

## Research Article

# Consensus Reaching Process in the Two-Rank Group Decision-Making with Heterogeneous Preference Information

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## ABSTRACT

This paper proposes a novel consensus reaching process (CRP) for the two-rank group decision-making (GDM) problems with heterogeneous preference information. The methods for deriving the individual and collective preference vector are provided. And the individual and collective two-rank vectors are obtained. Then, the feedback adjustment rules are proposed. Next, an algorithm is given to describe the two-rank CRP with heterogeneous preference information. Finally, we present a practical example to illustrate the feasibility of the proposed method.

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## 1. INTRODUCTION

Group decision-making (GDM) [1,2] as an important field in decision analysis has gained the increasing attentions since its first appearance. In real applications of GDM problems, due to the differences of the professional knowledge, cultural background, and social experience among different individuals, it is natural that different decision makers may utilize heterogeneous representation structures to express their preferences on the alternatives [3]. Recently, the study on consensus reaching process (CRP) has become an interesting and necessary area [4–7].

Over the last few decades, a mass of consensus models for the GDM with heterogeneous preference structures have been developed from different focuses: (i) Heterogeneous formats of expressions. For example, in the existing studies [8–14] it is assumed that different individuals will express the additive preference relations, multiplicative preference relations, preference orderings, and utility values, etc. (ii) Multi-granularity linguistic term information. For example, with the consideration of the individuals may use different scale of linguistic terms to express their preferences, various CRPs with the multi-granularity linguistic term information have been proposed [15–19]. (iii) The optimization-based models. The existing studies optimize some available standards such as cost, time,

adjustments, and information losses to achieve a better solution to increase the agreements among individuals [20–28].

Despite the research already conducted on the CRP with heterogeneous preference structures, there exists a major challenge that contradicts the real applications. Most of the existing CRPs are based on the individual and group rankings of alternatives to build a consensus, while in some real-life GDM problems in which individuals don't need to rank alternatives, and instead hope to divide the alternatives into two categories [1,17] (i.e., two-rank GDM problems). For example:

**Example 1.** In the problem of reviewing the projects approved by Natural science foundation of China, based on the novelty of topic, quality of writing, and previous performances, each expert will divide the candidate proposals into two categories, i.e., the approved proposals and the rejected proposals.

**Example 2.** In the problem of bank loaning, with the consideration of some factors such as income, deposit, profession, and history records, the bank will divide the individuals who applied for a loan into two categories, i.e., arranging loans for some individuals and rejecting loans for other individuals.

To overcome the weaknesses in the existing CRP with heterogeneous preference structures, the aim of this paper is to develop a novel CRP for the two-rank GDM problems with heterogeneous preference information. The rest of this study is organized as follows: Section 2 introduces the preliminaries regarding four kinds

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of heterogeneous preference representation structures, and formulates the two-rank GDM problems. In Section 3, we develop a novel CRP for the two-rank GDM with heterogeneous preference representation structures. Next, a practical example under the background of selecting the subway construction plan is given to show the feasibility of the proposed two-rank CRP in Section 4. Finally, conclusions are summarized in Section 5.

## 2. PRELIMINARIES

In this section, the basic knowledge regarding heterogeneous preference information is firstly introduced. Then, the formulation of two-rank GDM problems is presented.

### 2.1. Heterogeneous Preference Information in GD M

Let  $X = \{x_1, x_2, \dots, x_n\}$  be an alternative set, where  $x_i$  represents the  $i$ th alternative,  $i = 1, 2, \dots, n$ . Let  $D = \{d_1, d_2, \dots, d_m\}$  be the set of individuals, where  $d_k$  denotes the  $k$ th individual,  $k = 1, 2, \dots, m$ .

In practice, due to the differences in the professional knowledge, cultural background, social experience, and personality among different individuals, sometimes these people use different representation structures to express the preferences on the alternatives.

Let  $D^L$ ,  $D^U$ ,  $D^P$ , and  $D^A$  represent the subset of individuals who express their preferences using linguistic preference relations, utility functions, additive preference relations, and multiplicative preference relations, respectively, satisfying  $D^L \cup D^U \cup D^P \cup D^A = D$ ,  $D^k \cap D^q = \emptyset$ ,  $k, q = L, U, P, A$  and  $k \neq q$ . Let  $S = \{s_0, s_1, \dots, s_T\}$  be a linguistic term set with odd granularities.

Specifically, the heterogeneous preference information is introduced in the below:

**Case 1.**  $d_k \in D^L$ . In this case, the individual  $d_k$  will provide linguistic preference relations  $L^{(k)} = (l_{ij}^{(k)})_{n \times n}$  on  $X$ , where  $l_{ij}^{(k)} = s_a \in S$  and  $l_{ji}^{(k)} = s_{T-a}$  [10,29–31].

**Case 2.**  $d_k \in D^U$ . In this case, the individual  $d_k$  will give provide the utility vectors  $U^{(k)} = (u_1^{(k)}, u_2^{(k)}, \dots, u_n^{(k)})^T$  on  $X$ , where  $u_i^{(k)}$  represents the utility value of  $x_i$  given by the individual  $d_k$ ,  $u_i^{(k)} \in [0, 1]$ , for  $i = 1, 2, \dots, n$  [32].

**Case 3.**  $d_k \in D^P$ . In this case, the individual  $d_k$  will give additive preference relations  $P^{(k)} = (p_{ij}^{(k)})_{n \times n}$  on  $X$ , where  $p_{ij}^{(k)} + p_{ji}^{(k)} = 1$ ,  $p_{ij}^{(k)} \in [0, 1]$  [33].

**Case 4.**  $d_k \in D^A$ . In this case, the individual  $d_k$  will provide multiplicative preference relations  $A^{(k)} = (a_{ij}^{(k)})_{n \times n}$  on  $X$ , where  $a_{ij}^{(k)} \times a_{ji}^{(k)} = 1$ ,  $a_{ij}^{(k)} \geq 0$  [34].

### 2.2. Formulation of the Two-Rank GDM Problems

The main notations of the two-rank GDM problem are listed as follows:

$X = \{x_1, x_2, \dots, x_n\}$  : The set of alternatives;

$D = \{d_1, d_2, \dots, d_m\}$  : The set of individuals;

$D^L = \{d_1, d_2, \dots, d_{m_1}\}$  : The set of individuals expressing the linguistic preference relations;

$D^U = \{d_{m_1+1}, d_{m_1+2}, \dots, d_{m_2}\}$  : The set of individuals expressing the utility vectors;

$D^P = \{d_{m_2+1}, d_{m_2+2}, \dots, d_{m_3}\}$  : The set of individuals expressing the additive preference relations;

$D^A = \{d_{m_3+1}, d_{m_3+2}, \dots, d_m\}$  : The set of individuals expressing the multiplicative preference relations;

$w^{(k)} = (w_1^{(k)}, w_2^{(k)}, \dots, w_n^{(k)})^T$  : Individual preference vector on  $X$  associated with  $d_k$ ;

$w^{(k*)} = (w_1^{(k*)}, w_2^{(k*)}, \dots, w_n^{(k*)})^T$  : Standardized individual preference vector on  $X$  associated with  $d_k$ ;

$w^{(c)} = (w_1^{(c)}, w_2^{(c)}, \dots, w_n^{(c)})^T$  : Collective preference vector on  $X$ ;

The problem concerned in this paper is how to obtain the collective two-rank result with the acceptable consensus level based on the above notations and variables.

## 3. THE PROPOSED METHOD

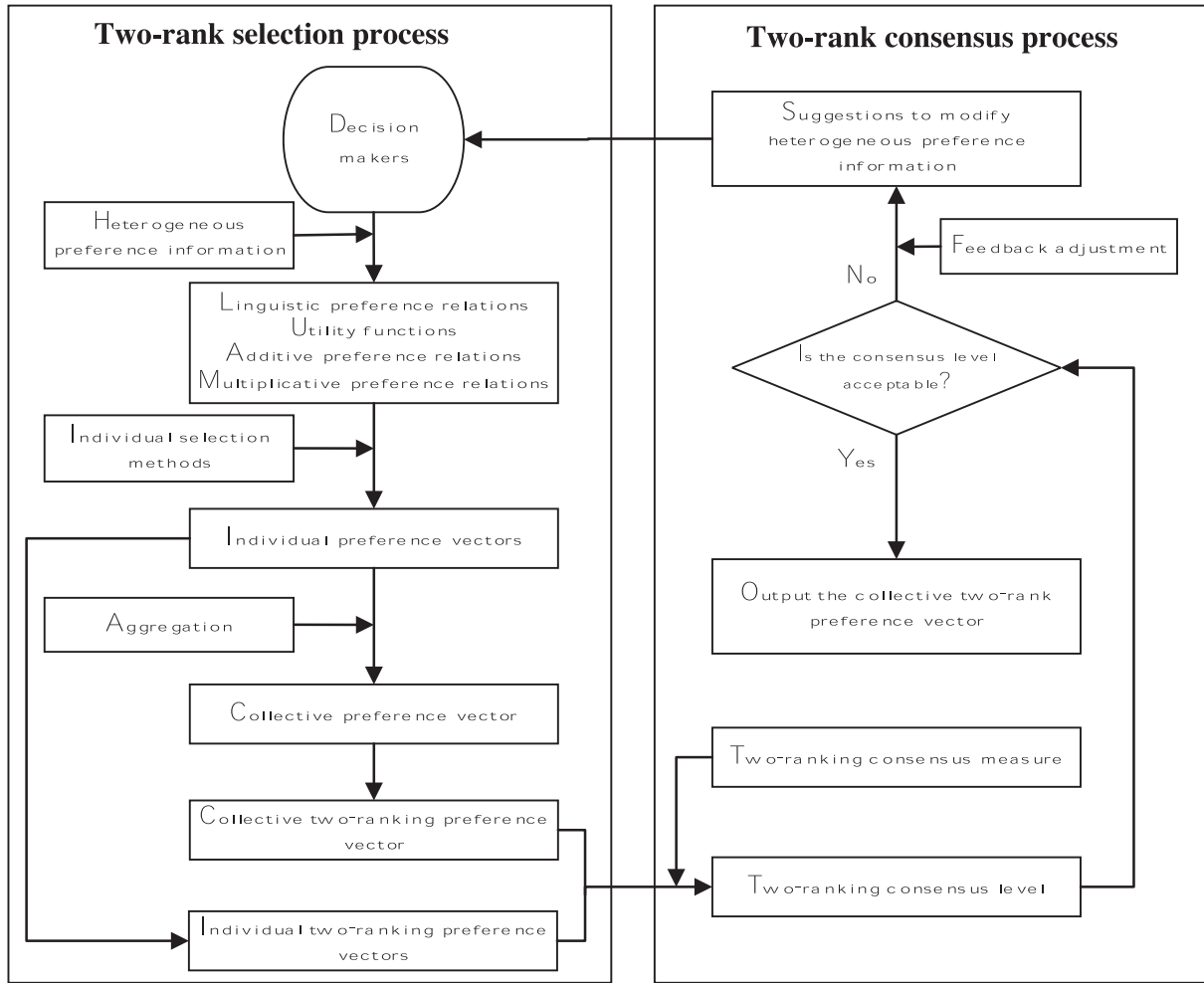
In this section, a novel consensus reaching framework for the two-rank GDM with heterogeneous preference information is firstly developed. Then, the detailed two-rank CRPs which comprises of selection process and feedback adjustments are given.

### 3.1. Two-Rank Consensus Reaching Framework

Inspired by the idea of the classical CRP, we build a novel consensus reaching framework for the two-rank GDM with heterogeneous preference information, which is depicted in Figure 1.

As shown in Figure 1, the two-rank selection process and the feedback adjustments are included in the framework.

- (i) Two-rank selection process. Firstly, the heterogeneous preference information expressed by the individuals is transformed into the individual preference vectors. Then, the collective preference vector is obtained by aggregating the individual preference vectors. Furthermore,
- (ii) Feedback adjustment. The consensus measure is further defined as the deviation between the individual preference vectors and collective preference vector. Once the consensus measure is not satisfied, it is necessary to use the feedback adjustment to improve the consensus level among different individuals. Based on the inherent characteristics of preference representation structures, the feedback adjustment rules are derived to provide the suggestions to adjust the opinions for each individual. The procedure of feedback adjustment is repeated until the consensus among individuals is achieved.



**Figure 1** | The consensus reaching process (CRP) framework of two-rank group decision-making (GDM) problems in a heterogeneous preference environment.

### 3.2. Two-Rank Selection Process

The procedure of two-rank selection process with heterogeneous preference information is given as follows:

#### Step 1: Obtaining the individual preference vectors

The individual preference vector  $w^{(k)} = (w_1^{(k)}, w_2^{(k)}, \dots, w_n^{(k)})^T$  ( $k \in M$ ) is determined by considering the following four cases:

##### Case 1. $d_k \in D^L$

In Case 1, the individual  $d_k$  provides his/her preference information on  $X$  using the linguistic preference relation  $L^{(k)} = (l_{ij}^{(k)})_{n \times n}$ , where  $l_{ij}^{(k)} \in S$ .

For computation convenience, by means of 2-tuple linguistic model, the linguistic term is usually transformed as a numerical scale. Thus, let  $B^{(k)} = (b_{ij}^{(k)})_{n \times n}$  be a numerical preference relation associated with  $L^{(k)} = (l_{ij}^{(k)})_{n \times n}$ , where

$$b_{ij}^{(k)} = \Delta^{-1} \left( l_{ij}^{(k)} \right) \quad (1)$$

From Eq. (1), it is easier to obtain that:  $b_{ii}^{(k)} = \frac{T}{2}$  and  $b_{ij}^{(k)} + b_{ji}^{(k)} = T$ .

Then, the preference on the alternative  $x_i$  by the individual  $d_k$  is calculated as follows:

$$w_i^{(k)} = \frac{1}{n} \sum_{j=1}^n b_{ij}^{(k)} \quad (2)$$

##### Case 2. $d_k \in D^U$

In Case 2, the individual  $d_k$  uses utility function  $U^{(k)} = (u_1^{(k)}, u_2^{(k)}, \dots, u_n^{(k)})^T$  to provide his/her preference information on alternatives  $X$ . Since the larger value  $u_i^{(k)}$  implies that the larger preference value  $w_i^{(k)}$ . Thus, the preference on the alternative  $x_i$  by the individual  $d_k$  is calculated as follows:

$$w_i^{(k)} = u_i^{(k)} \quad (3)$$

##### Case 3. $d_k \in D^P$

In Case 3, the individual  $d_k$  uses additive preference relation  $P^{(k)} = (p_{ij}^{(k)})_{n \times n}$  to provide his/her preference information on  $X$ .

Thus, the preference vector of the individual  $d_k$  on the alternative  $x_i$  is calculated by

$$w_i^{(k)} = \frac{1}{n} (p_{i1}^{(k)} + p_{i2}^{(k)} + \dots + p_{in}^{(k)}) \quad (4)$$

**Case 4.**  $d_k \in D^A$

In Case 4, the individual  $d_k$  provides his/her preference information on  $X$  by means of the multiplicative preference relation  $A^{(k)} = (a_{ij}^{(k)})_{n \times n}$ . Then, the row geometric mean method (RGMM) in AHP is used to derive the preference of the individual  $d_k$  on the alternative  $x_i$ .

Let  $w_i^{(k)}$  be as defined before, then the RGMM can be described as follows [35]:

$$\begin{cases} \min \sum_{i=1}^n \sum_{j>i}^n [\ln(a_{ij}^{(k)}) - (\ln(w_i^{(k)}) - \ln(w_j^{(k)}))]^2 \\ \text{s.t.} \begin{cases} \sum_{i=1}^n w_i^{(k)} = 1 \\ w_i^{(k)} \geq 0 \end{cases} \end{cases} \quad (5)$$

Crawford and Williams [35] have shown that the optimal solution to model (5) is written as the geometric means of the rows of matrix  $A^{(k)}$ :

$$w_i^{(k)} = \frac{\left( \prod_{j=1}^n a_{ij}^{(k)} \right)^{1/n}}{\sum_{i=1}^n \left( \prod_{j=1}^n a_{ij}^{(k)} \right)^{1/n}} \quad (6)$$

### Step 2: Obtaining the collective preference vectors

Due to the scale differences in the individual preference vectors of different decision makers, we first standardize the individual preference vector into the standardized preference vector.

Let  $w^{(k*)} = (w_1^{(k*)}, w_2^{(k*)}, \dots, w_n^{(k*)})^T$  be the standardized individual preference vectors of  $d_k$ , where

$$w_i^{(k*)} = \frac{w_i^{(k)}}{\sum_{i=1}^n w_i^{(k)}} \quad (7)$$

Then, let  $w^{(c)} = (w_1^{(c)}, w_2^{(c)}, \dots, w_n^{(c)})^T$  be the collective preference vector, where

$$w_i^{(c)} = \frac{1}{m} (w_i^{(1*)} + w_i^{(2*)} + \dots + w_i^{(m*)}) \quad (8)$$

### Step 3: Obtaining the individual and collective two-rank vectors

Let  $O^{(k)} = (o_1^{(k)}, o_2^{(k)}, \dots, o_n^{(k)})^T$  be the individual ranking vector of  $d_k$ , and let  $O^{(c)} = (o_1^{(c)}, o_2^{(c)}, \dots, o_n^{(c)})^T$  be collective ranking vector.

The ranking vector  $O^{(k)}$  and  $O^{(c)}$  can be determined by the preference vectors  $w^{(k*)}$  and  $w^{(c)}$ , respectively. Meanwhile, the larger values  $w_i^{(k*)}$  and  $w_i^{(c)}$  implies the smaller values for  $o_i^{(k*)}$  and  $o_i^{(c)}$ , respectively.

Then, let  $p$  be the required number in the first category. In practical two-rank GDM,  $p$  can be determined by the real demand of moderator. Let  $R^{(k)} = (r_1^{(k)}, r_2^{(k)}, \dots, r_n^{(k)})^T$  be the individual two-rank vector of  $d_k$  and let  $R^{(c)} = (r_1^{(c)}, r_2^{(c)}, \dots, r_n^{(c)})^T$  be collective two-rank vector, where

$$r_i^{(k)} = \begin{cases} 1, & o_i^{(k)} \leq p \\ 0, & \text{otherwise} \end{cases} \quad (9a)$$

and

$$r_i^{(c)} = \begin{cases} 1, & o_i^{(c)} \leq p \\ 0, & \text{otherwise} \end{cases} \quad (9b)$$

### 3.3. Feedback Adjustments

Before presenting the feedback adjustment rules, the consensus measure is defined as follows:

**Definition 1.** Let  $R^{(k)} = (r_1^{(k)}, r_2^{(k)}, \dots, r_n^{(k)})^T$  and  $R^{(c)} = (r_1^{(c)}, r_2^{(c)}, \dots, r_n^{(c)})^T$  be as defined before. Then, the two-rank consensus level among all the individuals is given by

$$CM \{d_1, d_2, \dots, d_m\} = \frac{1}{mp} \sum_{k=1}^m \sum_{i=1}^n t_i^{(k)} \quad (10)$$

where  $t_i^{(k)}$  indicates whether the individual  $d_k$  determines the same category on  $x_i$  as the collective two-rank vector, i.e.,

$$t_i^{(k)} = \begin{cases} 1, & r_i^{(k)} = 1 \text{ and } r_i^{(c)} = 1 \\ 0, & \text{otherwise} \end{cases} \quad (11)$$

If  $CM \{d_1, d_2, \dots, d_m\} = 1$ , then all the individuals have full consensus with the two-rank vectors. Otherwise, the larger  $CM \{d_1, d_2, \dots, d_m\}$  indicates the higher two-rank consensus level among the individuals.

Let  $\overline{CM}$  be the predetermined threshold for judging the acceptable consensus level. If  $CM \{d_1, d_2, \dots, d_m\} \geq \overline{CM}$ , then the acceptable two-rank consensus among the individuals is achieved.

Then, the aim of feedback adjustments is to provide individuals with adjustment suggestions for heterogeneous preference information, thus the consensus level among the individuals will be improved.

In this paper, we design a new feedback adjustment rules by considering the heterogeneous preference information. The feedback adjustment rules are composed of two key components: (i) Transforming the collective preference vector into the type of preference information expressed by each individual. For example, if the

individual  $d_k$  uses the linguistic preference information to express his/her preferences, then the collective preference vector should also be transformed into the linguistic preference relation. (ii) The individuals adjust their preferences based on the original preference information and transformed preference information.

Furthermore, with the consideration of the heterogeneous preference structures, the detailed feedback adjustment rules are introduced as follows:

**Case 1.**  $d_k \in D^L$ . In this case, the collective preference vector is transformed into a linguistic preference relation. Let  $B^{(c,k)} = (b_{ij}^{(c,k)})_{n \times n}$  be the transformed linguistic preference relation. According to 2-tuple linguistic representation model, we obtain the values of  $\Delta^{-1}(b_{ij}^{(c,k)})$  as follows:

$$\Delta^{-1}(b_{ij}^{(c,k)}) = T \frac{w_i^{(c*)}}{w_i^{(c*)} + w_j^{(c*)}} \quad (12)$$

Then, we encode the values of  $\Delta^{-1}(b_{ij}^{(c,k)})$  into  $b_{ij}^{(c,k)}$  as follows:

$$b_{ij}^{(c,k)} = \Delta(\Delta^{-1}(b_{ij}^{(c,k)})) = (r_{ij}^{(c,k)}, \alpha_{ij}^{(c,k)}) \quad (13)$$

Furthermore, let  $\bar{B}^{(k)} = (\bar{b}_{ij}^{(k)})_{n \times n}$  be the adjusted linguistic preference relation, where

$$\begin{cases} \bar{b}_{ij}^{(k)} \in [\min(b_{ij}^{(k)}, b_{ij}^{(c,k)}), \max(b_{ij}^{(k)}, b_{ij}^{(c,k)})], & i < j, \\ \bar{b}_{ij}^{(k)} = s_{T/2}, & i = j, \\ \bar{b}_{ij}^{(k)} = \boxminus(T - \boxminus^{-1}(b_{ji}^{(k)})), & i > j. \end{cases} \quad (14)$$

**Case 2.**  $d_k \in D^U$ . In this case, the collective preference vector is transformed into a utility preference vector. Let  $U^{(c,k)} = (u_1^{(c,k)}, u_2^{(c,k)}, \dots, u_n^{(c,k)})^T$  be the transformed utility vector, where

$$u_i^{(c,k)} = w_i^{(c*)} \sum_{i=1}^n u_i^{(k)} \quad (15)$$

Let  $\bar{U}^{(k)} = (\bar{u}_1^{(k)}, \bar{u}_2^{(k)}, \dots, \bar{u}_n^{(k)})^T$  be the adjusted utility preference vector, where

$$\bar{u}_i^{(k)} \in [\min(u_i^{(k)}, u_i^{(c,k)}), \max(u_i^{(k)}, u_i^{(c,k)})] \quad (16)$$

**Case 3.**  $d_k \in D^P$ . In this case, the collective preference vector is transformed into an additive preference relation. Let  $P^{(c,k)} = (p_{ij}^{(c,k)})_{n \times n}$  be a transformed additive preference relation, where

$$p_{ij}^{(c,k)} = \frac{w_i^{(c*)}}{w_i^{(c*)} + w_j^{(c*)}} \quad (17)$$

Let  $\bar{P}^{(k)} = (\bar{p}_{ij}^{(k)})_{n \times n}$  be the adjusted additive preference relation, where

$$\begin{cases} \bar{p}_{ij}^{(k)} \in [\min(p_{ij}^{(k)}, p_{ij}^{(c,k)}), \max(p_{ij}^{(k)}, p_{ij}^{(c,k)})], & i < j, \\ \bar{p}_{ij}^{(k)} = 0.5, & i = j, \\ \bar{p}_{ij}^{(k)} = 1 - \bar{p}_{ji}^{(k)}, & i > j. \end{cases} \quad (18)$$

**Case 4.**  $d_k \in D^A$ . In this case, the collective preference vector is transformed into a multiplicative preference relation. Let  $A^{(c,k)} = (a_{ij}^{(c,k)})_{n \times n}$  be a transformed multiplicative preference relation, where

$$a_{ij}^{(c,k)} = \frac{w_i^{(c*)}}{w_j^{(c*)}} \quad (19)$$

Let  $\bar{A}^{(k)} = (\bar{a}_{ij}^{(k)})_{n \times n}$  be the adjusted multiplicative preference relation, where

$$\begin{cases} \bar{a}_{ij}^{(k)} \in [\min(a_{ij}^{(k)}, a_{ij}^{(c,k)}), \max(a_{ij}^{(k)}, a_{ij}^{(c,k)})], & i < j, \\ \bar{a}_{ij}^{(k)} = 1, & i = j, \\ \bar{a}_{ij}^{(k)} = \frac{1}{\bar{a}_{ji}^{(k)}}, & i > j. \end{cases} \quad (20)$$

### 3.4. Two-rank Consensus Reaching Algorithm

Based on the proposed feedback adjustment rules, we design a two-rank consensus reaching algorithm with heterogeneous preference information.

Let  $z$  be the iteration number. Then the algorithm is given in the below.

**Input:** Maximum number of iterations  $\tau$ , the required number  $p$  in the first category, a predetermined consensus threshold  $\bar{CL}$ , and the original preference information  $\{L^{(1)}, \dots, L^{(m_1)}, U^{(m_1+1)}, \dots, U^{(m_2)}, P^{(m_2+1)}, \dots, P^{(m_3)}, A^{(m_3+1)}, \dots, A^{(m)}\}$  by individuals.

**Output:** The adjusted individual preference information  $\{\bar{L}^{(1)}, \dots, \bar{L}^{(m_1)}, \bar{U}^{(m_1+1)}, \dots, \bar{U}^{(m_2)}, \bar{P}^{(m_2+1)}, \dots, \bar{P}^{(m_3)}, \bar{A}^{(m_3+1)}, \dots, \bar{A}^{(m)}\}$ , the standardized collective preference vector  $\bar{w}^{(c*)}$ , the collective two-rank vector  $\bar{R}^{(c)} = (\bar{r}_1^{(c)}, \bar{r}_2^{(c)}, \dots, \bar{r}_n^{(c)})^T$ , and number of iterations.

**Step 1:** Let  $z = 0$ ,  $L_z^{(k)} = (l_{ij,z}^{(k)})_{n \times n} = (l_{ij}^{(k)})_{n \times n}$  ( $k = 1, 2, \dots, m_1$ ),  $U_z^{(k)} = (u_{1,z}^{(k)}, u_{2,z}^{(k)}, \dots, u_{n,z}^{(k)})^T = (u_1^{(k)}, u_2^{(k)}, \dots, u_n^{(k)})^T$  ( $k = m_1 + 1, m_1 + 2, \dots, m_2$ ),

$$P_z^{(k)} = \left( p_{ij,z}^{(k)} \right)_{n \times n} = \left( p_{ij}^{(k)} \right)_{n \times n} \quad (k = m_2 + 1, m_2 + 2, \dots, m_3) \text{ and } A_z^{(k)} = \left( a_{ij,z}^{(k)} \right)_{n \times n} = \left( a_{ij}^{(k)} \right)_{n \times n} \quad (k = m_3 + 1, m_3 + 2, \dots, m).$$

**Step 2:** Based on the two-rank selection process in Section 3.2, four cases are considered to obtain individual preference vectors.

**Case 1.**  $d_k \in D^L (k = 1, 2, \dots, m_1)$ .  $w_i^{(k,z)} = \frac{1}{n} \sum_{j=1}^n \Delta^{-1} \left( \left( l_{ij,z}^{(k)}, \alpha_{ij,z}^{(k)} \right) \right)$ .

**Case 2.**  $d_k \in D^U (k = m_1 + 1, m_1 + 2, \dots, m_2)$ .  $w_i^{(k,z)} = u_{i,z}^{(k)}$ .

**Case 3.** In this case,  $d_k \in D^P (k = m_2 + 1, m_2 + 2, \dots, m_3)$ .  $w_i^{(k,z)} = \frac{1}{n} \left( p_{i1,z}^{(k)} + p_{i2,z}^{(k)} + \dots + p_{in,z}^{(k)} \right)$ .

**Case 4.** In this case,  $d_k \in D^A (k = m_3 + 1, m_3 + 2, \dots, m)$ .  $w_i^{(k,z)} = \frac{\left( \prod_{j=1}^n a_{ij,z}^{(k)} \right)^{1/n}}{\sum_{i=1}^n \left( \prod_{j=1}^n a_{ij,z}^{(k)} \right)^{1/n}}$ .

**Step 3:** Transform  $w^{(k,z)} = \left( w_1^{(k,z)}, w_2^{(k,z)}, \dots, w_n^{(k,z)} \right)^T$  into  $w^{(k*,z)} = \left( w_1^{(k*,z)}, w_2^{(k*,z)}, \dots, w_n^{(k*,z)} \right)^T$  by using Eq. (7). Then, using Eq. (8) derives the collective preference vector  $w^{(c,z)} = \left( w_1^{(c,z)}, w_2^{(c,z)}, \dots, w_n^{(c,z)} \right)^T$ .

**Step 4:** According to the preference vectors  $w^{(k*,z)} = \left( w_1^{(k*,z)}, w_2^{(k*,z)}, \dots, w_n^{(k*,z)} \right)^T$  and  $w^{(c,z)} = \left( w_1^{(c,z)}, w_2^{(c,z)}, \dots, w_n^{(c,z)} \right)^T$ , we obtain the ranking vectors  $O_z^{(k)} = \left( o_{1,z}^{(k)}, o_{2,z}^{(k)}, \dots, o_{n,z}^{(k)} \right)^T$  and  $O_z^{(c)} = \left( o_{1,z}^{(c)}, o_{2,z}^{(c)}, \dots, o_{n,z}^{(c)} \right)^T$ . Next, utilizing the value of  $p$ , the two-rank vectors  $R_z^{(k)} = \left( r_{1,z}^{(k)}, r_{2,z}^{(k)}, \dots, r_{n,z}^{(k)} \right)^T$  and  $R_z^{(c)} = \left( r_{1,z}^{(c)}, r_{2,z}^{(c)}, \dots, r_{n,z}^{(c)} \right)^T$  are determined.

**Step 5:** Using Eq. (10) calculates the two-rank consensus level  $CM \{d_1, d_2, \dots, d_m\}$  among all the individuals. If  $CM \{d_1, d_2, \dots, d_m\} \geq \overline{CL}$ , go to Step 8; otherwise, go to Step 6.

**Step 6:** With the consideration of heterogeneous preference information, the feedback adjustment rules are proposed.

**Case 1.**  $d_k \in D^L (k = 1, 2, \dots, m_1)$ . Let  $B_z^{(c,k)} = \left( b_{ij,z}^{(c,k)} \right)_{n \times n}$ . Using Eq. (12) obtains the semantics  $\Delta^{-1} \left( b_{ij,z}^{(c,k)} \right)$ . Then, we encode the values of  $\Delta^{-1} \left( b_{ij,z}^{(c,k)} \right)$  into  $b_{ij}^{(c,k)} = \left( l_{ij,z}^{(c,k)}, \alpha_{ij,z}^{(c,k)} \right)$  by using Eq. (13). Next, the linguistic preference relations  $L_{z+1}^{(k)} = \left( l_{ij,z+1}^{(k)}, \alpha_{ij,z+1}^{(k)} \right)_{n \times n}$  are derived as:

$$\begin{cases} \left( l_{ij,z+1}^{(k)}, \alpha_{ij,z+1}^{(k)} \right) \in \left[ \min \left( l_{ij,z}^{(k)}, l_{ij,z}^{(c,k)} \right), \max \left( l_{ij,z}^{(k)}, l_{ij,z}^{(c,k)} \right) \right], & i < j, \\ l_{ij,z+1}^{(k)} = s_{T/2}, & i = j, \\ l_{ij,z+1}^{(k)} = \Delta \left( T - \Delta^{-1} \left( l_{ji,z+1}^{(k)} \right) \right), & i > j. \end{cases}$$

**Case 2.**  $d_k \in D^U (k = m_1 + 1, m_1 + 2, \dots, m_2)$ . Let  $U_z^{(c,k)} = \left( u_{1,z}^{(c,k)}, u_{2,z}^{(c,k)}, \dots, u_{n,z}^{(c,k)} \right)^T$ . Using Eq. (15) obtains  $U_z^{(c,k)}$ . Next, the utility preference vectors  $U_{z+1}^{(c,k)} = \left( u_{1,z+1}^{(c,k)}, u_{2,z+1}^{(c,k)}, \dots, u_{n,z+1}^{(c,k)} \right)^T$  are derived as:

$$u_{i,z+1}^{(k)} \in \left[ \min \left( u_{i,z}^{(k)}, u_{i,z}^{(c,k)} \right), \max \left( u_{i,z}^{(k)}, u_{i,z}^{(c,k)} \right) \right]$$

**Case 3.**  $d_k \in D^P (k = m_2 + 1, m_2 + 2, \dots, m_3)$ . Let  $P_z^{(c,k)} = \left( p_{ij,z}^{(c,k)} \right)_{n \times n}$ . Using Eq. (17) obtains  $P_z^{(c,k)}$ . Next, the additive preference relations  $P_{z+1}^{(k)} = \left( p_{ij,z+1}^{(k)} \right)_{n \times n}$  are derived as:

$$\begin{cases} p_{ij,z+1}^{(k)} \in \left[ \min \left( p_{ij,z}^{(k)}, p_{ij,z}^{(c,k)} \right), \max \left( p_{ij,z}^{(k)}, p_{ij,z}^{(c,k)} \right) \right], & i < j, \\ p_{ij,z+1}^{(k)} = 0.5, & i = j, \\ p_{ij,z+1}^{(k)} = 1 - p_{ji,z+1}^{(k)}, & i > j. \end{cases}$$

**Case 4.**  $d_k \in D^A (k = m_3 + 1, m_3 + 2, \dots, m)$ . Let  $A_z^{(c,k)} = \left( a_{ij,z}^{(c,k)} \right)_{n \times n}$ . Using Eq. (19) obtains  $A_z^{(c,k)}$ . Next, the multiplicative preference relations  $A_{z+1}^{(k)} = \left( a_{ij,z+1}^{(k)} \right)_{n \times n}$  are derived as:

$$\begin{cases} a_{ij,z+1}^{(k)} \in \left[ \min \left( a_{ij,z}^{(k)}, a_{ij,z}^{(c,k)} \right), \max \left( a_{ij,z}^{(k)}, a_{ij,z}^{(c,k)} \right) \right], & i < j, \\ a_{ij,z+1}^{(k)} = 1, & i = j, \\ a_{ij,z+1}^{(k)} = \frac{1}{a_{ji,z+1}^{(k)}}, & i > j. \end{cases}$$

**Step 7:** If  $z \geq \tau$ , go to step 8, otherwise let  $z = z + 1$ , then go to step 2.

**Step 8:** Let  $\overline{L}^{(k)} = \overline{L}_z^{(k)} (k = 1, 2, \dots, m_1)$ ,  $\overline{U}^{(k)} = \overline{U}_z^{(k)} (k = m_1 + 1, m_1 + 2, \dots, m_2)$ ,  $\overline{P}^{(k)} = \overline{P}_z^{(k)} (k = m_2 + 1, m_2 + 2, \dots, m_3)$ ,  $\overline{A}^{(k)} = \overline{A}_z^{(k)} (m_3 + 1, m_3 + 2, \dots, m)$ ,  $\overline{w}^{(c^*)} = \overline{w}^{(c^*,z)}$ ,  $\overline{R}^{(k)} = \overline{R}_z^{(k)}$  and  $\overline{R}^{(c)} = \overline{R}_z^{(c)}$ .

**Step 9:** Output adjusted individual preference information  $\{ \overline{L}^{(1)}, \dots, \overline{L}^{(m_1)}, \overline{U}^{(m_1+1)}, \dots, \overline{U}^{(m_2)}, \overline{P}^{(m_2+1)}, \dots, \overline{P}^{(m_3)}, \overline{A}^{(m_3+1)}, \dots, \overline{A}^{(m)} \}$ , the standardized collective preference vector  $\overline{w}^{(c*)}$ , the collective two-rank vector  $\overline{R}^{(c)} = \left( \overline{r}_1^{(c)}, \overline{r}_2^{(c)}, \dots, \overline{r}_n^{(c)} \right)^T$ , and the number of iterations  $z$ .

**Note 1.** Compared with the existing consensus reaching models, the proposed consensus reaching algorithms increase the agreements of individuals on the two categories of alternatives (i.e., two-rank). Meanwhile, the provided feedback adjustment rules facilitate the individuals adjust their own preference information in the premise of without transforming the preference representation structures.

## 4. NUMERICAL EXAMPLE

In this section, a practical example in the background of subway construction plan is given to show the feasibility of the proposed two-rank CRPs.

Nowadays, the subway has become one of the most popular transportation tools for people. Thus, in this context, many cities all over the world paid the consideration attention on the subway construction. B city is an important city in western China, which is also famous for its ancient culture. To ease the road congestion and contribute to better air quality, B city decides to construct a new subway. Due to the complicated terrain and heritage conservation, the government in B city obtains the subway construction plans of other cities via various legal channels. Then, some approved plans will provide the important references for determining the final subway construction plan.



Suppose B city obtains six subway construction plans  $X = \{x_1, x_2, \dots, x_6\}$  after preliminary investigations. B city government invites eight experts (i.e., decision makers)  $D = \{d_1, d_2, \dots, d_8\}$  to distinguish six plans into two categories: approved plans and disapproved plans. To facilitate the expressions of decision makers, B city government allows the decision makers can use their own means to express the preference information.

Suppose the decision makers  $d_1$  and  $d_2$  use the linguistic preference relations  $L^{(1)}$  and  $L^{(2)}$  to express preference information on  $X$ . Here, the term set used by the decision maker is  $S = \{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{moderate}, s_4 = \text{good}, s_5 = \text{very good}, s_6 = \text{extremely good}\}$ . Then,  $L^{(1)}$  and  $L^{(2)}$  are given as follows:

$$L^{(1)} = \begin{bmatrix} s_3 & s_4 & s_6 & s_5 & s_3 & s_6 \\ s_2 & s_3 & s_5 & s_6 & s_3 & s_5 \\ s_0 & s_1 & s_3 & s_3 & s_1 & s_4 \\ s_1 & s_0 & s_3 & s_3 & s_1 & s_4 \\ s_3 & s_3 & s_5 & s_5 & s_3 & s_5 \\ s_0 & s_1 & s_2 & s_2 & s_1 & s_3 \end{bmatrix}, L^{(2)} = \begin{bmatrix} s_3 & s_3 & s_5 & s_6 & s_2 & s_5 \\ s_3 & s_3 & s_5 & s_6 & s_2 & s_5 \\ s_1 & s_1 & s_3 & s_3 & s_1 & s_3 \\ s_0 & s_0 & s_3 & s_3 & s_0 & s_3 \\ s_4 & s_4 & s_5 & s_6 & s_3 & s_6 \\ s_1 & s_1 & s_3 & s_3 & s_0 & s_3 \end{bmatrix}$$

The decision makers  $d_3$  and  $d_4$  use the utility functions  $U^{(3)}$  and  $U^{(4)}$  to express preference information on  $X$ , i.e.,

$$U^{(3)} = (0.7, 0.8, 0.2, 0.4, 0.7, 0.2)^T, \\ U^{(4)} = (0.5, 0.9, 0.6, 0.2, 0.7, 0.1)^T.$$

The decision makers  $d_5$  and  $d_6$  use the additive preference relation  $P^{(5)}$  and  $P^{(6)}$  to express preference information on  $X$ , i.e.,

$$P^{(5)} = \begin{bmatrix} 0.5 & 0.54 & 0.83 & 0.89 & 0.55 & 0.92 \\ 0.46 & 0.5 & 0.8 & 0.82 & 0.58 & 0.91 \\ 0.17 & 0.2 & 0.5 & 0.55 & 0.16 & 0.73 \\ 0.11 & 0.18 & 0.45 & 0.5 & 0.12 & 0.75 \\ 0.45 & 0.42 & 0.84 & 0.88 & 0.5 & 0.86 \\ 0.08 & 0.09 & 0.27 & 0.25 & 0.14 & 0.5 \end{bmatrix}, \\ P^{(6)} = \begin{bmatrix} 0.5 & 0.45 & 0.85 & 0.85 & 0.53 & 0.87 \\ 0.55 & 0.5 & 0.93 & 0.9 & 0.65 & 0.96 \\ 0.15 & 0.07 & 0.5 & 0.49 & 0.15 & 0.52 \\ 0.15 & 0.1 & 0.51 & 0.5 & 0.16 & 0.55 \\ 0.47 & 0.35 & 0.85 & 0.84 & 0.5 & 0.84 \\ 0.13 & 0.04 & 0.48 & 0.45 & 0.16 & 0.5 \end{bmatrix}$$

The decision makers  $d_7$  and  $d_8$  use the multiplicative preference relation  $A^{(7)}$  and  $A^{(8)}$  to express preference information on  $X$ , i.e.,

$$A^{(7)} = \begin{bmatrix} 1 & 1/3 & 7 & 3 & 2 & 5 \\ 3 & 1 & 9 & 5 & 4 & 8 \\ 1/7 & 1/9 & 1 & 1/6 & 1/5 & 1/4 \\ 1/3 & 1/5 & 6 & 1 & 1/2 & 3 \\ 1/2 & 1/4 & 5 & 2 & 1 & 3 \\ 1/5 & 1/8 & 4 & 1/3 & 1/3 & 1 \end{bmatrix}, A^{(8)} = \begin{bmatrix} 1 & 5 & 6 & 1/4 & 1/4 & 7 \\ 1/5 & 1 & 5 & 1/5 & 1/6 & 7 \\ 1/6 & 1/5 & 1 & 1/6 & 1/7 & 2 \\ 4 & 5 & 6 & 1 & 3 & 9 \\ 4 & 6 & 7 & 1/3 & 1 & 8 \\ 1/7 & 1/7 & 1/2 & 1/9 & 1/8 & 1 \end{bmatrix}$$

In the following, we use the CRP in Section 3 to determine the approved plans.

For  $d_1, d_2 \in D^L$ , we determine the standardized individual preference vectors by using Eqs. (1) and (2) and (7), i.e.,

$$w^{(1*)} = (0.2500, 0.2222, 0.1111, 0.1111, 0.2222, 0.0834)^T, \\ w^{(2*)} = (0.2222, 0.2222, 0.1111, 0.0833, 0.2593, 0.1019)^T.$$

For  $d_3, d_4 \in D^U$ , we determine the standardized individual preference vectors by using Eqs. (3) and (7), i.e.,

$$w^{(3*)} = (0.2333, 0.2667, 0.0667, 0.1333, 0.2333, 0.0667)^T, \\ w^{(4*)} = (0.1667, 0.3000, 0.2000, 0.0667, 0.2333, 0.0333)^T.$$

For  $d_5, d_6 \in D^P$ , we determine the standardized individual preference vectors by using Eqs. (4) and (7), i.e.,

$$w^{(5*)} = (0.2350, 0.2261, 0.1283, 0.1172, 0.2194, 0.0740)^T, \\ w^{(6*)} = (0.2249, 0.2493, 0.1044, 0.1094, 0.2143, 0.0977)^T.$$

For  $d_7, d_8 \in D^A$ , we determine the standardized individual preference vectors by using Eqs. (6) and (7), i.e.,

$$w^{(7*)} = (0.2274, 0.4520, 0.0253, 0.1028, 0.1396, 0.0529)^T, \\ w^{(8*)} = (0.1616, 0.0826, 0.0359, 0.4048, 0.2911, 0.0240)^T.$$

Then, based on Eq. (8), the collective standardized preference vectors are determined by

$$w^{(c*)} = (0.2151, 0.2526, 0.0979, 0.1411, 0.2266, 0.0667)^T.$$

And the individual ranking vectors and collective ranking vectors are obtained as

$$o^{(1)} = (1, 2, 4, 5, 3, 6)^T, o^{(2)} = (2, 3, 4, 6, 1, 5)^T, \\ o^{(3)} = (2, 1, 5, 4, 3, 6)^T, o^{(4)} = (4, 1, 3, 5, 2, 6)^T, \\ o^{(5)} = (1, 2, 4, 5, 3, 6)^T, o^{(6)} = (2, 1, 5, 4, 3, 6)^T, \\ o^{(7)} = (2, 1, 6, 4, 3, 5)^T, o^{(8)} = (3, 4, 5, 1, 2, 6)^T, \\ o^{(c)} = (3, 1, 5, 4, 2, 6)^T.$$

Suppose the B city government wants to choose three approved plans, i.e.,  $p = 3$ . Based on Eqs. (9) and (10), we obtain the individual and collective two-rank vectors, i.e.,

$$R^{(1)} = (1, 1, 0, 0, 1, 0)^T, R^{(2)} = (1, 1, 0, 0, 1, 0)^T, \\ R^{(3)} = (1, 1, 0, 0, 1, 0)^T, R^{(4)} = (0, 1, 1, 0, 1, 0)^T, \\ R^{(5)} = (1, 1, 0, 0, 1, 0)^T, R^{(6)} = (1, 1, 0, 0, 1, 0)^T, \\ R^{(7)} = (1, 1, 0, 0, 1, 0)^T, R^{(8)} = (1, 0, 0, 1, 1, 0)^T, \\ R^{(c)} = (1, 1, 0, 0, 1, 0)^T.$$

Next, the two-rank consensus level is calculated as  $CM\{d_1, d_2, \dots, d_m\} = \frac{11}{12}$ . Suppose B city government establishes the threshold as:  $\overline{CM} = 0.95$ . Obviously,  $CM\{d_1, d_2, \dots, d_m\} < \overline{CM}$ , i.e., the acceptable consensus is not reached. Thus, it is necessary to carry out the feedback adjustment stage.

In the feedback adjustment stage, the collective preference vector into the type of preference information expressed by each individual.

For  $d_1, d_2 \in D^L$ , we utilize Eq. (12) to obtain the transformed linguistic preference relations:

$$\begin{aligned} B^{(c,1)} &= B^{(c,2)} \\ &= \begin{bmatrix} (s_3, 0.00) & (s_3, 0.16) & (s_4, -0.04) & (s_4, -0.06) & (s_3, 0.28) & (s_4, 0.36) \\ (s_3, -0.16) & (s_3, 0.00) & (s_4, -0.19) & (s_4, -0.21) & (s_3, 0.13) & (s_4, 0.23) \\ (s_2, 0.04) & (s_2, 0.19) & (s_3, 0.00) & (s_3, -0.02) & (s_2, 0.31) & (s_3, 0.47) \\ (s_2, 0.06) & (s_2, 0.21) & (s_3, 0.02) & (s_3, 0.00) & (s_2, 0.33) & (s_3, 0.49) \\ (s_3, -0.28) & (s_3, -0.13) & (s_4, -0.31) & (s_4, -0.33) & (s_3, 0.00) & (s_4, 0.12) \\ (s_2, -0.36) & (s_2, -0.23) & (s_3, -0.47) & (s_3, -0.49) & (s_2, -0.12) & (s_3, 0.00) \end{bmatrix}. \end{aligned}$$

For  $d_3, d_4 \in D^U$ , we utilize Eq. (15) to obtain the transformed utility preference vectors:

$$\begin{aligned} U^{(c,3)} &= (0.7241, 0.6515, 0.3742, 0.3793, 0.5985, 0.2725)^T, \\ U^{(c,4)} &= (0.7241, 0.6515, 0.3742, 0.3793, 0.5985, 0.2725)^T. \end{aligned}$$

For  $d_5, d_6 \in D^P$ , we utilize Eq. (17) to obtain the transformed additive preference relation:

$$\begin{aligned} P^{(c,5)} &= P^{(c,6)} \\ &= \begin{bmatrix} 0.5000 & 0.5270 & 0.6578 & 0.6572 & 0.5497 & 0.7258 \\ 0.4730 & 0.5000 & 0.6331 & 0.6325 & 0.5229 & 0.7038 \\ 0.3422 & 0.3669 & 0.5000 & 0.4994 & 0.3884 & 0.5793 \\ 0.3428 & 0.3675 & 0.5006 & 0.5000 & 0.3890 & 0.5799 \\ 0.4503 & 0.4771 & 0.6116 & 0.6110 & 0.5000 & 0.6843 \\ 0.2742 & 0.2962 & 0.4207 & 0.4201 & 0.3157 & 0.5000 \end{bmatrix}. \end{aligned}$$

For  $d_7, d_8 \in D^A$ , we utilize Eq. (19) to obtain the transformed multiplicative preference relation:

$$\begin{aligned} A^{(c,7)} &= A^{(c,8)} \\ &= \begin{bmatrix} 1.0000 & 1.1141 & 1.9222 & 1.9173 & 1.2209 & 2.6467 \\ 0.8976 & 1.0000 & 1.7254 & 1.7210 & 1.0959 & 2.3757 \\ 0.5202 & 0.5796 & 1.0000 & 0.9975 & 0.6352 & 1.3769 \\ 0.5216 & 0.5811 & 1.0025 & 1.0000 & 0.6368 & 1.3805 \\ 0.8191 & 0.9125 & 1.5744 & 1.5704 & 1.0000 & 2.1679 \\ 0.3778 & 0.4209 & 0.7262 & 0.7244 & 0.4613 & 1.0000 \end{bmatrix}. \end{aligned}$$

Then, we obtain the adjusted preference information of each individual as follows:

For  $d_1, d_2 \in D^L$ , we utilize Eq. (14) to derive their adjusted preference information as

$$\begin{aligned} \overline{L^{(1)}} &= \begin{bmatrix} (s_3, 0.00) & (s_4, -0.17) & (s_6, -0.41) & (s_5, -0.21) & (s_3, 0.23) & (s_6, -0.33) \\ (s_2, 0.17) & (s_3, 0.00) & (s_5, -0.24) & (s_6, -0.44) & (s_3, 0.10) & (s_5, -0.15) \\ (s_0, 0.41) & (s_1, 0.24) & (s_3, 0.00) & (s_3, 0.00) & (s_2, 0.05) & (s_4, -0.11) \\ (s_1, 0.21) & (s_0, 0.44) & (s_3, 0.00) & (s_3, 0.00) & (s_2, 0.06) & (s_4, -0.10) \\ (s_3, -0.23) & (s_3, -0.10) & (s_4, -0.05) & (s_4, -0.06) & (s_3, 0.00) & (s_5, -0.18) \\ (s_0, 0.33) & (s_1, 0.15) & (s_2, 0.11) & (s_2, 0.10) & (s_1, 0.18) & (s_3, 0.00) \end{bmatrix}, \end{aligned}$$

$\overline{L^{(2)}}$

$$= \begin{bmatrix} (s_3, 0.00) & (s_3, 0.13) & (s_5, -0.21) & (s_6, -0.41) & (s_3, 0.03) & (s_5, -0.13) \\ (s_3, -0.13) & (s_3, 0.00) & (s_5, -0.24) & (s_6, -0.44) & (s_3, -0.10) & (s_5, -0.15) \\ (s_1, 0.21) & (s_1, 0.24) & (s_3, 0.00) & (s_3, 0.00) & (s_2, 0.05) & (s_3, 0.38) \\ (s_0, 0.41) & (s_0, 0.44) & (s_3, 0.00) & (s_3, 0.00) & (s_2, -0.14) & (s_3, 0.39) \\ (s_3, -0.03) & (s_3, 0.10) & (s_4, -0.05) & (s_4, 0.14) & (s_3, 0.00) & (s_6, -0.38) \\ (s_1, 0.13) & (s_1, 0.15) & (s_3, -0.38) & (s_3, -0.39) & (s_0, 0.38) & (s_3, 0.00) \end{bmatrix}.$$

For  $d_3, d_4 \in D^U$ , we utilize Eq. (16) to obtain derive their adjusted preference information as

$$\begin{aligned} \overline{U^{(3)}} &= (0.7202, 0.7702, 0.3419, 0.3957, 0.6788, 0.2592)^T, \\ \overline{U^{(4)}} &= (0.6802, 0.8502, 0.5555, 0.3426, 0.6788, 0.2392)^T. \end{aligned}$$

For  $d_5, d_6 \in D^P$ , we utilize Eq. (18) to obtain derive their adjusted preference information as

$$\begin{aligned} \overline{P^{(5)}} &= \begin{bmatrix} 0.5000 & 0.5374 & 0.7956 & 0.8434 & 0.5499 & 0.8812 \\ 0.4626 & 0.5000 & 0.7666 & 0.7825 & 0.5686 & 0.8688 \\ 0.2044 & 0.2334 & 0.5000 & 0.5399 & 0.3428 & 0.6999 \\ 0.1566 & 0.2175 & 0.4601 & 0.5000 & 0.3352 & 0.7160 \\ 0.4501 & 0.4314 & 0.6572 & 0.6648 & 0.5000 & 0.8249 \\ 0.1188 & 0.1312 & 0.3001 & 0.2840 & 0.1751 & 0.5000 \end{bmatrix}, \end{aligned}$$

$\overline{P^{(6)}}$

$$= \begin{bmatrix} 0.5000 & 0.5116 & 0.8116 & 0.8114 & 0.5458 & 0.8412 \\ 0.4884 & 0.5000 & 0.8706 & 0.8465 & 0.6246 & 0.9088 \\ 0.1884 & 0.1294 & 0.5000 & 0.4975 & 0.3408 & 0.5674 \\ 0.1886 & 0.1535 & 0.5025 & 0.5000 & 0.3432 & 0.5739 \\ 0.4542 & 0.3754 & 0.6592 & 0.6568 & 0.5000 & 0.8169 \\ 0.1588 & 0.0912 & 0.4326 & 0.4261 & 0.1831 & 0.5000 \end{bmatrix}.$$

For  $d_7, d_8 \in D^A$ , we utilize Eq. (20) to obtain derive their adjusted preference information as

$$\overline{A^{(7)}} = \begin{bmatrix} 1 & 1 & 3 & 2 & 2 & 3 \\ 1 & 1 & 3 & 2 & 2 & 4 \\ 1/3 & 1/3 & 1 & 1/4 & 1/2 & 1 \\ 1/2 & 1/2 & 4 & 1 & 1/2 & 2 \\ 1/2 & 1/2 & 2 & 2 & 1 & 3 \\ 1/3 & 1/4 & 1 & 1/2 & 1/3 & 1 \end{bmatrix},$$

$$\overline{A^{(8)}} = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 4 \\ 1/2 & 1 & 2 & 1 & 1 & 3 \\ 1/3 & 1/2 & 1 & 1/3 & 1/2 & 2 \\ 1 & 1 & 3 & 1 & 1 & 3 \\ 1 & 1 & 2 & 1 & 1 & 3 \\ 1/4 & 1/3 & 1/2 & 1/3 & 1/3 & 1 \end{bmatrix}.$$



Next, based on the adjusted preference information, we obtain the corresponding standardized individual preference vectors as

$$\begin{aligned}w_1^{(1*)} &= (0.2419, 0.2170, 0.1260, 0.1261, 0.1976, 0.0914)^T, \\w_1^{(2*)} &= (0.2261, 0.2217, 0.1287, 0.1121, 0.2106, 0.1009)^T, \\w_1^{(3*)} &= (0.2275, 0.2433, 0.1080, 0.1250, 0.2144, 0.0819)^T, \\w_1^{(4*)} &= (0.2033, 0.2541, 0.1660, 0.1024, 0.2028, 0.0715)^T, \\w_1^{(5*)} &= (0.2282, 0.2194, 0.1400, 0.1325, 0.1960, 0.0839)^T, \\w_1^{(6*)} &= (0.2234, 0.2355, 0.1235, 0.1257, 0.1924, 0.0995)^T, \\w_1^{(7*)} &= (0.2409, 0.2836, 0.0919, 0.1181, 0.1875, 0.0780)^T, \\w_1^{(8*)} &= (0.2856, 0.1799, 0.1111, 0.1603, 0.2019, 0.0612)^T, \\w_1^{(c*)} &= (0.2346, 0.2318, 0.1244, 0.1253, 0.2004, 0.0835)^T.\end{aligned}$$

And the individual ranking vectors and collective ranking vectors are obtained as

$$\begin{aligned}\bar{o}_1^{(1)} &= (1, 2, 5, 4, 3, 6)^T, \bar{o}_1^{(2)} = (1, 2, 4, 5, 3, 6)^T, \\ \bar{o}_1^{(3)} &= (2, 1, 5, 4, 3, 6)^T, \bar{o}_1^{(4)} = (2, 1, 4, 5, 3, 6)^T, \\ \bar{o}_1^{(5)} &= (1, 2, 4, 5, 3, 6)^T, \bar{o}_1^{(6)} = (2, 1, 5, 4, 3, 6)^T, \\ \bar{o}_1^{(7)} &= (2, 1, 5, 4, 3, 6)^T, \bar{o}_1^{(8)} = (1, 3, 5, 4, 2, 6)^T, \\ \bar{o}_1^{(c)} &= (1, 2, 5, 4, 3, 6)^T.\end{aligned}$$

Furthermore, we obtain the individual and collective two-rank vectors, i.e.,

$$\begin{aligned}\bar{R}^{(1)} &= (1, 1, 0, 0, 1, 0)^T, \bar{R}^{(2)} = (1, 1, 0, 0, 1, 0)^T, \\ \bar{R}^{(3)} &= (1, 1, 0, 0, 1, 0)^T, \bar{R}^{(4)} = (1, 1, 0, 0, 1, 0)^T, \\ \bar{R}^{(5)} &= (1, 1, 0, 0, 1, 0)^T, \bar{R}^{(6)} = (1, 1, 0, 0, 1, 0)^T, \\ \bar{R}^{(7)} &= (1, 1, 0, 0, 1, 0)^T, \bar{R}^{(8)} = (1, 1, 0, 1, 0, 0)^T, \\ \bar{R}^{(c)} &= (1, 1, 0, 0, 1, 0)^T.\end{aligned}$$

Finally, the two-rank consensus level is calculated as  $CM\{d_1, d_2, \dots, d_m\} = 1 > 0.95$ . Thus, an acceptable consensus has been reached. From the obtained collective two-rank result, B city government determines three subway construction plans  $x_1$ ,  $x_2$ , and  $x_4$  as the approved subway construction plans, and the subway construction plan  $x_1$  is considered with the highest priority.

## 5. CONCLUSION

This paper develops a novel consensus reaching method for two-rank GDM with heterogeneous preference information. The main contribution of this paper is presented as follows:

- (1) With the consideration of the heterogeneous preference information, we propose the method for deriving the individual preferences vectors. Then, the collective preference vector is obtained by aggregating the individual preferences vectors. Furthermore, the individual and collective two-rank vectors are obtained by the corresponding preference vectors.

- (2) We design a new consensus measure for two-rank GDM. Then, the feedback adjustment rules are proposed to improve the consensus level among the individuals. The collective preference vector is transformed into the type of preference information expressed by each individual. On this basis, the suggestions for adjusting the preference information are provided.
- (3) We design an algorithm to describe the two-rank CRP with heterogeneous preference information. Then, we present a practical example to illustrate the feasibility of the proposed method.

In the future, we argue three interesting research topics:

- (1) In some practical GDM problems, the individuals are always in a social network [36–38]. This will strongly impact the communication among different individuals. Thus, it would be an interesting topic for extending the proposed method into a social network context.
- (2) It is assumed that small number of individuals participated in two-rank GDM. However, in many scenarios, the large number of individuals will be included in two-rank GDM [39–42]. It is also necessary to investigate the large-scale two-rank GDM.
- (3) Minimum adjustment which implies to optimize the adjusted cost for each individual, has been regarded as an important criterion for evaluating the performances of consensus reaching. Thus, it is important to develop a new consensus reaching model with minimum adjustments for two-rank GDM.

## CONFLICTS OF INTEREST

The authors declare no conflicts of interest regarding the publication for the paper.

## AUTHORS' CONTRIBUTIONS

Huali Tang: Conceptualization, Methodology, Writing - original draft. Shoufu Wan: Conceptualization, Methodology, Writing - review & editing. Cong-Cong Li: Conceptualization, Methodology, Writing - review & editing. Haiming Liang: Conceptualization, Methodology, Writing - original draft. Yucheng Dong: Conceptualization, Methodology, Funding acquisition, Writing - review & editing.

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## REFERENCES

- [1] D.S. Hochbaum, A. Levin, *Methodologies and algorithms for group-rankings decision*, *Manage. Sci.* 52 (2006), 1394–1408.
- [2] A. Altuzarra, J.M. Moreno-Jiménez, M. Salvador, *Consensus building in AHP-group decision making: a bayesian approach*, *Oper. Res.* 58 (2010), 1755–1773.

- [3] X. Chen, H.J. Zhang, Y.C. Dong, The fusion process with heterogeneous preference structures in group decision making: a survey, *Inf. Fusion*. 24 (2015), 72–83.
- [4] Z.S. Chen, X. Zhang, J.P. Chang, K.S. Chin, Bid evaluation for major construction projects under large-scale group decision-making environment and characterized expertise levels, *Int. J. Comput. Int. Syst.* 13 (2020), 1227–1242.
- [5] S.B. Yao, A new distance-based consensus reaching model for multi-attribute group decision-making with linguistic distribution assessments, *Int. J. Comput. Int. Sys.* 12 (2019), 395–409.
- [6] Y.J. Xu, C.Y. Li, X.W. Wen, Missing values estimation and consensus building for incomplete hesitant fuzzy preference relations with multiplicative consistency, *Int. J. Comput. Int. Syst.* 11 (2018), 101–119.
- [7] R.M. Rodríguez, Á. Labella, G. De Tré, L. Martínez, A large scale consensus reaching process managing group hesitation, *Knowl-Based. Syst.* 159 (2018), 86–97.
- [8] F. Chiclana, F. Herrera, E. Herrera-Viedma, Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations, *Fuzzy. Set. Syst.* 97 (1998), 33–48.
- [9] F. Chiclana, F. Herrera, E. Herrera-Viedma, Integrating multiplicative preference relations in a multipurpose decision-making model based on fuzzy preference relations, *Fuzzy. Set. Syst.* 122 (2001), 277–291.
- [10] Y.C. Dong, Y.F. Xu, S. Yu, Linguistic multiperson decision making based on the use of multiple preference relations, *Fuzzy. Set. Syst.* 160 (2009), 603–623.
- [11] Y.C. Dong, H.J. Zhang, Multiperson decision making with different preference representation structures: a direct consensus framework and its properties, *Knowl-Based. Syst.* 58 (2014), 45–57.
- [12] E. Herrera-Viedma, F. Herrera, F. Chiclana, A consensus model for multiperson decision making with different preference structures, *IEEE Trans. Syst. Man. CY-A*. 32 (2002), 394–402.
- [13] F. Herrera, E. Herrera-Viedma, L. Martínez, A fusion approach for managing multi-granularity linguistic term sets in decision making, *Fuzzy. Set. Syst.* 114 (2000), 43–58.
- [14] Y. Dong, N. Luo, H. Liang, Consensus building in multiperson decision making with heterogeneous preference representation structures: a perspective based on prospect theory, *Appl. Soft. Comput.* 35 (2015), 898–910.
- [15] Y.P. Jiang, Z.P. Fan, J. Ma, A method for group decision making with multi-granularity linguistic assessment information, *Inf. Sci.* 178 (2008), 1098–1109.
- [16] F. Mata, L. Martínez, E. Herrera-Viedma, An adaptive consensus support model for group decision-making problems in a multi-granular fuzzy linguistic context, *IEEE Trans. Fuzzy. Syst.* 17 (2009), 279–290.
- [17] H.J. Zhang, Y.C. Dong, X. Chen, The 2-rank consensus reaching model in the multigranular linguistic multiple-attribute group decision-making, *IEEE Trans. Syst. Man. CY-S*. 48 (2018), 2080–2094.
- [18] Z. Zhang, C.H. Guo, L. Martínez, Managing multigranular linguistic distribution assessments in large-scale multiattribute group decision making, *IEEE Trans. Syst. Man. CY-S*. 47 (2017), 3063–3076.
- [19] Z. Zhang, W.Y. Yu, L. Martínez, Y. Gao, Managing multigranular unbalanced hesitant fuzzy linguistic Information in multiattribute large-scale group decision making: a linguistic distribution-based approach, *IEEE Trans. Fuzzy. Syst.* 28 (2019), 2875–2889.
- [20] J. Ma, Y.P. Jiang, Y.H. Sun, L. Ma, A goal programming approach to group decision making based on multiplicative preference relations and fuzzy preference relations, *Eur. J. Oper. Res.* 174 (2006), 311–321.
- [21] C.C. Li, Y. Gao, Y.C. Dong, Managing ignorance elements and personalized individual under incomplete linguistic distribution context in group decision-making, *Group Decis. Negot.* (2020). in press.
- [22] Z.P. Fan, Y. Zhang, A goal programming approach to group decision-making with three formats of incomplete preference relations, *Soft. Comput.* 14 (2010), 1083–1090.
- [23] Z.P. Fan, Y.P. Jiang, J.Y. Mao, An optimization approach to multiperson decision making based on different formats of preference information, *IEEE Trans. Syst. Man. CY-A*. 36 (2006), 876–889.
- [24] J. Wu, F. Chiclana, E. Herrera-Viedma, Trust based consensus model for social network in an incomplete linguistic information context, *Appl. Soft. Comput.* 35 (2015), 827–839.
- [25] W.Y. Yu, Z. Zhang, Q.Y. Zhong, Consensus reaching for MAGDM with multi-granular hesitant fuzzy linguistic term sets: a minimum adjustment-based approach, *Ann. Oper. Res.* (2019), 1–24. in press.
- [26] Z. Zhang, Y. Gao, Z.L. Li, Consensus reaching for social network group decision making by considering leadership and bounded confidence, *Knowl. Based Syst.* 204 (2020), 106240.
- [27] Á. Labella, H.B. Liu, R.M. Rodríguez, L. Martínez, A cost consensus metric for consensus reaching processes based on a comprehensive minimum cost model, *Eur. J. Oper. Res.* 281 (2020), 316–331.
- [28] Á. Labella, Y. Liu, R.M. Rodríguez, L. Martínez, Analyzing the performance of classical consensus models in large scale group decision making: a comparative study, *Appl. Soft Comput.* 67 (2018), 677–690.
- [29] S. Alonso, *et al.*, Group decision making with incomplete fuzzy linguistic preference relations, *Int. J. Intell. Syst.* 24 (2009), 201–222.
- [30] R.M. Rodríguez, Á. Labella, L. Martínez, An overview on fuzzy modelling of complex linguistic preferences in decision making, *Int. J. Comput. Int. Syst.* 9 (2016), 81–94.
- [31] L. Martínez, D. Ruan, F. Herrera, Computing with words in decision support systems: an overview on models and applications, *Int. J. Comput. Int. Syst.* 3 (2010), 382–395.
- [32] T. Tanino, On group decision making under fuzzy preferences, in: J. Kacprzyk, M. Fedrizzi (Eds.), *Multiperson Decision Making Using Fuzzy Sets and Possibility Theory*, Kluwer, Dordrecht, The Netherlands, 1990, pp. 172–185.
- [33] S.A. Orlovsky, Decision-making with a fuzzy preference relation, *Fuzzy. Set. Syst.* 1 (1978), 155–167.
- [34] T.L. Saaty, *The Analytic Hierarchy Process*, McGraw-Hill, New York, NY, USA, 1980.
- [35] G. Crawford, C. Williams, A note on the analysis of subjective judgment matrices, *J. Math. Psychol.* 29 (1985), 387–405.
- [36] T. Wu, K. Zhang, X. Liu, C. Cao, A two-stage social trust network partition model for large-scale group decision-making problems, *Knowl-Based. Syst.* 163 (2019), 632–643.
- [37] N. Capuano, F. Chiclana, H. Fujita, E. Herrera-Viedma, V. Loia, Fuzzy group decision making with incomplete information guided by social influence, *IEEE Trans. Fuzzy. Syst.* 26 (2018), 1704–1718.

- [38] Y.C. Dong, Q.B. Zha, H.J. Zhang, G. Kou, H. Fujita, F. Chiclana, Consensus reaching in social network group decision making: research paradigms and challenges, *Knowl-Based. Syst.* 162 (2018), 3–13.
- [39] H.J. Zhang, J. Xiao, I. Palomares, H.M. Liang, Y.C. Dong, Linguistic distribution-based optimization approach for large-scale GDM with comparative linguistic information: an application on the selection of wastewater disinfection technology, *IEEE Trans. Fuzzy. Syst.* 28 (2020), 376–389.
- [40] X. Liu, Y.J. Xu, R. Montes, R.X. Ding, F. Herrera, Alternative ranking-based clustering and reliability index-based consensus reaching process for hesitant fuzzy large scale group decision making, *IEEE Trans. Fuzzy. Syst.* 27 (2019), 159–171.
- [41] I. Palomares, L. Martínez, F. Herrera, A consensus model to detect and manage noncooperative behaviors in large-scale group decision making, *IEEE Trans. Fuzzy. Syst.* 22 (2014), 516–530.
- [42] H.J. Zhang, Y.C. Dong, E. Herrera-Viedma, Consensus building for the heterogeneous large-scale GDM with the individual concerns and satisfactions, *IEEE Trans. Fuzzy. Syst.* 26 (2018), 884–898.