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Research Article

Affine Ricci Solitons of Three-Dimensional Lorentzian Lie Groups

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ABSTRACT

In this paper, we classify affine Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections and perturbed canonical connections and perturbed Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups with some product structure.

1. INTRODUCTION

The concept of the Ricci soliton is introduced by Hamilton in [9], which is a natural generalization of Einstein metrics. Study of Ricci soliton over different geometric spaces is one of interesting topics in geometry and mathematical physics. In particular, it has become more important after G. Perelman applied Ricci solitons to solve the long standing Poincare conjecture. In [10,13,15–18], Einstein manifolds associated to affine connections (especially semi-symmetric metric connections and semi-symmetric non-metric connections) were studied (see the definition 3.2 in [18] and the definition 3.1 in [10]). It is natural to study Ricci solitons associated to affine connections. Affine Ricci solitons had been introduced and studied, for example, see [6,8,11,12,14].

Our motivation is to find more examples of affine Ricci solitons. A three-dimensional Lie group $G_i(i = 1, \cdots, 7)$ is a sub-Riemannian manifold. In [1], Balogh, Tyson and Vecchi applied a Riemannian approximation scheme to get a Gauss-Bonnet theorem in the Heisenberg group $\mathbb{H}^3$. Let $T\mathbb{H}^3 = \text{span}\{e_1, e_2, e_3\}$, then they took the distribution $H = \text{span}\{e_1, e_2\}$ and $H^\perp = \text{span}\{e_3\}$ (for details, see [1]). Similarly in [20], for the affine group and the group of rigid motions of the Minkowski plane, we took the similar distributions. In [21], for the Lorentzian Heisenberg group, we also took the similar construction. Motivated by [1,20,21], we consider the similar distribution $H = \text{span}\{e_1, e_2\}$ and $H^\perp = \text{span}\{e_3\}$ for the three dimensional Lorentzian Lie group $G_i(i = 1, \cdots, 7)$. Then for the above distribution, we have a natural product structure J: $Je_1 = e_1$, $Je_2 = e_2$, $Je_3 = -e_3$. In [7], Etayo and Santamaria studied some affine connections on manifolds with the product structure or the complex structure. In particular, the canonical connection and the Kobayashi-Nomizu connection for a product structure were studied. So we consider the canonical connection and the Kobayashi-Nomizu connection associated to the above distribution on the G_i and get affine Ricci solitons associated to the canonical connection and the Kobayashi-Nomizu connection. In particular, from our results, we can get affine Einstein manifolds associated to the canonical connection and the Kobayashi-Nomizu connection. It is interesting to consider relations between affine Ricci solitons associated to the canonical connection and the Kobayashi-Nomizu connection and Ricci solitons associated to the Levi-Civita connection. It is also interesting to study affine Ricci solitons associated to other affine connections, for example, Schouten-Van Kampen connections and Vranceanu connections associated to the above product structure and semi-symmetric connections.

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By the canonical connection and the Kobayashi-Nomizu connection on three-dimensional Lorentzian Lie groups, we obtain some examples of affine Ricci solitons. But we find that the coefficient λ of the metric tensor g in the Ricci soliton equation (see (3.13) and (3.14)) is always zero for these obtained examples. In order to obtain more interesting examples with the non zero coefficient λ, we introduce perturbed canonical connections and perturbed Kobayashi-Nomizu connections in Section 4. Using these perturbed connections, we get some examples of affine Ricci solitons with the non zero coefficient λ.

In [3], Calvaruso studied three-dimensional generalized Ricci solitons, both in Riemannian and Lorentzian settings. He determined their homogeneous models, classifying left-invariant generalized Ricci solitons on three-dimensional Lie groups. Then it is natural to classify affine Ricci solitons on three-dimensional Lie groups. In [19], we introduced a particular product structure on three-dimensional Lorentzian Lie groups and we computed canonical connections and Kobayashi-Nomizu connections and their curvature on three-dimensional Lorentzian Lie groups with this product structure. We defined algebraic Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections. We classified algebraic Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups with this product structure. In this paper, we classify affine Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections and perturbed canonical connections and perturbed Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups with this product structure.

In Section 2, we recall the classification of three-dimensional Lorentzian Lie groups. In Section 3, we classify affine Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups with this product structure. In Section 4, we classify affine Ricci solitons associated to perturbed canonical connections and perturbed Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups with this product structure.

2. THREE-DIMENSIONAL LORENTZIAN LIE GROUPS

In this section, we recall the classification of three-dimensional Lorentzian Lie groups in [4,5](also see Theorems 2.1 and 2.2 in [2]).

Theorem 2.1. *Let (G, g) be a three-dimensional connected unimodular Lie group, equipped with a left-invariant Lorentzian metric. Then there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G is one of the following:*

($\mathfrak{g}_1$):

$$[e_1, e_2] = \alpha e_1 - \beta e_3, \quad [e_1, e_3] = -\alpha e_1 - \beta e_2, \quad [e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \quad \alpha \neq 0. \tag{2.1}$$

($\mathfrak{g}_2$):

$$[e_1, e_2] = \gamma e_2 - \beta e_3, \quad [e_1, e_3] = -\beta e_2 - \gamma e_3, \quad [e_2, e_3] = \alpha e_1, \quad \gamma \neq 0. \tag{2.2}$$

($\mathfrak{g}_3$):

$$[e_1, e_2] = -\gamma e_3, \quad [e_1, e_3] = -\beta e_2, \quad [e_2, e_3] = \alpha e_1. \tag{2.3}$$

($\mathfrak{g}_4$):

$$[e_1, e_2] = -e_2 + (2\eta - \beta) e_3, \quad \eta = 1 \text{ or } -1, \quad [e_1, e_3] = -\beta e_2 + e_3, \quad [e_2, e_3] = \alpha e_1. \tag{2.4}$$

Theorem 2.2. *Let (G, g) be a three-dimensional connected non-unimodular Lie group, equipped with a left-invariant Lorentzian metric. Then there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G is one of the following:*

($\mathfrak{g}_5$):

$$[e_1, e_2] = 0, \quad [e_1, e_3] = \alpha e_1 + \beta e_2, \quad [e_2, e_3] = \gamma e_1 + \delta e_2, \quad \alpha + \delta \neq 0, \quad \alpha\gamma + \beta\delta = 0. \tag{2.5}$$

($\mathfrak{g}_6$):

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3, \quad [e_2, e_3] = 0, \quad \alpha + \delta \neq 0, \quad \alpha\gamma - \beta\delta = 0. \tag{2.6}$$

($\mathfrak{g}_7$):

$$[e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3, \quad [e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3, \quad [e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3, \quad \alpha + \delta \neq 0, \quad \alpha\gamma = 0. \tag{2.7}$$

3. AFFINE RICCI SOLITONS ASSOCIATED TO CANONICAL CONNECTIONS AND KOBAYASHI-NOMIZU CONNECTIONS ON THREE-DIMENSIONAL LORENTZIAN LIE GROUPS

Throughout this paper, we shall by $\{G_i\}_{i=1,\ldots,7}$, denote the connected, simply connected three-dimensional Lie group equipped with a left-invariant Lorentzian metric g and having Lie algebra $\{\mathfrak{g}\}_{i=1,\ldots,7}$. Let ∇ be the Levi-Civita connection of G_i and R its curvature tensor, taken with the convention

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{3.1}$$

The Ricci tensor of (G_i, g) is defined by

$$\rho(X,Y) = -g(R(X,e_1)Y,e_1) - g(R(X,e_2)Y,e_2) + g(R(X,e_3)Y,e_3), \tag{3.2}$$

where $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis, with e_3 timelike. We define a product structure J on G_i by

$$Je_1 = e_1,\ Je_2 = e_2,\ Je_3 = -e_3, \tag{3.3}$$

then $J^2 = \mathrm{id}$ and $g(Je_j, Je_j) = g(e_j, e_j)$. By [7], we define the canonical connection and the Kobayashi-Nomizu connection as follows:

$$\nabla^0_X Y = \nabla_X Y - \frac{1}{2}(\nabla_X J)JY, \tag{3.4}$$

$$\nabla^1_X Y = \nabla^0_X Y - \frac{1}{4}[(\nabla_Y J)JX - (\nabla_{JY}J)X]. \tag{3.5}$$

We define

$$R^0(X,Y)Z = \nabla^0_X\nabla^0_Y Z - \nabla^0_Y\nabla^0_X Z - \nabla^0_{[X,Y]}Z, \tag{3.6}$$

$$R^1(X,Y)Z = \nabla^1_X\nabla^1_Y Z - \nabla^1_Y\nabla^1_X Z - \nabla^1_{[X,Y]}Z. \tag{3.7}$$

The Ricci tensors of (G_i, g) associated to the canonical connection and the Kobayashi-Nomizu connection are defined by

$$\rho^0(X,Y) = -g(R^0(X,e_1)Y,e_1) - g(R^0(X,e_2)Y,e_2) + g(R^0(X,e_3)Y,e_3), \tag{3.8}$$

$$\rho^1(X,Y) = -g(R^1(X,e_1)Y,e_1) - g(R^1(X,e_2)Y,e_2) + g(R^1(X,e_3)Y,e_3). \tag{3.9}$$

Let

$$\widetilde{\rho}^0(X,Y) = \frac{\rho^0(X,Y) + \rho^0(Y,X)}{2}, \tag{3.10}$$

and

$$\widetilde{\rho}^1(X,Y) = \frac{\rho^1(X,Y) + \rho^1(Y,X)}{2}. \tag{3.11}$$

Since $(L_V g)(Y,Z) := g(\nabla_Y V, Z) + g(Y, \nabla_Z V)$, we let

$$(L^j_V g)(Y,Z) := g(\nabla^j_Y V, Z) + g(Y, \nabla^j_Z V), \tag{3.12}$$

for $j = 0, 1$ and vector fields V, Y, Z.

Definition 3.1. *(G_i, g, J) is called the affine Ricci soliton associated to the connection ∇^0 if it satisfies*

$$(L^0_V g)(Y,Z) + 2\widetilde{\rho}^0(Y,Z) + 2\lambda g(Y,Z) = 0 \tag{3.13}$$

where λ is a real number and $V = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ and $\lambda_1, \lambda_2, \lambda_3$ are real numbers. (G_i, g, J) is called the affine Ricci soliton associated to the connection ∇^1 if it satisfies

$$(L^1_V g)(Y,Z) + 2\widetilde{\rho}^1(Y,Z) + 2\lambda g(Y,Z) = 0 \tag{3.14}$$

By (2.25) in [19], we have for (G_1, g, J, ∇^0)

$$\begin{aligned} &\widetilde{\rho}^0(e_1,e_1) = -\left(\alpha^2 + \frac{\beta^2}{2}\right), && \widetilde{\rho}^0(e_1,e_2) = 0, \\ &\widetilde{\rho}^0(e_1,e_3) = \frac{\alpha\beta}{4}, && \widetilde{\rho}^0(e_2,e_2) = -\left(\alpha^2 + \frac{\beta^2}{2}\right), \\ &\widetilde{\rho}^0(e_2,e_3) = \frac{\alpha^2}{2}, && \widetilde{\rho}^0(e_3,e_3) = 0. \end{aligned} \tag{3.15}$$

By Lemma 2.4 in [19] and (3.12), we have for (G_1, g, J, ∇^0, V)

$$(L^0_V g)(e_1,e_1) = 2\lambda_2\alpha, \qquad (L^0_V g)(e_1,e_2) = -\lambda_1\alpha \tag{3.16}$$

$$(L_V^0 g)(e_1,e_3) = -\frac{\beta}{2}\lambda_2, \quad (L_V^0 g)(e_2,e_2) = 0,$$
$$(L_V^0 g)(e_2,e_3) = \frac{\beta}{2}\lambda_1, \quad (L_V^0 g)(e_3,e_3) = 0.$$

If (G_1, g, J, V) is an affine Ricci soliton associated to the connection ∇^0, then by (3.13), we have

$$\begin{cases} 2\lambda_2\alpha - 2\alpha^2 - \beta^2 + 2\lambda = 0, \\ \lambda_1\alpha = 0, \\ -\beta\lambda_2 + \alpha\beta = 0, \\ -2\alpha^2 - \beta^2 + 2\lambda = 0, \\ \frac{\beta}{2}\lambda_1 + \alpha^2 = 0, \\ \lambda = 0. \end{cases} \tag{3.17}$$

Solve (3.17), we have

Theorem 3.2. *(G_1, g, J, V) is not an affine Ricci soliton associated to the connection ∇^0.*

By (2.33) in [19], we have for (G_1, g, J, ∇^1)

$$\widetilde{\rho}^1(e_1,e_1) = -\left(\alpha^2+\beta^2\right), \quad \widetilde{\rho}^1(e_1,e_2) = \alpha\beta,$$
$$\widetilde{\rho}^1(e_1,e_3) = -\frac{\alpha\beta}{2}, \quad \widetilde{\rho}^1(e_2,e_2) = -\left(\alpha^2+\beta^2\right),$$
$$\widetilde{\rho}^1(e_2,e_3) = \frac{\alpha^2}{2}, \quad \widetilde{\rho}^1(e_3,e_3) = 0. \tag{3.18}$$

By Lemma 2.8 in [19] and (3.12), we have for (G_1, g, J, ∇^1, V)

$$(L_V^1 g)(e_1,e_1) = 2\lambda_2\alpha, \quad (L_V^1 g)(e_1,e_2) = -\lambda_1\alpha,$$
$$(L_V^1 g)(e_1,e_3) = \lambda_1\alpha - \beta\lambda_2, \quad (L_V^1 g)(e_2,e_2) = 0,$$
$$(L_V^1 g)(e_2,e_3) = \beta\lambda_1 - \alpha\lambda_2 - \alpha\lambda_3, \quad (L_V^1 g)(e_3,e_3) = 0. \tag{3.19}$$

If (G_1, g, J, V) is an affine Ricci soliton associated to the connection ∇^1, then by (3.14), we have

$$\begin{cases} \lambda_2\alpha - \alpha^2 - \beta^2 + \lambda = 0, \\ -\lambda_1\alpha + 2\alpha\beta = 0, \\ \lambda_1\alpha - \beta\lambda_2 - \alpha\beta = 0, \\ -\alpha^2 - \beta^2 + \lambda = 0, \\ \beta\lambda_1 - \alpha\lambda_2 - \alpha\lambda_3 + \alpha^2 = 0, \\ \lambda = 0. \end{cases} \tag{3.20}$$

Solve (3.20), we have

Theorem 3.3. *(G_1, g, J, V) is not an affine Ricci soliton associated to the connection ∇^1.*

By (2.44) in [19], we have for (G_2, g, J, ∇^0)

$$\widetilde{\rho}^0(e_1,e_1) = -\left(\gamma^2 + \frac{\alpha\beta}{2}\right), \quad \widetilde{\rho}^0(e_1,e_2) = 0,$$
$$\widetilde{\rho}^0(e_1,e_3) = 0, \quad \widetilde{\rho}^0(e_2,e_2) = -\left(\gamma^2 + \frac{\alpha\beta}{2}\right),$$
$$\widetilde{\rho}^0(e_2,e_3) = \frac{\beta\gamma}{2} - \frac{\alpha\gamma}{4}, \quad \widetilde{\rho}^0(e_3,e_3) = 0. \tag{3.21}$$

By Lemma 2.14 in [19] and (3.12), we have for (G_2, g, J, ∇^0, V)

$$(L_V^0 g)(e_1,e_1) = 0, \quad (L_V^0 g)(e_1,e_2) = \lambda_2\gamma$$
$$(L_V^0 g)(e_1,e_3) = -\frac{\alpha}{2}\lambda_2, \quad (L_V^0 g)(e_2,e_2) = -2\gamma\lambda_1,$$
$$(L_V^0 g)(e_2,e_3) = \frac{\alpha}{2}\lambda_1, \quad (L_V^0 g)(e_3,e_3) = 0. \tag{3.22}$$

If (G_2, g, J, V) is an affine Ricci soliton associated to the connection ∇^0, then by (3.13), we have

$$\begin{cases} -\left(\gamma^2+\frac{\alpha\beta}{2}\right)+\lambda=0, \\ \lambda_2\gamma=0, \\ \alpha\lambda_2=0, \\ -\gamma\lambda_1-\left(\gamma^2+\frac{\alpha\beta}{2}\right)+\lambda=0, \\ \frac{\alpha}{2}\lambda_1+2\left(\frac{\beta\gamma}{2}-\frac{\alpha\gamma}{4}\right)=0, \\ \lambda=0. \end{cases} \tag{3.23}$$

Solve (3.23), we have

Theorem 3.4. *(G_2, g, J, V) is not an affine Ricci soliton associated to the connection ∇^0.*

By (2.54) in [19], we have for (G_2, g, J, ∇^1)

$$\begin{aligned} &\tilde{\rho}^1(e_1,e_1)=-\left(\beta^2+\gamma^2\right), && \tilde{\rho}^1(e_1,e_2)=0, \\ &\tilde{\rho}^1(e_1,e_3)=0, && \tilde{\rho}^1(e_2,e_2)=-\left(\gamma^2+\alpha\beta\right), \\ &\tilde{\rho}^1(e_2,e_3)=-\frac{\alpha\gamma}{2}, && \tilde{\rho}^1(e_3,e_3)=0. \end{aligned} \tag{3.24}$$

By Lemma 2.18 in [19] and (3.12), we have for (G_2, g, J, ∇^1, V)

$$\begin{aligned} &(L^1_V g)(e_1,e_1)=0, && (L^1_V g)(e_1,e_2)=\lambda_2\gamma, \\ &(L^1_V g)(e_1,e_3)=-\alpha\lambda_2+\gamma\lambda_3, && (L^1_V g)(e_2,e_2)=-2\gamma\lambda_1, \\ &(L^1_V g)(e_2,e_3)=\lambda_1\beta, && (L^1_V g)(e_3,e_3)=0. \end{aligned} \tag{3.25}$$

If (G_2, g, J, V) is an affine Ricci soliton associated to the connection ∇^1, then by (3.14), we have

$$\begin{cases} -\beta^2-\gamma^2+\lambda=0, \\ \lambda_2\gamma=0, \\ -\alpha\lambda_2+\gamma\lambda_3=0, \\ -\gamma\lambda_1-\left(\gamma^2+\alpha\beta\right)+\lambda=0, \\ \lambda_1\beta-\alpha\gamma=0, \\ \lambda=0. \end{cases} \tag{3.26}$$

Solve (3.26), we have

Theorem 3.5. *(G_2, g, J, V) is not an affine Ricci soliton associated to the connection ∇^1.*

By (2.64) in [19], we have for (G_3, g, J, ∇^0)

$$\begin{aligned} &\tilde{\rho}^0(e_1,e_1)=-\gamma a_3, && \tilde{\rho}^0(e_1,e_2)=0, \\ &\tilde{\rho}^0(e_1,e_3)=0, && \tilde{\rho}^0(e_2,e_2)=-\gamma a_3, \\ &\tilde{\rho}^0(e_2,e_3)=0, && \tilde{\rho}^0(e_3,e_3)=0, \end{aligned} \tag{3.27}$$

where $a_3=\frac{1}{2}(\alpha+\beta-\gamma)$. By Lemma 2.24 in [19] and (3.12), we have for (G_3, g, J, ∇^0, V)

$$\begin{aligned} &(L^0_V g)(e_1,e_1)=0, && (L^0_V g)(e_1,e_2)=0, \\ &(L^0_V g)(e_1,e_3)=-a_3\lambda_2, && (L^0_V g)(e_2,e_2)=0, \\ &(L^0_V g)(e_2,e_3)=a_3\lambda_1, && (L^0_V g)(e_3,e_3)=0. \end{aligned} \tag{3.28}$$

If (G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^0, then by (3.13), we have

$$\begin{cases} \gamma a_3=0, \\ \lambda_2 a_3=0, \\ \lambda_1 a_3=0, \\ \lambda=0. \end{cases} \tag{3.29}$$

Solve (3.29), we have

Theorem 3.6. *(G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^0 if and only if*

(i) $\lambda = 0, \alpha + \beta - \gamma = 0,$
(ii) $\lambda = 0, \alpha + \beta - \gamma \neq 0, \gamma = \lambda_1 = \lambda_2 = 0.$

By (2.69) in [19], we have for (G_3, g, J, ∇^1)

$$\begin{aligned} &\widetilde{\rho}^1(e_1,e_1) = \gamma(a_1 - a_3), && \widetilde{\rho}^1(e_1,e_2) = 0, \\ &\widetilde{\rho}^1(e_1,e_3) = 0, && \widetilde{\rho}^1(e_2,e_2) = -\gamma(a_2 + a_3), \\ &\widetilde{\rho}^1(e_2,e_3) = 0, && \widetilde{\rho}^1(e_3,e_3) = 0, \end{aligned} \tag{3.30}$$

where $a_1 = \frac{1}{2}(\alpha - \beta - \gamma),\ a_2 = \frac{1}{2}(\alpha - \beta + \gamma)$. By Lemma 2.27 in [19] and (3.12), we have for (G_3, g, J, ∇^1, V)

$$\begin{aligned} &(L^1_V g)(e_1,e_1) = 0, && (L^1_V g)(e_1,e_2) = 0, \\ &(L^1_V g)(e_1,e_3) = -(a_2 + a_3)\lambda_2, && (L^1_V g)(e_2,e_2) = 0, \\ &(L^1_V g)(e_2,e_3) = \lambda_1(a_3 - a_1), && (L^1_V g)(e_3,e_3) = 0. \end{aligned} \tag{3.31}$$

If (G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^1, then by (3.14), we have

$$\begin{cases} \gamma(a_1 - a_3) + \lambda = 0, \\ (a_2 + a_3)\lambda_2 = 0, \\ -\gamma(a_2 + a_3) + \lambda = 0, \\ \lambda_1(a_3 - a_1) = 0, \\ \lambda = 0. \end{cases} \tag{3.32}$$

Solve (3.32), we have

Theorem 3.7. *(G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^1 if and only if the following statements hold true*

(i) $\lambda = 0, \gamma \neq 0, \alpha = \beta = 0,$
(ii) $\lambda = 0, \gamma = 0, \alpha\lambda_2 = 0, \lambda_1\beta = 0.$

By (2.81) in [19], we have for (G_4, g, J, ∇^0)

$$\begin{aligned} &\widetilde{\rho}^0(e_1,e_1) = (2\eta - \beta)b_3 - 1, && \widetilde{\rho}^0(e_1,e_2) = 0, \\ &\widetilde{\rho}^0(e_1,e_3) = 0, && \widetilde{\rho}^0(e_2,e_2) = (2\eta - \beta)b_3 - 1, \\ &\widetilde{\rho}^0(e_2,e_3) = \frac{b_3 - \beta}{2}, && \widetilde{\rho}^0(e_3,e_3) = 0, \end{aligned} \tag{3.33}$$

where $b_3 = \frac{\alpha}{2} + \eta$. By Lemma 2.32 in [19] and (3.12), we have for (G_4, g, J, ∇^0, V)

$$\begin{aligned} &(L^0_V g)(e_1,e_1) = 0, && (L^0_V g)(e_1,e_2) = -\lambda_2, \\ &(L^0_V g)(e_1,e_3) = -b_3\lambda_2, && (L^0_V g)(e_2,e_2) = 2\lambda_1, \\ &(L^0_V g)(e_2,e_3) = b_3\lambda_1, && (L^0_V g)(e_3,e_3) = 0. \end{aligned} \tag{3.34}$$

If (G_4, g, J, V) is an affine Ricci soliton associated to the connection ∇^0, then by (3.13), we have

$$\begin{cases} (2\eta - \beta)b_3 - 1 + \lambda = 0, \\ \lambda_2 = 0, \\ \lambda_1 + (2\eta - \beta)b_3 - 1 + \lambda = 0, \\ \lambda_1 b_3 + b_3 - \beta = 0, \\ \lambda = 0. \end{cases} \tag{3.35}$$

Solve (3.35), we have

Theorem 3.8. *(G_4, g, J, V) is an affine Ricci soliton associated to the connection ∇^0 if and only if $\lambda = \lambda_1 = \lambda_2 = 0, \alpha = 0, \beta = \eta$.*

By (2.89) in [19], we have for (G_4, g, J, ∇^1)

$$\begin{aligned} &\widetilde{\rho}^1(e_1,e_1) = -[1 + (\beta - 2\eta)(b_3 - b_1)], && \widetilde{\rho}^1(e_1,e_2) = 0, \\ &\widetilde{\rho}^1(e_1,e_3) = 0, && \widetilde{\rho}^1(e_2,e_2) = -[1 + (\beta - 2\eta)(b_2 + b_3)], \\ &\widetilde{\rho}^1(e_2,e_3) = \frac{\alpha + b_3 - b_1 - \beta}{2}, && \widetilde{\rho}^1(e_3,e_3) = 0, \end{aligned} \tag{3.36}$$

where $b_1 = \frac{\alpha}{2} + \eta - \beta$, $b_2 = \frac{\alpha}{2} - \eta$. By Lemma 2.36 in [19] and (3.12), we have for (G_4, g, J, ∇^1, V)

$$\begin{aligned} &(L^1_V g)(e_1, e_1) = 0, && (L^1_V g)(e_1, e_2) = -\lambda_2, \\ &(L^1_V g)(e_1, e_3) = -(b_2 + b_3)\lambda_2 - \lambda_3, && (L^1_V g)(e_2, e_2) = 2\lambda_1, \\ &(L^1_V g)(e_2, e_3) = \lambda_1(b_3 - b_1), && (L^1_V g)(e_3, e_3) = 0. \end{aligned} \tag{3.37}$$

If (G_4, g, J, V) is an affine Ricci soliton associated to the connection ∇^1, then by (3.14), we have

$$\begin{cases} -[1 + (\beta - 2\eta)(b_3 - b_1)] + \lambda = 0, \\ \lambda_2 = 0, \\ -(b_2 + b_3)\lambda_2 - \lambda_3 = 0, \\ \lambda_1 - [1 + (\beta - 2\eta)(b_2 + b_3)] + \lambda = 0, \\ \lambda_1(b_3 - b_1) + (\alpha + b_3 - b_1 - \beta) = 0, \\ \lambda = 0. \end{cases} \tag{3.38}$$

Solve (3.38), we have

Theorem 3.9. *(G_4, g, J, V) is not an affine Ricci soliton associated to the connection ∇^1.*

By (3.5) in [19], we have for (G_5, g, J, ∇^0), $\widetilde{\rho}^0(e_i, e_j) = 0$, for $1 \le i, j \le 3$. By Lemma 3.3 in [19] and (3.12), we have for (G_5, g, J, ∇^0, V)

$$\begin{aligned} &(L^0_V g)(e_1, e_1) = 0, && (L^0_V g)(e_1, e_2) = 0, \\ &(L^0_V g)(e_1, e_3) = \frac{\beta - \gamma}{2}\lambda_2, && (L^0_V g)(e_2, e_2) = 0, \\ &(L^0_V g)(e_2, e_3) = -\frac{\beta - \gamma}{2}\lambda_1, && (L^0_V g)(e_3, e_3) = 0. \end{aligned} \tag{3.39}$$

If (G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^0, then by (3.13), we have

$$\begin{cases} \lambda = 0, \\ (\beta - \gamma)\lambda_2 = 0, \\ (\beta - \gamma)\lambda_1 = 0, \end{cases} \tag{3.40}$$

Solve (3.40), we have

Theorem 3.10. *(G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^0 if and only if one of the following cases occurs*

(i) $\lambda = \beta = \gamma = 0, \alpha + \delta \neq 0.$
(ii) $\lambda = 0, \beta \neq \gamma, \lambda_1 = \lambda_2 = 0, \alpha + \delta \neq 0, \alpha\gamma + \beta\delta = 0.$

By Lemma 3.7 in [19], we have for (G_5, g, J, ∇^1), $\widetilde{\rho}^1(e_i, e_j) = 0$, for $1 \le i, j \le 3$. By Lemma 3.6 in [19] and (3.12), we have for (G_5, g, J, ∇^1, V)

$$\begin{aligned} &(L^1_V g)(e_1, e_1) = 0, && (L^1_V g)(e_1, e_2) = 0, \\ &(L^1_V g)(e_1, e_3) = -\alpha\lambda_1 - \gamma\lambda_2, && (L^1_V g)(e_2, e_2) = 0, \\ &(L^1_V g)(e_2, e_3) = -\beta\lambda_1 - \delta\lambda_2, && (L^1_V g)(e_3, e_3) = 0. \end{aligned} \tag{3.41}$$

If (G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^1, then by (3.14), we have

$$\begin{cases} \lambda = 0, \\ \alpha\lambda_1 + \gamma\lambda_2 = 0, \\ \beta\lambda_1 + \delta\lambda_2 = 0. \end{cases} \tag{3.42}$$

Solve (3.42), we have

Theorem 3.11. *(G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^1 if and only if the following statements hold true*

(i) $\lambda = \lambda_1 = \lambda_2 = 0,$
(ii) $\lambda = 0, \lambda_1 \neq 0, \lambda_2 = 0, \alpha = \beta = 0, \delta \neq 0,$
(iii) $\lambda = 0, \lambda_1 = 0, \lambda_2 \neq 0, \delta = \gamma = 0, \alpha \neq 0.$

By (3.18) in [19], we have for (G_6, g, J, ∇^0)

$$\widetilde{\rho}^0(e_1, e_1) = \frac{1}{2}\beta(\beta - \gamma) - \alpha^2, \qquad \widetilde{\rho}^0(e_1, e_2) = 0, \tag{3.43}$$

$$\widetilde{\rho}^0(e_1,e_3)=0, \qquad \widetilde{\rho}^0(e_2,e_2)=\frac{1}{2}\beta(\beta-\gamma)-\alpha^2,$$
$$\widetilde{\rho}^0(e_2,e_3)=\frac{1}{2}[-\gamma\alpha+\frac{1}{2}\delta(\beta-\gamma)], \qquad \widetilde{\rho}^0(e_3,e_3)=0.$$

By Lemma 3.11 in [19] and (3.12), we have for (G_6,g,J,∇^0,V)

$$\begin{aligned} &(L^0_Vg)(e_1,e_1)=0, && (L^0_Vg)(e_1,e_2)=\alpha\lambda_2,\\ &(L^0_Vg)(e_1,e_3)=\frac{\gamma-\beta}{2}\lambda_2, && (L^0_Vg)(e_2,e_2)=-2\alpha\lambda_1,\\ &(L^0_Vg)(e_2,e_3)=\frac{\beta-\gamma}{2}\lambda_1, && (L^0_Vg)(e_3,e_3)=0. \end{aligned} \tag{3.44}$$

If (G_6,g,J,V) is an affine Ricci soliton associated to the connection ∇^0, then by (3.13), we have

$$\begin{cases} \frac{1}{2}\beta(\beta-\gamma)-\alpha^2+\lambda=0,\\ \alpha\lambda_2=0,\\ (\gamma-\beta)\lambda_2=0,\\ -\alpha\lambda_1+\frac{1}{2}\beta(\beta-\gamma)-\alpha^2+\lambda=0,\\ \frac{\beta-\gamma}{2}\lambda_1-\gamma\alpha+\frac{1}{2}\delta(\beta-\gamma)=0,\\ \lambda=0. \end{cases} \tag{3.45}$$

Solve (3.45), we have

Theorem 3.12. *(G_6,g,J,V) is an affine Ricci soliton associated to the connection ∇^0 if and only if*

(i) $\lambda=\lambda_1=\lambda_2=\gamma=\delta=0$, $\alpha\neq 0$, $\alpha^2=\frac{1}{2}\beta^2$,
(ii) $\lambda=\lambda_1=\lambda_2=\alpha=\beta=\gamma=0$, $\delta\neq 0$,
(iii) $\lambda=\lambda_2=0$, $\lambda_1\neq 0$, $\alpha=\beta=\gamma=0$, $\delta\neq 0$,
(iv) $\lambda=\lambda_2=0$, $\lambda_1\neq 0$, $\alpha=\beta=0$, $\delta\neq 0$, $\gamma\neq 0$, $\lambda_1=-\delta$,
(v) $\lambda=\alpha=\beta=\gamma=0$, $\lambda_2\neq 0$, $\delta\neq 0$.

By (3.23) in [19], we have for (G_6,g,J,∇^1)

$$\begin{aligned} &\widetilde{\rho}^1(e_1,e_1)=-(\alpha^2+\beta\gamma), && \widetilde{\rho}^1(e_1,e_2)=0,\\ &\widetilde{\rho}^1(e_1,e_3)=0, && \widetilde{\rho}^1(e_2,e_2)=-\alpha^2,\\ &\widetilde{\rho}^1(e_2,e_3)=0, && \widetilde{\rho}^1(e_3,e_3)=0. \end{aligned} \tag{3.46}$$

By Lemma 3.15 in [19] and (3.12), we have for (G_6,g,J,∇^1,V)

$$\begin{aligned} &(L^1_Vg)(e_1,e_1)=0, && (L^1_Vg)(e_1,e_2)=\lambda_2\alpha,\\ &(L^1_Vg)(e_1,e_3)=-\delta\lambda_3, && (L^1_Vg)(e_2,e_2)=-2\alpha\lambda_1,\\ &(L^1_Vg)(e_2,e_3)=-\gamma\lambda_1, && (L^1_Vg)(e_3,e_3)=0. \end{aligned} \tag{3.47}$$

If (G_6,g,J,V) is an affine Ricci soliton associated to the connection ∇^1, then by (3.14), we have

$$\begin{cases} -(\alpha^2+\beta\gamma)+\lambda=0,\\ \lambda_2\alpha=0,\\ \delta\lambda_3=0,\\ -\alpha\lambda_1-\alpha^2+\lambda=0,\\ \gamma\lambda_1=0,\\ \lambda=0. \end{cases} \tag{3.48}$$

Solve (3.48), we have

Theorem 3.13. *(G_6,g,J,V) is an affine Ricci soliton associated to the connection ∇^1 if and only if the following statements hold true*

(i) $\lambda=\alpha=\beta=\lambda_1=\lambda_3=0$, $\delta\neq 0$,
(ii) $\lambda=\alpha=\beta=\gamma=\lambda_3=0$, $\delta\neq 0$, $\lambda_1\neq 0$.

By (3.34) in [19], we have for (G_7,g,J,∇^0)

$$\widetilde{\rho}^0(e_1,e_1)=-\left(\alpha^2+\frac{\beta\gamma}{2}\right), \qquad \widetilde{\rho}^0(e_1,e_2)=0, \tag{3.49}$$

$$\widetilde{\rho}^0(e_1,e_3) = -\frac{1}{2}\left(\gamma\alpha + \frac{\delta\gamma}{2}\right), \quad \widetilde{\rho}^0(e_2,e_2) = -(\alpha^2 + \frac{\beta\gamma}{2}),$$
$$\widetilde{\rho}^0(e_2,e_3) = \frac{1}{2}\left(\alpha^2 + \frac{\beta\gamma}{2}\right), \quad \widetilde{\rho}^0(e_3,e_3) = 0.$$

By Lemma 3.20 in [19] and (3.12), we have for (G_7, g, J, ∇^0, V)

$$\begin{aligned} &(L^0_V g)(e_1,e_1) = -2\alpha\lambda_2, && (L^0_V g)(e_1,e_2) = \alpha\lambda_1 - \beta\lambda_2,\\ &(L^0_V g)(e_1,e_3) = (\beta - \frac{\gamma}{2})\lambda_2, && (L^0_V g)(e_2,e_2) = 2\beta\lambda_1,\\ &(L^0_V g)(e_2,e_3) = (\frac{\gamma}{2} - \beta)\lambda_1, && (L^0_V g)(e_3,e_3) = 0. \end{aligned} \tag{3.50}$$

If (G_7, g, J, V) is an affine Ricci soliton associated to the connection ∇^0, then by (3.13), we have

$$\begin{cases} -\alpha\lambda_2 - \left(\alpha^2 + \frac{\beta\gamma}{2}\right) + \lambda = 0,\\ \alpha\lambda_1 - \beta\lambda_2 = 0,\\ \left(\beta - \frac{\gamma}{2}\right)\lambda_2 - \left(\gamma\alpha + \frac{\delta\gamma}{2}\right) = 0,\\ \beta\lambda_1 - \left(\alpha^2 + \frac{\beta\gamma}{2}\right) + \lambda = 0,\\ \left(\frac{\gamma}{2} - \beta\right)\lambda_1 + \alpha^2 + \frac{\beta\gamma}{2} = 0,\\ \lambda = 0. \end{cases} \tag{3.51}$$

Solve (3.51), we have

Theorem 3.14. *(G_7, g, J, V) is an affine Ricci soliton associated to the connection ∇^0 if and only if the following statements hold true*

(i) $\lambda = \alpha = \beta = \gamma = 0, \delta \neq 0,$
(ii) $\lambda = \alpha = \beta = 0, \gamma \neq 0, \lambda_1 = 0, \lambda_2 = -\delta, \delta \neq 0,$
(iii) $\lambda = \alpha = \gamma = \lambda_1 = \lambda_2 = 0, \beta \neq 0.$

By (3.42) in [19], we have for (G_7, g, J, ∇^1)

$$\begin{aligned} &\widetilde{\rho}^1(e_1,e_1) = -\alpha^2, && \widetilde{\rho}^1(e_1,e_2) = \frac{1}{2}(\beta\delta - \alpha\beta),\\ &\widetilde{\rho}^1(e_1,e_3) = \beta(\alpha + \delta), && \widetilde{\rho}^1(e_2,e_2) = -(\alpha^2 + \beta^2 + \beta\gamma),\\ &\widetilde{\rho}^1(e_2,e_3) = \frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2), && \widetilde{\rho}^1(e_3,e_3) = 0. \end{aligned} \tag{3.52}$$

By Lemma 3.24 in [19] and (3.12), we have for (G_7, g, J, ∇^1, V)

$$\begin{aligned} &(L^1_V g)(e_1,e_1) = -2\alpha\lambda_2, && (L^1_V g)(e_1,e_2) = \alpha\lambda_1 - \beta\lambda_2,\\ &(L^1_V g)(e_1,e_3) = -\alpha\lambda_1 - \gamma\lambda_2 - \beta\lambda_3, && (L^1_V g)(e_2,e_2) = 2\beta\lambda_1,\\ &(L^1_V g)(e_2,e_3) = -\beta\lambda_1 - \delta\lambda_2 - \delta\lambda_3, && (L^1_V g)(e_3,e_3) = 0. \end{aligned} \tag{3.53}$$

If (G_7, g, J, V) is an affine Ricci soliton associated to the connection ∇^1, then by (3.14), we have

$$\begin{cases} -\alpha\lambda_2 - \alpha^2 + \lambda = 0,\\ \alpha\lambda_1 - \beta\lambda_2 + \beta\delta - \alpha\beta = 0,\\ -\alpha\lambda_1 - \gamma\lambda_2 - \beta\lambda_3 + 2\beta(\alpha + \delta) = 0,\\ \beta\lambda_1 - (\alpha^2 + \beta^2 + \beta\gamma) + \lambda = 0,\\ -\beta\lambda_1 - \delta\lambda_2 - \delta\lambda_3 + \beta\gamma + \alpha\delta + 2\delta^2 = 0,\\ \lambda = 0. \end{cases} \tag{3.54}$$

Solve (3.54), we have

Theorem 3.15. *(G_7, g, J, V) is an affine Ricci soliton associated to the connection ∇^1 if and only if*

(i) $\lambda = \alpha = \beta = \gamma = 0, \lambda_2 + \lambda_3 - 2\delta = 0, \delta \neq 0,$
(ii) $\lambda = \alpha = \beta = 0, \gamma \neq 0, \lambda_2 = 0, \lambda_3 = 2\delta, \delta \neq 0,$
(iii) $\lambda = \alpha = 0, \delta \neq 0, \beta \neq 0, \lambda_1 = \beta + \gamma, \lambda_2 = \delta, \lambda_3 = \frac{-\gamma\delta + 2\beta\delta}{\beta}, \gamma = \frac{\beta(\beta^2 + \delta^2)}{\delta^2}.$

4. AFFINE RICCI SOLITONS ASSOCIATED TO PERTURBED CANONICAL CONNECTIONS AND PERTURBED KOBAYASHI-NOMIZU CONNECTIONS ON THREE-DIMENSIONAL LORENTZIAN LIE GROUPS

We note that in our classifications in Section 2 always $\lambda = 0$. In order to get the affine Ricci soliton with non zero λ, we introduce perturbed canonical connections and perturbed Kobayashi-Nomizu connections in the following. Let e_3^* be the dual base of e_3. We define on $G_{i=1,\dots,7}$

$$\nabla^2_X Y = \nabla^0_X Y + \overline{\lambda} e_3^*(X) e_3^*(Y) e_3, \tag{4.1}$$

$$\nabla^3_X Y = \nabla^1_X Y + \overline{\lambda} e_3^*(X) e_3^*(Y) e_3, \tag{4.2}$$

where $\overline{\lambda}$ is a non zero real number. Then

$$\nabla^2_{e_3} e_3 = \overline{\lambda} e_3, \quad \nabla^2_{e_i} e_j = \nabla^0_{e_i} e_j; \tag{4.3}$$

$$\nabla^3_{e_3} e_3 = \overline{\lambda} e_3, \quad \nabla^3_{e_i} e_j = \nabla^1_{e_i} e_j. \tag{4.4}$$

where i or j does not equal 3. We let

$$(L^j_V g)(Y,Z) := g(\nabla^j_Y V, Z) + g(Y, \nabla^j_Z V), \tag{4.5}$$

for $j = 2, 3$ and vector fields V, Y, Z. Then we have for $G_{i=1,\cdots,7}$

$$(L^2_V g)(e_3, e_3) = -2\overline{\lambda}\lambda_3, \quad (L^2_V g)(e_j, e_k) = (L^0_V g)(e_j, e_k), \tag{4.6}$$

$$(L^3_V g)(e_3, e_3) = -2\overline{\lambda}\lambda_3, \quad (L^3_V g)(e_j, e_k) = (L^1_V g)(e_j, e_k), \tag{4.7}$$

where j or k does not equal 3.

Definition 4.1. *(G_i, g, J) is called the affine Ricci soliton associated to the connection ∇^2 if it satisfies*

$$(L^2_V g)(Y,Z) + 2\widetilde{\rho}^2(Y,Z) + 2\lambda g(Y,Z) = 0. \tag{4.8}$$

(G_i, g, J) is called the affine Ricci soliton associated to the connection ∇^3 if it satisfies

$$(L^3_V g)(Y,Z) + 2\widetilde{\rho}^3(Y,Z) + 2\lambda g(Y,Z) = 0. \tag{4.9}$$

For (G_1, ∇^2), similar to (3.15), we have

$$\widetilde{\rho}^2(e_2, e_3) = \frac{\alpha^2 + \overline{\lambda}\alpha}{2}, \quad \widetilde{\rho}^2(e_j, e_k) = \widetilde{\rho}^0(e_j, e_k), \tag{4.10}$$

for the pair $(j,k) \neq (2,3)$. If (G_1, g, J, V) is an affine Ricci soliton associated to the connection ∇^2, then by (4.8), we have

$$\begin{cases} 2\lambda_2\alpha - 2\alpha^2 - \beta^2 + 2\lambda = 0, \\ \lambda_1\alpha = 0, \\ -\beta\lambda_2 + \alpha\beta = 0, \\ -2\alpha^2 - \beta^2 + 2\lambda = 0, \\ \frac{\beta}{2}\lambda_1 + \alpha^2 + \overline{\lambda}\alpha = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.11}$$

Solve (4.11), we have

Theorem 4.2. *(G_1, g, J, V) is an affine Ricci soliton associated to the connection ∇^2 if and only if $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -\overline{\lambda}$, $\alpha = -\overline{\lambda}$, $\beta = 0$, $\lambda = \overline{\lambda}^2$.*

For (G_1, ∇^3), similar to (3.18), we have

$$\widetilde{\rho}^3(e_2, e_3) = \frac{\alpha^2 + \overline{\lambda}\alpha}{2}, \quad \widetilde{\rho}^3(e_j, e_k) = \widetilde{\rho}^1(e_j, e_k), \tag{4.12}$$

for the pair $(j,k) \neq (2,3)$. If (G_1, g, J, V) is an affine Ricci soliton associated to the connection ∇^3, then by (4.9), we have

$$\begin{cases} \lambda_2\alpha - \alpha^2 - \beta^2 + \lambda = 0, \\ -\lambda_1\alpha + 2\alpha\beta = 0, \\ \lambda_1\alpha - \beta\lambda_2 - \alpha\beta = 0, \\ -\alpha^2 - \beta^2 + \lambda = 0, \\ \beta\lambda_1 - \alpha\lambda_2 - \alpha\lambda_3 + \alpha^2 + \overline{\lambda}\alpha = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.13}$$

Solve (4.13), we have

Theorem 4.3. *(G_1, g, J, V) is not an affine Ricci soliton associated to the connection ∇^3.*

Proof. By the first and second and fourth equations in (4.13) and $\alpha \neq 0$, we get $\lambda_2 = 0$, $\lambda_1 = 2\beta$, $\lambda = \alpha^2 + \beta^2$, By the third equation in (4.13), we get $\lambda_1 = \lambda_2 = \beta = 0$, $\lambda = \alpha^2$. By the fifth equation in (4.13), we get $\lambda_3 = \alpha + \overline{\lambda}$. By the sixth equation in (4.13), we get $\alpha^2 + \overline{\lambda}\alpha + \overline{\lambda}^2 = 0$. Then $\overline{\lambda} = \alpha = 0$, this is a contradiction. □

For (G_2, ∇^2), similar to (3.21), we have

$$\widetilde{\rho}^2(e_1, e_3) = \frac{-\gamma\overline{\lambda}}{2}, \quad \widetilde{\rho}^2(e_j, e_k) = \widetilde{\rho}^0(e_j, e_k), \tag{4.14}$$

for the pair $(j, k) \neq (1, 3)$. If (G_2, g, J, V) is an affine Ricci soliton associated to the connection ∇^2, then by (4.8), we have

$$\begin{cases} -\left(\gamma^2 + \frac{\alpha\beta}{2}\right) + \lambda = 0, \\ \lambda_2\gamma = 0, \\ \alpha\lambda_2 + 2\gamma\overline{\lambda} = 0, \\ -\gamma\lambda_1 - \left(\gamma^2 + \frac{\alpha\beta}{2}\right) + \lambda = 0, \\ \frac{\alpha}{2}\lambda_1 + 2\left(\frac{\beta\gamma}{2} - \frac{\alpha\gamma}{4}\right) = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.15}$$

Solve (4.15), we have

Theorem 4.4. *(G_2, g, J, V) is not an affine Ricci soliton associated to the connection ∇^2.*

For (G_2, ∇^3), similar to (3.24), we have

$$\widetilde{\rho}^3(e_1, e_3) = \frac{-\gamma\overline{\lambda}}{2}, \quad \widetilde{\rho}^3(e_j, e_k) = \widetilde{\rho}^1(e_j, e_k), \tag{4.16}$$

for the pair $(j, k) \neq (1, 3)$. If (G_2, g, J, V) is an affine Ricci soliton associated to the connection ∇^3, then by (4.9), we have

$$\begin{cases} -\beta^2 - \gamma^2 + \lambda = 0, \\ \lambda_2\gamma = 0, \\ -\alpha\lambda_2 + \gamma\lambda_3 - \gamma\overline{\lambda} = 0, \\ -\gamma\lambda_1 - \left(\gamma^2 + \alpha\beta\right) + \lambda = 0, \\ \lambda_1\beta - \alpha\gamma = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.17}$$

Solve (4.17), we have

Theorem 4.5. *(G_2, g, J, V) is not an affine Ricci soliton associated to the connection ∇^3.*

For (G_3, ∇^2), we have $\widetilde{\rho}^2(e_j, e_k) = \widetilde{\rho}^0(e_j, e_k)$, for any pairs (j, k). If (G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^2, then by (4.8), we have

$$\begin{cases} -\gamma a_3 + \lambda = 0, \\ \lambda_2 a_3 = 0, \\ \lambda_1 a_3 = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.18}$$

Solve (4.18), we have

Theorem 4.6. *(G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^2 if and only if the following statements hold true*

(i) $a_3 \neq 0$, $\lambda_1 = \lambda_2 = 0$, $\lambda = \gamma a_3$, $\lambda_3 = -\frac{\gamma a_3}{\overline{\lambda}}$,
(ii) $a_3 = \lambda = \lambda_3 = 0$.

For (G_3, ∇^3), we have $\widetilde{\rho}^3(e_j, e_k) = \widetilde{\rho}^1(e_j, e_k)$, for any pairs (j, k). If (G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^3, then by (4.9), we have

$$\begin{cases} \gamma(a_1 - a_3) + \lambda = 0, \\ (a_2 + a_3)\lambda_2 = 0, \\ -\gamma(a_2 + a_3) + \lambda = 0, \\ \lambda_1(a_3 - a_1) = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.19}$$

Solve (4.19), we have

Theorem 4.7. *(G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^3 if and only if one of the following cases occurs*

(i) $\gamma = \lambda = \lambda_3 = 0, \alpha\lambda_2 = 0, \beta\lambda_1 = 0,$
(ii) $\gamma \neq 0, \alpha = \beta = \lambda = \lambda_3 = 0,$
(iii) $\gamma \neq 0, \alpha = \beta \neq 0, \lambda_1 = \lambda_2 = 0, \lambda = \alpha\gamma, \lambda_3 = -\frac{\alpha\gamma}{\overline{\lambda}}.$

For (G_4, ∇^2), we have

$$\widetilde{\rho}^2(e_1, e_3) = \frac{\overline{\lambda}}{2}, \qquad \widetilde{\rho}^2(e_j, e_k) = \widetilde{\rho}^0(e_j, e_k), \tag{4.20}$$

for the pair $(j, k) \neq (1, 3)$. If (G_4, g, J, V) is an affine Ricci soliton associated to the connection ∇^2, then by (4.8), we have

$$\begin{cases} (2\eta - \beta)b_3 - 1 + \lambda = 0, \\ \lambda_2 = 0, \\ -b_3\lambda_2 + \overline{\lambda} = 0, \\ \lambda_1 + (2\eta - \beta)b_3 - 1 + \lambda = 0, \\ \lambda_1 b_3 + b_3 - \beta = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.21}$$

Solve (4.21), we have

Theorem 4.8. *(G_4, g, J, V) is not an affine Ricci soliton associated to the connection ∇^2.*

For (G_4, ∇^3), we have

$$\widetilde{\rho}^3(e_1, e_3) = \frac{\overline{\lambda}}{2}, \qquad \widetilde{\rho}^3(e_j, e_k) = \widetilde{\rho}^1(e_j, e_k), \tag{4.22}$$

for the pair $(j, k) \neq (1, 3)$. If (G_4, g, J, V) is an affine Ricci soliton associated to the connection ∇^3, then by (4.9), we have

$$\begin{cases} -[1 + (\beta - 2\eta)(b_3 - b_1)] + \lambda = 0, \\ \lambda_2 = 0, \\ -(b_2 + b_3)\lambda_2 - \lambda_3 + \overline{\lambda} = 0, \\ \lambda_1 - [1 + (\beta - 2\eta)(b_2 + b_3)] + \lambda = 0, \\ \lambda_1(b_3 - b_1) + (\alpha + b_3 - b_1 - \beta) = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.23}$$

Solve (4.23), we have

Theorem 4.9. *(G_4, g, J, V) is not an affine Ricci soliton associated to the connection ∇^3.*

For (G_5, g, J, ∇^2), $\widetilde{\rho}^2(e_i, e_j) = 0$, for $1 \leq i, j \leq 3$. If (G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^2, then by (4.8), we have

$$\begin{cases} \lambda = 0, \\ (\beta - \gamma)\lambda_2 = 0, \\ (\beta - \gamma)\lambda_1 = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.24}$$

Solve (4.24), we have

Theorem 4.10. *(G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^2 if and only if the following statements hold true*

(i) $\gamma \neq \beta, \lambda = \lambda_1 = \lambda_2 = \lambda_3 = 0, \alpha + \delta \neq 0, \alpha\gamma + \beta\delta = 0,$
(ii) $\lambda = \beta = \gamma = 0, \alpha + \delta \neq 0, \lambda_3 = 0.$

For (G_5, g, J, ∇^3), $\widetilde{\rho}^3(e_i, e_j) = 0$, for $1 \le i, j \le 3$. If (G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^3, then by (4.9), we have

$$\begin{cases} \lambda = 0, \\ \alpha\lambda_1 + \gamma\lambda_2 = 0, \\ \beta\lambda_1 + \delta\lambda_2 = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.25}$$

Solve (4.25), we have

Theorem 4.11. *(G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^3 if and only if*

(i) $\lambda = \lambda_1 = \lambda_2 = \lambda_3 = 0$,
(ii) $\lambda = \lambda_2 = \lambda_3 = \alpha = \beta = 0$, $\lambda_1 \neq 0$, $\delta \neq 0$,
(iii) $\lambda = 0$, $\lambda_1 = \lambda_3 = 0$, $\lambda_2 \neq 0$, $\delta = \gamma = 0$, $\alpha \neq 0$.

For (G_6, ∇^2), we have

$$\widetilde{\rho}^2(e_1, e_3) = \frac{\delta\overline{\lambda}}{2}, \quad \widetilde{\rho}^2(e_j, e_k) = \widetilde{\rho}^0(e_j, e_k), \tag{4.26}$$

for the pair $(j, k) \neq (1, 3)$. If (G_6, g, J, V) is an affine Ricci soliton associated to the connection ∇^2, then by (4.8), we have

$$\begin{cases} \frac{1}{2}\beta(\beta - \gamma) - \alpha^2 + \lambda = 0, \\ \alpha\lambda_2 = 0, \\ (\gamma - \beta)\lambda_2 + 2\delta\overline{\lambda} = 0, \\ -\alpha\lambda_1 + \frac{1}{2}\beta(\beta - \gamma) - \alpha^2 + \lambda = 0, \\ \frac{\beta-\gamma}{2}\lambda_1 - \gamma\alpha + \frac{1}{2}\delta(\beta - \gamma) = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.27}$$

Solve (4.27), we have

Theorem 4.12. *(G_6, g, J, V) is an affine Ricci soliton associated to the connection ∇^2 if and only if the following statements hold true*

(i) $\alpha = \beta = 0$, $\delta \neq 0$, $\gamma \neq 0$, $\lambda = \lambda_3 = 0$, $\lambda_1 = -\delta$, $\lambda_2 = -\frac{2\delta\overline{\lambda}}{\gamma}$,
(ii) $\alpha \neq 0$, $\lambda_1 = \lambda_2 = \gamma = \delta = 0$, $\lambda = \alpha^2 - \frac{1}{2}\beta^2$, $\lambda_3 = -\frac{\lambda}{\overline{\lambda}}$.

For (G_6, ∇^3), we have

$$\widetilde{\rho}^3(e_1, e_3) = \frac{\delta\overline{\lambda}}{2}, \quad \widetilde{\rho}^3(e_j, e_k) = \widetilde{\rho}^1(e_j, e_k), \tag{4.28}$$

for the pair $(j, k) \neq (1, 3)$. If (G_6, g, J, V) is an affine Ricci soliton associated to the connection ∇^3, then by (4.9), we have

$$\begin{cases} -(\alpha^2 + \beta\gamma) + \lambda = 0, \\ \lambda_2\alpha = 0, \\ -\delta\lambda_3 + \delta\overline{\lambda} = 0, \\ -\alpha\lambda_1 - \alpha^2 + \lambda = 0, \\ \gamma\lambda_1 = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.29}$$

Solve (4.29), we have

Theorem 4.13. *(G_6, g, J, V) is an affine Ricci soliton associated to the connection ∇^3 if and only if $\alpha \neq 0$, $\lambda_1 = \lambda_2 = \gamma = \delta = 0$, $\lambda = \alpha^2$, $\lambda_3 = -\frac{\alpha^2}{\overline{\lambda}}$.*

For (G_7, ∇^2), we have

$$\widetilde{\rho}^2(e_1, e_3) = \frac{1}{2}(\beta\overline{\lambda} - \alpha\gamma - \frac{\delta\gamma}{2}), \quad \widetilde{\rho}^2(e_2, e_3) = \frac{1}{2}(\delta\overline{\lambda} + \alpha^2 + \frac{\beta\gamma}{2}), \quad \widetilde{\rho}^2(e_j, e_k) = \widetilde{\rho}^0(e_j, e_k), \tag{4.30}$$

for the pair $(j,k) \neq (1,3),(2,3)$. If (G_7, g, J, V) is an affine Ricci soliton associated to the connection ∇^2, then by (4.8), we have

$$\begin{cases} -\alpha\lambda_2 - \left(\alpha^2 + \frac{\beta\gamma}{2}\right) + \lambda = 0, \\ \alpha\lambda_1 - \beta\lambda_2 = 0, \\ \left(\beta - \frac{\gamma}{2}\right)\lambda_2 + \beta\overline{\lambda} - \left(\gamma\alpha + \frac{\delta\gamma}{2}\right) = 0, \\ \beta\lambda_1 - \left(\alpha^2 + \frac{\beta\gamma}{2}\right) + \lambda = 0, \\ \left(\frac{\gamma}{2} - \beta\right)\lambda_1 + \delta\overline{\lambda} + \alpha^2 + \frac{\beta\gamma}{2} = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.31}$$

Solve (4.31), we have

Theorem 4.14. *(G_7, g, J, V) is an affine Ricci soliton associated to the connection ∇^2 if and only if one of the following cases occurs*

(i) $\alpha = \beta = 0, \gamma \neq 0, \delta \neq 0, \lambda = 0, \lambda_1 = -\frac{2\delta\overline{\lambda}}{\gamma}, \lambda_2 = -\delta, \lambda_3 = 0,$
(ii) $\alpha \neq 0, \lambda_1 = \lambda_2 = \beta = \gamma = 0, \lambda = \alpha^2, \delta \neq 0, \overline{\lambda} = -\frac{\alpha^2}{\delta}, \lambda_3 = \delta.$

For (G_7, ∇^3), we have

$$\widetilde{\rho}^3(e_1,e_3) = \alpha\beta + \beta\delta + \frac{\beta\overline{\lambda}}{2}, \qquad \widetilde{\rho}^3(e_2,e_3) = \frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2 + \delta\overline{\lambda}), \qquad \widetilde{\rho}^3(e_j,e_k) = \widetilde{\rho}^1(e_j,e_k), \tag{4.32}$$

for the pair $(j,k) \neq (1,3),(2,3)$. If (G_7, g, J, V) is an affine Ricci soliton associated to the connection ∇^3, then by (4.9), we have

$$\begin{cases} -\alpha\lambda_2 - \alpha^2 + \lambda = 0, \\ \alpha\lambda_1 - \beta\lambda_2 + \beta\delta - \alpha\beta = 0, \\ -\alpha\lambda_1 - \gamma\lambda_2 - \beta\lambda_3 + 2\beta(\alpha + \delta + \frac{\overline{\lambda}}{2}) = 0, \\ \beta\lambda_1 - (\alpha^2 + \beta^2 + \beta\gamma) + \lambda = 0, \\ -\beta\lambda_1 - \delta\lambda_2 - \delta\lambda_3 + \beta\gamma + \alpha\delta + 2\delta^2 + \delta\overline{\lambda} = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.33}$$

Solve (4.33), we have

Theorem 4.15. *(G_7, g, J, V) is an affine Ricci soliton associated to the connection ∇^3 if and only if the following statements hold true*

(i) $\lambda = \alpha = \beta = \gamma = \lambda_3 = 0, \delta \neq 0, \lambda_2 = 2\delta + \overline{\lambda},$
(ii) $\alpha = \beta = \lambda = \lambda_2 = \lambda_3 = 0, \gamma \neq 0, \delta \neq 0, \overline{\lambda} = -2\delta,$
(iii) $\alpha = \lambda = \lambda_3 = 0, \beta \neq 0, \delta \neq 0, \lambda_1 = \beta + \gamma, \lambda_2 = \delta, \overline{\lambda} = \frac{\gamma\delta - 2\beta\delta}{\beta}, \gamma = \frac{\beta^3 + \beta\delta^2}{\delta^2},$
(iv) $\alpha \neq 0, \beta = \gamma = \delta = \lambda_1 = \lambda_2 = 0, \lambda = \alpha^2, \lambda_3 = -\frac{\alpha^2}{\overline{\lambda}},$
(v) $\alpha \neq 0, \beta = \gamma = \lambda_1 = \lambda_2 = 0, \lambda = \alpha^2, \delta \neq 0, \lambda_3 = \alpha + 2\delta + \overline{\lambda}, \overline{\lambda}^2 + (\alpha + 2\delta)\overline{\lambda} + \alpha^2 = 0.$

Proof. We know that $\alpha\gamma = 0$ and $\alpha + \delta \neq 0$.

Case i) $\alpha = 0$, then $\delta \neq 0$. By (4.33), we have $\lambda = \lambda_3 = 0$ and

$$\begin{cases} \beta(\lambda_2 - \delta) = 0, \\ -\gamma\lambda_2 + 2\beta\delta + \beta\overline{\lambda} = 0, \\ \beta\lambda_1 - (\beta^2 + \beta\gamma) = 0, \\ -\beta\lambda_1 - \delta\lambda_2 + \beta\gamma + 2\delta^2 + \delta\overline{\lambda} = 0. \end{cases} \tag{4.34}$$

Case i)-a) $\beta = 0$, then by (4.34), we have $\gamma\lambda_2 = 0$ and $\lambda_2 = 2\delta + \overline{\lambda}$.
Case i)-a)-1) $\gamma = 0$, we get (i).
Case i)-a)-2) $\gamma \neq 0$, we get $\lambda_2 = 0$ and $\overline{\lambda} = -2\delta$. So we have (ii).
Case i)-b) $\beta \neq 0$, then by (4.34), we have $\lambda_1 = \beta + \gamma, \lambda_2 = \delta, \overline{\lambda} = \frac{\gamma\delta - 2\beta\delta}{\beta}$. By the fourth equation in (4.34), we get $\gamma = \frac{\beta^3 + \beta\delta^2}{\delta^2}$ and this is (iii).
Case ii) $\alpha \neq 0$, so $\gamma = 0$.
Case ii)-a) $\beta = 0$, by (4.33), we get $\lambda_1 = \lambda_2 = 0, \lambda = \alpha^2, \delta\lambda_3 = \alpha\delta + 2\delta^2 + \delta\overline{\lambda}, \lambda_3 = -\frac{\alpha^2}{\overline{\lambda}}$.
Case ii)-a)-1) $\delta = 0$, we get (iv).
Case ii)-a)-2) $\delta \neq 0$, we get (v).

Case ii)-b) $\beta \neq 0$, we get $\alpha\lambda_2 + \beta\lambda_1 - \beta^2 = 0$ and $\alpha\lambda_1 - \beta\lambda_2 + \beta(\delta - \alpha) = 0$. So get

$$\begin{cases} \lambda_1 = \beta - \frac{\alpha\beta\delta}{\alpha^2+\beta^2}, \\ \lambda_2 = \frac{\beta^2\delta}{\alpha^2+\beta^2}, \\ \lambda = \frac{\alpha\beta^2\delta}{\alpha^2+\beta^2} + \alpha^2, \\ \lambda_3 = \overline{\lambda} + \alpha + 2\delta + \frac{\alpha^2\delta}{\alpha^2+\beta^2}. \end{cases} \tag{4.35}$$

Using (4.35) and the fifth equation in (4.33), we get $\beta^2(\alpha^2 - \alpha\delta + \delta^2 + \beta^2) + \alpha^2\delta^2 = 0$, so we get $\beta = 0$ and this is a contradiction. So we have no solutions in this case. □

CONFLICTS OF INTEREST

The author declares no conflicts of interest.

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