



Bayesian Estimation of System Reliability Models Using Monte-Carlo Technique of Simulation

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ABSTRACT

This paper discusses the problem of how Monte-Carlo simulation method is deal with Bayesian estimation of reliability of system of n s-independent two-state component. Time-to-failure for each component is assumed to have Weibull distribution with different parameters for each component. The shape parameter for each component is assumed to be known with the scale parameter distributed with a priori Rayleigh distribution with known parameters. Monte-Carlo simulation is used to generate the random deviates for the scale parameters and replicates for time-to-failure for each combination of scale parameters values are generated. Reliability is estimated as a function of time. Further, for the Bayes estimation of reliability we assume Poisson distribution with a priori time-shifted Rayleigh distribution. Finally, the robustness in the Bayesian estimation problem relative to changes in the assigned priori distribution is considered. We approximate the Bayes estimator of the reliability. The Bayes risk with respect to the priori time-shifted beta distribution is considered and at last approximate robustness of the Bayes estimator of reliability is examined with respect to the uniform priori. We have compared the maximum likelihood estimator of reliability with the Bayes estimator with prior uniform distribution. Finally, the method is illustrated by considering the illustrative example of vehicle system.

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1. INTRODUCTION

Bayesian estimation modelling is used for system identification, prediction and performance of reliability of the model. Bayesian inference requires the numerical approximation of the posteriors. Monte Carlo sampling is used for Bayesian Calculation. Modelling updating is used to improve the accuracy of predications of the reliability system (Bhattacharya [1], Box and Tiao [2–4]). Some researchers had introduced the two parameter generalization of Lindley distribution as a prolonged model of bathtub data alternative gamma, lognormal, Weibull, and exponential distribution. (Beck and Katafygiotis [5], Beck [6], Papadimitriou *et al.* [7], Nadarajah *et al.* [8]). Recently Wilson and Huzurbazar [9] had worked on the systems reliability. He had used the Bayesian networks for it. This model had binary outcomes and discrete data as multilevel form. Singh *et al.* [10] had used the generalized Lindley distribution under squared error and general entropy loss functions to develop the Bayesian estimation procedure in case f complete sample of observations. The estimating the reliability of a network through variations of Monte Carlo simulation is one of the active area for research [11–15].

In the present paper, Monte- Carlo simulation method is applied for Bayesian estimation of reliability of system of n s-independent two state components. Time-to-failure for each component is assumed to have Weibull distribution with different parameters for each component. The shape parameter for each component is assumed to be known with the scale parameter distributed with a priori Rayleigh distribution with known parameters. Monte-Carlo simulation is used to generate to generate the random deviates for the scale parameters and replicates for time-to-failure for each combination of scale parameters values are generated Reliability is estimated as a function of time. Further, for the Bayes estimation of reliability we assume Poisson distribution with a priori time-shifted Rayleigh distribution. Finally, the robustness in the Bayesian estimation problem relative to changes in the assigned priori distribution is considered. We approximate the Bayes estimator of the Reliability. The Bayes risk with respect to the priori time-shifted beta distribution is considered and at last approximate robustness of the Bayes estimator of reliability is examined with respect to the Uniform priori. We have compared the maximum likelihood estimator of reliability with the Bayes estimator with prior Uniform Distribution.

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2. ROBUSTNESS AND TIME-SHIFTED BETA PRIORI DISTRIBUTION IN THE BAYESIAN ESTIMATION

Consider the Rayleigh distribution with probability density function (pdf),

$$f(\sigma|\theta) = \frac{\theta^2}{\sigma^2} \exp\left(-\frac{\theta^2}{2\sigma^2}\right); \theta, \sigma > 0 \tag{1}$$

Let the reliability at time t be defined as the probability of zero failures occurring at time t , thus,

$$r(t|\sigma) = \int_t^\infty \frac{\theta}{\sigma^2} \exp\left(-\frac{\theta^2}{2\sigma^2}\right) d\theta = \exp\left(-\frac{t^2}{2\sigma^2}\right) \tag{2}$$

The likelihood function is

$$l(\sigma|\theta) = \frac{\left(\prod_{i=1}^n \theta_i\right)}{\sigma^{2n}} \exp\left(-\frac{s^2}{2\sigma^2}\right)$$

where $s = \sum_{i=1}^n x_i$. Now the MLE of reliability is

$$MLE \left[\widehat{r}(t) \right] = \exp\left(-\frac{m^2}{s^2}\right)$$

The uniform minimum variance unbiased estimator (MVUE) of reliability is

$$\widehat{r}(t) = \begin{cases} \left(1 - \frac{t^2}{s^2}\right)^{n-1} & ; 0 \leq t \leq s \\ 0 & ; \text{otherwise} \end{cases}$$

Suppose that the intensity parameter σ is a random variable. Assume, the time-shifted beta distribution as the priori of σ with pdf,

$$g_w(\sigma - \sigma_w) = \frac{1}{B(\mu, v)} (\sigma - \sigma_w)^{\mu-1} [1 - (\sigma - \sigma_w)]^{v-1}; (\mu, v) > 0, \sigma_w \geq \sigma$$

Let, $(\sigma - \sigma_w) = K$, hence

$$g_w(K) = \frac{1}{B(\mu, v)} (K)^{\mu-1} [1 - (K)]^{v-1}; (\mu, v) > 0, K \geq 0 \tag{3}$$

where (μ, v) are the parameters and σ_w is the wear-out failure time. Wear-out failure occurs as a result of deterioration process or mechanical wear and its probability of occurrence increase with time. A failure rate as a function of time decreases in an early failure and it increases in wear-out failure. The posterior of σ given θ is

$$h(\sigma/\theta) = \frac{l(\sigma/\theta) g_w(K)}{\int_0^\infty l(\sigma/\theta) g_w(K) dk}$$

$$h(\sigma/\theta) = \frac{K^{\mu-1} (1 - K)^{v-1} s^2}{\sigma^3 B(\mu, v)} \exp\left(-\frac{s^2}{2\sigma^2}\right) \tag{4}$$

Where

$$B(\mu, v) = \frac{\Gamma(\mu)\Gamma(v)}{\Gamma(\mu + v)} \tag{5}$$

Hence, the relation $r = \exp\left(-\frac{t^2}{2\sigma^2}\right)$ is a monotonically decreasing function of $\sigma > 0$ for fixed t , it defines a one-to-one corresponding between r and σ . Thus, we may use the posterior density of σ for determining the Bayes decisions function of reliability. Therefore, the Bayes estimator of reliability is given by

$$\tilde{r}(t) = E[r(t)/X] = \frac{K^{\mu-1}(1-K)^{\nu-1}s^2}{B(\mu, \nu)(t^2 + s^2)} \quad (6)$$

Similarly, the posterior variance is evaluated to be

$$\text{Var}\{\tilde{r}(t)/X\} = \frac{H^2(H-1)\sigma^3}{(t^2 + s^2)s^2}, \quad (7)$$

$$\text{where } H = \frac{K^{\mu-1}(1-K)^{\nu-1}s^2}{B(\mu, \nu)}.$$

Since, the Bayes risk is defined as the expectation of the posterior variance where expectation is taken over all possible random variables. Thus,

$$R[\tilde{r}(t)] = E_K\{E_\sigma \text{Var}[\tilde{r}(t)/X]\} \quad (8)$$

Now, with the help of Equation (7), Equation (8) can be written as

$$R[\tilde{r}(t)] = \frac{H^2(H-1)\sigma^4}{(t^2 + s^2)s^2\theta}$$

Now by taking expectation with respect K , for the priori distribution,

$$R[\tilde{r}(t)]_B = E_K\left[\frac{H^2(H-1)\sigma^4}{(t^2 + s^2)s^2\theta}\right]$$

$$R[\tilde{r}(t)]_B = \left[\frac{H^2(H-1)\sigma^4}{(t^2 + s^2)s^2\theta}\right] \quad (9)$$

where the subscript B denotes the time-shifted beta priori distribution.

3. ROBUSTNESS BY RAYLEIGH DISTRIBUTION IN THE BAYESIAN ESTIMATION

The robustness of the Bayes estimator, $\tilde{r}(t)$ with respect to the uniform priori,

$$u(\theta) = \frac{1}{\beta}; \quad 0 < \theta < \beta \quad (10)$$

In the determination of the risk, we note that Equation (8) is the result before taking expectations with respect to σ derived from the Bayes decision function (4), which is the consequence of the time-shifted beta priori (3). Thus, if the more appropriate priori for σ is the uniform (10), then by taking expectation of (8) with respect to σ for the uniform priori distribution, we determined the risk based on this distribution. Thus,

$$R[\tilde{r}(t)]_U = E_\sigma\left[\frac{H^2(H-1)\sigma^4}{(t^2 + s^2)s^2}\right]$$

$$R[\tilde{r}(t)]_U = \left[\frac{H^2(H-1)\beta^4}{(t^2 + s^2)s^2}\right] \quad (11)$$

Finally, for comparison the MLE of reliability over the uniform distribution is given by,

$$MLE[\tilde{r}(t)]_U = E_\sigma\{MLE\}[\tilde{r}(t)]$$

$$MLE [\tilde{r}(t)]_U = \frac{n\beta}{s^2} \left[\exp \left(\frac{n\beta^2}{s^2} \right) \right] \tag{12}$$

For numerical comparison, two ratios are formed and computed for various values of the indicated parameters. The first involves the ratio of the Bayes risk $\tilde{r}(t)$ relative to the uniform and beta prior's distribution. The comparison is made in this manner, because one expects the Bayes risk given by (9) and relative to the assigned priori to be smaller than that given by (11). The second involves the Bayes risk given by (11) and MLE of reliability with respect to the uniform priori given by (12). Thus, a comparison between these two will indicate the Bayes results are still superior to the corresponding classical estimators. The numeral calculations for the different parameters are indicated in the Tables 1 and 2.

3.1. Illustrate Example on Vehicle System Model Based on Above Results

The model for the system of vehicles, that is, all the passenger carrying vehicles in the system. The model is based on the following assumptions:

1. Only the operating vehicles are liable to fail and the vehicles on standby or maintenance are not subject to failure. This assumption results in assigning zero failure rates to the vehicles being maintained or in standby mode. The failure rate of a cold standby mode, that is, partially powered, this assumption is valid only so long as the failure rate in warm standby mode is small as compared with that of the operating vehicle.
2. The failure of a single vehicle in the retrieval mode causes the whole system of vehicles to be down. The duration of the down time of a vehicle is considered from the time it comes to a halt to time of resuming normal operation. The down time of all the vehicles, directly affected vehicle, and the vehicles coming to a halt as a result of blocking, is assumed to be the same. This assumption was made because of short headways, all the vehicles may not be equally affected and a correction to models may be needed.
3. A vehicle in a partial failure mode is assumed to be removed from the system as soon as possible after the occurrence of the failure and therefore the probability of another unit failing during this period or the partial mode passing into full mode is assumed zero.
4. So long as there one standby, a unit on which maintenance is completed will be interchanged with an operating or standby unit. When, however, no spares are left, the unit passes directly from maintenance into the operation mode, without going on standby.

Now, we illustrate the example based on the above model.

Table 1 Ratio of $R[\tilde{r}(t)]_U$ to $R[\tilde{r}(t)]_\beta$.

x = 5, x _w = 2, K = 3, s = 10						
T	σ	θ	μ = 3.5, v = 2, β = 2	σ	θ	μ = 3.5, v = 2, β = 0.5
1	1.5	3.5	2.765	1.5	3.5	0.172
2	1.5	4.5	3.555	1.5	4.5	0.222
3	1.5	5.5	4.345	1.5	5.5	0.271
4	1.5	6.5	5.135	1.5	6.5	0.320
5	1.5	7.5	5.925	1.5	7.5	0.370
6	1.5	8.5	6.716	1.5	8.5	0.419
7	1.5	9.5	7.506	1.5	9.5	0.469
8	1.5	10.5	8.296	1.5	10.5	0.518
9	1.5	11.5	9.086	1.5	11.5	0.567
10	1.5	12.5	7.601	1.5	12.5	0.475

Table 2 Ratio of $R[\tilde{r}(t)]_U$ to $MLE[\tilde{r}(t)]_U$.

x = 5, x _w = 2, K = 3, s = 10						
T	σ	θ	μ = 3.5, v = 2, β = 2	σ	θ	μ = 3.5, v = 2, β = 0.5
1	1.0	2.0	0.343	1.0	2.0	1.72
2	1.0	2.5	0.309	1.0	2.5	1.45
3	1.0	3.0	0.251	1.0	3.0	1.15
4	1.0	3.5	0.194	1.0	3.5	0.85
5	1.0	4.0	0.143	1.0	4.0	0.60
6	1.0	4.5	0.101	1.0	4.5	0.41
7	1.0	5.0	0.092	1.0	5.0	0.36
8	1.0	5.5	0.026	1.0	5.5	0.31
9	1.0	6.0	0.076	1.0	6.0	0.27
10	1.0	6.5	0.068	1.0	6.5	3.60

3.2. Vehicle System Example

The model described above has been implemented in a computer program.

The vehicle system data for this example is described in the above. The system consists of 12 vehicles out of which one are on maintenance and one kept as spares.

3.3. Vehicle System Data

Time to vehicle failure (permanent) $\sigma = 5000.00$ Hours

Wear-out time to vehicle failure (partial mode) $\sigma_w = 500.00$ Hours

Number of operating vehicle $s = 10$

Number of spare vehicle $\mu = 1$

Number of vehicle on maintenance $v = 1$

Vehicle exposure factor $\beta = 0.05$

Now, we construct the Table 3, for the comparison two ratios.

3.4. Concluding Remarks

It is apparent from these results that when the perturbed risk $R[\tilde{r}(t)]_U$ is compared to the Bayes risk $R[\tilde{r}(t)]\beta$ the predominant occurrence is increase in the later over the former depending, of course, on the data available. On the other hand, however, the perturbed risk $R[\tilde{r}(t)]_U$ is in all cases smaller than the corresponding average MLE of the reliability.

4. BAYESIAN ESTIMATION OF RELIABILITY WITH PRIORI TIME-SHIFTED RAYLEIGH DISTRIBUTION USING MONTE-CARLO SIMULATION

Consider the Poisson distribution, with parameter ‘a’ and probability mass function (pmf),

$$P(X; at_s) = \frac{(at_s)^x \exp(-at_s)}{\mathcal{L}x}; \quad x = 0, 1, 2, \dots \tag{13}$$

where X is the number of failures occurring in the time span t_s . Let the reliability at time t be defined as the probability of zero failures occurring upto time t . Thus,

$$r(t/x) = \exp(-at); \quad t > 0 \tag{14}$$

The MVUE of reliability is

$$\hat{r}(t) = \begin{cases} \left(1 - \frac{t}{t_s}\right)^x & ; t_s > t \\ 0 & ; t_s \leq t \end{cases} \tag{15}$$

Table 3 Ratio of $R[\tilde{r}(t)]_U$ to $R[\tilde{r}(t)]\beta, MLE[\tilde{r}(t)]_U$.

t (No of vehicles)	σ	θ	$\mu = 1.0, v = 1.0, \beta = 0.05$		
			$R[\tilde{r}(t)]_U$	$R[\tilde{r}(t)]\beta$	$MLE[\tilde{r}(t)]$
1			6.1262	980.198	6.420
2			5.9495	951.923	10.912
3			5.6766	908.256	16.487
4			5.3340	853.448	23.353
5			4.9499	791.999	31.755
6	1.0	0.001	4.5496	727.941	41.980
7			4.1526	664.429	54.365
8			3.7728	603.658	69.304
9			3.4185	546.961	87.258
10			3.0937	494.999	108.764

While the mean-squared error (MSE) of

$$MSE [\hat{r}(t)] = \exp(-2at) [\exp(at^2t_s) - 1] \tag{16}$$

Suppose that the intensity parameter 'a' is a random variable. Assume the time-shifted Rayleigh distribution as the priori distribution of "a" with probability distribution function is

$$g_w(\sigma - \sigma_w) = \frac{(a - a_w)}{\sigma^2} \exp\left(-\frac{(a - a_w)^2}{\sigma^2}\right); a \geq a_w, \sigma > 0 \tag{17}$$

where, σ is the parameter and a_w is the wear-out failure time. The posterior probability density function of "a" given x is

$$\begin{aligned} h(a/x) &= \frac{r(t/x) g_w(a - a_w)}{\int_0^\infty r(t/x) g_w(a - a_w) dw} \\ &= \frac{2\sigma^2 (a - a_w) \exp(-(a - a_w)^2/2\sigma^2)}{\left\{ 2(a - a_w) \exp(-a_w^2/2\sigma^2) - t \exp\left[\frac{(a - a_w)^2}{2\sigma^2} + at\right] [a_w^2 (a_w - 1)] \right\}} \end{aligned} \tag{18}$$

Since, the relation $r = \exp(-at)$ is a monotonically decreasing function of $a > 0$ for fixed t, it defines a one-to-one correspondence between r and a. Thus, we may use the posterior density for determining the Bayes decision function of reliability relative to a square-loss function. Briefly, the Bayes estimator of reliability is given by,

$$\tilde{r}(t) = \frac{CD - AB}{E}, \tag{19}$$

where

$$\begin{aligned} A &= \left\{ a_w^2 \exp\left(-\frac{a_w^2}{2\sigma^2}\right) - t \exp\left[\frac{(a - a_w)^2}{2\sigma^2} + at\right] [a_w^2 (a_w - 1)]^2 \right\} \\ B &= \left\{ \sigma^2 \exp\left(-\frac{a_w^2}{2\sigma^2}\right) + t a_w \exp\left(-\frac{a_w^2}{2\sigma^2}\right) \right\}; C = \left\{ a_w \exp\left[\frac{(a - a_w)^2}{2\sigma^2} + at\right] \right\} \\ D &= \left\{ 2a_w \exp\left(-\frac{a_w^2}{2\sigma^2}\right) - at \exp\left[\frac{(a - a_w)^2}{2\sigma^2} + at\right] [a_w^2 (a_w - 1)]^2 \right\} \\ E &= \left\{ a_w^2 \exp\left(-\frac{a_w^2}{2\sigma^2}\right) + t \exp\left[-\frac{(a - a_w)^2}{2\sigma^2} - at\right] [a_w^2 (a_w - 1)]^2 \right\} \end{aligned}$$

4.1. Monte-Carlo Simulation

To establish the usefulness of the Bayes estimators of reliability Monte-Carlo simulation is employed. There exist point epochs at which stochastic process regenerates itself and makes the future evaluation of the process conditionally independent of the past manners. This is an important possession, which is used in simulation experiments regarding the behavior of the system.

The procedure available for the generation of random deviations useful in reliability theory. The central concept is the generation of random numbers of a uniform random variable on (0, 1). By using the computer program, the random deviates for the arbitrary values of the parameters are generated. Further, the reliability for Weibull and negative exponential distributions for lifetime is calculated which is shown in Table 3. The importance of uniformly distributed variables for Weibull and negative exponential distributions is clear from the following results.

Let X be a random variable with pdf, $f_X(\cdot)$ and cumulative distribution $F_X(x) = \int_{-\infty}^x f_X(x) dx$.

The new random variable $F_X(x)$ is uniformly distributed in (0, 1). This property is used to generate random variables with arbitrary distribution. We have generated random number by using the different distributions as follows:

Table 4 Monte-Carlo simulation result.

Weibull Distribution			Negative Exponential Distribution		
$\sigma = 10.0; a_w = 2.0$					
$a = b = 5.0$			$a = b = 10.0$		
U	T	Bayes Reliability $\tilde{r}(t)$	U	T	Bayes Reliability $\tilde{r}(t)$
0.6459	3.5862	0.9874	0.9711	2.0891	0.8615
0.1957	4.6688	0.8546	0.9564	2.2717	0.7451
0.1993	4.6564	0.8052	0.9345	2.4700	0.6321
0.1635	4.7654	0.7456	0.7038	3.4328	0.5211
0.0794	5.0963	0.6542	0.7025	3.4364	0.5016
0.0623	5.1904	0.5321	0.6712	3.5208	0.4123
0.0616	5.1946	0.5021	0.5644	3.7844	0.3014
0.0022	6.0794	0.4562	0.5349	0.8529	0.2015
0.0010	6.2286	0.3154	0.5318	0.8601	0.1206
0.0004	6.3857	0.2154	0.3209	4.3416	0.1123

5. BAYESIAN ESTIMATION OF RELIABILITY WITH NEGATIVE EXPONENTIAL DISTRIBUTION USING MONTE-CARLO SIMULATION

Let T be a negative exponentially distributed random variable with pdf $\lambda e^{-\lambda t}$, $\lambda > 0$. Then, $F_T(t) = 1 - e^{-\lambda t}$. The random variable $U = F_T(t)$ is uniformly distributed in $(0, 1)$. Implying that $U = 1 - e^{-\lambda t}$ is also uniformly distributed in $(0, 1)$. We get,

$$T = \left(\frac{-1}{\lambda}\right) \log_e V \tag{20}$$

where V is uniformly distributed over $(0, 1)$. Thus using random number from uniform variable in $(0, 1)$ and using Equation (20), we generate the random number from exponential distribution is generated. We assume some random values of the parameters and access the T , which is shown in Table 4.

6. BAYESIAN ESTIMATION OF RELIABILITY WITH WEIBULL DISTRIBUTION USING MONTE-CARLO SIMULATION

Let T be a Weibull distribution with pdf

$$F_T(t) = \begin{cases} \frac{1}{a^b} a t^{b-1} \exp(-ta)^b & ; 0 \leq t \leq \infty \\ 0 & ; elsewhere \end{cases}$$

The cdf of T denoted as $F_T(t) = 1 - \exp(-t/a)^{1b}$

$$\text{Now, } T = a(-\log U)^{1b} \tag{21}$$

To generate T , generate the uniform variable U , and using relation (21). Thus, we generate the random number from uniform variable in $(0, 1)$ and using Equation (21). We assume some erratic values of the parameters and assess the T , which is shown in Table 4. Then the curves are plotted for the above values obtained in Table 4.

Then subsequently the Table 5, represents the values of the MVUE and Bayes reliability for the different values of T . Furthermore, the curves for the MSE, minimum unbiased estimator (MVUE), and Bayes reliability are drawn to show the fluctuations. This is useful for the system designers or engineers to design more reliable systems.

Curves for Exponential Distribution

Curves for Weibull Distribution

6.1. Conclusions

From the Figures 1 and 2, we perceive that as the time increases the reliability decreases and from the Figures 3 and 4, we conclude that for the exponential distribution as the time increases the MVUE of reliability and the Bayes reliability are numerically close. And from the Figures 5 and 6, we conclude that for the Weibull distribution as the time increases the MVUE of reliability and the Bayes reliability are numerically close.

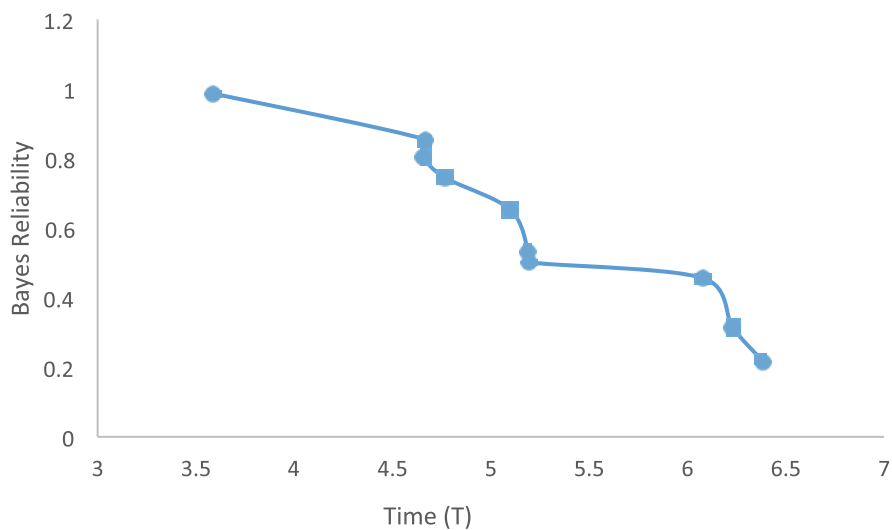


Figure 1 Bayes Reliability (Weibull Distribution).

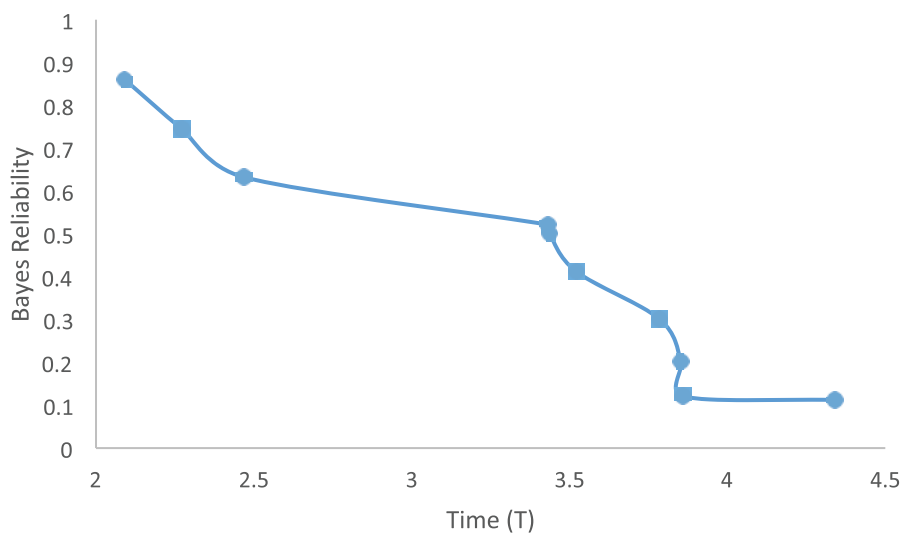


Figure 2 Bayes Reliability (Negative Exponential Distribution).

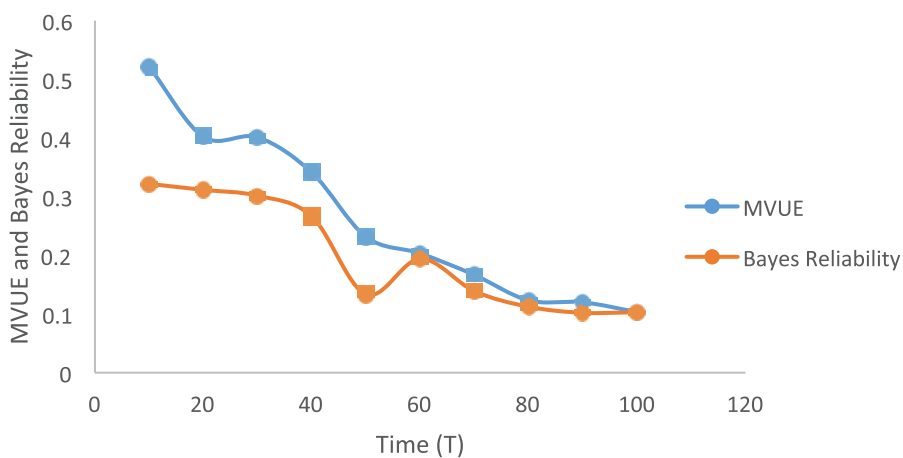


Figure 3 Minimum Unbiased Estimator (Negative Exponential Distribution).

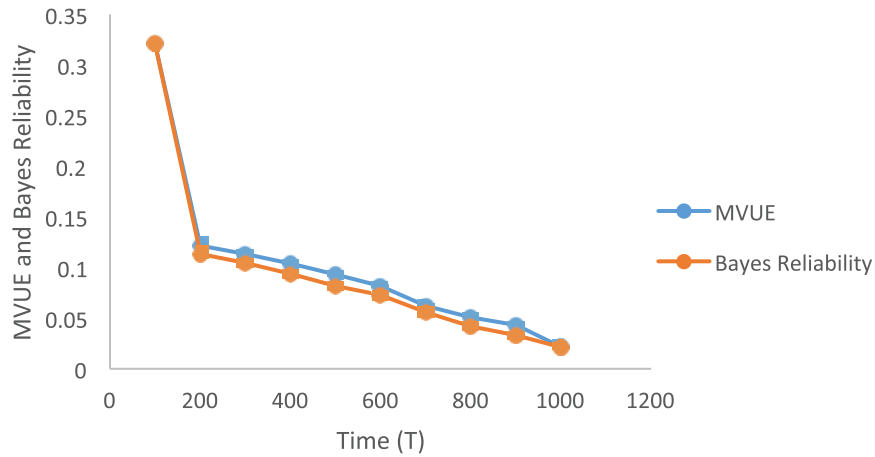


Figure 4 Minimum Unbiased Estimator (Negative Exponential Distribution).

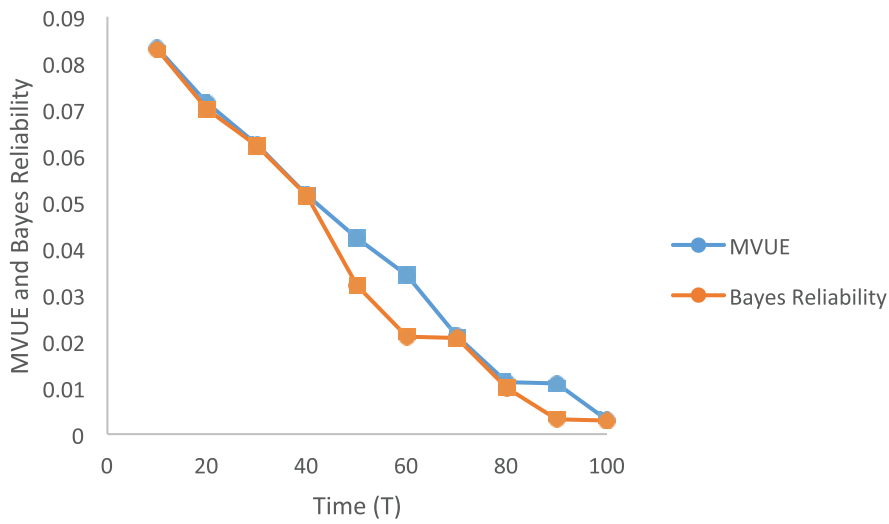


Figure 5 Minimum Unbiased Estimators (Weibull Distribution).

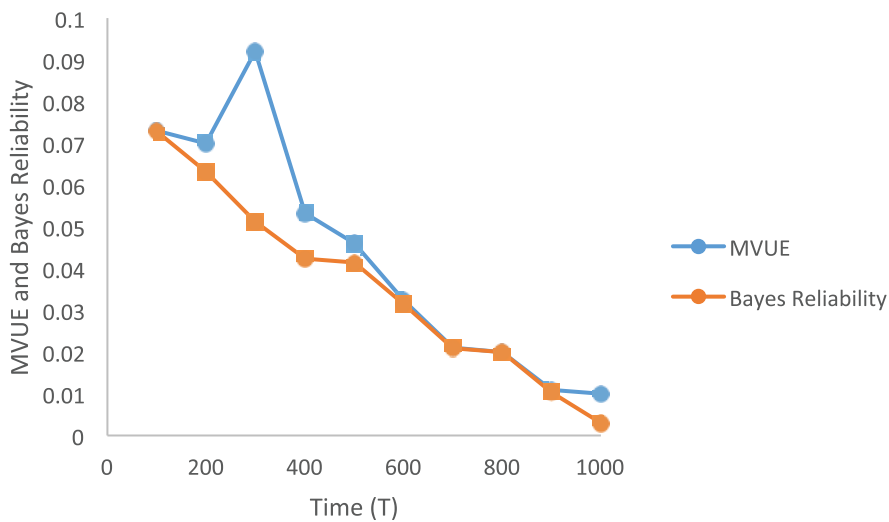


Figure 6 Minimum Unbiased Estimator (Weibull Distribution).

Table 5 For the values of MSE, MVUE, and Bayes reliability.

Negative Exponential Distribution							
$t_s = 100; a = 2.0; \sigma = 5.0; a_w = 0.05$				$t_s = 1000; a = 2.0; \sigma = 10.0; a_w = 0.05$			
<i>T</i>	MSE	MVUE	Bayes Reliability	<i>T</i>	MSE	MVUE	Bayes Reliability
10	0.0295	0.5213	0.3216	100	0.0318	0.3216	0.3212
20	0.0288	0.4021	0.3116	200	0.1216	0.1216	0.1134
30	0.0224	0.4011	0.3016	300	0.1132	0.1132	0.1046
40	0.0202	0.3412	0.2664	400	0.1042	0.1042	0.0936
50	0.0065	0.2316	0.1334	500	0.0932	0.0932	0.0819
60	0.0033	0.2033	0.1942	600	0.0816	0.0816	0.0732
70	0.0077	0.1666	0.1389	700	0.0621	0.0621	0.0561
80	0.0015	0.1232	0.1124	800	0.0511	0.0511	0.0421
90	0.0200	0.1200	0.1020	900	0.0432	0.0432	0.0333
100	0.0296	0.1034	0.1032	1000	0.0216	0.0216	0.0214

Weibull Distribution							
$t_s = 100; a = 2.0; \sigma = 5.0; a_w = 0.02$				$t_s = 1000; a = 5.0; \sigma = 10.0; a_w = 0.009$			
<i>T</i>	MSE	MVUE	Bayes Reliability	<i>T</i>	MSE	MVUE	Bayes Reliability
10	0.0021	0.0832	0.0830	100	0.0024	0.0731	0.0729
20	0.0642	0.0714	0.0700	200	0.055	0.070	0.0632
30	0.0114	0.0624	0.0621	300	0.012	0.0921	0.0512
40	0.0151	0.0515	0.0514	400	0.016	0.0532	0.0423
50	0.000	0.0423	0.0321	500	0.000	0.0461	0.0414
60	0.000	0.0342	0.0210	600	0.000	0.0324	0.0314
70	0.058	0.0212	0.0208	700	0.052	0.0211	0.0209
80	0.000	0.0112	0.0100	800	0.000	0.020	0.0199
90	0.015	0.0109	0.0032	900	0.016	0.0109	0.0103
100	0.002	0.0030	0.0029	1000	0.002	0.0100	0.0030

MSE, mean-squared error; MVUE, minimum variance unbiased estimator.

7. BAYESIAN ESTIMATION OF SYSTEM RELIABILITY WITH PRIORI RAYLEIGH DISTRIBUTION USING MONTE-CARLO SIMULATION

Consider the system consists of *n* s-independent two-state components. Let the time-to-failure for each component be describe by a Weibull Distribution with the pdf for *i*-th component failure time given by,

$$f_i(t\beta_i) = \beta_i t^{\alpha_i} \exp \left[\frac{[-\beta_i t^{\alpha_i+1}]}{(\alpha_i + 1)} \right]; \beta_j > 0, \alpha_j > -1 \tag{22}$$

The shape parameter α_i , is assumed to be known for each component while the scale parameter β_i is assumed to have an apriori Rayleigh distribution with pdf is given by

$$g_i(\beta_i) = \frac{\beta_i}{b_i^2} \exp \left(\frac{-\beta_i^2}{2b_i^2} \right); x_i, b_i > 0 \tag{23}$$

With known parameter b_i

Let one trail of the system yield the failure times, t_i , for the components that fail. Let I_f denotes the set of the components that fail. The posterior pdf of the scale parameter of the *i*-th component for $i \in I * [j]$ is given by

$$h_i(\beta_i, t_i) = \frac{f_i(t_i/\beta_i) g_i(\beta_i)}{\int_0^\infty f_i(t_i/\beta_i) g_i(\beta_i) d\beta_i}; i \in I * [j] \tag{24}$$

Substituting the values of Equations (22) and (23) in Equation (24), we get

$$h_i(\beta_i, t_i) = \frac{\frac{k_i t_i^{\alpha_i}}{b^2} \exp\left[\frac{-k_i t_i^{\alpha_i+1}}{\alpha_i+1} - \frac{k_i^2}{2b^2}\right] \left[\frac{\alpha_i+1}{(\alpha_i+1)-t_i^{\alpha_i+1}}\right]}{\left[\frac{2t_i^{\alpha_i+1}}{\alpha_i+1} - 2\right]}; i \in I * [j] \tag{25}$$

If $i \notin I * [j]$, the conditional probability to be used is the conditional reliability of the component for the system time $I * [j]$. The posterior pdf of the scale parameter for such a component is given by

$$h_i(\beta_i/\bar{t}_f) = \frac{R_i(\bar{t}_f/\beta_i) g_i(\beta_i)}{\int_0^\infty R_i(\bar{t}_f/\beta_i) g_i(\beta_i) d\beta_i}; i \in I * [j] \tag{26}$$

where \bar{t}_f indicates the components did not fail by the system failure time \bar{t}_f . Since, the conditional reliability for the i -th component is given by

$$R_i(t_f/\beta_i) = \exp\left[\frac{-\beta_i t_f^{\alpha_i+1}}{(\alpha_i+1)}\right] \tag{27}$$

The posterior pdf can be given by

$$h_i(\beta_i/\bar{t}_f) = \frac{1}{b_i} \exp\left[\frac{-\beta_i \bar{t}_f^{\alpha_i+1}}{(\alpha_i+1)}\right]; i \notin I * [j] \tag{28}$$

The posterior pdf of the scale parameter for these components that failed is evaluated by Equation (28) and the posterior pdf of the scale parameter for those components which does not failed is evaluated by the Equation (27).

7.1. Monte-Carlo Simulation

The posterior pdf of the scale parameter for the components can be evaluated by the above relation and can be converted to the reliability estimate for each component. We generate the values of random derivatives from the Weibull distribution. To generate β_i , generate a uniform random variable U , which is uniformly distributed between (0, 1) is given as follows:

$$\beta_i = \alpha(-\log U)^{1/C} \tag{29}$$

Replications of the combinations of the scale parameters generated may also be provided from the posterior pdf. Consider k_s replications of scale parameter combination and k_r replications of time-to-failure for each combination. The total replications are given by

$$N = k_s k_r$$

If $k_f(t)$ is assumed to be the number of replications that yield system failure for required operating time t , the reliability as a function of tm is estimated by,

$$\hat{R}(t) = [N - k_f(t)] / N \tag{30}$$

Now, we construct the Table 6, for generate the random derivatives by using Monte-Carlo Simulation, for different values of the parameters and we obtain the number of observations. And, we construct the Table 7, to evaluate failure time for the one trail with different parameters. In Table 8, we calculate the reliability using 200 replications for scale parameter combinations and 200 replications for time to failure given each combination, and then we estimate the reliability as the function of time. Next, we calculate the reliability using 100 replication for time to failure given each combination, and then we estimate the reliability as the function of time.

And next, we illustrate the curve to demonstrate the fluctuations between the values, which help the system designers to intend the more consistent and proficient systems.

Table 6 Random number deviations for different values of parameters.

Component No (i)	Random Deviations	$a = c = 0.5$			$a = c = 0.8$		
		Observations (β_i)	Shape Parameter (α_i)	Rayleigh-Priori Parameter (b_i)	Observations (β_i)	Shape Parameter (α_i)	Rayleigh-Priori Parameter (b_i)
1	0.123	2.223	1.0	2.0	2.033	3.0	1.0
2	0.651	0.091	1.5	2.2	0.276	3.5	1.2
3	0.868	0.009	2.0	2.4	0.068	4.0	1.4
4	0.729	0.049	2.5	2.6	0.188	4.5	1.6
5	0.798	0.025	3.0	2.8	0.123	5.0	1.8
6	0.073	3.400	3.5	3.0	2.650	5.5	2.0
7	0.490	0.253	4.0	3.2	0.523	6.0	2.2
8	0.454	0.310	4.5	3.4	0.593	6.5	2.4
9	0.107	2.493	5.0	3.6	2.183	7.0	2.6

Table 7 Results of one trail for failure time and posterior densities.

Component No (i)	Time t_i	Failure Time f_i	Time t_i	Failure Time f_i
1	0.8	Did not fail	0.1	Did not fail
2	0.9	0.2026	0.1	0.5772
3	1.0	0.0265	0.2	0.1799
4	1.2	0.2048	0.3	0.6562
5	1.3	0.1469	0.4	0.7487
6	1.4	Did not fail	0.5	Did not fail
7	1.5	Did not fail	0.6	Did not fail
8	1.6	0.0005	0.7	0.0022
9	1.7	0.0872	0.8	Did not fail

Table 8 Reliability estimation by simulation result.

$\alpha_i = \beta_i = 200$ Replications		$\alpha_i = \beta_i = 100$ Replications	
Time ti	Reliability	Time ti	Reliability
10	0.0821	0.1	0.0987
20	0.0754	0.2	0.0854
30	0.0635	0.3	0.0789
40	0.0524	0.4	0.0698
50	0.0456	0.5	0.0542
60	0.0400	0.6	0.0423
70	0.0325	0.7	0.0321
80	0.0302	0.8	0.0254
90	0.0215	0.9	0.0121
100	0.0155	1.0	0.0109

7.2. Conclusions

From Figures 7 and 8, we see that as the time increases the reliability decreases. Monte-Carlo simulation and Bayesian estimation can be combined together to estimate system reliability given systems with s-independent two-state components whose failure times are Weibull with known space parameters and Rayleigh priors for the scale parameters. With the modification can be extended to systems where shape parameters also have some priori distribution.

7.3. Necessity and Significance of the Study

The Bayesian approach is used whenever the available data for the study is too small and complex. Therefore, to resolve the objectives of the study, Bayesian estimation was used. In our study, Bayesian estimation approach is very well known and used in passenger carrying vehicle system.

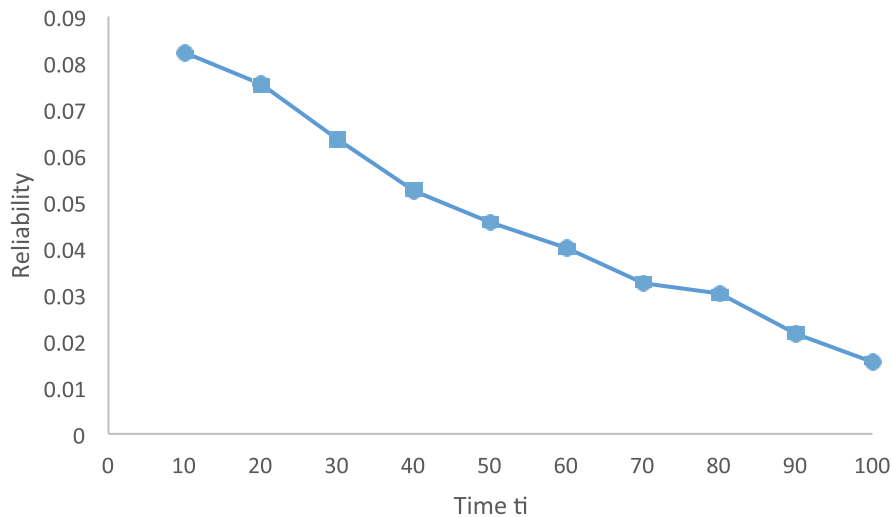


Figure 7 | Reliability Estimation.

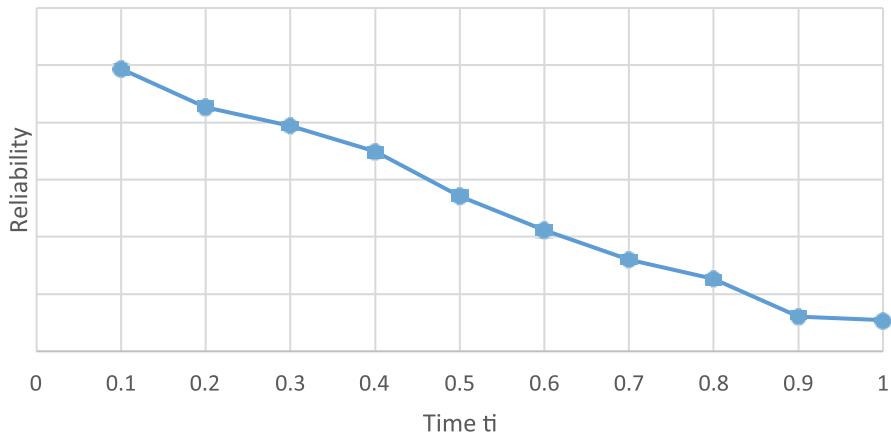


Figure 8 | Reliability Estimation.

In order to optimize failure rate of the vehicles, it is necessary to know the distribution of several variables. This model help to minimize the maintenance and failure rate of the passenger carrying system wherever the vehicles are in operation and standby mode.

CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest.

AUTHORS' CONTRIBUTIONS

Author 1 and 2 conceived of the presented idea, developed the models, and performed the analytic calculations and numerical simulations. Author 3 wrote the manuscript with support of Author 1. All authors discussed the results and contributed to the final manuscript.

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