International Journal of Computational Intelligence Systems
Vol. 14(1), 2021, pp. 617–624
DOI: https://doi.org/10.2991/ijcis.d.210114.002; ISSN: 1875-6891; eISSN: 1875-6883
https://www.atlantis-press.com/journals/ijcis/

Research Article

On Computing Domination Set in Intuitionistic Fuzzy Graph

A. Bozhenyuk[1,*], S. Belyakov[1], M. Knyazeva[1], I. Rozenberg[2]

[1] Information and Analytical Security Systems Department, Southern Federal University, 1, Engels Str., Taganrog, 347900, Russia
[2] Public Corporation "Research and Development Institute of Railway Engineers," 27/1, Nizhegorodskaya Str., Moscow, 109029, Russia

ARTICLE INFO

Article History
Received 26 Feb 2020
Accepted 08 Jan 2021

Keywords
Intuitionistic fuzzy set
Intuitionistic fuzzy graph
Minimal intuitionistic dominating vertex subset
Domination set

ABSTRACT

In this paper, the concept of minimal intuitionistic dominating vertex subset of an intuitionistic fuzzy graph was considered, and on its basis, the notion of a domination set as an invariant of the intuitionistic fuzzy graph was introduced. A method and an algorithm for finding all minimal intuitionistic dominating vertex subset and domination set was proposed. This method is the generalization of Maghout's method for fuzzy graphs. The example of finding the domination set of the intuitionistic fuzzy graph were considered as well.

1. INTRODUCTION

Nowadays, science and technology are characterized by complex processes and phenomena for which complete information is not always available. For such cases, mathematical models of various types of systems containing elements of uncertainty have been developed. Most of these models are based on the extension of the general set theory, namely, fuzzy sets. The concept of fuzzy sets was introduced by Zadeh [1] as a method of representing uncertainty and fuzziness. Since then, the theory of fuzzy sets has become an area of research in various disciplines.

In 1983, Atanassov [2] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. He introduced a new component to the definition of a fuzzy set, which determines the degree of nonmembership. Fuzzy sets consider the membership degree of an element in a given set (where the degree of nonmembership equals one minus the membership degree), while intuitionistic fuzzy sets operate both with a membership degree and a degree of nonmembership that are more or less independent of each other. The only restriction is that the sum of these two degrees does not exceed 1. Intuitionistic fuzzy sets are fuzzy sets of a higher order. Their application makes the solution procedure more complicated, but if the computational complexity or memory can be neglected, then a better result can be achieved.

The fuzzy graphs theory is finding an increasing number of applications for modeling real-time systems, where the level of information inherent in the system depends on different levels of accuracy. The original definition of a fuzzy graph [3] was based on fuzzy relations by Zadeh [4]. In article [5], fuzzy analogs of several basic graphical concepts were presented. In articles [6,7], the notion of fuzzy graph complement was defined, and some operations on fuzzy graphs were studied. The concepts of intuitionistic fuzzy relations and intuitionistic fuzzy graphs were introduced in articles [8,9], and some of their properties were investigated. In articles [10–12], the concepts of a dominating set, a regular independent set, a domination edge number of edges in intuitionistic fuzzy graphs were considered.

Different types of intuitionistic fuzzy graphs were considered in literature in order to cope with a diversity of practical cases: intuitionistic fuzzy competition graphs, intuitionistic fuzzy neighborhood graphs, intuitionistic fuzzy rough graphs, and others [13,14] were introduced to analyze the ecosystems and to represent the relations of competition among the species in the food web.

Some properties and features of intuitionistic fuzzy graphs were considered [15], for example, edge irregular intuitionistic fuzzy graphs, edge totally irregular intuitionistic fuzzy graphs, and others are introduced and the intuitionistic fuzzy genus graph with its genus value, strong and weak intuitionistic fuzzy genus graph are defined, as well as the isomorphism properties on intuitionistic fuzzy genus graph are discussed [16].

Some new operations on intuitionistic fuzzy graphs namely normal product and tensor product were introduced as well as the degree of the intuitionistic fuzzy graphs obtained by the operations Cartesian product, tensor product, normal product, and composition of intuitionistic fuzzy graphs [17]. Finally covering and paired domination in intuitionistic fuzzy graphs was investigates to meet different practical problems [18].

* Corresponding author. Email: avb002@yandex.ru

The idea of double dominating set in the intuitionistic fuzzy graph as well as lower- and upper-domination number was studied in the paper [19] and the notion of strong intuitionistic fuzzy graphs and intuitionistic fuzzy line graphs [20] were discussed.

Many practical results of applications of intuitionistic fuzzy graphs and hypergraphs were achieved in various fields of decision-making [21,22], support systems [23,24], operations research problems, for example, allocation problems, covering problems, identification of the best location, and modeling fuzzy relations between elements in graphs [25].

All the investigations show great importance and practical relevance of intuitionistic fuzzy graphs in modeling system architectures, relations between elements and data structures, solving decision-making problems [26].

However, when modeling various complex systems and processes by intuitionistic graphs, sometimes it is sufficient to use graphs that are "simpler" in their structure. Namely, the vertices of the graph are crisp and the edges are intuitionistic. Such graphs [27,28] were called intuitionistic graphs of the first kind. For example, when reflecting complex interactions and relationships between elements in a geographic information system (GIS), an intuitionistic fuzzy graph of the first kind formalizes fuzzy relations and connections between crisp objects, so that objects themselves remain static and unchanged [29,30]. The practical relevance of such an approach can be interpreted as a problem of optimal service centers allocation (finding domination set in the intuitionistic fuzzy graph) within the railway network transportation system.

Contributions. The major contributions of this paper are summarized as follows:

- Newly defined concepts termed minimal intuitionistic dominating vertex subset and domination set of the intuitionistic fuzzy graph of the first kind are formalized. These concepts are a generalization of the minimal dominating vertex subsets of a crisp graph [31] and the domination set of a fuzzy graph [32], respectively. Furthermore, a novel problem of mining of all minimal intuitionistic dominating vertex subsets from such graph is formalized.
- To address the proposed problem, a formal method for finding the minimal intuitionistic dominating vertex subset is presented. In addition, it has been proven that this method gives an exact solution (finds all minimal intuitionistic dominating vertex subsets).
- Based on the proposed method, an algorithm was developed that allows determining the domination set of the intuitionistic fuzzy graph.

The domination set and minimal intuitionistic dominating vertex subsets allow one to directly solve some optimization problems on an intuitionistic fuzzy graph. In particular, if the problem of locating centers at the vertices of an intuitionistic fuzzy graph is considered, then the domination set directly solves the problem of finding the number of centers with a given degree, and the minimal intuitionistic dominating vertex subsets allow solving the problem of optimal placement of a given number of centers with the highest degree.

2. BASIC CONCEPTS AND DEFINITIONS

Definition 1. [1] Let X be a nonempty set. A fuzzy set drawn A from X is defined as $A = \{\langle \mu_A(x), x\rangle | x \in X\}$, where $\mu_A : X \to [0,1]$ is the membership function of the fuzzy set A. A fuzzy set is a collection of objects with graded membership, that is, having degrees of membership.

Definition 2. [33] Let X be a nonempty set. An intuitionistic fuzzy set A in X is an object having the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\},$$

where the functions $\mu_A(x), \nu_A(x) : X \to [0,1]$ define respectively, the degree of membership and degree of nonmembership of the element $x \in X$ to the set A, which is a subset of X, and

$$(\forall x \in X)[\mu_A(x) + \nu_A(x) \leqslant 1].$$

Furthermore, value $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the intuitionistic fuzzy set index or hesitation margin of x in A. $\pi_A(x)$ is the degree of indeterminacy of x to the intuitionistic fuzzy set.

The intuitionistic fuzzy relation R on the set $X \times Y$ is an intuitionistic fuzzy set of the form

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle | (x, y) \in X \times Y\},$$

here $\mu_R : X \times Y \to [0,1]$ and $\nu_R : X \times Y \to [0,1]$.

The intuitionistic fuzzy relation R satisfies the condition

$$(\forall x, y \in X \times Y)[\mu_R(x, y) + \nu_R(x, y) \leqslant 1].$$

Definition 3. Let p and q be intuitionistic fuzzy variables that have the form $p = (\mu(p), \nu(p))$, $q = (\mu(q), \nu(q))$, here $\mu(p) + \nu(p) \leqslant 1$, and $\mu(q) + \nu(q) \leqslant 1$. Then the operations & and $\vee$ are defined as [34]

$$p \& q = (min(\mu(p), \mu(q)), max(\nu(p), \nu(q))), \quad (1)$$

$$p \vee q = (max(\mu(p), \mu(q)), min(\nu(p), \nu(q))). \quad (2)$$

We assume that $p < q$ if $\mu(p) < \mu(q)$ and $\nu(p) > \nu(p)$.

Definition 4. [5] A fuzzy graph is a triplet $\tilde{G} = (V, \sigma, \mu)$, where V is finite and nonempty vertex set, $\sigma : V \to [0,1]$ is a fuzzy subset of V, and $\mu : V \times V \to [0,1]$ is fuzzy relation on $X \times X$ such that

$$(\forall x, y \in V)[\mu(x, y) \leqslant min(\sigma(x), \sigma(y))].$$

This definition considers a fuzzy graph as a collection of fuzzy vertices and fuzzy edges. Another version of a fuzzy graph was proposed in [35,36] as a set of crisp vertices and fuzzy edges.

Definition 5. [35] A fuzzy graph is a pair $\tilde{G} = (V, R)$, where V is a crisp set of vertices and R is a fuzzy relation on V, in which the elements (edges) connecting the vertices V, have the membership function $\mu_R : V \times V \to [0,1]$.

Such a fuzzy graph in article [36] was called a fuzzy graph of the *first kind*.

Definition 6. [8,9] An intuitionistic fuzzy graph is a pair $\tilde{G} = (A, B)$, where $A = \langle V, \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy set on the set of vertices V, and $B = \langle V \times V, \mu_B, \nu_B \rangle$ is an intuitionistic fuzzy relation such that

$$\mu_B(x,y) \leqslant min(\mu_A(x), \mu_A(y)), \quad (3)$$
$$\nu_B(x,y) \leqslant max(\nu_A(x), \nu_A(y)).$$

and the following condition is fulfilled:

$$(\forall x, y \in V)[0 \leqslant \mu_B(x,y) + \nu_B(x,y) \leqslant 1].$$

It should be noted that Definition 5 is an extension of a fuzzy graph in the sense of Definition 3, in which the vertices and edges of the graph are considered not as fuzzy, but as intuitionistic sets. In the case of using a fuzzy graph in the sense of Definition 4, such a definition of an intuitionistic fuzzy graph does not make sense, since in the latter case the values $\mu_A(x) = \mu_A(y) = 1$, $\nu_A(x) = \nu_A(y) = 0$, and therefore, the quantity $\nu_B(x,y) = 0$. In this regard, we introduce the following definition:

Definition 7. An intuitionistic fuzzy graph of the first kind is a pair $\tilde{G} = (V, U)$, where V is a crisp set of vertices, $U = \langle V \times V, \mu, \nu \rangle$ is intuitionistic fuzzy relation (intuitionistic fuzzy edges) such that

$$(\forall x, y \in V)[0 \leqslant \mu(x,y) + \nu(x,y) \leqslant 1].$$

3. DOMINATION SET OF INTUITIONISTIC FUZZY GRAPH

Let $\tilde{G} = (V, U)$ be an intuitionistic fuzzy graph of the first kind. Let $p(x,y) = (\mu(x,y), \nu(x,y))$ be an intuitionistic fuzzy variable that determines the degree of adjacency and the degree of nonadjacency from vertex x to vertex y. Let X is an arbitrary subset of the vertices set V. For each vertex $y \in V \setminus X$, we define the volume

$$p_X(y) = \bigvee_{x \in X} p(x,y). \quad (4)$$

Definition 8. [27] We call the set X an *intuitionistic dominating vertex set for vertex y* with the intuitionistic degree of domination $p_X(y)$.

Definition 9. We call the set X an *intuitionistic dominating vertex set* for graph $\tilde{G}$ with the intuitionistic degree of domination:

$$\beta(X) = \underset{y \in V \setminus X}{\&} p_x(y) = \underset{y \in V \setminus X}{\&} \bigvee_{x \in X} p(x,y). \quad (5)$$

The operations & and $\vee$ are defined by (1) and (2) in expressions (4) and (5). Assuming that $(\forall y \in V)[p(y,y) = (1,0)]$, expression (5) can be rewritten as

$$\beta(X) = \underset{y \in V \setminus X}{\&} p_x(y) = \underset{y \in V}{\&} \bigvee_{x \in X} p(x,y). \quad (6)$$

Intuitionistic degree of domination $\beta(X) = (\mu(X), \nu(X))$ means that there is some vertex in subset $X \subseteq V$ that is adjacent to any other vertex of the graph with degree at least $\mu(X)$, and there is some vertex that is not adjacent to any vertex of the graph with degree to not more than $\nu(X)$.

Example 1. For the intuitionistic fuzzy graph $\tilde{G} = (V, U)$, and the subset $X = \{x_1, x_2\}$, shown in Figure 1, we define the values $p_X(x_3) = (0.5, 0.3)$, $p_X(x_4) = (0.5, 0.2)$, $p_X(x_5) = (0.6, 0.2)$. Consequently, intuitionistic degree of domination $\beta(X) = (0.5, 0.3)$.

Remark 1. If the graph $\tilde{G}$ is crisp, then the value $p(x,y) = (1,0)$, if the vertex y is adjacent to the vertex x, and $p(x,y) = (0,1)$ otherwise.

Remark 2. If the graph $\tilde{G}$ is crisp, then the value $\beta(X) = (1,0)$, if subset $X \subseteq V$ is a dominating subset of crisp graph [31], and $\beta(X) = (0,1)$ otherwise.

Definition 10. We call the subset $X \subseteq V$ a *minimal intuitionistic dominating vertex subset* with the degree $\beta(X)$, if the condition $\beta(X') < \beta(X)$ is true for any subset $X' \subseteq X$.

Example 2. For the intuitionistic fuzzy graph presented in Figure 1, the minimal intuitionistic dominating vertex subsets are $X_1 = \{x_2\}$ with $\beta(X_1) = (0.3, 0.3)$, and $X_2 = \{x_1, x_2\}$ with $\beta(X_2) = (0.5, 0.3)$.

Denote by $Y_k = \{X_{k1}, X_{k2}, \dots, X_{kl}\}$ the family of all minimal intuitionistic dominating vertex subsets with k vertices and degrees of domination $\beta_{k1}, \beta_{k2}, \dots, \beta_{kl}$ respectively. Let's $\beta_k^0 = \beta_{k1} \vee \beta_{k2} \vee \dots \vee \beta_{kl}$. It means that there is a minimal intuitionistic dominating vertex subset with k vertices with a degree of domination β_k^0 in the graph $\tilde{G}$ and there is no other intuitionistic dominating vertex subset with k vertices whose degree of domination would be greater than β_k^0.

Definition 11. An intuitionistic fuzzy set

$$\tilde{D} = \{\langle \beta_1^0/1 \rangle, \langle \beta_2^0/2 \rangle, \dots, \langle \beta_n^0/n \rangle\}$$

is called a *domination set* of graph $\tilde{G}$.

The domination set is an invariant of intuitionistic fuzzy graph $\tilde{G}$, since it does not change during the structural transformations of the graph.

Example 3. For the intuitionistic fuzzy graph presented in Figure 1, the domination set is

$$\tilde{D} = \{\langle (0.3, 0.3)/1 \rangle, \langle (0.5, 0.3)/2 \rangle, \langle (0.5, 0.3)/3 \rangle,$$
$$\langle (0.6, 0.2)/4 \rangle, \langle (1, 0)/5 \rangle\}.$$

Remark 3. The concept of domination set was introduced for the intuitionistic fuzzy graph of the first kind. However, taking into

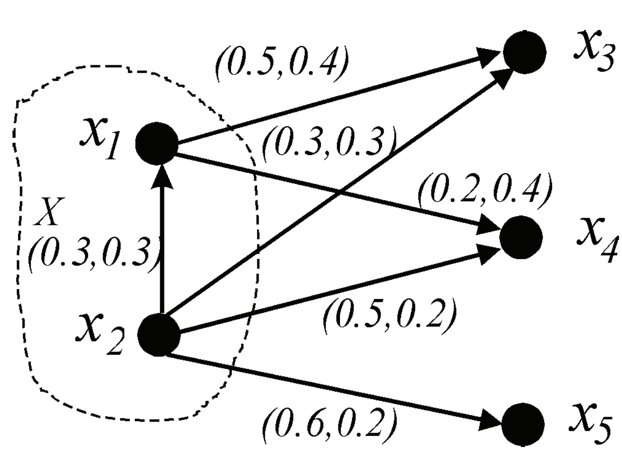

Figure 1 | Intuitionistic fuzzy graph, $X = \{x_1, x_2\}$

account inequalities (3), it also holds for the intuitionistic fuzzy graph of a general form (in the sense of Definition 6).

Property 1. For intuitionistic dominating set the following proposition is true:

$$(0,1) \leqslant \beta_1^0 \leqslant \beta_2^0 \leqslant, \dots, \leqslant \beta_n^0 = (1,0).$$

4. METHOD AND ALGORITHM FOR FINDING DOMINATION SET

We will consider the method of finding a family of all minimal intuitionistic dominating vertex subsets. The given method is similar to Maghout's method for the definition of all minimal fuzzy dominating vertex sets [32] for fuzzy graphs.

Let us assume that set X_β is an intuitionistic dominating vertex subset of the graph $\tilde{G}$ with the degree of domination $\beta = (\mu_\beta, \nu_\beta)$. Then for an arbitrary vertex $x_i \in V$, one of the following conditions must be true:

(a) $x_i \in X_\beta$.

(b) if $x_i \notin X_\beta$, then there is a vertex x_j so that it belongs to set X_β, while the vertex x_j is adjacent to vertex x_i with the degree $(\mu(x_j, x_i), \nu(x_j, x_i)) \geqslant \beta$.

In other words, the following statement is true:

$$(\forall x_i \in V)[x_i \in X_\beta \vee (\exists x_j \in X_\beta | \mu(x_j, x_i) \geqslant \mu_\beta \& \nu(x_j, x_i) \leqslant \nu_\beta)]. \tag{7}$$

To each vertex $x_i \in V$ we assign Boolean variable p_i that takes value 1, if $x_i \in X_\beta$ and 0 otherwise. We assign the intuitionistic variable $\xi_{ji} = (\mu_\beta, \nu_\beta)$ for the proposition $(\mu(x_j, x_i), \nu(x_j, x_i)) \geqslant (\mu_\beta, \nu_\beta)$. Passing from the quantifier form of proposition (7) to the form in terms of logical operations, we obtain a true logical proposition:

$$\Phi_D = \underset{i=\overline{1,n}}{\&} \left(p_i \vee \bigvee_{j=\overline{1,n}} (p_j \& \xi_{ji}) \right)$$

Here, $n = |V|$. Supposing $\xi_{ii} = (1,0)$ and considering that the equality $p_i V \bigvee_j (p_j \& \xi_{ji} = \bigvee_j (p_j \& \xi_{ji}$ is true for any vertex x_j, we finally obtain

$$\Phi_D = \underset{i=\overline{1,n}}{\&} \left(\bigvee_{j=\overline{1,n}} (p_j \& \xi_{ji}) \right). \tag{8}$$

We open the parentheses in the expression (8) and reduce the similar terms by following rules:

$$a \vee a\&b = a; a\&b \vee a\&\overline{b} = a; \\ (\xi_1 \geqslant \xi_2) \to (\xi_1 \& a \vee \xi_2 \& a \& b = \xi_1 \& a). \tag{9}$$

Here, $a, b \in \{0,1\}$ and $\xi_1, \xi_2 \in [(0,1),(1,0)]$.

Then the expression (8) may be presented as

$$\Phi_D = \bigvee_{i=\overline{1,l}} (p_{1i} \& p_{2i} \& \dots \& p_{ki} \& \beta_i). \tag{10}$$

We can prove the next property:

Property 2. Each disjunctive member in the expression (8) gives a minimum intuitionistic dominating vertex subset with the degree β_i.

Proof. Let's consider that further simplification is impossible in expression (10). Let, for definiteness, disjunctive member

$$(p_1 \& p_2 \& \dots \& p_k \& \beta) \tag{11}$$

is included in the expression (10). Here, $k < n$ and $(0,1) < \beta \leqslant (1,0)$.

We rewrite (8) as

$$\begin{aligned} \Phi_D = {} & ((1,0)p_1 \vee \xi_{2,1} p_2 \vee \dots \vee \xi_{n,1} p_n) \& \\ & \& (\xi_{1,2} p_1 \vee (1,0) p_2 \vee \dots \vee \xi_{n,2} p_n) \& \\ & \& \dots \& \\ & \& (\xi_{1,n} p_1 \vee \xi_{2,n} p_2 \vee \dots \vee (1,0) p_n \end{aligned} \tag{12}$$

Then in expression (12) the following statement should be fulfilled:

$$(\forall i = \overline{1,k})[\xi_{i,k+1} < \beta].$$

Therefore, all disjunctive members which do not contain variables $p_{k+1}, p_{k+2}, \dots, p_n$, necessarily contain coefficients of the smaller value β in expression (10). From there, the disjunctive member (11) is not included in the expression (10). The received contradiction proves that subset $X_\beta = \{x_1, x_2, \dots, x_k\}$ has degree β.

We now show that the disjunctive member (11) is the minimum member. We will assume the opposite. Then should be performed condition (a) or condition (b):

(a) There exists a vertex $x \in X_\beta$ such that $p(x, y) > \beta$ holds for any vertex $y \in V \setminus X_\beta$.

(b) There is a subset $X' \subset X_\beta$ such that for any vertex $y \in V \setminus X'$ there exists a vertex $x \in X'$ such that $p(x, y) = \beta$.

Let the condition (a) is satisfied. Then the next statement is true:

$$(\forall y \in V \setminus X_\beta)(\exists x \in X_\beta)[p(x, y) = \beta' > \beta].$$

Let's present expression Φ_D in the form (12). If logic multiplication of each bracket against each other is performed without rules of absorption (9) we receive n^2 disjunctive members. Moreover, each member contains exactly n elements, where each element comes from a separate bracket of decomposition (12). We will choose one of n^2 disjunctive members as follows:

- element $(1,0)p_1$ is selected from the first bracket;
- element $(1,0)p_2$ is selected from the second bracket;
- etc.;
- element $(1,0)p_k$ is selected from the bracket k;
- from the bracket $(k+1)$ we will select element $\xi_{i_1,k+1} \& p_{i_1}$ such, that index $i_1 \in [1,k]$, and $\xi_{i_1,k+1} \geqslant \beta'$;

- from the bracket $(k+2)$ we will select element $\xi_{i_2,k+2}\&p_{i_2}$ such, that index $i_2 \in [1,k]$, and $\xi_{i_2,k+2} \geqslant \beta'$;
- etc.;
- from the bracket n we will select element $\xi_{i_{n-k},n}\&p_{i_{n-k}}$ such, that index $i_{n-k} \in [1,k]$, and $\xi_{i_{n-k},n} \geqslant \beta'$.

Using rules of absorption (9), the resulting disjunctive member can be represented as $(p_1\&p_2\&\dots\&p_k\&\beta')$, where $\beta' = \xi_{i_1,k+1}\&\xi_{i_2,k+2}\&\dots\&\xi_{i_{n-k},n} > \beta$ and which will be necessarily absorbed by a disjunctive member (11).

We obtained a contradiction, which proves the impossibility of case (a).

Now suppose that condition (b) is satisfied. Let's for definiteness $X' = \{x_1, x_2, \dots, x_{k-1}\}$. Considering expression Φ_D in the form (12), we will choose a disjunctive member as follows:

- element $(1,0)p_1$ is selected from the first bracket;
- element $(1,0)p_2$ is selected from the second bracket;
- etc.;
- element $(1,0)p_{k-1}$ is selected from the $(k-1)$ bracket;
- from the bracket (k) we will select element $\xi_{i_1,k}\&p_{i_1}$ such, that index $i_1 \in [1,k-1]$, and $\xi_{i_1,k+1} \geqslant \beta$;
- from the bracket $(k+1)$ we will select element $\xi_{i_2,k+1}\&p_{i_2}$ such, that index $i_2 \in [1,k-1]$, and $\xi_{i_2,k} \geqslant \beta$;
- etc.;
- from the bracket n we will select element $\xi_{i_{n-k+1},n}\&p_{i_{n-k+1}}$ such, that index $i_{n-k+1} \in [1,k-1]$, and $\xi_{i_{n-k+1},n} \geqslant \beta$.

Using rules of absorption (9), the resulting disjunctive member can be represented as $(p_1\&p_2\&\dots\&p_{k-1}\&\beta')$, where $\beta' = \xi_{i_1,k}\&\xi_{i_2,k+1}\&\dots\&\xi_{i_{n-k+1},n} \geqslant \beta$ and which will be necessarily absorbed by a disjunctive member (11). We obtained a contradiction, which proves the impossibility of case (b).

Hence, Property 2 is proved.

The following algorithm for finding of intuitionistic dominating set may be proposed on the base of Property 2:

- We write proposition (8) for given intuitionistic fuzzy graph $\tilde{G}$.
- We simplify proposition (8) by proposition (9) and present it as a proposition (10).
- We define all minimum intuitionistic dominating vertex subsets, which correspond to the disjunctive members of proposition (10).
- We define the domination set of graph $\tilde{G}$.

To construct the expression (10) we rewrite the expression (8) as follows:

$$\Phi_D = \underset{i=\overline{1,n}}{\&}(a_{i1}p_1 \vee a_{i2}p_2 \vee \dots \vee a_{in}p_n). \tag{13}$$

We convert pair $a_{ij}\&p_j$ from expression (13) to weighted binary vector $a_{ij}\&\overline{P}_j$. Here $\overline{P}_j = ||p_i^{(j)}||$ is a binary vector that has a dimension of n. The elements of vector $\overline{P}_j$ are defined as

$$p_i^{(j)} = \begin{cases} 1 & \text{if } i = j\,, \\ 0 & \text{if } i \neq j\,. \end{cases}$$

The conjunction of (a_1p_1) and (a_2p_2) from expression (13) corresponds the conjunction of two weighted binary vectors $a_1\overline{P}_1$ and $a_2\overline{P}_2$, $\overline{P}_1 = ||p_i^{(1)}||$, $\overline{P}_2 = ||p_i^{(2)}||$, $i = \overline{1,n}$, $a_1, a_2 \in [(0,1),(1,0)]$. In a vector space the conjunction is defined as $a_1\overline{P}_1\&a_2\overline{P}_2 = a\overline{P}$, where $a = \min\{a_1, a_2\}$, $\overline{P} = ||p_i||$, $p_i = \max\{p_i^{(1)}, p_i^{(2)}\}$, $i = \overline{1,n}$.

We define the operation $\leqslant$ "less or equal" between binary vectors. Binary vector $\overline{P}_1$ is less or equal than $\overline{P}_2$ if and only if each element of $\overline{P}_1$ is less or equal than the corresponding element of vector $\overline{P}_2$. Or

$$(\overline{P}_1 \leqslant \overline{P}_2) \Leftrightarrow (\forall i = \overline{1,n})[p_i^{(1)} \leqslant p_i^{(2)}].$$

Considering the algebra in space of weighted binary vectors, we can make a rule of absorption:

$$(a_1 \geqslant a_2\&\overline{P}_1 \leqslant \overline{P}_2) \Rightarrow a_1\overline{P}_1 \vee a_2\overline{P}_2 = a_1\overline{P}_1. \tag{14}$$

Now we can construct statement (10) using the conjunction operation and the rule of absorption of weighted binary vectors by Algorithm 1.

The idea of Algorithm 1 is as follows. Parentheses set of expression (13) is passed as an input. Each element of parentheses is converted to a weighted binary vector. In a loop, the vectors are multiplied together, according to the Conjunction Function (Algorithm 2), where absorption rule 14 is applied. The temporary result vector replaces the 1st vector and is used in the next iteration. In the end, the final result is taken from the 1st vector, which will determine the expression (10). That is, each element of the 1st vector defines a minimum intuitionistic dominating vertex subset with a calculated degree.

The time complexity of an algorithm is measured by the number of successive steps for its execution and is denoted by T. A step can be interpreted as an operation of binary comparison of elements.

All operations of Algorithm 2 require $N_j \times M_j \times (n+2)$ comparisons. Here N_j is the number of elements in the vector $\overline{V}_1$; M_j is the number of compared elements in the vector $\overline{V}_2$ at the j-th step of calling $(j = \overline{2,n})$ Algorithm 2 from Algorithm 1; $(n+2)$ is the number of comparisons in each cycle of Algorithm 2; n is the number of vertices in the graph $\tilde{G}$.

Let $\rho(x)$ be the half-degree of entry of the vertex $x \in V$ (the number of edges that "enter" the vertex x). Let us denote by $\rho_{max} = \max_{x \in V} \rho(x)$. If we estimate the complexity of the algorithm "from above," assuming that there will be no absorptions in Algorithm 2, then we get: $N_2 \leqslant \rho_{max}$, $N_3 \leqslant \rho_{max}^2$, ..., $N_n \leqslant \rho_{max}^{n-1}$, $(\forall j = \overline{2,n})M_j \leqslant \rho_{max}$.

This implies $T \leqslant (\rho_{max}^2 + \rho_{max}^3 + \rho_{max}^n) \times (n+2) \times \tau$, where τ is the binary comparison time. As the result, the estimation of the

Algorithm 1: Expression (10) minimization

Data: bracketed expressions from (13)
Result: minimized expression (10)

$\overline{V}_1 \leftarrow ||v_i^{(1)}||, i = \overline{1,n}$ `/* Each element of the first bracketed expression is converted to weighted binary vector. */;`
$j \leftarrow 2$;
while $j \leqslant n$ **do**
  $\overline{V}_2 \leftarrow ||v_i^{(j)}||, i = \overline{1,n}$ `/* Each element of the bracketed expression j is converted to weighted binary vector. */;`
  $\overline{V}_1 \leftarrow$ `Conjunction(`$\overline{V}_1$, $\overline{V}_2$`)` ;
  $j \leftarrow j+1$;
end
Build expression (10) from binary vectors in $\overline{V}_1$;

time complexity of the considered algorithm can be represented as $O(n\rho_{max}^n)$.

This way we have all minimal intuitionistic dominating vertex subsets of graph $\tilde{G}$.

5. NUMERICAL EXAMPLE

Let's consider an example of finding the domination set of graph $\tilde{G}$ shown in Figure 2.

The adjacent matrix of intuitionistic fuzzy graph $\tilde{G}$ looks like this:

$$R_X = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} (1,0) & (0,1) & (0.8,0.1) & (0.5,0.4) & (0,1) & (0,1) \\ (0.5,0.5) & (1,0) & (0.3,0.5) & (0,1) & (0,1) & (0,1) \\ (0,1) & (0.4,0.4) & (1,0) & (0,1) & (0,1) & (0.3,0.6) \\ (0.6,0.3) & (0,1) & (0,1) & (1,0) & (0,1) & (0,1) \\ (0,1) & (0.7,0.2) & (0,1) & (0,1) & (1,0) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0,1) & (0.6,0.3) & (1,0) \end{pmatrix} \end{array}$$

The corresponding expression (8) for this graph has the following form:

$$\begin{aligned} \Phi_D = {} & [(1,0)p_1 \vee (0.5,0.5)p_2 \vee (0.6,0.3)p_4]\& \\ & \&[(1,0)p_2 \vee (0.4,0.4)p_3 \vee (0.7,0.2)p_5]\& \\ & \&[(0.8,0.1)p_1 \vee (0.3,0.5)p_2 \vee (1,0)p_3]\& \\ & \&[(0.5,0.4)p_1 \vee (1,0)p_4]\& \\ & \&[(1,0)p_5 \vee (0.6,0.3)p_6]\& \\ & \&[(0.3,0.6)p_3 \vee (1,0)p_6]. \end{aligned}$$

Vectors $\overline{V}_1$ and $\overline{V}_2$ have the following form before the first iteration:

Algorithm 2: Function for vector conjunction

Function `Conjunction(`$\overline{V}_1$, $\overline{V}_2$`)`
  $\overline{V}_{res} \leftarrow []$;
  $N \leftarrow length(\overline{V}_1)$;
  $M \leftarrow length(\overline{V}_2)$;
  for $i = 1$ **to** N **do**
    for $j = 1$ **to** M **do**
      $\mu^1, \nu^1, p^1 \leftarrow \overline{V}_1[i]$;
      $\mu^2, \nu^2, p^2 \leftarrow \overline{V}_2[j]$;
      if $\mu^2 = 0$ **then**
        continue ;
      end
      `/* New element is a multiplication of` $\overline{V}_1[i]$ `and` $\overline{V}_2[j]$ `*/ ;`
      $\mu_{new} \leftarrow \min(\mu^1, \mu^2)$;
      $\nu_{new} \leftarrow \max(\nu^1, \nu^2)$;
      $p_{new} \leftarrow p^1 \vee p^2$;
      $K \leftarrow length(V_{res})$;
      $\overline{V}_{buf} \leftarrow [(\mu_{new}, \nu_{new}, p_{new})]$;
      for $k = 1$ **to** K **do**
        $\mu*, \nu*, p* \leftarrow \overline{V}_{res}[k]$;
        if $\mu* \geqslant \mu_{new} \& \nu* \leqslant \nu_{new} \& p* \leqslant p_{new}$ **then** `/* new element is absorbed according to rule (14) */`
          $\overline{V}_{buf} \leftarrow \overline{V}_{res}$;
          break ;
        end
        if $\mu* \leqslant \mu_{new} \& \nu* \geqslant \nu_{new} \& p* \geqslant p_{new}$ **then** `/* element` $\overline{V}_{res}[k]$ `is absorbed according to rule (14) */`
          continue ;
        end
        $\overline{V}_{buf}.append(\overline{V}_{res}[k])$;
      end
      $\overline{V}_{res} \leftarrow \overline{V}_{buf}$;
    end
  end
  return $\overline{V}_{res}$
end

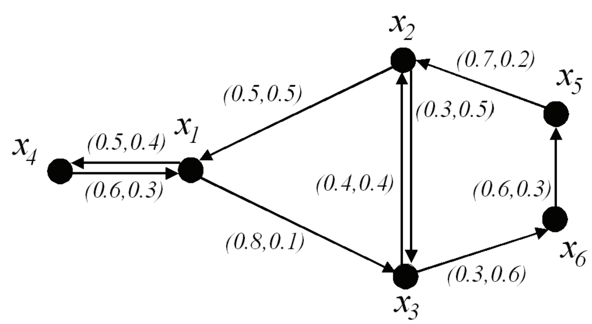

Figure 2 Intuitionistic fuzzy graph $\tilde{G}$.

$$\overline{V}_1 = \left| \begin{array}{c} (1.0, 0.0)(100000) \\ (0.5, 0.5)(010000) \\ (0.0, 1.0)(001000) \\ (0.6, 0.3)(000100) \\ (0.0, 1.0)(000010) \\ (0.0, 1.0)(000001) \end{array} \right|,$$

$$\overline{V}_2 = \left| \begin{array}{c} (0.0, 1.0)(100000) \\ (1.0, 0.0)(010000) \\ (0.4, 0.4)(001000) \\ (0.0, 1.0)(000100) \\ (0.7, 0.2)(000010) \\ (0.0, 1.0)(000001) \end{array} \right|.$$

After the first iteration of the algorithm, vectors $\overline{V}_1$ and $\overline{V}_2$ have the following forms:

$$\overline{V}_1 = \left| \begin{array}{c} (1.0, 0.0)(110000) \\ (0.4, 0.4)(101000) \\ (0.7, 0.2)(100010) \\ (0.5, 0.5)(010000) \\ (0.6, 0.3)(010100) \\ (0.4, 0.4)(001100) \\ (0.6, 0.3)(000110) \end{array} \right|,$$

$$\overline{V}_2 = \left| \begin{array}{c} (0.8, 0.1)(100000) \\ (0.3, 0.5)(010000) \\ (1.0, 0.0)(001000) \\ (0.0, 0.0)(000100) \\ (0.0, 0.0)(000010) \\ (0.0, 0.0)(000001) \end{array} \right|.$$

After completing the iterations, finally we have

$$\overline{V}_1 = \left| \begin{array}{c} (1.0, 0.0)(111111) \\ (0.8, 0.1)(110111) \\ (0.7, 0.2)(100111) \\ (0.6, 0.3)(011101) \\ (0.6, 0.3)(110101) \\ (0.6, 0.3)(001111) \\ (0.5, 0.4)(110001) \\ (0.3, 0.6)(101010) \\ (0.4, 0.4)(101001) \\ (0.5, 0.4)(100011) \\ (0.5, 0.5)(010101) \\ (0.4, 0.4)(001101) \end{array} \right|.$$

So, the formula (10) for this graph has the form

$$\begin{aligned} \Phi_D = {} & \underline{(0.5, 0.4)p_1p_2p_6} \vee (0.3, 0.6)p_1p_3p_5 \vee \\ & \vee (0.4, 0.4)p_1p_3p_6 \vee (0.5, 0.4)p_1p_5p_6 \vee \\ & \vee (0.4, 0.4)p_3p_4p_6 \vee (0.6, 0.3)p_1p_2p_4p_6 \vee \\ & \vee \underline{(0.7, 0.2)p_1p_4p_5p_6} \vee (0.6, 0.3)p_2p_3p_4p_6 \vee \\ & \vee (0.6, 0.3)p_3p_4p_5p_6 \vee \underline{(0.8, 0.1)p_1p_2p_4p_5p_6} \vee \\ & \vee (0.5, 0.5)p_2p_4p_6 \vee \underline{(1, 0)p_1p_2p_3p_4p_5p_6} \end{aligned}$$

It follows from the last equality that graph $\tilde{G}$ has 12 minimal intuitionistic dominating vertex subsets, and the domination set is defined as

$$\begin{aligned} \tilde{D} = {} & \{\langle (0.5, 0.4)/3\rangle, \langle (0.7, 0.2)/4\rangle, \langle (0.8, 0.1)/5\rangle, \\ & \langle (1, 0)/6\rangle\}. \end{aligned}$$

This domination set shows, in particular, that there is a subset in the graph ($X = \{x_1, x_4, x_5, x_6\}$), consisting of 4 vertices such that all other vertices of the graph ($V \setminus X = \{x_2, x_3\}$) are adjacent to at least one vertex of the subset X with the degree at least 0.7, and not adjacent with the degree at most 0.2.

6. CONCLUSION

In this paper, we considered the concepts of fuzzy minimal intuitionistic dominating vertex subsets and a domination set of the first kind intuitionistic fuzzy graph. The method for finding all fuzzy minimal dominating vertex subsets was developed and it has been proven that this method gives an exact solution. This method is the generalization of Maghout's method for a fuzzy graph. The domination set allows estimating any intuitionistic fuzzy graph from the point of view of its invariants. It should be noted that the considered method is a method of complete ordered search since such problems can be reduced to coverage problems, that is, these problems are NP compete. However, the proposed method can be effectively used for intuitionistic fuzzy graphs without a large number of vertices.

In future research, the considered method and algorithm can be used to find other invariants, in particular, externally stable set, intuitionistic antibase, and coloring of the intuitionistic fuzzy graphs.

CONFLICTS OF INTEREST

The authors have no conflicts of interest.

AUTHORS' CONTRIBUTIONS

A. Bozhenyuk - Statement of the problem, idea of the solution method; S. Belyakov - development of an algorithm, estimation of the complexity of the algorithm; M. Knyazeva - Literature review, solution example; I. Rosenberg - Proof of the solution method.

ACKNOWLEDGMENTS

The reported study was funded by the Russian Foundation for Basic Research according to the research project N18-01-00023 and project N20-01-00197.

REFERENCES

[1] L.A. Zadeh, Fuzzy sets, Inf. Control. 8 (1965), 338–353.
[2] K.T. Atanassov, Intuitionistic fuzzy sets, in: VII ITKR's Session, Sofia, Central Science and Technical Library, Bulgarian Academy of Sciences, 1697/84, In: Sgurev, V. (ed.), Bulgarian Academy of Sciences, Sofia; 1983.
[3] A. Kaufmann, Introduction a la theorie des sous-ensemles flous, Masson, Paris, France, 1977.
[4] L.A. Zadeh, Similarity relations and fuzzy orderings, Inf. Sci. 3 (1971), 177–200.
[5] A. Rosenfeld, Fuzzy graphs, in: L. A. Zadeh, K. S. Fu, M. Shimura (Eds.), Fuzzy Sets and Their Applications, Academic Press, New York, NY, USA, 1975, pp. 77–95.
[6] J.N. Mordeson, P.S. Nair, Fuzzy Graphs and Fuzzy Hypergraphs, Springer, Heidelberg, Germany, 2000.
[7] M.S. Sunitha, A.V. Kumar, Complement of a fuzzy graph, Indian J. Pure Appl. Math. 33 (2002), 1451–1464.
[8] A. Shannon, K.T. Atanassov, A first step to a theory of the intuitionistic fuzzy graphs, in: D. Lakov (Ed.), Proceeding of the FUBEST, Bulgarian Academy of Sciences, Sofia, Bulgaria, 1994, pp. 59–61.
[9] A. Shannon, K.T. Atanassov, Intuitionistic fuzzy graphs from α-, β- and (α, β)-levels, Notes Intuit. Fuzzy Sets. 1 (1995), 32–35.
[10] M.G. Karunambigai, S. Sivasankar, K. Palanivel, Different types of domination in intuitionistic fuzzy graph, Ann. Pure Appl. Math. 14 (2017), 87–101.
[11] R. Parvathi, G. Thamizhendhi, Domination in intuitionistic fuzzy graphs, Notes Intuit. Fuzzy Sets. 16 (2010), 39–49.
[12] S. Velammal, Edge domination in intuitionistic fuzzy graphs, Int. J. Comput. Sci. Math. 4 (2012), 159–165.
[13] S. Sahoo, M. Pal, Intuitionistic fuzzy competition graph, Appl. Math. Comput. 52 (2016), 37–57.
[14] M. Akram, F. Zafar, Hybrid Soft Computing Models Applied to Graph Theory, Studies in Fuzziness and Soft Computing, vol. 380, Springer, Cham, Switzerland, 2020, pp. 1–430.
[15] S. Sahoo, M. Pal, Certain types of edge irregular intuitionistic fuzzy graphs, Intell. Fuzzy Syst. 34 (2018), 295–305.
[16] S. Sahoo, G. Ghorai, M. Pal, Embedding and genus of intuitionistic fuzzy graphs on spheres, J. Multi. Valued Logic Soft Comput. 31 (2018), 139–154.
[17] S. Sahoo, M. Pal, Product of intuitionistic fuzzy graphs and degree, Intell. Fuzzy Syst. 32 (2017), 1059–1067.
[18] S. Sahoo, M. Pal, H. Rashmanlou, R.A. Borzooei, Covering and paired domination in intuitionistic fuzzy graphs, Intell. Fuzzy Syst. 33 (2017), 4007–4015.
[19] A. Nagoorgani, M. Akram, S. Anupriya, Double domination on intuitionistic fuzzy graphs, Appl. Math. Comput. 52 (2016), 515–528.
[20] M. Akram, B. Davvaz, Strong intuitionistic fuzzy graphs, FILOMAT. 26 (2012), 177–195.
[21] C. Jana, M. Pal, G. Wei, Multiple attribute decision making method based on intuitionistic Dombi operators and its application in mutual fund evaluation, Arch. Control Sci. 30 (2020), 437–470.
[22] C. Jana, M. Pal, Application of bipolar intuitionistic fuzzy soft sets in decision making problem, Int. J. Fuzzy Syst. Appl. 7 (2018), 32–55.
[23] M. Akram, M. Arshad, M.M. Sarwar, Novel applications of intuitionistic fuzzy digraphs in decision support systems, Sci. World J. 2014 (2014), 904606.
[24] M. Akram, W.A. Dudek, Intuitionistic fuzzy hypergraphs with applications, Inf. Sci. 218 (2013), 182–193.
[25] M. Akram, M. Arshad, M. Shumaiza, Fuzzy rough graph theory with applications, Int. J. Comput. Intell. Syst. 12 (2018), 90–107.
[26] M. Pal, S. Samanta, G. Ghorai, Modern Trends in Fuzzy Graph Theory, Springer, Singapore, 2020.
[27] A. Bozhenyuk, M. Knyazeva, I. Rozenberg, Algorithm for finding domination set in intuitionistic fuzzy graph, in 11th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2019), Atlantis Studies in Uncertainty Modelling, Prague; 2019, pp. 72–76.
[28] A. Bozhenyuk, S. Belyakov, J. Kacprzyk, M. Knyazeva, The method of finding the base set of intuitionistic fuzzy graph, in: C. Kahraman, S. Cevik Onar, B. Oztaysi, I. Sari, S. Cebi, A. Tolga (Eds.), Intelligent and Fuzzy Techniques: Smart and Innovative Solutions, Advances in Intelligent Systems and Computing, vol. 1197, Cham, Switzerland, Springer, 2021, pp. 18–25.
[29] A. Bozhenyuk, S. Belyakov, M. Knyazeva, Modeling objects and processes in GIS by fuzzy temporal graphs, in: S. Shahbazova, J. Kacprzyk, V. Balas, V. Kreinovich (Eds.), Recent Developments and the New Direction in Soft-Computing Foundations and Applications, Studies in Fuzziness and Soft Computing, Recent Developments and the New Direction in Soft-Computing Foundations and Applications, vol. 393, Springer, Cham, Switzerland, 2020, pp. 277–286.
[30] A. Bozhenyuk, S. Belyakov, M. Knyazeva, J. Kacprzyk, Allocation centers problem on fuzzy graphs with largest vitality degree, in: I. Batyrshin, M. Martínez-Villaseñor, H. Ponce Espinosa (Eds.), Advances in Soft Computing, Lecture Notes in Computer Science, vol. 11288, Springer, Cham, Switzerland, 2018, pp. 379–390.
[31] O. Ore, Theory of Graphs, American Mathematical Society, Providence, RI, USA, 1962.
[32] L.S. Bershtein, A.V. Bozhenuk, Maghout method for determination of fuzzy independent, dominating vertex sets and fuzzy graph kernels, Int. J. Gen. Syst. 30 (2001), 45–52.
[33] K.T. Atanassov, Intuitionistic Fuzzy Sets: Theory and Application, Springer, Physica-Verlag, Heidelberg, Germany, 1999.
[34] K. Atanassov, G. Gargov, Elements of intuitionistic fuzzy logic. Part I, Fuzzy Sets Syst. 95 (1998), 39–52.
[35] R.T. Yeh, S. Bang, Fuzzy relations, fuzzy graphs, and their applications to clustering analysis, in: L.A. Zadeh, K.S. Fu, M. Shimura (Eds.), Fuzzy Sets and Their Applications, Academic Press, New York, NY, USA, 1975, pp. 125–149.
[36] L. Bershtein, A. Bozhenyuk, Fuzzy graphs and fuzzy hypergraphs, in: J. Dopico, J. de la Calle, A. Sierra (Eds.), Encyclopedia of Artificial Intelligence, Information SCI, Hershey, New York, NY, USA, 2008, pp. 704–709.