{aligned}$$

Hence, $(A \times B)_{(r,t)}^{(\hat{r},\hat{t})} = A_{(r,t)}^{(\hat{r},\hat{t})} \times B_{(r,t)}^{(\hat{r},\hat{t})}$.

Theorem 4.3. Let $A \times B$ be CIFSR of ring $S_1 \times S_2$. Then $(A \times B)_{(r,t)}^{(\hat{r},\hat{t})}$ is a subring of ring $S_1 \times S_2$, for all $r, \hat{r} \in [0, 1]$, and $t, \hat{t} \in [0, 2\pi]$, where $\eta_A(0, 0') \geq r$, $\varphi_A(0, 0') \geq t$, $\hat{\eta}_A(0, 0') \leq \hat{r}$, and $\hat{\varphi}_A(0, 0') \leq \hat{t}$ also $(0, 0')$ is identity element of $S_1 \times S_2$.

Proof. Note that $(A \times B)_{(r,t)}^{(\hat{r},\hat{t})}$ is nonempty, as $(0, 0') \in (A \times B)_{(r,t)}^{(\hat{r},\hat{t})}$. Let $(m,n), (x,y) \in (A \times B)_{(r,t)}^{(\hat{r},\hat{t})}$ be any two elements. Then

$$\begin{aligned} \eta_{A \times B}(m,n) &\geq r, \varphi_{A \times B}(m,n) \geq t, \hat{\eta}_{A \times B}(m,n) \\ &\leq \hat{r}, \hat{\varphi}_{A \times B}(m,n) \leq \hat{t}, \\ \eta_{A \times B}(x,y) &\geq r, \varphi_{A \times B}(x,y) \geq t, \hat{\eta}_{A \times B}(x,y) \\ &\leq \hat{r}, \hat{\varphi}_{A \times B}(x,y) \leq \hat{t}. \end{aligned}$$

Consider $\eta_{A \times B}((m,n) - (x,y)) \geq \hat{r}, \hat{\eta}_{A \times B}((m,n) - (x,y)) \leq \hat{t}$.

$$\begin{aligned} e^{i\varphi_{A \times B}((m,n)-(x,y))} &= \theta_{A \times B}((m,n) - (x,y)) \\ &\geq \min\{\theta_{A \times B}(m,n), \theta_{A \times B}(x,y)\} \end{aligned}$$

$$\begin{aligned} &= \min\{\eta_{A \times B}(m,n) e^{i\varphi_{A \times B}(m,n)}, \\ &\quad \eta_{A \times B}(x,y) e^{i\varphi_{A \times B}(x,y)}\} \\ &= \min\{\eta_{A \times B}(m,n), \eta_{A \times B}(x,y)\} \\ &\quad e^{i \min\{\varphi_{A \times B}(m,n), \varphi_{A \times B}(x,y)\}} \end{aligned}$$

(As $A \times B$ is homogeneous)

$$\eta_{A \times B}((m,n) - (x,y)) \geq \min\{\eta_{A \times B}(m,n), \eta_{A \times B}(x,y)\} = \min\{r, r\} = r.$$

$$\begin{aligned} \varphi_{A \times B}((m,n) - (x,y)) &\geq \min\{\varphi_{A \times B}(m,n), \varphi_{A \times B}(x,y)\} \\ &= \min\{t, t\} = t. \end{aligned}$$

Moreover, $\hat{\eta}_{A \times B}((m,n) - (x,y))$

$$\begin{aligned} e^{i\hat{\varphi}_{A \times B}((m,n)-(x,y))} &= \hat{\theta}_{A \times B}((m,n) - (x,y)) \\ &\leq \max\{\hat{\theta}_{A \times B}(m,n), \hat{\theta}_{A \times B}(x,y)\} \\ &= \max\{\hat{\eta}_{A \times B}(m,n) e^{i\hat{\varphi}_{A \times B}(m,n)}, \\ &\quad \hat{\eta}_{A \times B}(x,y) e^{i\hat{\varphi}_{A \times B}(x,y)}\} \\ &= \max\{\hat{\eta}_{A \times B}(m,n), \hat{\eta}_{A \times B}(x,y)\} \\ &\quad e^{i \max\{\hat{\varphi}_{A \times B}(m,n), \hat{\varphi}_{A \times B}(x,y)\}}. \end{aligned}$$

(By homogeneous property)

$$\hat{\eta}_{A \times B}((m,n) - (x,y)) \leq \max\{\hat{\eta}_{A \times B}(m,n), \hat{\eta}_{A \times B}(x,y)\} = \max\{\hat{r}, \hat{r}\} = \hat{r},$$

$$\begin{aligned} \hat{\varphi}_{A \times B}((m,n) - (x,y)) &\leq \max\{\hat{\varphi}_{A \times B}(m,n), \hat{\varphi}_{A \times B}(x,y)\} \\ &= \max\{\hat{t}, \hat{t}\} = \hat{t}. \end{aligned}$$

Implies that (m,n)

$$- (x,y) \in (A \times B)_{(r,t)}^{(\hat{r},\hat{t})}.$$

Also, $\eta_{A \times B}((m,n)(x,y))$

$$\begin{aligned} e^{i\varphi_{A \times B}((m,n)(x,y))} &= \theta_{A \times B}((m,n)(x,y)) \\ &\geq \min\{\theta_{A \times B}(m,n), \theta_{A \times B}(x,y)\} \\ &= \min\{\eta_{A \times B}(m,n) e^{i\varphi_{A \times B}(m,n)}, \\ &\quad \eta_{A \times B}(x,y) e^{i\varphi_{A \times B}(x,y)}\} \\ &= \min\{\eta_{A \times B}(m,n), \eta_{A \times B}(x,y)\} \\ &\quad e^{i \min\{\varphi_{A \times B}(m,n), \varphi_{A \times B}(x,y)\}} \end{aligned}$$

(By homogeneous property)

$$\eta_{A \times B}((m,n)(x,y)) \geq \min\{\eta_{A \times B}(m,n), \eta_{A \times B}(x,y)\} = \min\{r, r\} = r,$$

$$\text{and, } \varphi_{A \times B}((m,n)(x,y)) \geq \min\{\varphi_{A \times B}(m,n), \varphi_{A \times B}(x,y)\} = \min\{t, t\} = t,$$

$$\begin{aligned} \hat{\eta}_{A \times B}((m,n)(x,y)) e^{i\hat{\varphi}_{A \times B}((m,n)(x,y))} &= \hat{\theta}_{A \times B}((m,n)(x,y)) \\ &\leq \max\{\hat{\theta}_{A \times B}(m,n), \hat{\theta}_{A \times B}(x,y)\} \\ &= \max\{\hat{\eta}_{A \times B}(m,n) e^{i\hat{\varphi}_{A \times B}(m,n)}, \\ &\quad \hat{\eta}_{A \times B}(x,y) e^{i\hat{\varphi}_{A \times B}(x,y)}\} \\ &= \max\{\hat{\eta}_{A \times B}(m,n), \hat{\eta}_{A \times B}(x,y)\} \\ &\quad e^{i \max\{\hat{\varphi}_{A \times B}(m,n), \hat{\varphi}_{A \times B}(x,y)\}}. \end{aligned}$$

(As $A \times B$ is homogeneous)

$$\hat{\eta}_{A \times B}((m,n)(x,y)) \leq \max\{\hat{\eta}_{A \times B}(m,n), \hat{\eta}_{A \times B}(x,y)\} = \max\{\hat{r}, \hat{r}\} = \hat{r},$$

$$\text{and, } \hat{\varphi}_{A \times B}((m,n)(x,y)) \leq \max\{\hat{\varphi}_{A \times B}(m,n), \hat{\varphi}_{A \times B}(x,y)\} = \max\{\hat{t}, \hat{t}\} = \hat{t}.$$

Implies that $(m,n)(x,y) \in (A \times B)_{(r,t)}^{(\hat{r},\hat{t})}$. Hence $(A \times B)_{(r,t)}^{(\hat{r},\hat{t})}$ is subring. $\square$

Theorem 4.4. Let $(A \times B)_{(r,t)}^{(\hat{r},\hat{t})}$ be a subring of ring $S_1 \times S_2$, then $A \times B$ is CIFSR of $S_1 \times S_2$, for all $r \in [0, 1]$, and $t \in [0, 2\pi]$, where $\eta_A(0, 0') \geq r$, $\varphi_A(0, 0') \geq t$, $\hat{\eta}_A(0, 0') \leq \hat{r}$, $\hat{\varphi}_A(0, 0') \leq \hat{t}$, also $(0, 0')$ is identity element of $S_1 \times S_2$.

Proof. Assume that $\min\{\eta_{A \times B}(m, n), \eta_{A \times B}(x, y)\} = r$, $\min\{\varphi_{A \times B}(m, n), \varphi_{A \times B}(x, y)\} = t$, $\max\{\hat{\eta}_{A \times B}(m, n), \hat{\eta}_{A \times B}(x, y)\} = \hat{r}$ and $\max\{\hat{\varphi}_{A \times B}(m, n), \hat{\varphi}_{A \times B}(x, y)\} = \hat{t}$. Then we have

$$\begin{aligned} \eta_{A \times B}(m, n) &\geq r, \varphi_{A \times B}(m, n) \geq t, \hat{\eta}_{A \times B}(m, n) \\ &\leq \hat{r}, \hat{\varphi}_{A \times B}(m, n) \leq \hat{t}, \\ \eta_{A \times B}(x, y) &\geq r, \varphi_{A \times B}(x, y) \geq t, \hat{\eta}_{A \times B}(x, y) \\ &\leq \hat{r} \text{ and } \hat{\varphi}_{A \times B}(x, y) \leq \hat{t}. \end{aligned}$$

This implies that $(m, n) \in (A \times B)_{(r,t)}^{(\hat{r},\hat{t})}$ and $(x, y) \in (A \times B)_{(r,t)}^{(\hat{r},\hat{t})}$.

As $(A \times B)_{(r,t)}^{(\hat{r},\hat{t})}$ is subring. So $(m, n) - (x, y), (m, n)(x, y) \in (A \times B)_{(r,t)}^{(\hat{r},\hat{t})}$. Then we have

$$\eta_{A \times B}((m, n) - (x, y)) \geq r, \varphi_{A \times B}((m, n) - (x, y)) \geq t,$$

$$\eta_{A \times B}((m, n)(x, y)) \geq r, \varphi_{A \times B}((m, n)(x, y)) \geq t,$$

$$\hat{\eta}_{A \times B}((m, n) - (x, y)) \leq \hat{r}, \hat{\varphi}_{A \times B}((m, n) - (x, y)) \leq \hat{t},$$

$$\hat{\eta}_{A \times B}((m, n)(x, y)) \leq \hat{r}, \hat{\varphi}_{A \times B}((m, n)(x, y)) \leq \hat{t}.$$

Implies that $\eta_{A \times B}((m, n) - (x, y)) \geq \min\{\eta_{A \times B}(m, n), \eta_{A \times B}(x, y)\}$

$$\varphi_{A \times B}((m, n) - (x, y)) \geq \min\{\varphi_{A \times B}(m, n), \varphi_{A \times B}(x, y)\},$$

$$\eta_{A \times B}((m, n)(x, y)) \geq \min\{\eta_{A \times B}(m, n), \eta_{A \times B}(x, y)\},$$

$$\varphi_{A \times B}((m, n)(x, y)) \geq \min\{\varphi_{A \times B}(m, n), \varphi_{A \times B}(x, y)\},$$

$$\hat{\eta}_{A \times B}((m, n) - (x, y)) \leq \max\{\hat{\eta}_{A \times B}(m, n), \hat{\eta}_{A \times B}(x, y)\},$$

$$\hat{\varphi}_{A \times B}((m, n) - (x, y)) \leq \max\{\hat{\varphi}_{A \times B}(m, n), \hat{\varphi}_{A \times B}(x, y)\},$$

$$\hat{\eta}_{A \times B}((m, n)(x, y)) \leq \max\{\hat{\eta}_{A \times B}(m, n), \hat{\eta}_{A \times B}(x, y)\},$$

$$\hat{\varphi}_{A \times B}((m, n)(x, y)) \leq \max\{\hat{\varphi}_{A \times B}(m, n), \hat{\varphi}_{A \times B}(x, y)\},$$

$$\text{Consider } \theta_{A \times B}((m, n) - (x, y)) = \eta_{A \times B}((m, n)(x, y))e^{i\eta_{A \times B}((m, n)(x, y))}$$

$$\geq \min\{\eta_{A \times B}(m, n), \eta_{A \times B}(x, y)\} e^{i \min\{\varphi_{A \times B}(m, n), \varphi_{A \times B}(x, y)\}}$$

$$= \min\{\eta_{A \times B}(m, n)e^{i\varphi_{A \times B}(m, n)}, \eta_{A \times B}(x, y)e^{i\varphi_{A \times B}(x, y)}\}$$

$$\theta_{A \times B}((m, n) - (x, y)) \geq \min\{\theta_{A \times B}(m, n), \theta_{A \times B}(x, y)\}$$

$$\text{Further, } \theta_{A \times B}((m, n)(x, y)) = \eta_{A \times B}((m, n)(x, y))e^{i\eta_{A \times B}((m, n)(x, y))}$$

$$\geq \min\{\eta_{A \times B}(m, n), \eta_{A \times B}(x, y)\} e^{i \min\{\varphi_{A \times B}(m, n), \varphi_{A \times B}(x, y)\}}$$

$$= \min\{\eta_{A \times B}(m, n)e^{i\varphi_{A \times B}(m, n)}, \eta_{A \times B}(x, y)e^{i\varphi_{A \times B}(x, y)}\}$$

$$\theta_{A \times B}((m, n)(x, y)) \geq \min\{\theta_{A \times B}(m, n), \theta_{A \times B}(x, y)\}$$

$$\text{Moreover, } \hat{\theta}_{A \times B}((m, n) - (x, y)) = \hat{\eta}_{A \times B}((m, n)(x, y))e^{i\hat{\eta}_{A \times B}((m, n)(x, y))}$$

$$\leq \max\{\hat{\eta}_{A \times B}(m, n), \hat{\eta}_{A \times B}(x, y)\} e^{i \max\{\hat{\varphi}_{A \times B}(m, n), \hat{\varphi}_{A \times B}(x, y)\}}$$

$$= \max\{\hat{\eta}_{A \times B}(m, n)e^{i\hat{\varphi}_{A \times B}(m, n)}, \hat{\eta}_{A \times B}(x, y)e^{i\hat{\varphi}_{A \times B}(x, y)}\}$$

$$\hat{\theta}_{A \times B}((m, n) - (x, y)) \leq \max\{\hat{\theta}_{A \times B}(m, n), \hat{\theta}_{A \times B}(x, y)\}$$

$$\text{Also, } \hat{\theta}_{A \times B}((m, n)(x, y)) = \hat{\eta}_{A \times B}((m, n)(x, y))e^{i\hat{\eta}_{A \times B}((m, n)(x, y))}$$

$$\leq \max\{\hat{\eta}_{A \times B}(m, n), \hat{\eta}_{A \times B}(x, y)\} e^{i \max\{\hat{\varphi}_{A \times B}(m, n), \hat{\varphi}_{A \times B}(x, y)\}}$$

$$= \max\{\hat{\eta}_{A \times B}(m, n)e^{i\hat{\varphi}_{A \times B}(m, n)}, \hat{\eta}_{A \times B}(x, y)e^{i\hat{\varphi}_{A \times B}(x, y)}\}$$

$$\hat{\theta}_{A \times B}((m, n)(x, y)) \leq \max\{\hat{\theta}_{A \times B}(m, n), \hat{\theta}_{A \times B}(x, y)\}.$$

Corollary 4.5. Let $A \times B$ be a CIFSR of $S_1 \times S_2$, then the level subsets $(A \times B)_{\hat{r}}$ and $(A \times B)_{\hat{t}}$ is a subring of ring $S_1 \times S_2$, for all $r, \hat{r} \in [0, 1]$, and $t, \hat{t} \in [0, 2\pi]$, where $\eta_A(0, 0') \geq r$, $\varphi_A(0, 0') \geq t$, $\hat{\eta}_{A \times B}(m, n) \leq \hat{r}$, and $\hat{\varphi}_{A \times B}(m, n) \leq \hat{t}$ and $(0, 0')$ is identity element of $S_1 \times S_2$.

5. HOMOMORPHISM OF DIRECT PRODUCT COMPLEX INTUITIONISTIC FUZZY SUBRING

In this section, we define the homomorphic image and preimage of the direct product of CIFSR. We prove some results of the direct product of CIFSR under ring homomorphism.

Definition 5.1. Let $f : S_1 \times S_2 \rightarrow R_1 \times R_2$ be a ring homomorphism form $S_1 \times S_2$ to $R_1 \times R_2$. Let $A \times B$ and $C \times D$ be two CIFSR of rings $S_1 \times S_2$ and $R_1 \times R_2$, respectively. For all $(x, y) \in S_1 \times S_2$ and for all $(m, n) \in R_1 \times R_2$. The set

$f(A \times B) = \{(m, n), f(\theta_{A \times B})(m, n), f(\hat{\theta}_{A \times B})(m, n)\}$ is said to be image of $A \times B$, where

$$= \begin{cases} \sup\{\theta_{A \times B}(x, y), & \text{if } f(x, y) = (m, n)\} \\ 0, & \text{otherwise} \end{cases}, \quad f^{-1}(m, n) \neq \emptyset$$

$$= \begin{cases} \inf\{\hat{\theta}_{A \times B}(x, y), & \text{if } f(x, y) = (m, n)\} \\ 1, & \end{cases}, \quad f^{-1}(m, n) \neq \emptyset$$

The set $f^{-1}(B_1 \times B_2) = \{(x, y), f^{-1}(\theta_{C \times D})(x, y), f^{-1}(\hat{\theta}_{C \times D})(x, y)\}$ is said to be preimage of $C \times D$, where

$$f^{-1}(\theta_{C \times D})(x, y) = \theta_{C \times D}(f(x, y)),$$

$$f^{-1}(\hat{\theta}_{C \times D})(x, y) = \hat{\theta}_{C \times D}(f(x, y)).$$

Theorem 5.2. [9] $f : S \rightarrow R$ be ring homomorphism from S to R . Let A be IF SR of S and B be IF SR of R . Then $f(A)$ is IF SR of R and $f^{-1}(B)$ is IF SR of S .

Lemma 5.3. Let $f : S_1 \times S_2 \rightarrow R_1 \times R_2$ be a homomorphism from group $S_1 \times S_2$ to group $R_1 \times R_2$. Let $A \times B$ and $C \times D$ be two CIFSR. Then

1. $f(\theta_{A \times B})(m, n) = f(\eta_{A \times B})(m, n) e^{if(\varphi_{A \times B})(m, n)}$, for all $(m, n) \in R_1 \times R_2$
2. $f(\hat{\theta}_{A \times B})(m, n) = f(\hat{\eta}_{A \times B})(m, n) e^{if(\hat{\varphi}_{A \times B})(m, n)}$
3. $f^{-1}(\theta_{C \times D})(x, y) = f^{-1}(\eta_{C \times D})(x, y) e^{if^{-1}(\varphi_{C \times D})(x, y)}$, for all $(x, y) \in S_1 \times S_2$
4. $f^{-1}(\hat{\theta}_{C \times D})(x, y) = f^{-1}(\hat{\eta}_{C \times D})(x, y) e^{if^{-1}(\hat{\varphi}_{C \times D})(x, y)}$

Proof. 1.

$$\begin{aligned} \text{Consider } f(\theta_{A \times B})(m, n) &= \max\{(\theta_{A \times B})(x, y), \\ &\quad \text{if } f(x, y) = (m, n) \\ &= \max\{(\eta_{A \times B})(x, y) e^{if(\varphi_{A \times B})(x, y)}, \\ &\quad \text{if } f(x, y) = (m, n) \\ &= \max\{(\eta_{A \times B})(x, y), \quad \text{if } f(x, y) = (m, n)\} \\ &\quad e^{i \max\{(\varphi_{A \times B})(x, y), \quad \text{if } f(x, y) = (m, n)\}} \\ &= f(\eta_{A \times B})(m, n) e^{if(\varphi_{A \times B})(m, n)}. \end{aligned}$$

$$\text{Hence, } f(\theta_{A \times B})(m, n) = f(\eta_{A \times B})(m, n) e^{if(\varphi_{A \times B})(m, n)}.$$

2.

$$\begin{aligned} \text{Suppose that } f(\hat{\theta}_{A \times B})(m, n) &= \max\{(\hat{\theta}_{A \times B})(x, y), \\ &\quad \text{if } f(x, y) = (m, n) \\ &= \max\{(\hat{\eta}_{A \times B})(x, y) e^{if(\hat{\varphi}_{A \times B})(x, y)}, \\ &\quad \text{if } f(x, y) = (m, n) \\ &= \max\{(\hat{\eta}_{A \times B})(x, y), \\ &\quad \text{if } f(x, y) = (m, n) \\ &\quad e^{i \max\{(\hat{\varphi}_{A \times B})(x, y), \quad \text{if } f(x, y) = (m, n)\}} \\ &= f(\hat{\eta}_{A \times B})(m, n) e^{if(\hat{\varphi}_{A \times B})(m, n)}. \end{aligned}$$

$$\text{As a result, } f(\hat{\theta}_{A \times B})(m, n) = f(\hat{\eta}_{A \times B})(m, n) e^{if(\hat{\varphi}_{A \times B})(m, n)}.$$

3.

$$\begin{aligned} \text{Assume that, } f^{-1}(\theta_{C \times D})(x, y) &= \theta_{C \times D}(f(x, y)) \\ &= \eta_{C \times D}(f(x, y)) e^{i\varphi_{C \times D}(f(x, y))} \\ &= f^{-1}(\eta_{C \times D})(x, y) \times e^{if^{-1}(\varphi_{C \times D})(x, y)}. \end{aligned}$$

$$\text{Consequently, } f^{-1}(\theta_{C \times D})(x, y) = f^{-1}(\eta_{C \times D})(x, y) e^{if^{-1}(\varphi_{C \times D})(x, y)}.$$

4.

$$\begin{aligned} \text{Consider, } f^{-1}(\hat{\theta}_{C \times D})(x, y) &= \hat{\theta}_{C \times D}(f(x, y)) \\ &= \hat{\eta}_{C \times D}(f(x, y)) e^{i\hat{\varphi}_{C \times D}(f(x, y))} \\ &= f^{-1}(\hat{\eta}_{C \times D})(x, y) \times e^{if^{-1}(\hat{\varphi}_{C \times D})(x, y)}. \end{aligned}$$

$$\text{Consequently, } f^{-1}(\hat{\theta}_{C \times D})(x, y) = f^{-1}(\hat{\eta}_{C \times D})(x, y) e^{if^{-1}(\hat{\varphi}_{C \times D})(x, y)}.$$

□

Theorem 5.4. Let $f : S_1 \times S_2 \rightarrow R_1 \times R_2$ be a homomorphism from ring $S_1 \times S_2$ to ring $R_1 \times R_2$. Let $A \times B$ be CIFSR of $S_1 \times S_2$. Then $f(A \times B)$ is CIFSR of $R_1 \times R_2$.

Proof. Obviously, $\{(p, q), \eta_{A \times B}(p, q), \hat{\eta}_{A \times B}(p, q)\}, (p, q) \in S_1 \times S_2\}$ and $\{(p, q), \varphi_{A \times B}(p, q), \hat{\varphi}_{A \times B}(p, q), (p, q) \in S_1 \times S_2\}$ are IF SR and π -IF SR, respectively. From Theorem 2.9 and Theorem 5.2 the homomorphic image of $\{(p, q), \eta_{A \times B}(p, q), \hat{\eta}_{A \times B}(p, q)\}, (p, q) \in S_1 \times S_2\}$ and $\{(p, q), \varphi_{A \times B}(p, q), \hat{\varphi}_{A \times B}(p, q), (p, q) \in S_1 \times S_2\}$ are IF SR and π -IF SR, respectively, for all $(m, n), (x, y) \in R_1 \times R_2$. Then we have

$$\begin{aligned}
f(\eta_{A \times B})((m, n) - (x, y)) &\geq \min\{f(\eta_{A \times B})(m, n), f(\eta_{A \times B})(x, y)\}, \\
f(\eta_{A \times B})((m, n)(x, y)) &\geq \min\{f(\eta_{A \times B})(m, n), f(\eta_{A \times B})(x, y)\}, \\
f(\hat{\eta}_{A \times B})((m, n) - (x, y)) &\leq \max\{f(\hat{\eta}_{A \times B})(m, n), f(\hat{\eta}_{A \times B})(x, y)\}, \\
f(\hat{\eta}_{A \times B})((m, n)(x, y)) &\leq \max\{f(\hat{\eta}_{A \times B})(m, n), f(\hat{\eta}_{A \times B})(x, y)\}, \\
f(\varphi_{A \times B})((m, n) - (x, y)) &\geq \min\{f(\varphi_{A \times B})(m, n), f(\varphi_{A \times B})(x, y)\}, \\
f(\varphi_{A \times B})((m, n)(x, y)) &\geq \min\{f(\varphi_{A \times B})(m, n), f(\varphi_{A \times B})(x, y)\}, \\
f(\hat{\varphi}_{A \times B})((m, n) - (x, y)) &\leq \max\{f(\hat{\varphi}_{A \times B})(m, n), f(\hat{\varphi}_{A \times B})(x, y)\}, \\
f(\hat{\varphi}_{A \times B})((m, n)(x, y)) &\leq \max\{f(\hat{\varphi}_{A \times B})(m, n), f(\hat{\varphi}_{A \times B})(x, y)\},
\end{aligned}$$

From Lemma 5.3, we have

$$\begin{aligned}
f(\theta_{A \times B})((m, n) - (x, y)) &= f(\eta_{A \times B})((m, n) - (x, y))e^{if(\varphi_{A \times B})((m, n) - (x, y))}, \\
&\text{for all } (m, n), (x, y) \in R_1 \times R_2 \\
&\geq \min\{f(\eta_{A \times B})(m, n), f(\eta_{A \times B})(x, y)\} e^{if(\varphi_{A \times B})(m, n), f(\varphi_{A \times B})(x, y))} \\
&= \min\{f(\eta_{A \times B})(m, n)e^{if(\varphi_{A \times B})(m, n)}, f(\eta_{A \times B})(y_2)e^{if(\varphi_{A \times B})(x, y))}\} \\
&= \min\{f(\theta_{A \times B})(m, n), f(\theta_{A \times B})(x, y)\}
\end{aligned}$$

Consequently, $f(\theta_{A \times B})((m, n) - (x, y)) \geq \min\{f(\theta_{A \times B})(m, n), f(\theta_{A \times B})(x, y)\}$

$$\begin{aligned}
\text{Further, } f(\theta_{A \times B})((m, n)(x, y)) &= f(\eta_{A \times B})((m, n)(x, y))e^{if(\varphi_{A \times B})((m, n)(x, y))}, \\
&\text{for all } (m, n), (x, y) \in R_1 \times R_2 \\
&\geq \min\{f(\eta_{A \times B})(m, n), f(\eta_{A \times B})(x, y)\} e^{if(\varphi_{A \times B})(m, n), f(\varphi_{A \times B})(x, y))} \\
&= \min\{f(\eta_{A \times B})(m, n)e^{if(\varphi_{A \times B})(m, n)}, f(\eta_{A \times B})(y_2)e^{if(\varphi_{A \times B})(x, y))}\} \\
&= \min\{f(\theta_{A \times B})(m, n), f(\theta_{A \times B})(x, y)\}.
\end{aligned}$$

As result, $f(\theta_{A \times B})((m, n)(x, y)) \geq \min\{f(\theta_{A \times B})(m, n), f(\theta_{A \times B})(x, y)\}$.

In the view of Lemma 5.3, we know that

$$\begin{aligned}
f(\hat{\theta}_{A \times B})((m, n) - (x, y)) &= f(\hat{\eta}_{A \times B})((m, n) - (x, y))e^{if(\hat{\varphi}_{A \times B})((m, n) - (x, y))}, \\
&(m, n), (x, y) \in R_1 \times R_2 \\
&\leq \max\{f(\hat{\eta}_{A \times B})(m, n), f(\hat{\eta}_{A \times B})(x, y)\} e^{if(\hat{\varphi}_{A \times B})(m, n), f(\hat{\varphi}_{A \times B})(x, y))} \\
&= \max\{f(\hat{\eta}_{A \times B})(m, n)e^{if(\hat{\varphi}_{A \times B})(m, n)}, f(\hat{\eta}_{A \times B})(y_2)e^{if(\hat{\varphi}_{A \times B})(x, y))}\} \\
&= \max\{f(\hat{\theta}_{A \times B})(m, n), f(\hat{\theta}_{A \times B})(x, y)\}
\end{aligned}$$

Consequently, $f(\hat{\theta}_{A \times B})((m, n) - (x, y)) \leq \max\{f(\hat{\theta}_{A \times B})(m, n), f(\hat{\theta}_{A \times B})(x, y)\}$.

$$\begin{aligned}
\text{Moreover, } f(\hat{\theta}_{A \times B})((m, n)(x, y)) &= f(\hat{\eta}_{A \times B})((m, n)(x, y))e^{if(\hat{\varphi}_{A \times B})((m, n)(x, y))}, \quad (m, n), (x, y) \in R_1 \times R_2 \\
&\leq \max\{f(\hat{\eta}_{A \times B})(m, n), f(\hat{\eta}_{A \times B})(x, y)\} e^{if(\hat{\varphi}_{A \times B})(m, n), f(\hat{\varphi}_{A \times B})(x, y))} \\
&= \max\{f(\hat{\eta}_{A \times B})(m, n)e^{if(\hat{\varphi}_{A \times B})(m, n)}, f(\hat{\eta}_{A \times B})(y_2)e^{if(\hat{\varphi}_{A \times B})(x, y))}\} \\
&= \max\{f(\hat{\theta}_{A \times B})(m, n), f(\hat{\theta}_{A \times B})(x, y)\}
\end{aligned}$$

Therefore, $f(\hat{\theta}_{A \times B})((m, n) - (x, y)) \leq \max\{f(\hat{\theta}_{A \times B})(m, n), f(\hat{\theta}_{A \times B})(x, y)\}$.

This establishes the proof. $\square$

Theorem 5.5. Let $f : S_1 \times S_2 \rightarrow R_1 \times R_2$ be a homomorphism from ring $S_1 \times S_2$ to ring $R_1 \times R_2$. Let $C \times D$ be CIFSR of $R_1 \times R_2$. Then $f^{-1}(C \times D)$ is CIFSR of $S_1 \times S_2$.

Proof. Obviously, $\{(p, q), \eta_{C \times D}(p, q), \hat{\eta}_{C \times D}(p, q)\}, (p, q) \in R_1 \times R_2\}$ and $\{(p, q), \varphi_{C \times D}(p, q), \hat{\varphi}_{C \times D}(p, q), (p, q) \in R_1 \times R_2\}$ are IFSR and π -IFSR respectively. Then from Theorems 2.9 and 5.2 the inverse image of $\{(p, q), \eta_{C \times D}(p, q), \hat{\eta}_{C \times D}(p, q)\}, (p, q) \in R_1 \times R_2\}$ and

$\{(p, q), \varphi_{C \times D}(p, q), \hat{\varphi}_{C \times D}(p, q), (p, q) \in R_1 \times R_2\}$ are IFSR and π -IFSR, respectively, for all $(m, n), (x, y) \in S_1 \times S_2$. Then we have

$$\begin{aligned}
 & f^{-1}(\eta_{C \times D})((m, n) - (x, y)) \\
 & \geq \min\{f^{-1}(\eta_{C \times D})(m, n), f^{-1}(\eta_{C \times D})(x, y)\}, \\
 & f^{-1}(\eta_{C \times D})((m, n)(x, y)) \\
 & \geq \min\{f^{-1}(\eta_{C \times D})(m, n), f^{-1}(\eta_{C \times D})(x, y)\}, \\
 & f^{-1}(\hat{\eta}_{C \times D})((m, n) - (x, y)) \\
 & \leq \max\{f^{-1}(\hat{\eta}_{C \times D})(m, n), f^{-1}(\hat{\eta}_{C \times D})(x, y)\}, \\
 & f^{-1}(\hat{\eta}_{C \times D})((m, n)(x, y)) \\
 & \leq \max\{f^{-1}(\hat{\eta}_{C \times D})(m, n), f^{-1}(\hat{\eta}_{C \times D})(x, y)\}, \\
 & f^{-1}(\varphi_{C \times D})((m, n) - (x, y)) \\
 & \geq \min\{f^{-1}(\varphi_{C \times D})(m, n), f^{-1}(\varphi_{C \times D})(x, y)\}, \\
 & f^{-1}(\varphi_{C \times D})((m, n)(x, y)) \\
 & \geq \min\{f^{-1}(\varphi_{C \times D})(m, n), f^{-1}(\varphi_{C \times D})(x, y)\}, \\
 & f^{-1}(\hat{\varphi}_{C \times D})((m, n) - (x, y)) \\
 & \leq \max\{f^{-1}(\hat{\varphi}_{C \times D})(m, n), f^{-1}(\hat{\varphi}_{C \times D})(x, y)\}, \\
 & f^{-1}(\hat{\varphi}_{C \times D})((m, n)(x, y)) \\
 & \leq \max\{f^{-1}(\hat{\varphi}_{C \times D})(m, n), f^{-1}(\hat{\varphi}_{C \times D})(x, y)\}.
 \end{aligned}$$

By Lemma 5.3(3), we have

$$\begin{aligned}
 & f^{-1}(\theta_{C \times D})((m, n) - (x, y)) \\
 & = f^{-1}(\eta_{C \times D})((m, n) - (x, y))e^{if^{-1}(\varphi_{C \times D})((m, n) - (x, y))}, \\
 & \text{for all } (m, n), (x, y) \in S_1 \times S_2 \\
 & \geq \min\left\{f^{-1}(\eta_{C \times D})(m, n), f^{-1}(\eta_{C \times D})(x, y)\right. \\
 & \quad \left.e^{if^{-1}(\varphi_{C \times D})(m, n)}, f^{-1}(\varphi_{C \times D})(x, y)\right\} \\
 & = \min\left\{f^{-1}(\eta_{C \times D})(m, n)e^{if^{-1}(\varphi_{C \times D})(m, n)},\right. \\
 & \quad \left.f^{-1}(\eta_{C \times D})(x, y)e^{if^{-1}(\varphi_{C \times D})(x, y)}\right\} \\
 & \quad \min\{f^{-1}(\theta_{C \times D})(m, n), f^{-1}(\theta_{C \times D})(x, y)\}.
 \end{aligned}$$

Therefore, $f^{-1}(\theta_{C \times D})((m, n) - (x, y))$

$$\geq \min\{f^{-1}(\theta_{C \times D})(m, n), f^{-1}(\theta_{C \times D})(x, y)\}.$$

Further, $f^{-1}(\theta_{C \times D})((m, n)(x, y)) = f^{-1}(\eta_{C \times D})((m, n)(x, y))e^{if^{-1}(\varphi_{C \times D})((m, n)(x, y))}$, for all $(m, n), (x, y) \in S_1 \times S_2$.

$$\begin{aligned}
 & \geq \min\{f^{-1}(\eta_{C \times D})(m, n), f^{-1}(\eta_{C \times D})(x, y)\} \\
 & \quad e^{if^{-1}(\varphi_{C \times D})(m, n)}, f^{-1}(\varphi_{C \times D})(x, y)\} \\
 & = \min\{f^{-1}(\eta_{C \times D})(m, n)e^{if^{-1}(\varphi_{C \times D})(m, n)}, \\
 & \quad f^{-1}(\eta_{C \times D})(x, y)e^{if^{-1}(\varphi_{C \times D})(x, y)}\} \\
 & = \min\{f^{-1}(\theta_{C \times D})(m, n), f^{-1}(\theta_{C \times D})(x, y)\}.
 \end{aligned}$$

Therefore,

$$f^{-1}(\theta_{C \times D})((m, n)(x, y)) \geq \min\{f^{-1}(\theta_{C \times D})(m, n), f^{-1}(\theta_{C \times D})(x, y)\}$$

Consider $f^{-1}(\hat{\theta}_{C \times D})((m, n) - (x, y)) = f^{-1}(\hat{\eta}_{C \times D})((m, n) - (x, y))e^{if^{-1}(\hat{\varphi}_{C \times D})((m, n) - (x, y))}$, for all $(m, n), (x, y) \in S_1 \times S_2$

$$\begin{aligned}
 & \leq \max\{f^{-1}(\hat{\eta}_{C \times D})(m, n), f^{-1}(\hat{\eta}_{C \times D})(x, y)\} e^{if^{-1}(\hat{\varphi}_{C \times D})(m, n)}, f^{-1}(\hat{\varphi}_{C \times D})(x, y)\} \\
 & = \max\{f^{-1}(\hat{\eta}_{C \times D})(m, n)e^{if^{-1}(\hat{\varphi}_{C \times D})(m, n)}, f^{-1}(\hat{\eta}_{C \times D})(x, y)e^{if^{-1}(\hat{\varphi}_{C \times D})(x, y)}\} \\
 & = \max\{f^{-1}(\hat{\theta}_{C \times D})(m, n), f^{-1}(\hat{\theta}_{C \times D})(x, y)\}
 \end{aligned}$$

Therefore, $f^{-1}(\hat{\theta}_{C \times D})((m, n) - (x, y)) \leq \max\{f^{-1}(\hat{\theta}_{C \times D})(m, n), f^{-1}(\hat{\theta}_{C \times D})(x, y)\}$. Further, $f^{-1}(\hat{\theta}_{C \times D})((m, n)(x, y))$

$$\begin{aligned}
 & = f^{-1}(\hat{\eta}_{C \times D})((m, n)(x, y)) e^{if^{-1}(\hat{\varphi}_{C \times D})((m, n)(x, y))}, \\
 & \text{for all } (m, n), (x, y) \in S_1 \times S_2 \\
 & \leq \max\{f^{-1}(\hat{\eta}_{C \times D})(m, n), f^{-1}(\hat{\eta}_{C \times D})(x, y)\} \\
 & \quad e^{if^{-1}(\hat{\varphi}_{C \times D})(m, n)}, f^{-1}(\hat{\varphi}_{C \times D})(x, y)\} \\
 & = \max\{f^{-1}(\hat{\eta}_{C \times D})(m, n)e^{if^{-1}(\hat{\varphi}_{C \times D})(m, n)}, \\
 & \quad f^{-1}(\hat{\eta}_{C \times D})(x, y)e^{if^{-1}(\hat{\varphi}_{C \times D})(x, y)}\} \\
 & = \max\{f^{-1}(\hat{\theta}_{C \times D})(m, n), f^{-1}(\hat{\theta}_{C \times D})(x, y)\}
 \end{aligned}$$

Consequently, $f^{-1}(\hat{\theta}_{C \times D})((m, n)(x, y)) \leq \max\{f^{-1}(\hat{\theta}_{C \times D})(m, n), f^{-1}(\hat{\theta}_{C \times D})(x, y)\}$. This concluded the proof. $\square$

6. CONCLUSION

In this paper, we have introduced direct product of CIFSR and we have proved that the direct product of two CIFSR is CIFSR. We have innovated the concept of level subset of the direct product of two CIFSR. We have also explained the some fundamental properties of the level subset of the product of two CIFSR. Moreover, we have investigated that homomorphic image (preimage) of the direct product of CIFSR under ring homomorphism.

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this paper.

AUTHORS' CONTRIBUTIONS

Conceptualization and Investigation by M. Gulzar and M. Haris Mateen; Validation, Yu-Ming Chu, D. Alghazzawi; Writing original draft, Ghazanfar Abbas; Writing review, editing and Funding, Yu-Ming Chu. All authors have read and agreed to the published version of the manuscript.

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