Research Article

Similarity Measures and Multi-person TOPSIS Method Using m-polar Single-Valued Neutrosophic Sets

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ABSTRACT

In this paper, we give a new notion of the m-polar single-valued neutrosophic sets (m-PSVNSs) which is a hybrid of the single-valued neutrosophic sets (SVNSs) and the m-polar fuzzy sets (m-PFSs) and study several of the structure operations including subset, equal, union, intersection, and complement. Subsequently, we present the basic definitions, theorems, and examples on m-PSVNSs. Also, we define the certain distance between two m-PSVNSs and a novel similarity measure for m-PSVNSs based on distances. A multi criteria decision-making (MCDM) problem is animated for m-PSVNS data that takes into account the distances for the best alternative (solution) by an application of similarity measure for m-PSVNSs in brand recognition. Finally, we construct a new methodology to extend the TOPSIS to m-PSVNS and illustrate its applicability via a numerical example.

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1. INTRODUCTION

Zadeh [1] defined the concept of fuzzy set (FS) as the generalization of crisp set theory and highlighted the complications of some models (e.g., educational mathematical and crisp set theory, which are unable to handle the difficulty of the statistics containing doubts) with the approach of membership. Smarandache [2,3] defined the neutrosophic set (NS) and gave several operations on NS. Wang *et al.* [4] defined the single valued neutrosophic set (SVNS) by separately assigning the degree of truth membership, the degree of indeterminacy membership, and the degree of falsity membership. Zhang [5] proposed the concept of bipolar fuzzy sets (BFSs) and relations. Chen *et al.* [6] presented the concept of m-polar fuzzy set (m-PFS) as a simplified type of BFS. The notions of m-PF graph, characterization of m-PF graph, and metrics in m-PF graph were introduced in [7–10]. A multi-criteria decision-making (MCDM) method on m-PF soft rough sets, hybrid m-PF models with m-PF ELECTRE-I were presented in [11–13].

A similarity measure (SM) for the fuzzy system plays a very considerable role in handling problems that comprehend ambiguous information, but unable to deal with the unclearness and uncertainty of the problems having normal information. The idea of SM of two sets (i.e., fuzzy values and vague sets) were discussed in [14,15], but it had failed to hold in some problems. To resolve this problem, Hong and Kim [16] brought into light some modified measures. Several SM based on SVNSs were proposed in [17–20]. Akram and Waseem [21] implemented a m-PFS and m-PF soft set on SMs to medical diagnosis. Yong [22] gave the new approach of the selection of location for plantation under the terms of the linguistic environment through graded mean reparation based on fuzzy TOPSIS. Adeel *et al.* [23] animates m-PF through the extension of order preference of MCGDM-TOPSIS for the best alternative. Shih *et al.* [24] described the technique of order preference through similarity using MADM in a group decision environment based on group preference for TOPSIS. Saeed *et al.* [25] gave an application in SM on multipolar neutrosophic soft sets structure in decision-making for medical diagnosis. Kang *et al.* [26] presented an application in BCK/BCI-algebras on multipolar IFS with finite degree. Akram [27,28] defined the notions of single-valued neutrosophic graphs and m-polar fuzzy graphs. There are many published papers on TOPSIS in different fields (for instance, [29–33]).

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In normal practice, the problem of SVN information occurs (i.e., which cannot be elaborated well using the existing approaches), while an m-polar single-valued neutrosophic set (m-PSVNS) is used to resolve the uncertain and more variant data, specifically in the SVNS form with m-PFS. Our motivation is to be able to find a solution of many daily life problems enhanced through distance-based SM with the m-PSVNS. It increases the number of applications in various fields, including electronic optimization, industries, and forensic facial portrait. Moreover, a new approach for the best m-PSVNS alternatives based on distance SMs in MCDM is animated.

This article is structured as follows: In Section 2, a brief overview of some fundamental concepts is provided. In Section 3, the concept of m-PSVNS and its basic operations are defined. In Section 4, distance measure formulas are presented on m-PSVNS. Further, the m-PSVNS is used to investigate a problem involving distance-based SMs with an algorithm. In Section 5, the MCDM for m-PSVN data is described with an algorithm for the best solution (alternative). The article is concluded with a review and further work outlook.

2. PRELIMINARIES

We give a short survey of concepts of BFSs, m-PFSs, and SVNSs as indicated below.

2.1. BFSs and m-PFSs

Definition 1. (cf. [5]). Suppose that Z (i.e., $Z = \{z_1, z_2, \dots, z_r | r = 1, 2, \dots, n\}, n \in N$ (natural Numbers)) be a set of elements. A BFS on Z is a pair (α, β) , where

$$\alpha : Z \rightarrow [0, 1] \text{ and } \beta : Z \rightarrow [-1, 0]$$

are two mappings. We note that BFS are an extension of FSs whose membership degree range is $[-1, 1]$.

Definition 2. (cf. [6]). An m-PFS on Z (i.e., $Z = \{z_1, z_2, \dots, z_r\}$) be a set of elements, is a mapping

$$\alpha : Z \rightarrow [0, 1]^m.$$

Example 1.

Suppose that 3-PFS on Z (i.e., $Z = \{z_1, z_2, z_3\}$) is defined by

$$\alpha(z_1) = \langle 0.45, 0.42, 0.59 \rangle, \alpha(z_2) = \langle 0.51, 0.52, 0.7 \rangle, \text{ and } \alpha(z_3) = \langle 0.4, 0.53, 0.9 \rangle.$$

2.2. SVNSs

Definition 3. (cf. [4]). A SVNS $\Phi = (\alpha, \beta, \gamma)$ on Z (i.e., $Z = \{z_1, z_2, \dots, z_r\}$) be a set of elements, is a mapping

$$(\alpha, \beta, \gamma) : Z \rightarrow [0, 1] \times [0, 1] \times [0, 1],$$

where $\alpha : Z \rightarrow [0, 1]$ (i.e., the degree of truth membership), $\beta : Z \rightarrow [0, 1]$ (i.e., the degree of indeterminacy membership), and $\gamma : Z \rightarrow [0, 1]$ (i.e., the degree of falsity membership) are FSs over Z such that $0 \leq \alpha_\Phi(z_r) + \beta_\Phi(z_r) + \gamma_\Phi(z_r) \leq 3$ ($\forall z_r \in Z, r = 1, 2, \dots, n$)

or

$$\Phi = \left\{ \frac{(\alpha_\Phi(z_r), \beta_\Phi(z_r), \gamma_\Phi(z_r))}{z_r} \middle| z_r \in Z \right\}.$$

Definition 4. (cf. [4]). Let

$$\Phi = \left\{ \frac{(\alpha_\Phi(z_r), \beta_\Phi(z_r), \gamma_\Phi(z_r))}{z_r} \middle| z_r \in Z \right\}.$$

and

$$\Psi = \left\{ \frac{(\xi_\Psi(z_r), \eta_\Psi(z_r), \epsilon_\Psi(z_r))}{z_r} \middle| z_r \in Z \right\}$$

are SVNSs. The following five operations are defined by

- (1) (Complement) $\Phi^c = \left\{ \frac{(\gamma_\Phi(z_r), 1 - \beta_\Phi(z_r), \alpha_\Phi(z_r))}{z_r} \mid z_r \in Z \right\}$.
- (2) (Inclusion) $\Phi \subseteq \Psi \Leftrightarrow \alpha_\Phi(z_r) \leq \xi_\Psi(z_r), \beta_\Phi(z_r) \geq \eta_\Psi(z_r), \text{ and } \gamma_\Phi(z_r) \geq \varepsilon_\Psi(z_r) (\forall z_r \in Z)$.
- (3) (Equal) $\Phi = \Psi \Leftrightarrow \Phi \subseteq \Psi \text{ and } \Psi \subseteq \Phi$.
- (4) (Union) $\Phi \cup \Psi = \left\{ \frac{(\alpha_\Phi(z_r) \vee \xi_\Psi(z_r), \beta_\Phi(z_r) \wedge \eta_\Psi(z_r), \gamma_\Phi(z_r) \wedge \varepsilon_\Psi(z_r))}{z_r} \mid z_r \in Z \right\}$.
- (5) (Intersection) $\Phi \cap \Psi = \left\{ \frac{(\alpha_\Phi(z_r) \wedge \xi_\Psi(z_r), \beta_\Phi(z_r) \vee \eta_\Psi(z_r), \gamma_\Phi(z_r) \vee \varepsilon_\Psi(z_r))}{z_r} \mid z_r \in Z \right\}$.

3. AN m-PSVNSs

We will introduce the concept of m-PSVNS and study several definitions, theorems, and examples as indicated below.

Definition 5. An m-PSVNS Φ on Z (i.e., $Z = \{z_1, z_2, \dots, z_r\}$ be a set of elements), is a mapping

$$\Phi : Z \rightarrow [0, 1]^m \times [0, 1]^m \times [0, 1]^m.$$

It can be written as

$$\Phi = \left\{ \frac{(\langle p_1 \circ \alpha_\Phi(z_r), p_1 \circ \beta_\Phi(z_r), p_1 \circ \gamma_\Phi(z_r) \rangle, \langle p_2 \circ \alpha_\Phi(z_r), p_2 \circ \beta_\Phi(z_r), p_2 \circ \gamma_\Phi(z_r) \rangle, \dots, \langle p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r) \rangle)}{z} \mid z_r \in Z \right\},$$

where $(p_k \circ \alpha_\Phi, p_k \circ \beta_\Phi, p_k \circ \gamma_\Phi) : ([0, 1] \times [0, 1] \times [0, 1]) \rightarrow [0, 1] \times [0, 1] \times [0, 1]$ is the k th projection mapping. $p_k \circ \alpha$ is the k th truth membership value, $p_k \circ \beta$ is the k th indeterminacy membership value, and $p_k \circ \gamma$ is the k th falsity membership value of element in $\Phi = (\langle p_1 \circ \alpha_\Phi(z_r), p_1 \circ \beta_\Phi(z_r), p_1 \circ \gamma_\Phi(z_r) \rangle, \langle p_2 \circ \alpha_\Phi(z_r), p_2 \circ \beta_\Phi(z_r), p_2 \circ \gamma_\Phi(z_r) \rangle, \dots, \langle p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r) \rangle)$ such that $0 \leq p_k \circ \alpha_\Phi(z_r) + p_k \circ \beta_\Phi(z_r) + p_k \circ \gamma_\Phi(z_r) \leq 3$.

Example 2. Suppose that an 2-PSVNS on three of elements $Z = \{z_1, z_2, z_3\}$ is defined by

$$\Phi = \left\{ \frac{(\langle 0.6, 0.3, 0.7 \rangle, \langle 0.4, 0.2, 0.8 \rangle)}{z_1}, \frac{(\langle 0.1, 0.7, 0.9 \rangle, \langle 0.4, 0.3, 0.8 \rangle)}{z_2}, \frac{(\langle 0.1, 0.3, 0.7 \rangle, \langle 0.2, 0.4, 0.7 \rangle)}{z_3} \right\}.$$

Definition 6. Assume that Φ and Ψ are m-PSVNS on a set of elements Z , where

$$\Phi = \left\{ \frac{(\langle p_1 \circ \alpha_\Phi(z_r), p_1 \circ \beta_\Phi(z_r), p_1 \circ \gamma_\Phi(z_r) \rangle, \langle p_2 \circ \alpha_\Phi(z_r), p_2 \circ \beta_\Phi(z_r), p_2 \circ \gamma_\Phi(z_r) \rangle, \dots, \langle p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r) \rangle)}{z} \mid z_r \in Z \right\},$$

and

$$\Psi = \left\{ \frac{(\langle p_1 \circ \xi_\Psi(z_r), p_1 \circ \eta_\Psi(z_r), p_1 \circ \varepsilon_\Psi(z_r) \rangle, \langle p_2 \circ \xi_\Psi(z_r), p_2 \circ \eta_\Psi(z_r), p_2 \circ \varepsilon_\Psi(z_r) \rangle, \dots, \langle p_k \circ \xi_\Psi(z_r), p_k \circ \eta_\Psi(z_r), p_k \circ \varepsilon_\Psi(z_r) \rangle)}{z} \mid z_r \in Z \right\}.$$

Then, $\Phi \subseteq \Psi$ (i.e., Φ is an m-PSVN subset of Ψ) if for all $z_r \in Z$, $p_k \circ \alpha_\Phi(z_r) \leq p_k \circ \xi_\Psi(z_r)$, $p_k \circ \beta_\Phi(z_r) \geq p_k \circ \eta_\Psi(z_r)$ and $p_k \circ \gamma_\Phi(z_r) \geq p_k \circ \varepsilon_\Psi(z_r)$.

Example 3. (Continued from Example 2). Suppose that Ψ be an 2-PSVNS on three of elements $Z = \{z_1, z_2, z_3\}$ is defined by

$$\Psi = \left\{ \frac{(\langle 0.7, 0.3, 0.4 \rangle, \langle 0.5, 0.1, 0.5 \rangle)}{z_1}, \frac{(\langle 0.2, 0.6, 0.4 \rangle, \langle 0.5, 0.2, 0.4 \rangle)}{z_2}, \frac{(\langle 0.6, 0.1, 0.6 \rangle, \langle 0.4, 0.4, 0.5 \rangle)}{z_3} \right\}.$$

Hence, $\Phi \subseteq \Psi (\forall z_r \in Z, r = 1, 2, 3)$.

Definition 7. Assume that Φ and Ψ are m-PSVNSs on a set of elements Z , where

$$\Phi = \left\{ \frac{(\langle p_1 \circ \alpha_\Phi(z_r), p_1 \circ \beta_\Phi(z_r), p_1 \circ \gamma_\Phi(z_r) \rangle, \langle p_2 \circ \alpha_\Phi(z_r), p_2 \circ \beta_\Phi(z_r), p_2 \circ \gamma_\Phi(z_r) \rangle, \dots, \langle p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r) \rangle)}{z} \mid z_r \in Z \right\},$$

and

$$\Psi = \left\{ \frac{(\langle p_1 \circ \zeta_\Psi(z_r), p_1 \circ \eta_\Psi(z_r), p_1 \circ \varepsilon_\Psi(z_r) \rangle, \langle p_2 \circ \zeta_\Psi(z_r), p_2 \circ \eta_\Psi(z_r), p_2 \circ \varepsilon_\Psi(z_r) \rangle, \dots, \langle p_k \circ \zeta_\Psi(z_r), p_k \circ \eta_\Psi(z_r), p_k \circ \varepsilon_\Psi(z_r) \rangle)}{z} \middle| z_r \in Z \right\}.$$

Then, $\Phi = \Psi$ (i.e., Φ is an m-PSVN equal of Ψ) if $\Phi \subseteq \Psi$ and $\Phi \supseteq \Psi$.

Definition 8. Assume that Φ be an m-PSVNS on a set of elements Z , where

$$\Phi = \left\{ \frac{(\langle p_1 \circ \alpha_\Phi(z_r), p_1 \circ \beta_\Phi(z_r), p_1 \circ \gamma_\Phi(z_r) \rangle, \langle p_2 \circ \alpha_\Phi(z_r), p_2 \circ \beta_\Phi(z_r), p_2 \circ \gamma_\Phi(z_r) \rangle, \dots, \langle p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r) \rangle)}{z} \middle| z_r \in Z \right\}.$$

Then,

(1) Φ is called an m-PSVN null set (denoted by $\hat{\emptyset}$), is defined by

$$\hat{\emptyset} = \left\{ \frac{(0, 1, 1)}{z_r} \middle| z_r \in Z \right\}.$$

(2) Φ is called an m-PSVN absolute set (denoted by $\hat{Z}$), is defined by

$$\hat{Z} = \left\{ \frac{(1, 0, 0)}{z_r} \middle| z_r \in Z \right\}.$$

Example 4. (Continued from Example 2). Then, an 2-PSVNSs $\hat{\emptyset}$ and $\hat{Z}$ are defined by

$$\hat{\emptyset} = \left\{ \frac{(\langle 0, 1, 1 \rangle, \langle 0, 1, 1 \rangle)}{z_1}, \frac{(\langle 0, 1, 1 \rangle, \langle 0, 1, 1 \rangle)}{z_2}, \frac{(\langle 0, 1, 1 \rangle, \langle 0, 1, 1 \rangle)}{z_3} \right\}$$

and

$$\hat{Z} = \left\{ \frac{(\langle 1, 0, 0 \rangle, \langle 1, 0, 0 \rangle)}{z_1}, \frac{(\langle 1, 0, 0 \rangle, \langle 1, 0, 0 \rangle)}{z_2}, \frac{(\langle 1, 0, 0 \rangle, \langle 1, 0, 0 \rangle)}{z_3} \right\}.$$

Definition 9. Assume that Φ and Ψ are m-PSVNSs on a set of elements Z , where

$$\Phi = \left\{ \frac{(\langle p_1 \circ \alpha_\Phi(z_r), p_1 \circ \beta_\Phi(z_r), p_1 \circ \gamma_\Phi(z_r) \rangle, \langle p_2 \circ \alpha_\Phi(z_r), p_2 \circ \beta_\Phi(z_r), p_2 \circ \gamma_\Phi(z_r) \rangle, \dots, \langle p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r) \rangle)}{z} \middle| z_r \in Z \right\},$$

and

$$\Psi = \left\{ \frac{(\langle p_1 \circ \zeta_\Psi(z_r), p_1 \circ \eta_\Psi(z_r), p_1 \circ \varepsilon_\Psi(z_r) \rangle, \langle p_2 \circ \zeta_\Psi(z_r), p_2 \circ \eta_\Psi(z_r), p_2 \circ \varepsilon_\Psi(z_r) \rangle, \dots, \langle p_k \circ \zeta_\Psi(z_r), p_k \circ \eta_\Psi(z_r), p_k \circ \varepsilon_\Psi(z_r) \rangle)}{z} \middle| z_r \in Z \right\}.$$

(1) The intersection $\Phi \cap \Psi$, is defined as

$$\Phi \cap \Psi = \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \wedge p_k \circ \zeta_\Psi(z_r), p_k \circ \beta_\Phi(z_r) \vee p_k \circ \eta_\Psi(z_r), p_k \circ \gamma_\Phi(z_r) \vee p_k \circ \varepsilon_\Psi(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\}.$$

(2) The union $\Phi \cup \Psi$, is defined as

$$\Phi \cup \Psi = \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \vee p_k \circ \zeta_\Psi(z_r), p_k \circ \beta_\Phi(z_r) \wedge p_k \circ \eta_\Psi(z_r), p_k \circ \gamma_\Phi(z_r) \wedge p_k \circ \varepsilon_\Psi(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\}.$$

Example 5. (Continued from Examples 2 and 3). Then, the intersection and union of an m -PSVNSs, respectively, given by

$$\Phi \cap \Psi = \left\{ \frac{(\langle 0.6, 0.3, 0.7 \rangle, \langle 0.4, 0.2, 0.8 \rangle)}{z_1}, \frac{(\langle 0.1, 0.7, 0.9 \rangle, \langle 0.4, 0.3, 0.8 \rangle)}{z_2}, \frac{(\langle 0.1, 0.3, 0.7 \rangle, \langle 0.2, 0.4, 0.7 \rangle)}{z_3} \right\}$$

and

$$\Phi \cup \Psi = \left\{ \frac{(\langle 0.7, 0.3, 0.4 \rangle, \langle 0.5, 0.1, 0.5 \rangle)}{z_1}, \frac{(\langle 0.2, 0.6, 0.4 \rangle, \langle 0.5, 0.2, 0.4 \rangle)}{z_2}, \frac{(\langle 0.6, 0.1, 0.6 \rangle, \langle 0.4, 0.4, 0.5 \rangle)}{z_3} \right\}.$$

Theorem 1. (Identity law of m -PSVNSs) Assume that Φ be m -PSVNS, $\hat{\emptyset}$ be m -PSVN null set, and $\hat{Z}$ be m -PSVN absolute set. Then the followings hold:

- (1) $\Phi \cup \hat{\emptyset} = \Phi$,
- (2) $\Phi \cap \hat{Z} = \Phi$.

Proof.

- (1) $\Phi \cup \hat{\emptyset}$

$$\begin{aligned} &= \left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r), p_k \circ \beta_{\Phi}(z_r), p_k \circ \gamma_{\Phi}(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \cup \left\{ \frac{(0, 1, 1)}{z_r} \middle| z_r \in Z \right\} \\ &= \left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r) \vee 0, p_k \circ \beta_{\Phi}(z_r) \wedge 1, p_k \circ \gamma_{\Phi}(z_r) \wedge 1)}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \\ &= \left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r), p_k \circ \beta_{\Phi}(z_r), p_k \circ \gamma_{\Phi}(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \\ &= \Phi. \end{aligned}$$

- (2) $\Phi \cap \hat{Z}$

$$\begin{aligned} &= \left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r), p_k \circ \beta_{\Phi}(z_r), p_k \circ \gamma_{\Phi}(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \cap \left\{ \frac{(1, 0, 0)}{z_r} \middle| z_r \in Z \right\} \\ &= \left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r) \wedge 1, p_k \circ \beta_{\Phi}(z_r) \vee 0, p_k \circ \gamma_{\Phi}(z_r) \vee 0)}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \\ &= \left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r), p_k \circ \beta_{\Phi}(z_r), p_k \circ \gamma_{\Phi}(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \\ &= \Phi. \end{aligned}$$

Theorem 2. (Domination law of m -PSVNSs) Assume that Φ be m -PSVNS, $\hat{\emptyset}$ be m -PSVN null set, and $\hat{Z}$ be m -PSVN absolute set. Then the followings hold:

- (1) $\Phi \cup \hat{Z} = \hat{Z}$,
- (2) $\Phi \cap \hat{\emptyset} = \hat{\emptyset}$.

Proof.

$$(1) \quad \Phi \cup \hat{Z}$$

$$\begin{aligned} &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \cup \left\{ \frac{(1, 0, 0)}{z_r} \middle| z_r \in Z \right\} \\ &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \vee 1, p_k \circ \beta_\Phi(z_r) \wedge 0, p_k \circ \gamma_\Phi(z_r) \wedge 0)}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \\ &= \left\{ \frac{(1, 0, 0)}{z_r} \middle| z_r \in Z \right\} \\ &= \hat{Z}. \end{aligned}$$

$$(2) \quad \Phi \cap \hat{Z}$$

$$\begin{aligned} &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \cap \left\{ \frac{(0, 1, 1)}{z_r} \middle| z_r \in Z \right\} \\ &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \wedge 0, p_k \circ \beta_\Phi(z_r) \vee 1, p_k \circ \gamma_\Phi(z_r) \vee 1)}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \\ &= \left\{ \frac{(0, 1, 1)}{z_r} \middle| z_r \in Z \right\} \\ &= \hat{Z}. \end{aligned}$$

Theorem 3. (Idempotent law of m -PSVNSs) Assume that Φ be m -PSVNS. Then the followings hold:

$$(1) \quad \Phi \cup \Phi = \Phi,$$

$$(2) \quad \Phi \cap \Phi = \Phi.$$

Proof.

$$(1) \quad \Phi \cup \Phi = \Phi$$

$$\begin{aligned} &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \cup \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \\ &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \vee p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r) \wedge p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r) \wedge p_k \circ \gamma_\Phi(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \\ &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \\ &= \Phi. \end{aligned}$$

$$(2) \quad \Phi \cap \Phi$$

$$\begin{aligned} &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \cap \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \\ &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \wedge p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r) \vee p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r) \vee p_k \circ \gamma_\Phi(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \\ &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \middle| z_r \in Z, k = 1, 2, \dots, m \right\} \\ &= \Phi. \end{aligned}$$

Theorem 4. (Commutative law of m -PSVNSs) Assume that Φ and Ψ are m -PSVNSs. Then the followings hold:

- (1) $\Phi \cap \Psi = \Psi \cap \Phi$,
- (2) $\Phi \cup \Psi = \Psi \cup \Phi$.

Proof.

- (1) $\Phi \cap \Psi$

$$\begin{aligned}
 &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \cap \left\{ \frac{(p_k \circ \zeta_\Psi(z_r), p_k \circ \eta_\Psi(z_r), p_k \circ \varepsilon_\Psi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \wedge p_k \circ \zeta_\Psi(z_r), p_k \circ \beta_\Phi(z_r) \vee p_k \circ \eta_\Psi(z_r), p_k \circ \gamma_\Phi(z_r) \vee p_k \circ \varepsilon_\Psi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &= \left\{ \frac{(p_k \circ \zeta_\Psi(z_r) \wedge p_k \circ \alpha_\Phi(z_r), p_k \circ \eta_\Psi(z_r) \vee p_k \circ \beta_\Phi(z_r), p_k \circ \varepsilon_\Psi(z_r) \vee p_k \circ \gamma_\Phi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &= \left\{ \frac{(p_k \circ \zeta_\Psi(z_r), p_k \circ \eta_\Psi(z_r), p_k \circ \varepsilon_\Psi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \cap \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &= \Psi \cap \Phi.
 \end{aligned}$$

- (2) $\Phi \cup \Psi$

$$\begin{aligned}
 &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \cup \left\{ \frac{(p_k \circ \zeta_\Psi(z_r), p_k \circ \eta_\Psi(z_r), p_k \circ \varepsilon_\Psi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \vee p_k \circ \zeta_\Psi(z_r), p_k \circ \beta_\Phi(z_r) \wedge p_k \circ \eta_\Psi(z_r), p_k \circ \gamma_\Phi(z_r) \wedge p_k \circ \varepsilon_\Psi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &= \left\{ \frac{(p_k \circ \zeta_\Psi(z_r) \vee p_k \circ \alpha_\Phi(z_r), p_k \circ \eta_\Psi(z_r) \wedge p_k \circ \beta_\Phi(z_r), p_k \circ \varepsilon_\Psi(z_r) \wedge p_k \circ \gamma_\Phi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &= \left\{ \frac{(p_k \circ \zeta_\Psi(z_r), p_k \circ \eta_\Psi(z_r), p_k \circ \varepsilon_\Psi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \cup \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &= \Psi \cup \Phi.
 \end{aligned}$$

Theorem 5. (Associative law of m -PSVNSs) Assume that Φ, Ψ , and Ω are m -PSVNSs. Then the followings hold:

1. $(\Phi \cap \Psi) \cap \Omega = \Phi \cap (\Psi \cap \Omega)$,
2. $(\Phi \cup \Psi) \cup \Omega = \Phi \cup (\Psi \cup \Omega)$.

Proof.

1. $(\Phi \cap \Psi) \cap \Omega$

$$\begin{aligned}
 &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \wedge p_k \circ \zeta_\Psi(z_r), p_k \circ \beta_\Phi(z_r) \vee p_k \circ \eta_\Psi(z_r), p_k \circ \gamma_\Phi(z_r) \vee p_k \circ \varepsilon_\Psi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &\cap \left\{ \frac{(p_k \circ \rho_\Omega(z_r), p_k \circ \lambda_\Omega(z_r), p_k \circ \delta_\Omega(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \wedge p_k \circ \zeta_\Psi(z_r) \wedge p_k \circ \rho_\Omega(z_r), p_k \circ \beta_\Phi(z_r) \vee p_k \circ \eta_\Psi(z_r) \vee p_k \circ \lambda_\Omega(z_r), p_k \circ \gamma_\Phi(z_r) \vee p_k \circ \varepsilon_\Psi(z_r) \vee p_k \circ \delta_\Omega(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &\cap \left\{ \frac{(\zeta_\Psi(z_r) \wedge p_k \circ \rho_\Omega(z_r), \eta_\Psi(z_r) \vee p_k \circ \lambda_\Omega(z_r), p_k \circ \varepsilon_\Psi(z_r) \vee p_k \circ \delta_\Omega(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &= \Phi \cap (\Psi \cap \Omega).
 \end{aligned}$$

2. $(\Phi \cup \Psi) \cup \Omega$

$$\begin{aligned}
&= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \vee p_k \circ \zeta_\Psi(z_r), p_k \circ \beta_\Phi(z_r) \wedge p_k \circ \eta_\Psi(z_r), p_k \circ \gamma_\Phi(z_r) \wedge p_k \circ \varepsilon_\Psi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
&\cup \left\{ \frac{(p_k \circ \rho_\Omega(z_r), p_k \circ \lambda_\Omega(z_r), p_k \circ \delta_\Omega(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
&= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \vee p_k \circ \zeta_\Psi(z_r) \vee p_k \circ \rho_\Omega(z_r), p_k \circ \beta_\Phi(z_r) \wedge p_k \circ \eta_\Psi(z_r) \wedge p_k \circ \lambda_\Omega(z_r), p_k \circ \gamma_\Phi(z_r) \wedge p_k \circ \varepsilon_\Psi(z_r) \wedge p_k \circ \delta_\Omega(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
&= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
&\cup \left\{ \frac{(\zeta_\Psi(z_r) \vee p_k \circ \rho_\Omega(z_r), \eta_\Psi(z_r) \wedge p_k \circ \lambda_\Omega(z_r), p_k \circ \varepsilon_\Psi(z_r) \wedge p_k \circ \delta_\Omega(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
&= \Phi \cup (\Psi \cup \Omega).
\end{aligned}$$

Theorem 6. (Distributive law of m -PSVNSs) Assume that Φ, Ψ , and Ω are m -PSVNSs. Then the followings hold:

1. $\Phi \cap (\Psi \cup \Omega) = (\Phi \cap \Psi) \cup (\Phi \cap \Omega)$,
2. $\Phi \cup (\Psi \cap \Omega) = (\Phi \cup \Psi) \cap (\Phi \cup \Omega)$.

Proof.

1. $\Phi \cap (\Psi \cup \Omega)$

$$\begin{aligned}
&= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \cap \left\{ \frac{(p_k \circ \zeta_\Psi(z_r) \vee p_k \circ \rho_\Omega(z_r), p_k \circ \eta_\Psi(z_r) \wedge p_k \circ \lambda_\Omega(z_r), p_k \circ \varepsilon_\Psi(z_r) \wedge p_k \circ \delta_\Omega(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
&= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \wedge [p_k \circ \zeta_\Psi(z_r) \vee p_k \circ \rho_\Omega(z_r)], p_k \circ \beta_\Phi(z_r) \wedge [p_k \circ \eta_\Psi(z_r) \wedge p_k \circ \lambda_\Omega(z_r)], p_k \circ \gamma_\Phi(z_r) \wedge [p_k \circ \varepsilon_\Psi(z_r) \wedge p_k \circ \delta_\Omega(z_r)])}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
&= \left\{ \frac{([p_k \circ \alpha_\Phi(z_r) \wedge p_k \circ \zeta_\Psi(z_r)] \vee [p_k \circ \alpha_\Phi(z_r) \wedge p_k \circ \rho_\Omega(z_r)], [p_k \circ \beta_\Phi(z_r) \wedge p_k \circ \eta_\Psi(z_r)] \vee [p_k \circ \beta_\Phi(z_r) \wedge p_k \circ \lambda_\Omega(z_r)], [p_k \circ \gamma_\Phi(z_r) \wedge p_k \circ \varepsilon_\Psi(z_r)] \vee [p_k \circ \gamma_\Phi(z_r) \wedge p_k \circ \delta_\Omega(z_r)])}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
&= (\Phi \cap \Psi) \cup (\Phi \cap \Omega).
\end{aligned}$$

2. $\Phi \cup (\Psi \cap \Omega)$

$$\begin{aligned}
&= \left\{ \frac{([p_k \circ \alpha_\Phi(z_r) \vee p_k \circ \zeta_\Psi(z_r)] \wedge [p_k \circ \alpha_\Phi(z_r) \vee p_k \circ \rho_\Omega(z_r)], [p_k \circ \beta_\Phi(z_r) \wedge p_k \circ \eta_\Psi(z_r)] \vee [p_k \circ \beta_\Phi(z_r) \wedge p_k \circ \lambda_\Omega(z_r)], [p_k \circ \gamma_\Phi(z_r) \wedge p_k \circ \varepsilon_\Psi(z_r)] \vee [p_k \circ \gamma_\Phi(z_r) \wedge p_k \circ \delta_\Omega(z_r)])}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
&= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \vee [p_k \circ \zeta_\Psi(z_r) \wedge p_k \circ \rho_\Omega(z_r)], p_k \circ \beta_\Phi(z_r) \wedge [p_k \circ \eta_\Psi(z_r) \vee p_k \circ \lambda_\Omega(z_r)], p_k \circ \gamma_\Phi(z_r) \wedge [p_k \circ \varepsilon_\Psi(z_r) \vee p_k \circ \delta_\Omega(z_r)])}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
&= \left\{ \frac{([p_k \circ \alpha_\Phi(z_r) \vee p_k \circ \zeta_\Psi(z_r)] \wedge [p_k \circ \alpha_\Phi(z_r) \vee p_k \circ \rho_\Omega(z_r)], [p_k \circ \beta_\Phi(z_r) \wedge p_k \circ \eta_\Psi(z_r)] \vee [p_k \circ \beta_\Phi(z_r) \wedge p_k \circ \lambda_\Omega(z_r)], [p_k \circ \gamma_\Phi(z_r) \wedge p_k \circ \varepsilon_\Psi(z_r)] \vee [p_k \circ \gamma_\Phi(z_r) \wedge p_k \circ \delta_\Omega(z_r)])}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
&= (\Phi \cup \Psi) \cap (\Phi \cup \Omega).
\end{aligned}$$

Definition 10. Assume that Φ be m -PSVNS on a set of elements Z , where

$$\Phi = \left\{ \frac{(\langle p_1 \circ \alpha_\Phi(z_r), p_1 \circ \beta_\Phi(z_r), p_1 \circ \gamma_\Phi(z_r) \rangle, \langle p_2 \circ \alpha_\Phi(z_r), p_2 \circ \beta_\Phi(z_r), p_2 \circ \gamma_\Phi(z_r) \rangle, \dots, \langle p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r) \rangle)}{z_r} \mid z_r \in Z \right\}$$

Then, the complement Φ^c of Φ , is defined by

$$\Phi^c = \left\{ \frac{(\langle p_1 \circ \gamma_\Phi(z_r), 1 - p_1 \circ \beta_\Phi(z_r), p_1 \circ \alpha_\Phi(z_r) \rangle, \langle p_2 \circ \gamma_\Phi(z_r), 1 - p_2 \circ \beta_\Phi(z_r), p_2 \circ \alpha_\Phi(z_r) \rangle, \dots, \langle p_k \circ \gamma_\Phi(z_r), 1 - p_k \circ \beta_\Phi(z_r), p_k \circ \alpha_\Phi(z_r) \rangle)}{z_r} \mid z_r \in Z \right\}$$

Example 6. (Continued from Example 2). The complement Φ^c of Φ , is calculated by

$$\Phi^c = \left\{ \frac{(\langle 0.7, 0.7, 0.6 \rangle, \langle 0.8, 0.8, 0.4 \rangle)}{z_1}, \frac{(\langle 0.9, 0.3, 0.1 \rangle, \langle 0.8, 0.7, 0.4 \rangle)}{z_2}, \frac{(\langle 0.7, 0.7, 0.1 \rangle, \langle 0.7, 0.6, 0.2 \rangle)}{z_3} \right\}.$$

Theorem 7. (Complementation and double complementation of m -PSVNSs) Assume that Φ be m -PSVNSs, $\hat{\emptyset}$ be m -PSVN null set, and $\hat{Z}$ be m -PSVN absolute set. Then the followings hold:

1. $\hat{\emptyset}^c = \hat{Z}$,
2. $\hat{Z}^c = \hat{\emptyset}$,
3. $(\Phi^c)^c = \Phi$.

Proof.

1. $\hat{\emptyset}^c = \left(\left\{ \frac{(0,1,1)}{z_r} \mid z_r \in Z \right\} \right)^c = \left\{ \frac{(1,0,0)}{z_r} \mid z_r \in Z \right\} = \hat{Z}$.
2. $\hat{Z}^c = \left(\left\{ \frac{(1,0,0)}{z_r} \mid z_r \in Z \right\} \right)^c = \left\{ \frac{(0,1,1)}{z_r} \mid z_r \in Z \right\} = \hat{\emptyset}$.
3. $(\Phi^c)^c = \left(\left\{ \frac{(p_k \circ \gamma_{\Phi}(z_r), 1 - p_k \circ \beta_{\Phi}(z_r), p_k \circ \alpha_{\Phi}(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \right)^c$
 $= \left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r), 1 - (1 - p_k \circ \beta_{\Phi}(z_r)), p_k \circ \gamma_{\Phi}(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\}$
 $= \left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r), p_k \circ \beta_{\Phi}(z_r), p_k \circ \gamma_{\Phi}(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\}$
 $= \Phi$.

Remark 1. By the following Example 7, we will show the equality of $\Phi \cup \Phi^c = \hat{X}$ and $\Phi \cap \Phi^c = \hat{\emptyset}$ does not hold.

Example 7. (Continued from Examples 2 and 6). Then $\Phi \cup \Phi^c$ and $\Phi \cap \Phi^c$ are calculated by

$$\Phi \cup \Phi^c = \left\{ \frac{(\langle 0.7, 0.3, 0.6 \rangle, \langle 0.8, 0.2, 0.4 \rangle)}{z_1}, \frac{(\langle 0.9, 0.3, 0.1 \rangle, \langle 0.8, 0.3, 0.4 \rangle)}{z_2}, \frac{(\langle 0.7, 0.3, 0.1 \rangle, \langle 0.7, 0.4, 0.2 \rangle)}{z_3} \right\}$$

and

$$\Phi \cap \Phi^c = \left\{ \frac{(\langle 0.6, 0.7, 0.7 \rangle, \langle 0.4, 0.8, 0.8 \rangle)}{z_1}, \frac{(\langle 0.7, 0.7, 0.9 \rangle, \langle 0.4, 0.7, 0.8 \rangle)}{z_2}, \frac{(\langle 0.1, 0.7, 0.7 \rangle, \langle 0.2, 0.6, 0.7 \rangle)}{z_3} \right\}.$$

This show $\Phi \cup \Phi^c \neq \hat{X}$ and $\Phi \cap \Phi^c \neq \hat{\emptyset}$

Theorem 8. (De Morgan's laws of m -PSVNSs) Assume that Φ, Ψ , and Ω are m -PSVNSs. Then the followings hold:

1. $(\Phi \cup \Psi)^c = \Phi^c \cap \Psi^c$,
2. $(\Phi \cap \Psi)^c = \Phi^c \cup \Psi^c$.

Proof.

1. $(\Phi \cup \Psi)^c = \left(\left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r), p_k \circ \beta_{\Phi}(z_r), p_k \circ \gamma_{\Phi}(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \cup \left\{ \frac{(p_k \circ \alpha_{\Psi}(z_r), p_k \circ \beta_{\Psi}(z_r), p_k \circ \gamma_{\Psi}(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \right)^c$
 $= \left(\left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r) \vee p_k \circ \alpha_{\Psi}(z_r), p_k \circ \beta_{\Phi}(z_r) \wedge p_k \circ \beta_{\Psi}(z_r), p_k \circ \gamma_{\Phi}(z_r) \wedge p_k \circ \gamma_{\Psi}(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \right)^c$
 $= \left\{ \frac{(p_k \circ \gamma_{\Phi}(z_r) \wedge p_k \circ \gamma_{\Psi}(z_r), 1 - (p_k \circ \beta_{\Phi}(z_r) \wedge p_k \circ \beta_{\Psi}(z_r)), p_k \circ \alpha_{\Phi}(z_r) \vee p_k \circ \alpha_{\Psi}(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\}$
 $= \left\{ \frac{(p_k \circ \gamma_{\Phi}(z_r) \wedge p_k \circ \gamma_{\Psi}(z_r), (1 - p_k \circ \beta_{\Phi}(z_r)) \vee (1 - p_k \circ \beta_{\Psi}(z_r)), p_k \circ \alpha_{\Phi}(z_r) \vee p_k \circ \alpha_{\Psi}(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\}$
 $= \left\{ \frac{(p_k \circ \gamma_{\Phi}(z_r), 1 - p_k \circ \beta_{\Phi}(z_r), p_k \circ \alpha_{\Phi}(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \cap \left\{ \frac{(p_k \circ \gamma_{\Psi}(z_r), 1 - p_k \circ \beta_{\Psi}(z_r), p_k \circ \alpha_{\Psi}(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\}$
 $= \Phi^c \cap \Psi^c$.

2. $(\Phi \cap \Psi)^c$

$$\begin{aligned}
 &= \left(\left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \cap \left\{ \frac{(p_k \circ \zeta_\Psi(z_r), p_k \circ \eta_\Psi(z_r), p_k \circ \varepsilon_\Psi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \right)^c \\
 &= \left(\left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \wedge p_k \circ \zeta_\Psi(z_r), p_k \circ \beta_\Phi(z_r) \vee p_k \circ \eta_\Psi(z_r), p_k \circ \gamma_\Phi(z_r) \vee p_k \circ \varepsilon_\Psi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \right)^c \\
 &= \left\{ \frac{(p_k \circ \gamma_\Phi(z_r) \vee p_k \circ \varepsilon_\Psi(z_r), 1 - (p_k \circ \beta_\Phi(z_r) \vee p_k \circ \eta_\Psi(z_r)), p_k \circ \alpha_\Phi(z_r) \wedge p_k \circ \zeta_\Psi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &= \left\{ \frac{(p_k \circ \gamma_\Phi(z_r) \vee p_k \circ \varepsilon_\Psi(z_r), (1 - p_k \circ \beta_\Phi(z_r)) \wedge (1 - p_k \circ \eta_\Psi(z_r)), p_k \circ \alpha_\Phi(z_r) \wedge p_k \circ \zeta_\Psi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &= \left\{ \frac{(p_k \circ \gamma_\Phi(z_r), 1 - p_k \circ \beta_\Phi(z_r), p_k \circ \alpha_\Phi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \cup \left\{ \frac{(p_k \circ \varepsilon_\Psi(z_r), 1 - p_k \circ \eta_\Psi(z_r), p_k \circ \zeta_\Psi(z_r))}{z_r} \mid z_r \in Z, k = 1, 2, \dots, m \right\} \\
 &= \Phi^c \cup \Psi^c.
 \end{aligned}$$

4. DISTANCE MEASURE AND SM FOR m-PSVNSs

We present a few new concepts of distances measure and SM for m-PSVNSs as indicated below.

Definition 11. Assume that Φ and Ψ are m-PSVNSs on a set of elements Z (i.e., $Z = \{z_1, z_2, \dots, z_r \mid r = 1, 2, \dots, n\}$),

where

$$\Phi = \left\{ \frac{(\langle p_1 \circ \alpha_\Phi(z_r), p_1 \circ \beta_\Phi(z_r), p_1 \circ \gamma_\Phi(z_r) \rangle, \langle p_2 \circ \alpha_\Phi(z_r), p_2 \circ \beta_\Phi(z_r), p_2 \circ \gamma_\Phi(z_r) \rangle, \dots, \langle p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r) \rangle)}{z} \mid z_r \in Z \right\},$$

and

$$\Psi = \left\{ \frac{(\langle p_1 \circ \zeta_\Psi(z_r), p_1 \circ \eta_\Psi(z_r), p_1 \circ \varepsilon_\Psi(z_r) \rangle, \langle p_2 \circ \zeta_\Psi(z_r), p_2 \circ \eta_\Psi(z_r), p_2 \circ \varepsilon_\Psi(z_r) \rangle, \dots, \langle p_k \circ \zeta_\Psi(z_r), p_k \circ \eta_\Psi(z_r), p_k \circ \varepsilon_\Psi(z_r) \rangle)}{z} \mid z_r \in Z \right\}$$

The distances measure between Φ and Ψ are defined by

(1) Hamming distance:

$$d_1(\Phi, \Psi) = \frac{1}{m} \left\{ \sum_{k=1}^m \sum_{r=1}^n \left| \frac{p_k \circ \alpha_\Phi(z_r) + p_k \circ \beta_\Phi(z_r) + p_k \circ \gamma_\Phi(z_r)}{2} - \frac{p_k \circ \zeta_\Psi(z_r) + p_k \circ \eta_\Psi(z_r) + p_k \circ \varepsilon_\Psi(z_r)}{2} \right| \right\}$$

(2) Normalized Hamming distance:

$$d_2(\Phi, \Psi) = \frac{1}{mn} \left\{ \sum_{k=1}^m \sum_{r=1}^n \left| \frac{p_k \circ \alpha_\Phi(z_r) + p_k \circ \beta_\Phi(z_r) + p_k \circ \gamma_\Phi(z_r)}{2} - \frac{p_k \circ \zeta_\Psi(z_r) + p_k \circ \eta_\Psi(z_r) + p_k \circ \varepsilon_\Psi(z_r)}{2} \right| \right\}$$

(3) Euclidean distance:

$$d_3(\Phi, \Psi) = \sqrt{\frac{1}{m} \left\{ \sum_{k=1}^m \sum_{r=1}^n \left(\frac{p_k \circ \alpha_\Phi(z_r) + p_k \circ \beta_\Phi(z_r) + p_k \circ \gamma_\Phi(z_r)}{2} - \frac{p_k \circ \zeta_\Psi(z_r) + p_k \circ \eta_\Psi(z_r) + p_k \circ \varepsilon_\Psi(z_r)}{2} \right)^2 \right\}}$$

(4) Normalized Euclidean distance:

$$d_4(\Phi, \Psi) = \sqrt{\frac{1}{mn} \left\{ \sum_{k=1}^m \sum_{r=1}^n \left(\frac{p_k \circ \alpha_\Phi(z_r) + p_k \circ \beta_\Phi(z_r) + p_k \circ \gamma_\Phi(z_r)}{2} - \frac{p_k \circ \zeta_\Psi(z_r) + p_k \circ \eta_\Psi(z_r) + p_k \circ \varepsilon_\Psi(z_r)}{2} \right)_2^2 \right\}}.$$

Theorem 9. The distances between Im-PFSs Φ and Ψ satisfy the following four inequalities:

1. $d_1(\Phi, \Psi) \leq n$,
2. $d_2(\Phi, \Psi) \leq 1$,
3. $d_3(\Phi, \Psi) \leq \sqrt{n}$,
4. $d_4(\Phi, \Psi) \leq 1$.

Theorem 10. The distance functions d_1, d_2, d_3 , and d_4 defined from m PSVNS(Z) $\rightarrow R^+$, are metric distances.

Proof. Suppose that Φ, Ψ , and Ω are three m-PSVNSs on Z . Then

1. $d_1(\Phi, \Psi) \geq 0$.
2. Let

$$\begin{aligned} d_1(\Phi, \Psi) = 0 &\Leftrightarrow \frac{1}{m} \left\{ \sum_{k=1}^m \sum_{r=1}^n \left| \frac{p_k \circ \alpha_\Phi(z_r) + p_k \circ \beta_\Phi(z_r) + p_k \circ \gamma_\Phi(z_r)}{2} - \frac{p_k \circ \zeta_\Psi(z_r) + p_k \circ \eta_\Psi(z_r) + p_k \circ \varepsilon_\Psi(z_r)}{2} \right| \right\} = 0 \\ &\Leftrightarrow \left| \frac{p_k \circ \alpha_\Phi(z_r) + p_k \circ \beta_\Phi(z_r) + p_k \circ \gamma_\Phi(z_r)}{2} - \frac{p_k \circ \zeta_\Psi(z_r) + p_k \circ \eta_\Psi(z_r) + p_k \circ \varepsilon_\Psi(z_r)}{2} \right| = 0 \\ &\Leftrightarrow \frac{p_k \circ \alpha_\Phi(z_r) + p_k \circ \beta_\Phi(z_r) + p_k \circ \gamma_\Phi(z_r)}{2} = \frac{p_k \circ \zeta_\Psi(z_r) + p_k \circ \eta_\Psi(z_r) + p_k \circ \varepsilon_\Psi(z_r)}{2}, 1 \leq k \leq m, 1 \leq r \leq n \\ &\Leftrightarrow \Phi = \Psi. \end{aligned}$$

3. $d_1(\Phi, \Psi) = d_1(\Psi, \Phi)$.
4. For any three m-PSVNSs Φ, Ψ , and Ω ,

$$\begin{aligned} &= \left| \frac{p_k \circ \alpha_\Phi(z_r) + p_k \circ \beta_\Phi(z_r) + p_k \circ \gamma_\Phi(z_r)}{2} - \frac{p_k \circ \zeta_\Psi(z_r) + p_k \circ \eta_\Psi(z_r) + p_k \circ \varepsilon_\Psi(z_r)}{2} \right| \\ &= \left| \frac{p_k \circ \alpha_\Phi(z_r) + p_k \circ \beta_\Phi(z_r) + p_k \circ \gamma_\Phi(z_r)}{2} - \frac{p_k \circ \rho_\Omega(z_r) + p_k \circ \lambda_\Omega(z_r) + p_k \circ \delta_\Omega(z_r)}{2} \right. \\ &\quad \left. + \frac{p_k \circ \rho_\Omega(z_r) + p_k \circ \lambda_\Omega(z_r) + p_k \circ \delta_\Omega(z_r)}{2} - \frac{p_k \circ \zeta_\Psi(z_r) + p_k \circ \eta_\Psi(z_r) + p_k \circ \varepsilon_\Psi(z_r)}{2} \right| \\ &\leq \left| \frac{p_k \circ \alpha_\Phi(z_r) + p_k \circ \beta_\Phi(z_r) + p_k \circ \gamma_\Phi(z_r)}{2} - \frac{p_k \circ \rho_\Omega(z_r) + p_k \circ \lambda_\Omega(z_r) + p_k \circ \delta_\Omega(z_r)}{2} \right| \\ &\quad + \left| \frac{p_k \circ \rho_\Omega(z_r) + p_k \circ \lambda_\Omega(z_r) + p_k \circ \delta_\Omega(z_r)}{2} - \frac{p_k \circ \zeta_\Psi(z_r) + p_k \circ \eta_\Psi(z_r) + p_k \circ \varepsilon_\Psi(z_r)}{2} \right|. \end{aligned}$$

Hence, $d_1(\Phi, \Psi) \leq d_1(\Phi, \Omega) + d_1(\Omega, \Psi)$. $\square$

Definition 12. The SM of two m-PSVNSs Φ and Ψ , defined by

$$SM(\Phi, \Psi) = \frac{1}{1 + d(\Phi, \Psi)} \quad (5)$$

where $d(\Phi, \Psi)$ is any of the above four distances in Definition 11.

Definition 13. The two m-PSVNSs Φ and Ψ are σ similar if and only if $SM(\Phi, \Psi) \geq \sigma$, i.e.,

$$\Phi \approx^\sigma \Psi \Leftrightarrow SM(\Phi, \Psi) \geq \sigma, \sigma \in (0, 1) \quad (6)$$

Φ and Ψ are significantly similar if $SM(\Phi, \Psi) \geq \frac{1}{2}$.

Example 8. Suppose that an 2-PSVNSs on two of elements $Z = \{z_1, z_2\}$ are defined by

$$\Phi = \left\{ \frac{(\langle 0.4, 0.3, 0.9 \rangle, \langle 0.5, 0.2, 0.8 \rangle)}{z_1}, \frac{(\langle 0.6, 0.3, 0.7 \rangle, \langle 0.5, 0.5, 0.6 \rangle)}{z_2} \right\}$$

and

$$\Psi = \left\{ \frac{(\langle 0.6, 0.2, 0.8 \rangle, \langle 0.1, 0.4, 0.7 \rangle)}{z_1}, \frac{(\langle 0.3, 0.2, 0.6 \rangle, \langle 0.7, 0.1, 0.8 \rangle)}{z_2} \right\}.$$

Then, the Hamming distance is $d_1(\Phi, \Psi) = 0.2$ and the SM is $SM(\Phi, \Psi) = 0.833$. It shows Φ is significantly similar to Ψ .

Theorem 11. The SM of two m-PSVNSs Φ and Ψ satisfy the following three properties:

- (1) $0 \leq SM(\Phi, \Psi) \leq 1$,
- (2) $SM(\Phi, \Psi) = SM(\Psi, \Phi)$,
- (3) $SM(\Phi, \Psi) = 1 \Leftrightarrow \Psi = \Phi$.

4.1. An Application of SM for m-PSVNSs in Brand Recognition

Based on the notion of Euclidean distance (i.e., Equation (3)) by SM, we will use m-PSVNSs information to solve brand recognition problem.

In the following, we will build an Algorithm 1 to solve brand recognition problem (i.e., to apply m-PSVNS in brand recognition problem).

Algorithm 1

Determination of how an observer finds similarity to an unknown watch.

First step: Suppose that there are n watches represented by m-PSVNS T_e ($e = 1, 2, \dots, n$) in feature space Z (i.e., $Z = \{z_1, z_2, \dots, z_r\}$).

Second step: Consider an m-PSVNS ψ is another unknown watch, which is to be recognized.

Third step: Convert the given m-PSVNS data into m-PFS by using the following formulas

$$p_k \circ \alpha'_{T_e}(z_r) = \frac{p_k \circ \alpha_{T_e}(z_r) + p_k \circ \beta_{T_e}(z_r) + p_k \circ \gamma_{T_e}(z_r)}{3} \quad (7)$$

and

$$p_k \circ \alpha''_{\psi}(z_r) = \frac{p_k \circ \alpha_{\psi}(z_r) + p_k \circ \beta_{\psi}(z_r) + p_k \circ \gamma_{\psi}(z_r)}{3} \quad (8)$$

Fourth step: Calculate d_1, d_2, d_3 , and d_4 distance between T_e and ψ .

Fifth step: Calculate $SM(T_e, \psi)$ between T_e and ψ by using the following formula

$$SM(T_e, \psi) = \frac{1}{1 + d(T_e, \psi)} \quad (9)$$

Sixth step: Evaluate the result by choosing the T_e , which has the greatest SM with unknown ψ .

In the following example, we will explain and apply the above six steps of Algorithm 1.

Example 9. Suppose that there are four brands of watches (i.e., T_1, T_2, T_3 , and T_4) and let $Z = \{z_1, z_2, z_3, z_4\}$, where z_1 is “Material,” z_2 is “Glass kind,” z_3 is “Water Resistance,” and z_4 is “Beautiful Finishing” be feature space of watches. The data of 2-PSVNSs of four brands in the first step of the Algorithm 1 are given in Table 1:

Also, the 2-PSVNS of unknown watch in the second step of the Algorithm 1 is given by

$$\psi = \left\{ \frac{(\langle 0.6, 0.3, 0.4 \rangle, \langle 0.4, 0.2, 0.5 \rangle)}{z_1}, \frac{(\langle 0.1, 0.7, 0.8 \rangle, \langle 0.4, 0.3, 0.6 \rangle)}{z_2}, \frac{(\langle 0.1, 0.3, 0.9 \rangle, \langle 0.2, 0.4, 0.8 \rangle)}{z_3}, \frac{(\langle 0.5, 0.1, 0.5 \rangle, \langle 0.3, 0.2, 0.6 \rangle)}{z_4} \right\}.$$

By the third step of the Algorithm 1 and by Equation (7), we convert 2-PSVNSs of four brands (i.e., T_1, T_2, T_3 , and T_4) into 2-PFSs as shown in the following Table 2:

Table 1 Data of 2-PSVNSs of four brands.

Brands	z_1	z_2	z_3	z_4
T_1	$(\langle 0.4, 0.5, 0.7 \rangle, \langle 0.7, 0.2, 0.8 \rangle)$	$(\langle 0.2, 0.6, 0.4 \rangle, \langle 0.3, 0.1, 0.8 \rangle)$	$(\langle 0.8, 0.2, 0.7 \rangle, \langle 0.5, 0.5, 0.6 \rangle)$	$(\langle 0.3, 0.2, 0.7 \rangle, \langle 0.4, 0.1, 0.8 \rangle)$
T_2	$(\langle 0.3, 0.1, 0.7 \rangle, \langle 0.5, 0.3, 0.6 \rangle)$	$(\langle 0.7, 0.1, 0.6 \rangle, \langle 0.4, 0.2, 0.9 \rangle)$	$(\langle 0.9, 0.1, 0.3 \rangle, \langle 0.6, 0.1, 0.4 \rangle)$	$(\langle 0.4, 0.3, 0.5 \rangle, \langle 0.2, 0.2, 0.7 \rangle)$
T_3	$(\langle 0.6, 0.1, 0.5 \rangle, \langle 0.4, 0.2, 0.6 \rangle)$	$(\langle 0.5, 0.2, 0.6 \rangle, \langle 0.4, 0.1, 0.8 \rangle)$	$(\langle 0.6, 0.2, 0.7 \rangle, \langle 0.5, 0.3, 0.6 \rangle)$	$(\langle 0.4, 0.1, 0.9 \rangle, \langle 0.4, 0.2, 0.8 \rangle)$
T_4	$(\langle 0.2, 0.6, 0.7 \rangle, \langle 0.1, 0.4, 0.6 \rangle)$	$(\langle 0.3, 0.4, 0.5 \rangle, \langle 0.4, 0.6, 0.5 \rangle)$	$(\langle 0.2, 0.4, 0.6 \rangle, \langle 0.1, 0.5, 0.5 \rangle)$	$(\langle 0.7, 0.2, 0.4 \rangle, \langle 0.2, 0.1, 0.7 \rangle)$

Table 2 2-PFSs of four brands.

Brands	z_1	z_2	z_3	z_4
T_1	(0.53, 0.56)	(0.4, 0.4)	(0.56, 0.53)	(0.4, 0.43)
T_2	(0.36, 0.46)	(0.46, 0.5)	(0.43, 0.36)	(0.4, 0.36)
T_3	(0.4, 0.4)	(0.43, 0.43)	(0.5, 0.46)	(0.46, 0.46)
T_4	(0.5, 0.36)	(0.4, 0.5)	(0.4, 0.36)	(0.43, 0.33)

By Equation (8) (i.e., the 2-PSVNS ψ of unknown watch), we obtain 2-PFSs as follows:

$$\psi = \left\{ \frac{(0.43, 0.36)}{z_1}, \frac{(0.53, 0.43)}{z_2}, \frac{(0.43, 0.46)}{z_3}, \frac{(0.36, 0.36)}{z_4} \right\}.$$

Then, by Equation (3) (i.e., Euclidean distance), we compute the Euclidean distance measure of T_e ($e = 1, 2, 3, 4$) and ψ in the fourth step of the Algorithm 1 as follows:

$$d_3(T_1, \psi) = 0.145, \quad d_3(T_2, \psi) = 0.039, \quad d_3(T_3, \psi) = 0.083, \quad d_3(T_4, \psi) = 0.058.$$

By Equation (9) (i.e., SM), we compute the SM of T_e ($e = 1, 2, 3, 4$) and ψ in the fifth step of the Algorithm 1 as follows:

$$SM(T_1, \psi) = \frac{1}{1 + d_3(T_1, \psi)} = \frac{1}{1 + 0.145} = 0.873,$$

$$SM(T_2, \psi) = \frac{1}{1 + d_3(T_2, \psi)} = \frac{1}{1 + 0.039} = 0.962,$$

$$SM(T_3, \psi) = \frac{1}{1 + d_3(T_3, \psi)} = \frac{1}{1 + 0.083} = 0.923,$$

$$SM(T_4, \psi) = \frac{1}{1 + d_3(T_4, \psi)} = \frac{1}{1 + 0.058} = 0.945.$$

Finally, although for every brand SM is greater than 0.5, SM of T_2 is the highest. Hence the unknown watch ψ is closest in similarity to brand T_2 .

5. A NOVEL METHODOLOGY TO EXTEND THE TOPSIS TO m-PSVNSs

We construct a new methodology to extend the TOPSIS to m-PSVNSs (i.e., this process is very applicable to deal with the group decision-making problem under m-PSVNS system).

Now, we propose an Algorithm 2 of the multi-decision maker multi-criteria decision-making of m-PSVNSs as follows:

Algorithm 2

Determination of the optimal decision based on m-PSVNSs.

First step: Assume that there is a group of n persons decision-makers to evaluate the ratings of alternatives Z_r ($r = 1, 2, \dots, p$) concerning criteria C_s ($s = 1, 2, \dots, q$) in single valued neutrosophic value form where the information about criterion weights is known.

Second step: Create an m-PSVNS decision (i.e., a multi-criteria group decision-making problem) which can be represented in matrix form as

$$\mathcal{M}_{p \times q} = \begin{pmatrix} \vartheta_{11} & \vartheta_{12} & \cdots & \vartheta_{1q} \\ \vartheta_{21} & \vartheta_{22} & \cdots & \vartheta_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \vartheta_{p1} & \vartheta_{p2} & \cdots & \vartheta_{pq} \end{pmatrix}.$$

where p (i.e., the number of alternatives), q (i.e., the number of criteria), and $\vartheta_{rs} \forall r (r = 1, 2, \dots, p), \forall s (s = 1, 2, \dots, q)$; represents the ratings of r -alternatives concerning the s -criteria in the single valued neutrosophic value. The multiple data of n persons of decision-maker group for rating ϑ_{rs} can be expressed as

$$\vartheta_{rs} = (p_k \circ \alpha_{rs}, p_k \circ \beta_{rs}, p_k \circ \gamma_{rs}) (k = 1, 2, \dots, m).$$

Third step: Convert the m-PSVNS value data of alternatives into m-PF n -single-value data by using the following formula

$$p_k \circ \gamma_{rs} = \frac{p_k \circ \alpha_{rs} + p_k \circ \beta_{rs} + p_k \circ \gamma_{rs}}{2} \quad (10)$$

and we obtain rating of alternatives as

$$\hat{\gamma}_{rs} = (p_1 \circ \gamma_{rs}, p_2 \circ \gamma_{rs}, \dots, p_m \circ \gamma_{rs}) (k = 1, 2, \dots, m)$$

and the weights of criterion is expressed in information as

$$\hat{w}_s = (p_1 \circ w_s, p_2 \circ w_s, \dots, p_m \circ w_s) (k = 1, 2, \dots, m; s = 1, 2, \dots, q).$$

Fourth step: By normalization (i.e., the normalization process is to preserve the property of m-PSVNS that the ranges of membership of elements is $[0, 1]$) we get the normalized m-PSVNS to m-PF decision matrix (i.e., fuzzy numbers belong to $[0, 1]$), denoted by $\hat{E}$, is defined by

$$\hat{E} = (\mathcal{E}_{rs})_{p \times q},$$

where

$$\mathcal{E}_{rs} = (p_k \circ \hat{\mathcal{E}}_{rs} = \frac{p_k \circ \gamma_{rs}}{\sqrt{\sum_{r=1}^p (p_k \circ \gamma_{rs})^2}}) (k = 1, 2, \dots, m; s = 1, 2, \dots, q) \quad (11)$$

Fifth step: Construct the weighted normalized m-PSVNS decision matrix, denoted by $\hat{F}$, is defined by

$$\hat{F} = (f_{rs})_{p \times q},$$

such that

$$f_{rs} = \mathcal{E}_{rs}(\cdot) \hat{w}_s = (p_k \circ \mathcal{E}_{rs}(\cdot) p_k \circ \hat{w}_s) (k = 1, 2, \dots, m; s = 1, 2, \dots, q) \quad (12)$$

where the elements $f_{rs} (r = 1, 2, \dots, p; s = 1, 2, \dots, q)$ are normalized the given m values to belong their ranges to from closed interval $[0, 1]$.

Sixth step: Determine the m-fuzzy positive ideal solution (denoted by A^*) and m-fuzzy negative ideal solution (denoted by A^-), are defined by

$$\mathcal{A}^* = (f_1^*, f_2^*, \dots, f_q^*) \text{ and } \mathcal{A}^- = (f_1^-, f_2^-, \dots, f_q^-),$$

where

$$f_s^* = p_k \circ \hat{f}_s^* = \begin{cases} (1, 1, \dots, 1), & s \in \mathcal{B} \\ (0, 0, \dots, 0), & s \in \mathcal{C} \end{cases}, f_s^- = p_k \circ \hat{f}_s^- = \begin{cases} (0, 0, \dots, 0), & s \in \mathcal{B} \\ (1, 1, \dots, 1), & s \in \mathcal{C} \end{cases}$$

for all $k = 1, 2, \dots, m; s = 1, 2, \dots, q$ and $\mathcal{B}$ denotes the benefit criteria and $\mathcal{C}$ denotes the cost criteria.

Seventh step: Compute the separation measure (i.e., Euclidean distance) of each alternative from A^* and A^- , respectively, as follows :

$$\mathcal{S}_r^* = d(f_{rs}, f_s^*) = \sqrt{\frac{1}{m} \left\{ \sum_{k=1}^m \sum_{s=1}^q (p_k \circ \hat{f}_{rs} - p_k \circ \hat{f}_s^*)^2 \right\}} \quad (13)$$

and

$$\mathcal{S}_r^- = d(f_{rs}, f_s^-) = \sqrt{\frac{1}{m} \left\{ \sum_{k=1}^m \sum_{s=1}^q (p_k \circ \hat{f}_{rs} - p_k \circ \hat{f}_s^-)^2 \right\}} \quad (14)$$

Eighth step: Compute the closeness coefficient (denoted by, $\overline{Cf_r}$) of each alternative Z_r ($r = 1, 2, \dots, p$) (i.e., as alternative Z_r is near to A^* and far from A^- as closeness coefficient $\overline{Cf_r}$ go closer to 1), is defined by

$$\overline{Cf_r} = \frac{\delta_r^-}{\delta_r^- + \delta_r^*} \quad (15)$$

Ninth step: According to the $\overline{Cf_r}$, we can give the rank to all alternatives and select the best one of them that nearer to 1.

In the following example, we will explain and apply the above nine steps of Algorithm 2.

Example 10. Let us consider an investment company, which wants to invest some money in the best alternative. The company forms a committee of three members for the selection of best alternative (i.e., Z_r ($r = 1, 2, 3, 4$)) among a panel of four possible alternatives to invest the money, where Z_1 is “a medicine company,” Z_2 is “an electronics company,” Z_3 is “a food company,” and Z_4 is “a telecommunication company.” The committee shall make a decision concerning the following four criteria (i.e., C_s ($s = 1, 2, 3, 4$)): C_1 is “the environmental impact analysis,” C_2 is “the risk analysis,” C_3 is “the social-political impact analysis,” and C_4 is “the growth analysis.” The data of 3-PSVN decision matrix (in the range (0,10)) and 3-polar weights by the committee of three decision-makers p_k ($k = 1, 2, 3$) evaluate all four possible alternatives (i.e., Z_1, Z_2, Z_3, Z_4) in the first and the second steps of Algorithm 2 are given by

$$\mathcal{M} = \left(\begin{array}{c|cccc} & C_1 & C_2 & C_3 & C_4 \\ \hline Z_1 & ((3, 4, 5), (2, 4, 5), (2, 3, 4)) & ((1, 1, 1), (4, 5, 4), (2, 4, 5)) & ((1, 1, 1), (2, 5, 2), (3, 2, 6)) & ((2, 4, 6), (4, 4, 1), (3, 4, 4)) \\ Z_2 & ((2, 1, 5), (3, 2, 7), (1, 1, 3)) & ((1, 2, 2), (3, 5, 6), (1, 3, 5)) & ((4, 6, 6), (2, 1, 3), (1, 5, 5)) & ((2, 4, 6), (1, 3, 3), (1, 3, 5)) \\ Z_3 & ((1, 2, 2), (1, 3, 5), (2, 3, 6)) & ((2, 4, 4), (2, 2, 7), (1, 2, 4)) & ((3, 4, 4), (2, 4, 4), (5, 4, 5)) & ((1, 3, 3), (2, 2, 4), (1, 3, 4)) \\ Z_4 & ((1, 3, 4), (2, 2, 4), (4, 4, 4)) & ((2, 2, 2), (2, 3, 3), (3, 4, 6)) & ((1, 5, 5), (3, 3, 6), (3, 6, 7)) & ((2, 2, 4), (1, 3, 3), (3, 4, 4)) \\ \hat{w} & (0.35, 0.26, 0.38) & (0.42, 0.31, 0.25) & (0.26, 0.45, 0.15) & (0.20, 0.36, 0.24) \end{array} \right).$$

By Equation (10) in the third step of Algorithm 2, we obtain the following 3-polar decision matrix as follows:

$$\mathcal{M}' = \left(\begin{array}{c|cccc} & C_1 & C_2 & C_3 & C_4 \\ \hline Z_1 & (6, 5.5, 4.5) & (2.5, 6.5, 5.5) & (2.5, 4.5, 5.5) & (6, 4.5, 5.5) \\ Z_2 & (4, 6, 2.5) & (2.5, 7, 4.5) & (8, 3, 5.5) & (6, 3.5, 4.5) \\ Z_3 & (2.5, 4.5, 5.5) & (5, 5.5, 3.5) & (5.5, 5, 7) & (3.5, 4, 4) \\ Z_4 & (4, 4, 6) & (3, 4, 6.5) & (5.5, 6, 8) & (4, 3.5, 5.5) \end{array} \right).$$

By Equation (11) in the fourth step of Algorithm 2, we normalized the 3-polar decision matrix to get the normalized 3-PF decision matrix as follows:

$$\mathcal{M}'' = \left(\begin{array}{c|cccc} & C_1 & C_2 & C_3 & C_4 \\ \hline Z_1 & (0.69, 0.54, 0.47) & (0.36, 0.55, 0.53) & (0.21, 0.47, 0.41) & (0.59, 0.57, 0.55) \\ Z_2 & (0.46, 0.59, 0.26) & (0.36, 0.59, 0.43) & (0.69, 0.31, 0.41) & (0.59, 0.44, 0.45) \\ Z_3 & (0.29, 0.44, 0.57) & (0.73, 0.46, 0.34) & (0.48, 0.32, 0.53) & (0.34, 0.51, 0.40) \\ Z_4 & (0.46, 0.39, 0.62) & (0.43, 0.34, 0.63) & (0.48, 0.63, 0.60) & (0.39, 0.44, 0.55) \end{array} \right).$$

By Equation (12) in the fifth step of the Algorithm 2, we obtain the weighted normalized 3-PSVNS decision matrix as follows:

$$\mathcal{M}''' = \left(\begin{array}{c|cccc} & C_1 & C_2 & C_3 & C_4 \\ \hline Z_1 & (0.24, 0.14, 0.17) & (0.15, 0.17, 0.13) & (0.05, 0.21, 0.06) & (0.11, 0.20, 0.14) \\ Z_2 & (0.16, 0.15, 0.09) & (0.15, 0.18, 0.10) & (0.17, 0.13, 0.06) & (0.11, 0.15, 0.10) \\ Z_3 & (0.10, 0.11, 0.21) & (0.30, 0.14, 0.08) & (0.12, 0.14, 0.07) & (0.06, 0.18, 0.09) \\ Z_4 & (0.16, 0.10, 0.23) & (0.12, 0.10, 0.15) & (0.12, 0.28, 0.09) & (0.07, 0.15, 0.13) \end{array} \right).$$

As C_1 and C_2 (resp., C_3 and C_4) are cost criteria (resp., are benefit criteria). Then the 3-fuzzy positive ideal solution A^* and 3-fuzzy negative ideal solution A^- in sixth step of the Algorithm 2 are given as follows:

$$A^* = ((0, 0, 0), (0, 0, 0), (1, 1, 1), (1, 1, 1))$$

$$A^- = ((1, 1, 1), (1, 1, 1), (0, 0, 0), (0, 0, 0)).$$

In the seventh step of the Algorithm 2, by computing separation measure (S_r^*, S_r^-) between each attribute with A^* and A^- by using Equations (13) and (14), respectively, we get the following results as follows ($r = 1, 2, 3, 4$):

$$S_1^*(Z_1, A^*) = 1.25, S_2^*(Z_2, A^*) = 1.26, S_2^*(Z_3, A^*) = 1.28, S_2^*(Z_4, A^*) = 1.23$$

and

$$S_1^-(Z_1, A^-) = 1.19, S_2^-(Z_2, A^-) = 1.23, S_2^-(Z_3, A^-) = 1.20, S_2^-(Z_4, A^-) = 1.23.$$

In the eighth step of the Algorithm 2, by computing closeness coefficient $\overline{Cf_r}$ of each alternative Z_r ($r = 1, 2, 3, 4$) and by Equation (15), we get the following results as follows:

$$\overline{Cf_r}(Z_1) = 0.48, \overline{Cf_r}(Z_2) = 0.49, \overline{Cf_r}(Z_3) = 0.48, \overline{Cf_r}(Z_4) = 0.5.$$

Finally, as $\overline{Cf_r}(Z_4) = 0.5$ is the highest among others, Z_4 is the best ideal solution for this multi-person TOPSIS method. Consequently, the outcome of the decision-making committee of the company is to invest money in a medicine company to obtain the best return of investment.

6. CONCLUSION

We presented the concept of the m-PSVNS as a new m-PFS model. We studied the several structure operations of the m-PSVNS and also discussed the basic properties of the m-PSVNS. Then, we introduced novel concepts of distances measure and SM for m-PSVNSs. Further, we constructed an Algorithm 1 to solve brand recognition problem by applying the m-PSVNS. Moreover, a new approach for the best m-PSVNS alternatives based on distance SMs in MCDM is animated. Finally, a new methodology to extend the TOPSIS to m-PSVNSs is proposed and its applicability was illustrated through a numerical example. In future, an m-PSVNS will definitely open new ways to apply with or without restriction existing results of m-F soft set, m-F soft rough set. Also, by combining the m-PFS and the other sets (e.g., interval-valued FSs [34], picture FSs [35], and spherical FSs [36]) we can extend our work to obtain more novel m-PFS models.

CONFLICTS OF INTEREST

The authors declare no conflicts of interest.

AUTHORS' CONTRIBUTIONS

Author 1, author 5 and author 6 sourced the funding; author 2 drafted the manuscript; author 3 revised the manuscript; author 4 supervised the work.

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