

Research Article

A Study on Semi-directed Graphs for Social Media Networks

Sovan Samanta^{1,*}, Madhumangal Pal², Rupkumar Mahapatra², Kousik Das³, Robin Singh Bhadoria⁴

¹Department of Mathematics, Tamralipta Mahavidyalaya, West Bengal, 721636, India

²Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore, 721102, India

³Department of Mathematics, D.J.H. School, Dantan, West Bengal, 721451, India

⁴Department of Computer Science and Engineering, Birla Institute of Applied Sciences

ARTICLE INFO

Article History

Received 15 Jul 2020

Accepted 11 Feb 2021

Keywords

Semi-directed graphs
 Incidence number
 Counter-isomorphism
 Matrices
 Incidence centrality
 Social media networks

ABSTRACT

In the literature of graph theory, networks are represented as directed graphs or undirected graphs and a mixed of both combinations. In today's era of computing, networks like brain and facebook that do not belong to any of the mentioned networks category and in fact, it belongs to the combination of both networks which have connections as directed as well as undirected. To represent such networks, semi-directed graphs have been studied in this paper that provides the detailed mathematical fundamentals related to better understand the conceptualization for social media networks. This paper also discusses the suitable matrices analyze for the representation of the graphs. Few new terminologies like incidence number, complete-incidence related to semi-directed graphs and counter isomorphism of semi-directed graphs have been inculcated. A centrality measure, namely incidence centrality, has also been proposed based on incidence number on neighbors in social media networks.

© 2021 The Authors. Published by Atlantis Press B.V.

This is an open access article distributed under the CC BY-NC 4.0 license (<http://creativecommons.org/licenses/by-nc/4.0/>).

1. INTRODUCTION

Graph Theory is eventually the study of relationships. For a set of nodes and connections, graph theory contributes a helpful tool to quantify and simplify the many evolving parts of dynamic methods. Swiss mathematician Leonhard Euler first originated the fundamental idea of graphs in the 18th century. After that, the branch is advanced rapidly and at present graph theory has several branches for researchers.

Analyzing brain network is a popular research area of network sciences. However, the representation is done by undirected graphs. Though, researchers on network science believe that the brain network might be a mixture of direction and undirection [1]. The nodes of the network denote the brain regions, and the links describe the connections or information flow among them. In the literature of graph theory, there is nothing to represent such mixture networks accurately. There are many instances in the current world, indicating that semi-direction of networks. For example, the current Facebook setting [2] has mixed direction. It can be noted that two friends on Facebook may be connected without following themselves. In that case, the linkage between the two friends may be undirected. If they follow each other, then along with undirected edge, there are directed edges too. In the literature, Sotkov and Tanaev [3], discussed coloring of mixed graphs. In mixed graphs, a link is either an edge or an arc (directed edge). But, the link between

two vertices may be edge or arc or both. To measure the connectivity between two nodes properly, semi-directed graphs have been introduced in this paper.

Representation of such mixed graphs has some literature. Liu and Li introduced Hermitian-adjacency matrices of mixed graphs. After that, Adiga *et al.* [4] described the adjacency matrix of a mixed graph. In that paper, the matrix was defined as a real matrix. Parallel edges combining both undirected and directed edges should have a single value representing adjacency. In this study, the adjacency matrix is defined in a more convenient way to describe such semi-directed graphs.

Isomorphism in graph theory is another interesting topic. Whitney [5] described some properties on isomorphic graphs in 1933. After that, several works on graph isomorphism can be found. But directed networks are also required to be represented, and their similarity is also desired. This is solved by Berztsi [6]. The isomorphism in a mixed graph is introduced in [7]. In all these cases, mixed graphs have been assumed as there exists directed edges or undirected edges. In our proposed graphs, these edges may appear simultaneously. A new term incidence number of a vertex is defined as a single value of the degree of semi-directed graphs. Also, isomorphism is defined as per assumption on mixed graphs. The opposite of the orientation of edges has been made in counter-isomorphic graphs. Barbosa *et al.* [8,9] introduced augmenting health assistant applications with social media. Readers can access the idea of mixed graphs from [10–15].

*Corresponding author. Email: ssamantavu@gmail.com

Mahapatra et al. [16–19] introduced the edge colouring of a fuzzy graph and radio k colouring in fuzzy graphs. There are different types of centralities available in the literature. But measuring influence can be made proper by their dominance in nature. Some persons do not dominate others in their relationship. These are assumed as undirected edges. Directed edges make the dominances and influences of relationship. In this study, a centrality measure, incidence centrality, has been defined. At last, the area of application is described before the conclusion has been made.

1.1. Gist of Contributions

- This study develops the properties (completeness, influential neighbors) on semi-directed graphs.
- Three types of matrices are proposed.
- Isomorphism and counter-isomorphism on semi-directed graphs are proposed.
- Areas of application on semi-directed graphs are proposed.

2. PRELIMINARIES

Definition 2.1. [9] Let V be a nonempty set of elements, called vertices or nodes. Also let, $E = E_1 \cup E_2$ where $E_1 \subset V \times V$ is a set of unordered pairs of vertices, i.e., $E_1 = \{(u, v) | u, v \in V\}$, called a set of undirected edges and $\overline{E_2} \subset V \times V$ is a set of ordered pair of vertices $\overline{E_2} = \{(\overline{a, b}) | a, b \in V\}$, called a set of directed edges. Here $G = (V, E_1, \overline{E_2})$ is said to be a semi-directed graph.

Example 2.1. Let $V = \{a, b, c, d, e, f\}$, $E_1 = \{(a, b), (b, d), (c, e)\}$ and $\overline{E_2} = \{(\overline{a, f}), (\overline{f, a}), (\overline{b, a}), (\overline{b, e}), (\overline{e, d}), (\overline{f, c}), (\overline{d, c})\}$. Here $G = (V, E_1, \overline{E_2})$ is a semi-directed graph (shown in Figure 1).

Definition 2.2. [9] Degree of a vertex u is denoted as triplet $D(u) = (d(u), d^+(u), d^-(u))$ where $d(u)$ is the sum of all incident edges of E_1 , $d^+(u)$ is the number of out-directed edges of E_2 from the vertex u and $d^-(u)$ is the number of in-directed edges of E_2 toward the vertex u . Now, the **incidence number of a vertex** u is denoted as $in(u)$ and defined as $in(u) = d(u) + d^+(u) - d^-(u)$.

Example 2.2. The degree of a vertex a of the graph, shown in Figure 1, is given as $d(a) = (1, 1, 2)$. Now the incidence number is $in(a) = 1 + 1 - 2 = 0$.

Note 2.1. The sum of the incidence numbers in a semi-directed graph is always even.

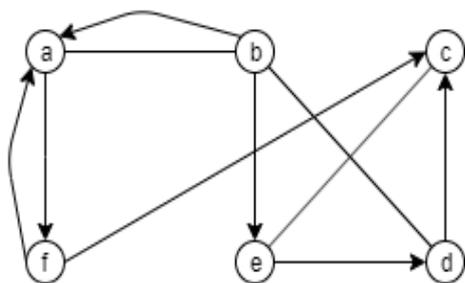


Figure 1 | Example of semi-directed graph.

Definition 2.3. A undirected walk in a semi-directed graph $G = (V, E_1, \overline{E_2})$ is a sequence: $v_1 e_1 v_2 e_2 v_3 \dots e_k v_{k+1}$ where $v_1, v_2, v_3, \dots, v_{k+1}$ are the vertices and e_1, e_2, \dots, e_k are edges respectively such that all the edges are undirected. A undirected walk from vertex u to v is said to be a undirected path of length m if there exist exactly m edges in the walk and no vertices, no edges repeat.

A directed walk in a semi-directed graph $G = (V, E_1, \overline{E_2})$ is a sequence: $v_1 \overline{e_1} v_2 \overline{e_2} v_3 \dots \overline{e_k} v_{k+1}$ where $v_1, v_2, v_3, \dots, v_{k+1}$ are the vertices and $\overline{e_1}, \overline{e_2}, \dots, \overline{e_k}$ are edges respectively such that all the edges are directed. A directed walk from vertex u to v is said to be a directed path of length m if there exist exactly m edges in the walk and no vertices, no edges repeat except u, v .

A path in a semi-directed graph $G = (V, E_1, \overline{E_2})$ is said to cycle if initial and end vertices are the same. If a semi-directed graph does not contain a cycle, then the graph is acyclic.

Example 2.3. In semi-directed graph (Figure 2), $v_5 e_4 v_4 e_3 v_3 e_1 v_1$ undirected a path and $v_4 \overline{e_5} v_2 \overline{e_2} v_1$ is a directed path and there is no cycle.

Definition 2.4. Let $G = (V, E_1, \overline{E_2})$ be a semi-directed graph. Then a graph $H = (V', E'_1, \overline{E'_2})$ is said to be a subgraph of G if $V' \subset V, E'_1 \subset E_1, \overline{E'_2} \subset \overline{E_2}$.

Example 2.5. In Figure 2, $G = (V, E_1, \overline{E_2})$ where $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E_1 = \{e_4, e_3\}$, $\overline{E_2} = \{e_1, e_2, e_5\}$ is a semi-directed graph. Consider $H = (V', E'_1, \overline{E'_2})$ where $V' = \{v_2, v_4, v_5\}$, $E'_1 = \{e_4\}$, $\overline{E'_2} = \{\overline{e_5}\}$. Then H is subgraph of G .

3. COMPLETENESS PROPERTY OF SEMI-DIRECTED GRAPHS

If there exist all three types of connections, i.e., out-directed edges, in-directed edges and undirected edges between every pair of vertices, then the graph is called a **complete semi-directed graph**. Without having all three types of edges, sometimes graphs may be called complete depending on incidence number and connections between every pair of vertices.

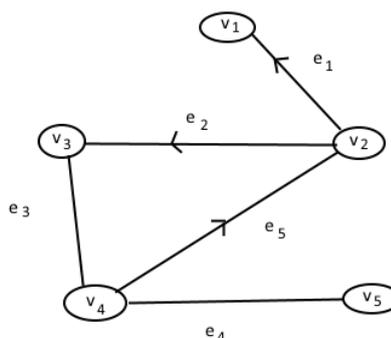


Figure 2 | An example of a semi-directed graph.

Definition 3.1. A semi-directed graph is said to be **complete-incidence semi-directed graph** if every pair of vertices is connected by at least one edge (undirected or directed) and the incidence number of all vertices are equal.

Example 3.1. Since in the above graph (Figure 3), there are the connections (undirected or directed) between every pair of vertices and the incidence number of each of the vertices is 3. Hence it is a complete-incidence semi-directed graph.

Note that, every complete semi-directed graph is complete-incidence semi-directed graphs.

Definition 3.2. A semi-directed graph is called regular if the incidence numbers of each vertex are equal.

Lemma 1. Every complete-incidence semi-directed graph is regular.

Proof. Let $G = (V, E_1, \overline{E}_2)$ be a complete-incidence semi-directed graph. Then there is at least one connection between others vertices and the incidence numbers of all the vertices are equal. Thus obviously, it is a regular semi-directed graph.

Definition 3.3. Let $G = (V, E_1, \overline{E}_2)$ be a semi-directed graph. If the vertex set V partitioned into two nonempty subsets V_1 and V_2 such that

- i) $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$,
- ii) Every vertex of V_1 is connected (undirected or directed) to the vertices of V_2 and conversely,
- iii) There are no edges between the vertices of V_1 itself or V_2 itself.

Then the graph is a bipartite semi-directed graph.

Definition 3.4. A bipartite semi-directed graph is said to be complete-incidence if there exist a partition V_1 and V_2 such that

- i) $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$,
- ii) Every vertex of V_1 is connected (undirected or directed) to all the vertices of V_2 and conversely and the incidence numbers of each vertex of V_1 (V_2) are same.
- iii) There are no edges between the vertices of V_1 itself or V_2 itself.

Example 3.2. Let us consider a complete-incidence bipartite graph, as shown in Figure 4. Here two sets of vertices are shown as

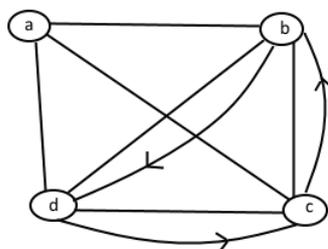


Figure 3 | An example of a complete-incidence semi-directed graph.

$\{a, b\}$ and $\{c, d, e\}$ which have no connections among themselves and each vertex of a set is connected to the vertices of another set.

Definition 3.5. Let $G = (V, E_1, \overline{E}_2)$ be a semi-directed graph and let $C_V = (V, E, \overline{E})$ be a complete semi-directed graph. The complement of G is denoted as G^c and defined as $G^c = (V, E_1^c, \overline{E}_2^c)$ where $E_1^c \cup E_1 = E, E_1^c \cap E_1 = \emptyset$ and $\overline{E}_2^c \cup \overline{E}_2 = \overline{E}, \overline{E}_2^c \cap \overline{E}_2 = \overline{E}$.

Example 3.3. In Figure 5, a semi-directed graph, and its complement graph is shown.

4. INFLUENTIAL NEIGHBORHOODS OF SEMI-DIRECTED GRAPHS

In semi-directed graphs, the 1-step neighbor is measured depending on the out-directed edge and undirected edge. Thus, influential neighborhoods are classified as followers and following. Incidence number is the parameter for influential neighbors.

Incidence number of a vertex (x) due to the node (y) is denoted as $in_y(x) = (\text{number of an undirected edge between } x \text{ and } y) + (\text{number of out-directed edge from } x \text{ to } y) - (\text{number of edge in-directed edge from } y \text{ to } x)$. If $in_y(x) > 0$, then x has a follower “ y ,” and the same will be represented as $in_y^+(x)$. Similarly, if $in_y(x) < 0$ then x has a precursor “ y ,” and the same will be represented as $n_y^-(x)$.

Thus set of followers of a node x , is denoted $F(x) = \{y \in V : in_y(x) > 0\}$ and set of precursors is denoted by $P(x) = \{z \in V : in_y(x) < 0\}$. It can be noted that a node x having both out-directed and in-directed edges to another node y , i.e., $in_y(x) = 0$, y is not an influential neighborhood, and hence y does not belong to $F(x)$ or $P(x)$.

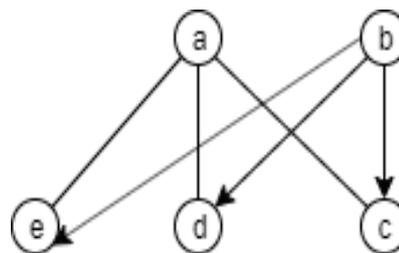


Figure 4 | A complete-incidence bipartite graph.

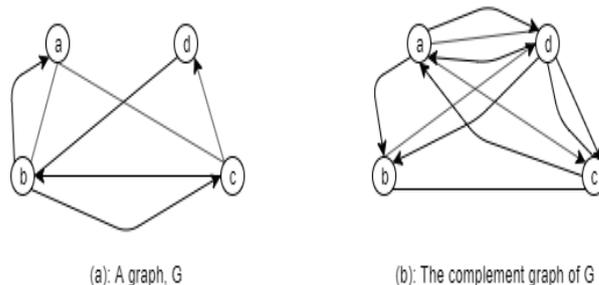


Figure 5 | A semi-directed graph and its complement.

Example 4.1. In Figure 6, the node *a* has one follower, namely *b*. The node *b* has three followers, namely *d*, *e*, *c* and one precursor as *a*.

5. MATRIX REPRESENTATION OF SEMI-DIRECTED GRAPH

5.1. Adjacency Matrix Representation of Semi-directed Graphs

Adjacent matrix is one of the important matrices. It is used to represent vertices which are adjacent to others, i.e., there is an edge connecting vertices in a graph. The adjacency matrix of a semi-directed graph $G = (V, E_1, \overline{E_2})$ with n vertices is a $n \times n$ symmetric binary matrix $A(G) = (a_{ij}, b_{ij})$, where

$$a_{ij} = \begin{cases} \text{in}_j(i), & \text{if there is an edge (directed or undirected)} \\ & \text{i}^{\text{th}} \text{ and } \text{j}^{\text{th}} \text{ vertices} \\ 0, & \text{if there is no edge between them.} \end{cases}$$

Example 5.1. The adjacency matrix (depending on incidence value) of the semi-directed graph, shown in Figure 1, is given as follows:

	a	b	c	d	e	f
a	-	0	0	0	0	0
b	2	-	0	1	1	0
c	0	0	-	-1	1	-1
d	0	1	1	-	-1	0
e	0	-1	1	1	-	0
f	0	0	1	0	0	-

5.2. Adjacency Matrix Representation of Semi-directed Graphs Depending on Incidence Value

In this article, a new kind of adjacency matrix is introduced and termed as “adjacency matrix depending on incidence value.” The

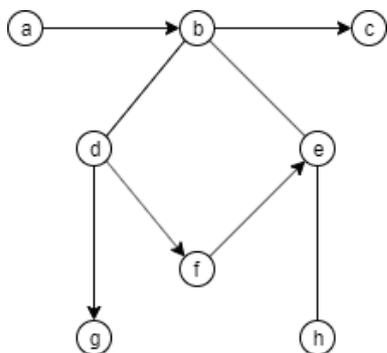


Figure 6 | A semi-directed graph with set of followers.

adjacency matrix of a semi-directed graph $G = (V, E_1, \overline{E_2})$ with n vertices is a $n \times n$ symmetric binary matrix $A(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} \text{in, if there is an edge of } E_1 \text{ between} \\ & \text{i}^{\text{th}} \text{ and } \text{j}^{\text{th}} \text{ vertices} \\ 0, & \text{if there is no edge between them.} \end{cases}$$

$$b_{ij} = \begin{cases} 1, & \text{if there is an edge of } E_2 \text{ directed from} \\ & \text{i}^{\text{th}} \text{ vertex to } \text{j}^{\text{th}} \text{ vertex} \\ 0, & \text{if there is no edge between them.} \end{cases}$$

Example 5.2. The adjacency matrix of the semi-directed graph, shown in Figure 1, is given as follows:

	a	b	c	d	e	f
a	-	(1, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 1)
b	(1, 1)	-	(0, 0)	(1, 0)	(0, 1)	(0, 0)
c	(0, 0)	(0, 0)	-	(0, 0)	(1, 0)	(0, 0)
d	(0, 0)	(1, 0)	(0, 1)	-	(0, 0)	(0, 0)
e	(0, 0)	(0, 0)	(1, 0)	(0, 1)	-	(0, 0)
f	(0, 1)	(0, 0)	(0, 1)	(0, 0)	(0, 0)	-

5.3. Incidence Matrix Representation of Semi-directed Graphs

Let $G = (V, E_1, \overline{E_2})$ be a semi-directed graph with n vertices and m edges. The **incidence matrix** of G is a $n \times m$ matrix $B(G) = (c_{ij}, d_{ij}, e_{ij})$, where

$$c_{ij} = \begin{cases} 1, & \text{if } \text{i}^{\text{th}} \text{ vertex is incident with } \text{j}^{\text{th}} \text{ edge of } E_1 \\ 0, & \text{else.} \end{cases}$$

$$d_{ij} = \begin{cases} 1, & \text{if the } \text{j}^{\text{th}} \text{ edge of } \overline{E_2} \text{ is directed away} \\ & \text{from the } \text{i}^{\text{th}} \text{ vertex} \\ 0, & \text{else.} \end{cases}$$

$$e_{ij} = \begin{cases} -1, & \text{if the } \text{j}^{\text{th}} \text{ edge of } \overline{E_2} \text{ is directed} \\ & \text{towards to the } \text{i}^{\text{th}} \text{ vertex} \\ 0, & \text{else.} \end{cases}$$

Example 5.3. The incidence matrix of the semi-directed graph, shown in Figure 7, is given as follows:

6. ISOMORPHISM OF SEMI-DIRECTED GRAPHS

Let $G_1 = (V_1, E_1, \overline{E_2})$ and $G_2 = (V_2, E_3, \overline{E_4})$ be two semi-directed graphs. Then G_1 and G_2 are said to be isomorphic if there

	e1	e2	e3	e4	e5	e6	e7	e8	e9
a	(0, 1, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(1, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
b	(0, 0, -1)	(0, 1, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(1, 0, 0)	(0, 0, 0)	(0, 0, 0)
c	(0, 0, 0)	(0, 0, -1)	(1, 0, 0)	(0, 0, 0)	(0, 0, 0)	(1, 0, 0)	(0, 0, 0)	(0, 0, 0)	(1, 0, 0)
d	(0, 0, 0)	(0, 0, 0)	(0, 0, -1)	(0, 1, 0)	(0, 0, 0)	(0, 0, 0)	(1, 0, 0)	(1, 0, 0)	(1, 0, 0)
e	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, -1)	(1, 0, 0)	(1, 0, 0)	(0, 0, 0)	(1, 0, 0)	(0, 0, 0)

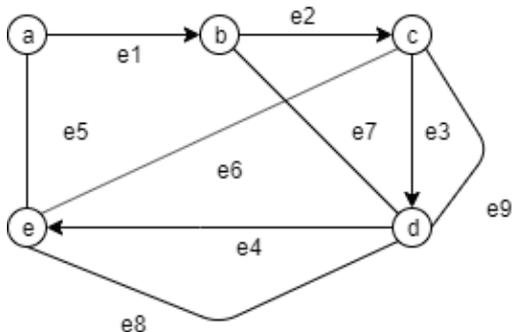


Figure 7 | A semi-directed graph.

exists a one-one correspondence between the vertex sets V_1 and V_2 , between undirected edge sets E_1 and E_3 and between directed edge sets $\overrightarrow{E_2}$ and $\overrightarrow{E_4}$ such that if any two vertices $a, b \in V_1$ are connected by an edge $(a, b) \in E_1$ or $(\overleftarrow{a, b}) \in \overrightarrow{E_2}$ if and only if corresponding vertices $c, d \in V_2$ are also connected by an undirected edge $(c, d) \in E_3$ or directed edge $(\overleftarrow{c, d}) \in \overrightarrow{E_4}$. Alternatively, the following definition is defined below.

Definition 6.1. Let $G_1 = (V_1, E_1, \overrightarrow{E_2})$ and $G_2 = (V_2, E_3, \overrightarrow{E_4})$ be two semi-directed graphs. Then G_1 and G_2 are said to be isomorphic if there exists a bijective mapping $f: V_1 \rightarrow V_2$ such that there exists an edge $(f(a), f(b)) \in E_3$ (or $(\overleftarrow{f(a), f(b)}) \in \overrightarrow{E_4}$) if and only if there exist edges $(a, b) \in E_1$ (or $(\overleftarrow{a, b}) \in \overrightarrow{E_2}$).

It follows from the definition that two isomorphic semi-directed graphs have

- same number of vertices
- same number of undirected edges
- same number of directed edges
- same degree sequences
- same adjacency matrices
- same incidence matrices
- same incidence number of corresponding vertices.

Example 6.1. Consider two pairs of semi-directed graphs (Figure 8). We see that there is one to one correspondence between vertices

$$a \leftrightarrow w, b \leftrightarrow x, c \leftrightarrow u, d \leftrightarrow v$$

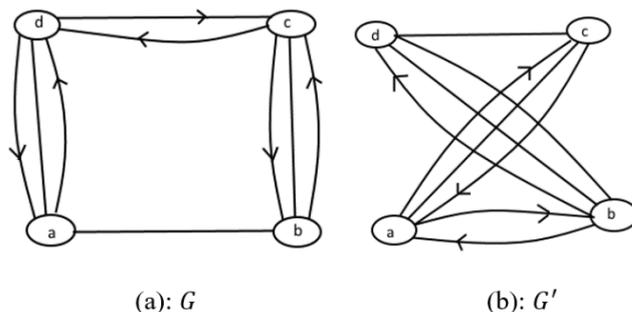


Figure 8 | Two semi-directed graphs.

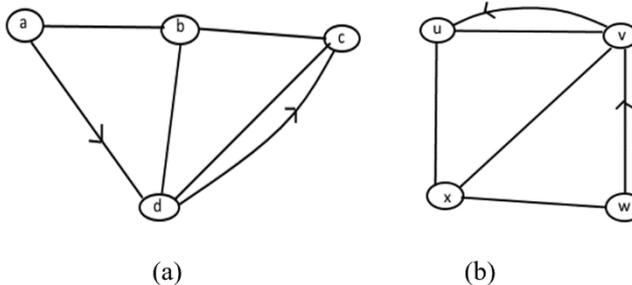


Figure 9 | Two semi-directed graphs.

and one-to-one correspondence between edges

$$(a, b) \leftrightarrow (w, x), (b, c) \leftrightarrow (x, u), (\overleftarrow{d, c}) \leftrightarrow (\overleftarrow{v, u}),$$

$$(d, c) \leftrightarrow (v, u), (b, d) \leftrightarrow (x, v), (\overleftarrow{a, d}) \leftrightarrow (\overleftarrow{w, v}).$$

Also, the adjacency matrix and incidence matrix both are same for two graphs. Hence the graphs are isomorphic.

Now consider two semi-directed graphs (Figure 9). These two graphs are like similar but not isomorphic since vertex c corresponds to vertex f and vertex b corresponds to vertex g but the edge $(\overleftarrow{c, b})$ does not correspond to the edge $(\overleftarrow{f, g})$ because of the opposite direction.

Lemma 6.1. Two semi-directed graphs are isomorphic if and only if its complements are isomorphic.

Definition 6.2. If a semi-directed graph is isomorphic to its complement graph, then the graph is called self-complement semi-directed graph.

Example 6.2. Consider the semi-directed graph G (Figure 10a) and hence find its complement G' (Figure 10b). See that there is one to one correspondence between vertices and edges correspondence are given by:

Tables drawn below represents the vertices and its edge combinations:

Vertex of G	Vertex of G'
a	d
b	c
c	b
d	a

Edges of G	Edges of G'
(a, b)	(d, c)
(b, c)	(c, b)
(b, c)	(c, b)
(c, b)	(b, c)
(c, d)	(b, a)
(d, c)	(a, b)
(a, d)	(d, a)
(a, d)	(d, a)
(d, a)	(a, d)

Theorem 6.1. The number of edges in a semi-directed self-complement graph with n vertices is $3/2 nC_2$.

Proof. Let G be any semi-directed graph with n vertices and G' be the self-complement graph of G . Since the graph contains n edges, then the total number of edges in a complete semi-directed graph is $3 nC_2$,

Again, the number of edges in G + number of edges in $G' = 3nC_2$.

Since self-complement graphs are isomorphic to itself, then the number of edges in $G =$ number of edges in G' .

Therefore, the number of edges in G is $3/2 nC_2$

6.1. Counter-Isomorphic Semi-directed Graphs

Definition 6.3. Let $G_1 = (V_1, E_1, \overline{E_2})$ and $G_2 = (V_2, E_3, \overline{E_4})$ be two semi-directed graphs. Then G_1 and G_2 are said to be counter-isomorphic if there exists a bijective mapping $f : V_1 \rightarrow V_2$ such

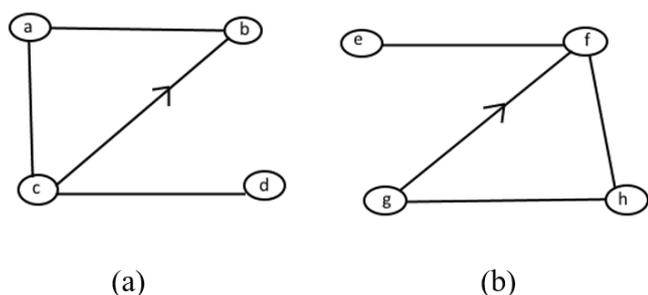


Figure 10 | Self-complementary semi-directed graph.

that there exists an edge $(f(b), f(a))$ in $\overline{E_4}$ (or $(f(a), f(b))$ in E_3) if and only if there exist edges (a, b) in $\overline{E_2}$ ((a, b) in E_1).

Example 6.3. In Figure 11, two graphs have been shown. The part (b) of the figure is counter isomorphic to the part (a).

Theorem 6.2. Incidence number of a vertex is differed by twice the difference of out-degree and in-degree of the vertex to its corresponding vertex in a counter-isomorphic graph.

Proof. Let $G_1 = (V_1, E_1, \overline{E_2})$ and $G_2 = (V_2, E_3, \overline{E_4})$ be two counter-isomorphic semi-directed graphs. Let a vertex $v_1 \in V_1$ and its corresponding vertex be $v_2 \in V_2$. It is clear that the incidence number of v_1 is expressed as $in(v_1) = d(v_1) + d^+(v_1) - d^-(v_1)$ and incidence number of v_2 is expressed as $in(v_2) = d(v_2) + d^+(v_2) - d^-(v_2)$.

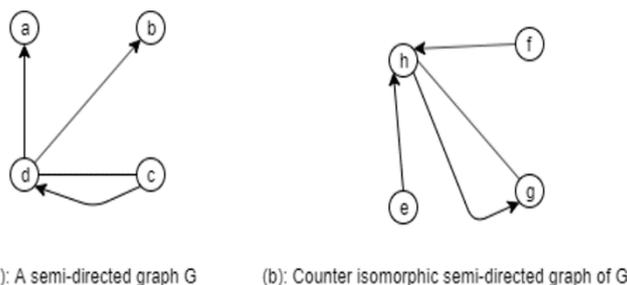
By the definition of a counter-isomorphic graph, the in-degree is replaced by out-degree in incidence number for the corresponding vertex and vice versa. Thus $d(v_1) = d(v_2), d^+(v_1) = d^-(v_2), d^-(v_1) = d^+(v_2)$ is true. Hence, $in(v_1) \sim in(v_2) = 2(d^+(v_1) \sim d^-(v_1)) = 2(d^-(v_2) \sim d^+(v_2))$ the result is true.

7. APPLICATION AREA OF SEMI-DIRECTED GRAPHS

There can be numerous applications that follows and utilizes the concept of semi-directed graph or network where direct and indirect point of connection can be established.

One major application that involves the prediction of epidemic spread (Figure 12) of a disease like flu (Corona Virus Disease 2019) or measles into human society. This can be used to inspect the impact of directed or indirect point of contacts. Such application can be investigate for the case of uncorrelated semi-directed networks [12].

Secondly, the basic reproduction rate of bacteria can be obtained using semi-directed graph in which the rate of growth for a particular cell can be determine for the infection rate to its adjacent cell. This phenomenon is termed as “Kinetic Modeling of Bacteria Growth in Cells.” This examination is very useful in Tumor Cell Morphology [13].



(a): A semi-directed graph G (b): Counter isomorphic semi-directed graph of G

Figure 11 | An example of counter-isomorphic graphs.

Note: The number of vertices and number edges are same in counter

Thirdly, in automated driving system (ADS), the communication among various vehicles can be based upon directed and undirected connection where each vehicle can trace its path for possible number of links based upon availability of roads. This concept of semi-directed networks can be used to not only improve vehicle safety, but also to recover vehicle efficiency and travel times prediction [14].

Fourth, one of the popular application of these semi-directed graph is user identification into social media network. It can be beneficial for finding vulnerabilities of cross platform or advertisement of sale over social sites. The study of user behavior for analyzing information diffusion and mapping of purchase pattern across social e-commerce site can impact direct and indirect connection for its examination [11].

Measuring central persons in large networks is an essential task nowadays. The central persons are generally the most influential persons in the networks. It can be a common point for spreading information like news or any useful value to all overall the network at a very rapid pace. Thus, finding such persons in a large network is an interesting research area. Several definitions of centrality are available in the literature [11]. All these definitions are based on directed or undirected graphs.

Suppose a network is considered as an undirected edge links two connected persons if they do not influence each other. If one person/node (a) influences another person/node (b), then there will exist one directed edge from a to b. Now, for influential measurements of a node (a) is important to find the collection of all such nodes (x) to whom a is connected by undirected edges and all the nodes (y) to whom influences, i.e., there exists directed edges from a to y. Also, the nodes which influence the target node is a kind of negative influences on the target node. Thus, the incidence number is a perfect measurement of such 1-step neighbors. As a measure of influence is not only the subject matters of 1-step neighbors, it can be extended up to the desired steps of neighbors. This measurement is termed as incidence centrality and defined as follows.

Definition 7.1. Let $G = (V, E_1, \overline{E}_2)$ be a semi-directed graph representation of a network. Incidence centrality $C_{in}(v)$ of a node, v is

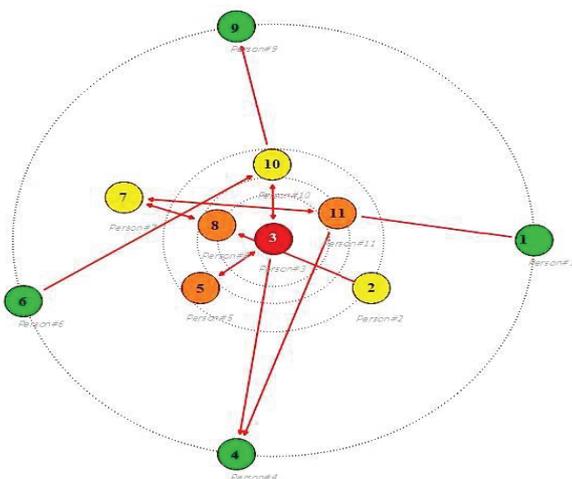


Figure 12 | Node Analysis into Semi-directed Network.

given by

$$C_{in}(v) = in(v) + p \left[\sum_{\{u \in F(v)\}} in_v(u) + \sum_{\{u \in P(v)\}} in_v(u) \right]$$

where $in(v)$ denotes the incidence number of the node v , $F(v)$ is the set of followers of v , $P(v)$ denotes the set of precursor nodes of v , and p is a parameter $\in (0, 1)$. By default, the value of the parameter will be taken as 0.5. This parameter is a parameter, and its value is chosen by the nature and problem of the network.

The incidence centrality considers the incidence number of a node and its neighborhood in a network.

Example 7.1. In Figure 13, a small network is considered.

The incidence centrality of {b} is calculated as

$$\begin{aligned} C_{in}(b) &= in(b) + 0.5 \left[\sum_{\{u \in F(b)\}} in_b(u) + \sum_{\{u \in P(b)\}} in_b(u) \right] \\ &= (-2) + 0.5 [1 - 1 - 1 - 1] \\ &= -3 \end{aligned}$$

The incidence centrality of {d},

$$\begin{aligned} C_{in}(d) &= in(d) + 0.5 \left[\sum_{\{u \in F(d)\}} in_d(u) + \sum_{\{u \in P(d)\}} in_d(u) \right] \\ &= (+2) + 0.5 [4 - 2] = +3 \end{aligned}$$

8. CONCLUSION WITH FUTURE WORKS

This paper introduced new properties of semi-directed graphs. These properties can be the backbone for any social media network representations. In the considered applications, a new centrality measure, incidence centrality, has been introduced. In such definition, direct neighborhood and 1-step neighborhoods have been considered and for indirect neighborhoods, the consideration is subject to the nature of such social media networks. With such network properties analysis, the semi-directed graph will get a new direction of representations. In future, this study can be extended to fuzzy semi-directed graphs, neutrosophic semi-directed graphs and related studies. It will be helpful to analysis of online social networks.

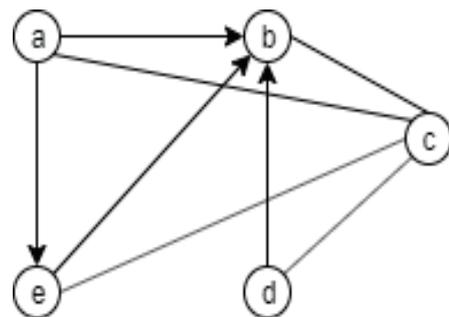


Figure 13 | A small network.

CONFLICTS OF INTEREST

There are no conflicts of interests among the authors.

AUTHORS' CONTRIBUTIONS

Sovan Samanta introduced the idea of the paper. Kousik Das constructed the paper. Madhumangal Pal and Rupkumar Mahapatra reviewed the article and put important suggestions. Robin Singh Bhadoria helped in the application part.

ACKNOWLEDGMENTS

The authors did not receive any funding from any sources for this work. The presented and discussed work herewith are efforts made by all the authors.

REFERENCES

- [1] O. Sporns, Graph theory methods: applications in brain networks, *Dialogues Clin. Neurosci.* 20 (2018), 111–121.
- [2] C. Yang, Y. Lee, Interactants and activities on Facebook, Instagram, and Twitter: associations between social media use and social adjustment to college, *Appl. Dev. Sci.* 24 (2020), 62–78.
- [3] Y.N. Sotskov, V.S. Tanaev, Chromatic polynomial of a mixed graph, *Vestsi Akademii Navuk BSSR, Seriya Fiz.-Mat Navuk.* 6 (1976), 20–23.
- [4] C. Adiga, B.R. Rakshith, W. So, On the mixed adjacency matrix of a mixed graph, *Linear Algebra Appl.* 495 (2016), 223–241.
- [5] H. Whitney, 2-isomorphic graphs, *Am. J. Math.* 55 (1933), 245–254.
- [6] A.T. Bertziss, A backtrack procedure for isomorphism of directed graphs, *J. ACM.* 20 (1973), 365–377.
- [7] A.M. Mohammed, Mixed graph representation and mixed graph isomorphism, *J. Sci.* 30 (2017), 303–310.
- [8] J.L.V. Barbosa, J.E.R. Tavares, I.G. Cardoso, B. Mota, B.G. Martini, TrailCare: an indoor and outdoor context-aware system to assist wheelchair users, *Int. J. Hum.-Comput. Stud.* 116 (2018), 1–14.
- [9] H.D. Vianna, J.L.V. Barbosa, Pompilos, a model for augmenting health assistant applications with social media content, *J. Univ. Comput. Sci.* 26 (2020), 4–32.
- [10] J. Liu, X. Li, Hermitian-adjacency matrices and Hermitian energies of mixed graphs, *Linear Algebra Appl.* 466 (2015), 182–207.
- [11] K. Das, S. Samanta, M. Pal, Study on centrality measures in social networks: a survey, *Soc. Netw. Anal. Mining.* 8 (2018), 1–11.
- [12] X. Zhang, G.Q. Sun, Y.X. Zhu, J. Ma, Z. Jin, Epidemic dynamics on semi-directed complex networks, *Math. Biosci.* 246 (2013), 242–251.
- [13] J. Yin, J. Redovich, Kinetic modeling of virus growth in cells, *Microbiol. Mol. Biol. Rev.* 82 (2018), e00066-17.
- [14] Y. Emzivat, J. Ibanez-Guzman, P. Martinet, O.H. Roux, Dynamic driving task fallback for an automated driving system whose ability to monitor the driving environment has been compromised, in *IEEE Intelligent Vehicles Symposium*, Los Angeles, CA, USA, 2017, pp. 1841–1847.
- [15] S. Samanta, G. Muhiudin, A.M. Alanazi, K. Das, A mathematical approach on representations of competitions: competition cluster hypergraphs, *Math. Probl. Eng.* 2020 (2020), 1–10.
- [16] R. Mahapatra, S. Samanta, M. Pal, Applications of edge colouring of fuzzy graphs, *Informatica*, 31 (2020), pp. 313–330.
- [17] R. Mahapatra, S. Samanta, T. Allahviranloo, M. Pal, Radio fuzzy graphs and assignment of frequency in radio stations, *Comput. App. Mathematics*, 38, (2019), pp. 1–20.
- [18] R. Mahapatra, S. Samanta, M. Pal, Q. Xin, RSM index: A new way of link prediction in social networks, *J. of Intel. and Fuzzy Sys.* 37 (2019), pp. 2137–2151.
- [19] R. Mahapatra, S. Samanta, M. Pal, Q. Xin, Link prediction in social networks by neutrosophic graph, *Inter. J. of Comput. Intel. Sys.* 13 (2020), pp. 1699–1713.