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Progressively Censored N-H Exponential Distribution

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ABSTRACT

An extended version of exponential distribution is considered in this paper. This lifetime distribution has increasing, decreasing and constant hazard rates. So it can be considered as another useful two-parameter extension/generalization of the exponential distribution. It can be used as an alternative to the gamma, Weibull and exponentiated exponential distributions. Maximum likelihood and Bayes estimates for two parameter of N-H exponential distribution are obtained based on a progressive type II censored samples. Bayesian estimates are obtained using squared error loss function. These Bayesian estimates are evaluated by applying the Lindely's approximation method.

1. INTRODUCTION

The exponential distribution is a popular choices for analyzing lifetime data, but the constant hazard rate makes it not suited in several practical situations. In such situations the Weibull and gamma models are frequently employed. The either increasing or decreasing hazard rate of these distributions marked them unfit for the analysis of many lifetime data sets where the hazard rate is U-shaped. This led some authors to introduce extensions or generalization of the exponential distribution as alternative models. In [1] Gupta and Kundu introduced the exponentiated (or generalized) exponential distribution and in [2] Nadarajah and Kotz introduced a generalization of exponential distribution referred to as the beta exponential distribution which is generated from the logit of a beta random variable. Also, Gupta and Kundu [3] introduced a shape parameter to an exponential model using the idea of Azzalini and called it weighted exponential distribution. Nadarajah and Haghighi [4] introduced another extension of the exponential distribution which was later called the N-H exponential distribution. Here, we will be concerned with the N-H exponential distribution. This distribution has survival function given by

$$S(t) = e^{1-(1+\lambda t)^{\alpha}} \tag{1}$$

The corresponding cdf, pdf and hazard function are given by

$$F(x) = 1 - e^{1-(1+\lambda t)^{\alpha}} \tag{2}$$

$$f(x) = \alpha\lambda(1+\lambda t)^{\alpha-1} e^{1-(1+\lambda t)^{\alpha}} \tag{3}$$

and

$$h(x) = \alpha\lambda(1+\lambda t)^{\alpha-1} \tag{4}$$

some properties of this distribution derived by [4] and they discussed the estimation by method of moments and maximum likelihood in the case of complete samples. They also provided formulas for the associated Fisher information matrix. They discussed the maximum likelihood estimation (mle) in the general case of multicensored data which include type I and type II censoring as particular cases.

The conventional type I and type II censoring allow for units to be removed or lost from the test at prefixed time and prefixed no of failures, respectively. These types of censoring are common censoring schemes but they do not have the facility of allowing removal of units at points

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other than the terminal points of the test. In this article a progressive sample scheme as popularized by [5], in particular progressive type II right censoring scheme will be considered.

Under progressive type II right censoring scheme, n units are placed on a test at time 0. At the first failure, R_1 surviving units are randomly removed from the remaining $n-1$ surviving units. At the second failure, R_2 surviving units are randomly removed from the remaining $n-2-R_1$ units etc. This process continues until the mth failure recorded and at this time all remaining units ($R_m = n-m-R_1-R_2-...-R_{m-1}$) are removed. In this censoring scheme, R_i and m are previously fixed. Note that if $R_1 = R_2 = ... = R_{m-1} = 0$ i.e., $R_m = n-m$, this scheme reduces to conventional type II one stage right censoring scheme. Also, one can note that if $R_1 = R_2 = ... = R_{m-1} = R_m = 0$ i.e., $m = n$, this scheme reduces to the case of complete sample(i.e., the case of noncensoring) [5]

In this paper, we discuss the mle of the two unknown parameters of N-H exponential with approximate confidence interval under progressive type II right censoring.

2. MAXIMUM LIKELIHOOD ESTIMATION

Suppose we have $X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}$ a progressive type II censored sample taken from N-H exponential distribution given by (2) and let $R_1, R_2, ..., R_m$ be the progressive censoring scheme. In this case the likelihood function take the form

$$L(\alpha, \lambda) = C\prod_{i=1}^{m} f(x_{i:m:n}; \alpha, \lambda)[1 - F(x_{i:m:n}; \alpha, \lambda)]^{R_i} \tag{5}$$

where $C = n(n-1-R_1)(n-2-R_2)...(n-m-R_1-...-R_{m-1})$; and $f(x)$ and $F(x)$ take the form given by formulas (3) and (2), respectively. Substituting (2) and (3) into (5), the likelihood function becomes

$$L(\alpha, \lambda|x) \propto \alpha^m \lambda^m \prod_{i=1}^{m} (1 + \lambda x_i)^{\alpha-1} (e^{1-(1+\lambda x_i)^{\alpha}})^{R_i+1} \tag{6}$$

and the natural logarithm of likelihood function is

$$logL \propto mlog\alpha + mlog\lambda + (\alpha - 1)\sum_{i=1}^{m} log(1 + \lambda x_i) + \sum_{i=1}^{m}(R_i + 1)(1 - (1 + \lambda x_i)^{\alpha}) \tag{7}$$

Differentiating Equation (7) with respect to α and λ and equating each to zero, we get

$$\frac{\partial logL}{\partial \alpha} \equiv \frac{m}{\alpha} + \sum_{i=1}^{m} log(1 + \lambda x_i) - \sum_{i=1}^{m}(R_i + 1)(1 - (1 + \lambda x_i)^{\alpha})log(1 + \lambda x_i) = 0 \tag{8}$$

and

$$\frac{\partial logL}{\partial \lambda} \equiv \frac{m}{\lambda} + (\alpha - 1)\sum_{i=1}^{m} \frac{x_i}{(1 + \lambda x_i)} - \alpha \sum_{i=1}^{m}(R_i + 1)(1 - (1 + \lambda x_i)^{\alpha-1}x_i = 0 \tag{9}$$

Solving Equations (8) and (9), we obtain $\hat{\alpha}$ and $\hat{\lambda}$. These equations will be normally solved numerically and some software packages utilize for this purpose.

3. BAYESIAN ESTIMATION

Now we will discuss parameter estimation via Bayesian viewpoint. Assuming that the two parameters α and λ are unknown, we propose to use independent gamma priors for both α and λ with pdfs

$$g_1(\alpha) \propto \alpha^{a-1} e^{-b\alpha}, \quad \alpha > 0, \quad a, b > 0$$
$$g_2(\lambda) \propto \lambda^{c-1} e^{-d\lambda}, \quad \lambda > 0, \quad c, d > 0$$

respectively. Two reasons for choosing gamma priors; one is its mathematical tractability and the other is due to the fact that if we known from experience some information about the parameter of interest, like its mean and its variance, it would be fairly easy to calculate the hyper-parameters of the gamma priors. The joint prior distribution for α and λ is

$$g(\alpha, \lambda) \propto \alpha^{a-1} \lambda^{c-1} e^{-(b\alpha + d\lambda)} \tag{10}$$

and

The joint posterior distribution of α and λ is obtained as

$$\pi(\alpha, \lambda \mid x) \propto \alpha^{m+a-1} \lambda^{m+c-1} e^{-m(b\alpha+d\lambda)} \prod_{i=1}^{m} (1 + \lambda x_i)^{\alpha-1} e^{\sum_{i=1}^{m}(R_i-1)(1-(1+\lambda x_i)^{\alpha})}$$

i.e.,

$$\pi(\alpha, \lambda \mid x) = K\alpha^{m+a-1} \lambda^{m+c-1} e^{-m(b\alpha+d\lambda)} \prod_{i=1}^{m} (1 + \lambda x_i)^{\alpha-1} e^{\sum_{i=1}^{m}(R_i-1)(1-(1+\lambda x_i)^{\alpha})}$$

where

$$K = \left[\int_0^{\infty} \int_0^{\infty} \alpha^{m+a-1} \lambda^{m+c-1} e^{-m(b\alpha+d\lambda)} \prod_{i=1}^{m} (1 + \lambda x_i)^{\alpha-1} e^{\sum_{t=1}^{m}(R_i-1)(1-(1+\lambda x_i)^{\alpha})} d\alpha d\lambda\right]^{-1}$$

Therefore, the Bayes estimate of any function of α and λ say $\theta(\alpha, \lambda)$ squared error loss function would be

$$\tilde{\theta}_{Bayes} = K \int_0^{\infty} \int_0^{\infty} \left[\theta(\alpha, \lambda)\alpha^{m+a-1} \lambda^{m+c-1} e^{-m(b\alpha+d\lambda)} * \prod_{i=1}^{m} (1 + \lambda x_i)^{\alpha-1} e^{\sum_{i=1}^{m}(R_i-1)(1-(1+\lambda x_i)^{\alpha})}\right] d\alpha d\lambda.$$

It is not easy to compute this integral, so one can use one of the approximation methods such as Lindely's approximation. Using this method we have the approximate Bayes estimate of α and λ under squared error loss function as follows, respectively

$$\tilde{\alpha}_{Lindely} = \hat{\alpha} + 1/2[2\rho_{\alpha}\sigma_{\alpha\alpha} + 2\rho_{\lambda}\sigma_{\alpha\lambda} + \sigma_{\alpha\alpha}^2 L_{\alpha\alpha\alpha} + 2\sigma_{\alpha\alpha}\sigma_{\lambda\alpha}L_{\lambda\alpha\lambda} + \sigma_{\lambda\alpha}\sigma_{\alpha\alpha}L_{\alpha\alpha\lambda} + 2\sigma_{\lambda\alpha}^2 L_{\lambda\alpha\lambda} + \sigma_{\alpha\alpha}\sigma_{\lambda\lambda}L_{\lambda\lambda\alpha} + \sigma_{\alpha\lambda}\sigma_{\lambda\lambda}L_{\lambda\lambda\lambda}] \quad (11)$$

$$\tilde{\lambda}_{Lindely} = \hat{\lambda} + 1/2[2\rho_{\lambda}\sigma_{\lambda\lambda} + 2\rho_{\alpha}\sigma_{\lambda\alpha} + \sigma_{\lambda\lambda}^2 L_{\lambda\lambda\lambda} + 2\sigma_{\alpha\lambda}^2 L_{\alpha\lambda\alpha} + \sigma_{\alpha\lambda}\sigma_{\alpha\alpha}L_{\alpha\alpha\alpha} + \sigma_{\lambda\lambda}\sigma_{\alpha\alpha}L_{\alpha\alpha\lambda} + 2\sigma_{\lambda\lambda}\sigma_{\alpha\lambda}L_{\alpha\alpha\lambda} + 2\sigma_{\lambda\lambda}\sigma_{\alpha\lambda}L_{\alpha\lambda\lambda}] \quad (12)$$

where $\hat{\alpha}$ and $\hat{\lambda}$ are the mle of α and λ while ρ_{α} and ρ_{λ} are the derivatives of the logarithm of the joint prior distribution and σ, L obtained from differentiating log L. (see details in Appendix)

4. A NUMERICAL STUDY

Here, we present some numerical results to show the behavior of the proposed method for various sample sizes (n = 20, 25, 30, 50, 70, 100); different effective sample sizes(m = 5, 10, 15, 20, 25, 30, 50); two different priors (noninformative prior and informative prior); two different censoring schemes (details of the schemes are given in Tables 1and 2).

Table 1 Different censoring schemes for the simulation study.

n	m	R1,…,Rm	Scheme Number
20	5	(3 * 5)[a]	[1]
		(5, 0, 5, 0, 5)	[2]
		(0, 5, 2, 5, 3)	[3]
		(0 * 4, 15)	[4]
20	10	(1 * 2, 0, 2 * 2, 0, 2, 1 * 2)	[5]
		(10, 0 * 9)	[6]
		(0 * 9, 10)	[7]
		(2 * 2, 0 * 2, 0 * 3, 2 * 2)	[8]

(*Continued*)

Table 1 Different censoring schemes for the simulation study. (*Continued*)

n	m	R1,…,Rm	Scheme Number
25	10	(2 * 2, 0 * 2, 1 * 4, 5, 2)	[9]
		(0 * 5, 5 * 3, 0 * 2)	[10]
		(1 * 4, 2 * 3, 0 * 2, 5)	[11]
		(0 * 4, 15, 0 * 5)	[12]
25	15	(1, 2, 0 * 2, 1, 2, 0 * 3, 1, 0 * 3, 2)	[13]
		(0 * 10, 2 * 5)	[14]
		(1 * 10, 0 * 5)	[15]
		(1 * 5, 0 * 5, 1 * 5)	[16]
30	15	(0 * 14, 15)	[17]
		(1 * 10, 0 * 4, 5)	[18]
		(0 * 5, 1 * 5, 2 * 5)	[19]
		(2 * 5, 0 * 9, 5)	[20]
30	20	(3, 0 * 4, 3, 0 * 4, 1, 0 * 4)	[21]
		(1 * 10, 0 * 10)	[22]
		(1 * 5, 0 * 5, 1 * 5, 0 * 5)	[23]
		(2 * 5, 0 * 15)	[24]
50	20	(1 * 5, 0 * 14, 25)	[25]
		(10, 0 * 4, 1 * 10, 2 * 5)	[26]
		(0 * 5, 2 * 10, 0 * 3, 5 * 2)	[27]
		(1 * 10, 0 * 5, 5 * 4)	[28]
50	25	(0 * 5, 1 * 5, 0 * 5, 2 * 3, 3 * 2, 4 * 2, 0 * 3)	[29]
		(1 * 25)	[30]
		(0 * 24, 25)	[31]
		(1 * 5, 0 * 16, 4 * 5)	[32]
100	30	(2 * 5, 0 * 5, 5 * 5, 0 * 5, 6 * 5, 1 * 5)	[33]
		(0 * 10, 2 * 10, 0 * 5, 10 * 5)	[34]
		(5 * 5, 0 * 20, 9 * 5)	[35]
		(2 * 5, 0 * 5, 10 * 5, 0 * 5, 2 * 5)	[36]
100	50	(0 * 15, 1 * 5, 0 * 5, 5 * 5, 0 * 15, 4 * 5)	[37]
		(2 * 5, 1 * 10, 0 * 10, 5 * 5, 0 * 15, 1 * 5)	[38]
		(0 * 35, 5 * 5, 0 * 5, 5 * 5)	[39]
		(2 * 10, 0 * 20, 1 * 10, 2 * 20)	[40]

(a) This 3 * 5 denotes that 3, 3, 3, 3, 3. This table is adapted from Dey and Dey [6].

Table 2 Different censoring schemes from poisson with mean 3 for the simulation study.

n	m	R1,…,Rm	Scheme Number
20	5	(1, 4, 1 * 3[a])	[1]
		(4, 1, 2 * 2, 3)	[2]
		(1, 0, 2, 5 * 2)	[3]
		(2, 1, 3 * 3)	[4]
		(2, 1, 4, 3 * 2)	[5]
		(2, 4, 9, 5, 3)	[6]
50	5	(1, 4, 1 * 3)	[7]
		(4, 1, 2 * 2, 3)	[8]
		(1, 0, 2, 5 * 2)	[9]
		(2, 1, 3 * 3)	[10]
		(2, 1, 4, 3 * 2)	[11]
		(2, 4, 9, 5, 3)	[12]

(*Continued*)

Table 2 Different censoring schemes from poisson with mean 3 for the simulation study. (*Continued*)

n	m	R1,…,Rm	Scheme Number
50	10	(1, 5, 1, 3, 2, 1, 3, 0, 3 * 2)	[13]
		(1, 8, 5, 1, 4 * 4, 2, 4)	[14]
		(3, 2, 5, 2, 3, 4, 2, 1, 3, 4)	[15]
		(1, 3, 6, 0 * 2, 6, 2, 3, 2, 0)	[16]
		(4, 3, 1, 2, 1, 2, 1 * 2, 2, 6)	[17]
		(4, 1, 3, 5, 4, 7, 2, 5, 3, 1)	[18]
70	10	(1, 8, 5, 1, 4 * 4, 2, 4)	[19]
		(1, 8, 5, 1, 4 * 4, 2, 4)	[20]
		(3, 2, 5, 2, 3, 4, 2, 1, 3, 4)	[21]
		(1, 3, 6, 0 * 2, 6, 2, 3, 2, 0)	[22]
		(4, 3, 1, 2, 1, 2, 1 * 2, 2, 6)	[23]
		(4, 1, 3, 5, 4, 7, 2, 5, 3, 1)	[24]
70	15	(1, 3, 0 * 2, 3, 2, 3 * 2, 2, 1, 2 * 2, 1, 3, 2)	[25]
		(2, 3 * 2, 4 * 3, 1, 3, 0, 2 * 2, 3, 1, 5, 1)	[26]
		(2, 3 * 2, 4 * 2, 1, 0, 2, 3 * 2, 2, 4 * 3, 5)	[27]
		(5, 3 * 2, 1, 2, 5, 4 * 2, 1, 3, 2, 3, 4, 2, 1)	[28]
		(4, 1, 4, 3, 6, 3, 2, 1, 3, 4, 2 * 2, 7, 1, 2)	[29]
		(1, 5, 2, 5, 1, 3, 6, 2, 4, 3, 5, 4, 1, 6, 3)	[30]
100	15	(1, 3, 0 * 2, 3, 2, 3 * 2, 2, 1, 2 * 2, 1, 3, 2)	[31]
		(2, 3 * 2, 4 * 3, 1, 3, 0, 2 * 2, 3, 1, 5, 1)	[32]
		(2, 3 * 2, 4 * 2, 1, 0, 2, 3 * 2, 2, 4 * 3, 5)	[33]
		(5, 3 * 2, 1, 2, 5, 4 * 2, 1, 3, 2, 3, 4, 2, 1)	[34]
		(4, 1, 4, 3, 6, 3, 2, 1, 3, 4, 2 * 2, 7, 1, 2)	[35]
		(1, 5, 2, 5, 1, 3, 6, 2, 4, 3, 5, 4, 1, 6, 3)	[36]
100	20	(4, 5, 2 * 2, 1, 4, 2, 1, 4, 5 * 2, 4, 3, 2, 5, 3, 1, 6, 1, 3)	[37]
		(5, 3, 1, 3, 4 * 2, 3 * 2, 0, 3 * 2, 1 * 2, 5, 4, 2, 3, 5, 6, 2)	[38]
		(1, 4, 2, 3, 5 * 2, 6, 3 * 2, 1, 3, 5, 3, 4, 3, 2 * 2, 4, 6, 3)	[39]
		(4, 2, 4, 2, 1, 9, 3, 5, 4, 2, 5, 1, 3, 5, 6, 3, 4 * 2, 1 * 2)	[40]
		(3, 4, 2, 1, 2, 5, 4, 6, 2, 4, 2 * 2, 0, 2 * 2, 6, 2, 1, 3 * 2)	[41]
		(1 * 2, 4, 3 * 2, 7, 2, 1, 2 * 3, 0, 3 * 2, 2 * 2, 4, 2, 3)	[42]

(a) This 1 * 3 denotes that 1, 1, 1.

For $\alpha = 0.5$, $\lambda = 1$ and given n, m and a sampling scheme, we generate a samples for a given censoring scheme. we compute the mles for the two unknown parameters and also we compute approximate Bayes estimates by using two different priors. The first prior is noninformative prior (Prior0) i.e., we take the hyper-parameters value as $a = b = c = d = 0$. The second prior is informative prior (Prior1) and we take the hyper-parameters as $a = 2, b = c = d = 1$ which chosen in such a way that the prior mean became the expected value of the corresponding parameter (see [7]). 1000 samples are generated and we compute the average bias and the corresponding mean squared error (MSE). Results are listed in Tables 3–6 where $\hat{\alpha}$, $\hat{\lambda}$ represents the mle estimates and $\tilde{\alpha}$, $\tilde{\lambda}$ represents the approximate Bayes estimates based on Lindely's approximation.

From Tables 4–6, one can note that for fixed sample size n, as effective sample size m increases, the implementations improve in terms of bias and MSEs in both procedures. Bayes estimates under Lindely's method is similar to the mle estimates as expected. In case of Bayes estimates, the results of Prior0 is similar to Prior1.

Table 3 The absolute value of bias and the mean squared error for the mle and Bayes estimates of α are listed for various sample sizes and sampling schemes.

				Prior 0	Prior 1
n	**m**	**Scheme**	$\hat{\alpha}$	$\tilde{\alpha}$	$\tilde{\alpha}$
20	5	[1]	0.4971915(0.528149)	0.4971415(0.528073)	0.4971915(0.528037)
		[2]	0.4829925(0.514042)	0.4829210(0.513973)	0.4829030(0.513955)
		[3]	0.5218350(0.539559)	0.5217500(0.539468)	0.5217300(0.539450)
		[4]	0.5191600(0.548442)	0.5190850(0.548365)	0.5190650(0.548347)
20	10	[5]	0.3415230(0.319913)	0.3415040(0.319884)	0.3414970(0.319880)
		[6]	0.2649890(0.242477)	0.2649600(0.242462)	0.2649545(0.242459)
		[7]	0.2499280(0.204724)	0.2498940(0.204707)	0.2498885(9.204704)
		[8]	0.3022270(0.282316)	0.3021885(0.282293)	0.3021795(0.282287)
25	10	[9]	0.3588315(0.324342)	0.3587875(0.324310)	0.3587775(0.324303)
		[10]	0.3572780(0.335403)	0.3572250(0.335365)	0.3572175(0.335360)
		[11]	0.3200430(0.293386)	0.3199970(0.293357)	0.3199845(0.293349)
		[12]	0.3178110(0.324944)	0.3178705(0.324918)	0.3178645(0.324915)
25	15	[13]	0.1958475(0.150822)	0.1958060(0.150805)	0.1957970(0.150802)
		[14]	0.2375360(0.188130)	0.2374980(0.188111)	0.2374890(0.188107)
		[15]	0.2051600(0.158191)	0.2051200(0.158175)	0.2051120(0.158171)
		[16]	0.2297045(0.180323)	0.2296675(0.180306)	0.2296595(0.180303)
30	15	[17]	0.6506570(0.556114)	0.6505650(0.555973)	0.6505600(0.555970)
		[18]	0.6489950(0.565434)	0.648880(0.565299)	0.648880(0.565297)
		[19]	0.2339490(0.191414)	0.2339100(0.191396)	0.2338995(0.191391)
		[20]	0.2132375(0.167186)	0.2131960(0.167168)	0.2131865(0.167164)
30	20	[21]	0.1545180(0.103939)	0.1544675(0.103924)	0.1544590(0.103921)
		[22]	0.1380175(0.092865)	0.1379555(0.0902693)	0.1379460(0.0902667)
		[23]	0.1512940(0.101957)	0.1512420(0.101942)	0.1512330(0.101939)
		[24]	0.1240075(0.0777336)	0.1239540(0.0777203)	0.1239440(0.0777178)
50	20	[25]	0.2211845(0.164537)	0.2211475(0.164521)	0.2211360(0.164516)
		[26]	0.2100615(0.156362)	0.2100215(0.156345)	0.210012(0.156341)
		[27]	0.1987010(0.155447)	0.1986630(0.155431)	0.1986510(0.155427)
		[28]	0.2080380(0.157063)	0.208000(0.157047)	0.2079890(0.157042)
50	25	[29]	0.1816730(0.125538)	0.1816190(0.125518)	0.1816090(0.125514)
		[30]	0.1649060(0.110223)	0.1648620(0.110209)	0.1648530(0.110206)
		[31]	0.1656070(0.121050)	0.1655740(0.121040)	0.1655630(0.121036)
		[32]	0.1638900(0.112809)	0.1638530(0.112797)	0.1638420(0.112793)
100	30	[33]	0.1778810(0.119835)	0.1778280(0.119816)	0.1778150(0.119812)
		[34]	0.1483560(0.114650)	0.1483260(0.114641)	0.1483120(0.114637)
		[35]	0.1623790(0.116393)	0.1623460(0.116382)	0.1623330(0.116378)
		[36]	0.1595940(0.112488)	0.1595460(0.112472)	0.1595330(0.112468)
100	50	[37]	0.0976598(0.0564262)	0.0976067(0.0564159)	0.097594(0.0564135)
		[38]	0.0736086(0.0384041)	0.0734818(0.0383854)	0.0734641(0.0383828)
		[39]	0.0942280(0.0610751)	0.0941882(0.061076)	0.0941766(0.0610654)
		[40]	0.0857108(0.04392267)	0.0856541(0.048214)	0.0856418(0.0439158)

MLE, maximum likelihood estimation.

5. A CASE OF REAL DATA

The data set in Table 7 represents the remission times (in months) of a sample of size 128 bladder cancer patients reported in [8]. The plot in Figure 1 depicts that N-H exponential distribution gives a reasonable fit to the data set. Using this data, the mle of the unknown parameters is $\hat{\alpha} = 0.846$, $\hat{\lambda} = 0.128$ and the popular Kolmogorov-Smirnov goodness of fit test was carried out at level of significance. The result is the N-H exponential distribution is fit to this data with p-value 0.310952. we generate progressively type II censored data from the above data

Table 4 The absolute value of bias and the mean squared error for the mle and Bayes estimates of λ are listed for various sample sizes and sampling schemes.

				Prior 0	Prior 1
n	**m**	**Scheme**	$\hat{\lambda}$	$u(\alpha, \lambda)$	$\tilde{\lambda}$
20	5	[1]	0.1159850(2.440330)	0.1159210(2.440330)	0.1159860(2.440330)
		[2]	0.1664680(2.877200)	0.1663370(2.877670)	0.1664760(2.877720)
		[3]	0.0521421(2.208430)	0.0520099(2.208420)	0.0521356(2.208430)
		[4]	0.1428650(2.925990)	0.1427380(2.925950)	0.1428660(2.925990)
20	10	[5]	0.0468047(1.475010)	0.0468103(1.475010)	0.4672770(1.475010)
		[6]	0.1489790(1.847030)	0.1489070(1.847010)	0.1490000(1.847040)
		[7]	0.0996390(1.687940)	0.0995607(1.687920)	0.0996578(1.687940)
		[8]	0.1553390(1.709930)	0.1552530(1.709900)	0.1553590(1.709930)
25	10	[9]	0.1151180(1.894140)	0.1150350(1.894120)	0.1151330(1.894140)
		[10]	0.00097442(1.27014)	0.0097176(1.27014)	0.00980259(1.27014)
		[11]	0.1550760(1.930400)	0.1549820(1.930370)	0.1551000(1.930400)
		[12]	0.0614795(1.129176)	0.0614042(1.29176)	0.0614893(1.29177)
25	15	[13]	0.1042340(1.083920)	0.1041470(1.083910)	0.1042640(1.083930)
		[14]	0.0970686(1.266200)	0.0969918(1.266180)	0.0970923(1.266200)
		[15]	0.0998006(1.173930)	0.0997249(1.173910)	0.0998328(1.173930)
		[16]	0.0980328(1.323670)	0.0979575(1.323650)	0.0980556(1.323670)
30	15	[17]	0.2909370(0.592421)	0.2909970(0.592457)	0.2909670(0.592439)
		[18]	0.2861400(0.628215)	0.2861990(0.628249)	0.2861690(0.628232)
		[19]	0.1142530(1.281150)	0.1151730(1.281130)	0.1152810(1.281150)
		[20]	0.1052180(1.131820)	0.1051320(1.131800)	0.1052460(1.131820)
30	20	[21]	0.0397746(0.722620)	0.0396972(0.722614)	0.0398096(0.722623)
		[22]	0.0163531(0.645947)	0.0162683(0.645945)	0.0163926(0.645949)
		[23]	0.0528775(0.718214)	0.0527984(0.718206)	0.0529175(0.718278)
		[24]	0.0919885(0.879429)	0.0919003(0.879413)	0.0920422(0.879439)
50	20	[25]	0.1060660(1.094770)	0.1059960(1.094750)	0.1060980(1.094770)
		[26]	0.0639464(0.918429)	0.0638751(0.918420)	0.0639725(0.918433)
		[27]	0.1027290(1.003060)	0.1026540(1.003050)	0.1027620(1.003070)
		[28]	0.0921558(0.996051)	0.0920827(0.996037)	0.092190(0.996057)
50	25	[29]	0.0226088(0.513429)	0.0226722(0.513432)	0.0225785(0.513428)
		[30]	0.0412888(0.746001)	0.0412199(0.745995)	0.0413179(0.746004)
		[31]	0.1573980(1.070030)	0.1573300(1.070010)	0.1574360(1.070050)
		[32]	0.0939640(0.882445)	0.0938659(0.882432)	0.0939705(0.882451)
100	30	[33]	0.0221276(0.492459)	0.0221921(0.462462)	0.0220961(0.492458)
		[34]	0.1468470(0.877965)	0.1467860(0.877947)	0.1468920(0.877978)
		[35]	0.1084710(0.777371)	0.1084100(0.777358)	0.1085090(0.777379)
		[36]	0.0209500(0.707031)	0.0208773(0.707029)	0.0209798(0.499895)
100	50	[37]	0.0221536(0.365813)	0.0220891(0.365810)	0.0221887(0.365815)
		[38]	0.00189239(0.266265)	0.00180332(0.266265)	0.00195382(0.266265)
		[39]	0.0798200(0.450539)	0.0797579(0.450530)	0.7985540(0.450545)
		[40]	0.0351824(0.391901)	0.0351142(0.0482149)	0.0352210(0.391903)

MLE, maximum likelihood estimation.

set and compute the estimates of the two unknown parameters and listed in Table 8. Approximate confidence intervals are computed in the case of mle.

On the other hand, one can compute an approximate Bayes estimates for the two unknown parameters using the Gibbs sampling procedure which generates samples from the posterior distribution. Here, we obtain the approximate Bayes estimates under the assumptions of noninformative prior. A set of 10000 Gibbs samples was generated after a "burn-in-sample" of size 1000. Using these generated samples

Table 5 The average bias and the mean squared error for the mle and Bayes estimates of α and λ are listed for various sample sizes and sampling schemes (random).

				Prior 0	Prior 1
n	**m**	**Scheme**	$\hat{\alpha}$	$\tilde{\alpha}$	$\tilde{\alpha}$
20	5	[1]	0.3264670(0.174211)	0.3263265(0.174119)	0.326554(0.174268)
		[2]	0.3125085(0.175525)	0.3109845(0.174575)	0.3107235(0.174413)
		[3]	0.2932695(0.156102)	0.2928570(0.155861)	0.2927425(0.155793)
		[4]	0.2964170(0.165958)	0.2951265(0.165194)	0.2948700(0.165043)
		[5]	0.3033180(0.170888)	0.3027190(0.170527)	0.3025795(0.170442)
		[6]	0.3056350(0.171403)	0.3053230(0.171213)	0.3052520(0.171169)
50	5	[7]	0.2326710(0.104817)	0.2326680(0.104816)	0.2328510(0.104901)
		[8]	0.227333(0.0996646)	0.227330(0.0996633)	0.227517(0.0997482)
		[9]	0.2312230(0.101513)	0.2312220(0.101512)	0.2314180(0.101603)
		[10]	0.2448950(0.108718)	0.2448930(0.108717)	0.2450840(0.108810)
		[11]	0.2361100(0.102681)	0.2361080(0.102681)	0.2362990(0.102771)
		[12]	0.2281270(0.102811)	0.2281250(0.102811)	0.2283180(0.102899)
50	10	[13]	0.2023535(0.098225)	0.2023340(0.098217)	0.2024420(0.098261)
		[14]	0.2194495(0.113864)	0.2189520(0.113646)	0.2188260(0.113591)
		[15]	0.1995715(0.100056)	0.1993410(0.0999638)	0.1995460(0.100046)
		[16]	0.188810(0.0991293)	0.1887925(0.0991227)	0.188901(0.0991636)
		[17]	0.1891650(0.0951523)	0.1891255(0.0951374)	0.1892485(0.0951839)
		[18]	0.2115465(0.107693)	0.164785(0.0900948)	0.162992(0.0895071)
70	10	[19]	0.196174(0.0991527)	0.196152(0.0991441)	0.196262(0.0991872)
		[20]	0.2076240(0.109312)	0.2071970(0.109135)	0.2070780(0.109086)
		[21]	0.1925230(0.0981574)	0.1923310(0.0980836)	0.1925220(0.0981572)
		[22]	0.178040(0.0892947)	0.178025(0.0892892)	0.178025(0.0892892)
		[23]	0.181694(0.0942027)	0.181657(0.0941896)	0.181779(0.0942337)
		[24]	0.2236670(0.113502)	0.1999880(0.103470)	0.1987630(0.102982)
70	15	[25]	0.1525525(0.074663)	0.1525445(0.0746605)	0.1526105(0.0746808)
		[26]	0.1610365(0.078941)	0.160040(0.0789305)	0.161088(0.0789577)
		[27]	0.1604285(0.0792977)	0.1594515(0.0789853)	0.159724(0.0790723)
		[28]	0.153904(0.0788255)	0.153777(0.0787864)	0.159724(0.0790723)
		[29]	0.160811(0.0844711)	0.160308(0.0843096)	0.1605155(0.0843761)
		[30]	0.173772(0.0840183)	0.1730975(0.0837843)	0.172953(0.0837344)
100	15	[31]	0.131363(0.0640767)	0.131362(0.0640763)	0.131421(0.0640919)
		[32]	0.137894(0.0678622)	0.137892(0.0678616)	0.137954(0.0678788)
		[33]	0.127235(0.0637183)	0.127232(0.0637175)	0.127301(0.245602)
		[34]	0.140579(0.068427)	0.140577(0.0684264)	0.140641(0.0684443)
		[35]	0.142327(0.0668081)	0.142325(0.0668075)	0.142389(0.0668256)
		[36]	0.133757(0.0670613)	0.133751(0.0670596)	0.133827(0.0670801)
100	20	[37]	0.619528(0.061622)	0.115463(0.0616065)	0.1195515(0.0616277)
		[38]	0.124516(0.0623036)	0.124466(0.0622911)	0.124598(0.0632512)
		[39]	0.133008(0.0679955)	0.1322985(0.0678074)	0.132512(0.0678639)
		[40]	0.137380(0.0704051)	0.1368745(0.0702666)	0.1370595(0.0703172)
		[41]	0.139426(0.0687665)	0.1394075(0.0687612)	0.139470(0.0687788)
		[42]	0.120810(0.0583184)	0.1207935(0.0583143)	0.120857(0.0583296)

MLE, maximum likelihood estimation.

Table 6 The average bias and the mean squared error for the mle and Bayes estimates of λ are listed for various sample sizes and sampling schemes (random).

				Prior 0	Prior 1
n	m	Scheme	$\hat{\lambda}$	$\tilde{\lambda}$	$\tilde{\lambda}$
20	5	[1]	0.1450170(0.414535)	0.1539360(0.414510)	0.1540930(0.414558)
		[2]	0.1222220(0.595943)	0.1225950(0.596034)	0.1221080(0.595915)
		[3]	0.0959207(0.553236)	0.0961172(0.553274)	0.0958474(0.553222)
		[4]	0.0977636(0.500359)	0.0981262(0.500431)	0.9762980(0.500333)
		[5]	0.1127710(0.486695)	0.1130020(0.486747)	0.1126970(0.486678)
		[6]	0.1295870(0.516272)	0.1297450(0.516313)	0.1295510(0.516263)
50	5	[7]	0.0638504(0.339343)	0.3393430(0.582532)	0.3393460(0.582534)
		[8]	0.0764455(0.305708)	0.0764358(0.305706)	0.0764867(0.305714)
		[9]	0.0891549(0.300518)	0.0891353(0.300515)	0.0892121(0.300528)
		[10]	0.0753222(0.338732)	0.0753097(0.338730)	0.0753691(0.338739)
		[11]	0.0953258(0.298049)	0.0953122(0.298046)	0.0953706(0.298057)
		[12]	0.0628087(0.316734)	0.0627925(0.316732)	0.0628660(0.316741)
50	10	[13]	0.0846792(0.362508)	0.0846487(0.362503)	0.0847311(0.362517)
		[14]	0.0865706(0.407572)	0.0867438(0.407602)	0.0864742(0.407556)
		[15]	0.0345237(0.423932)	0.0344032(0.423924)	0.0346810(0.423943)
		[16]	0.0276860(0.386191)	0.0276545(0.386189)	0.0277470(0.386194)
		[17]	0.0441424(0.396225)	0.0440963(0.396221)	0.0442209(0.396232)
		[18]	0.0894087(0.389970)	0.0906514(0.390193)	0.0876544(0.389659)
70	10	[19]	0.0660680(0.390865)	0.0665740(0.390660)	0.0666644(0.390872)
		[20]	0.0429737(0.465414)	0.0431430(0.465429)	0.0428696(0.465405)
		[21]	0.0507068(0.369911)	0.0505973(0.369900)	0.050848(0.369926)
		[22]	0.0436478(0.369652)	0.0436186(0.369650)	0.0436186(0.369650)
		[23]	0.0342054(0.389776)	0.0341602(0.389773)	0.0342852(0.389782)
		[24]	0.0115685(0.376568)	0.116635(0.376789)	0.114665(0.376333)
70	15	[25]	0.0297971(0.305182)	0.0297808(0.305181)	0.0298339(0.305185)
		[26]	0.0550084(0.316372)	0.0549709(0.312447)	0.0550635(0.316378)
		[27]	0.0591451(0.335305)	0.0589468(0.335281)	0.0593916(0.335334)
		[28]	0.0193960(0.368465)	0.0193132(0.368462)	0.0194974(0.368469)
		[29]	0.0209317(0.381710)	0.0207688(0.381703)	0.0211117(0.381718)
		[30]	0.0646878(0.324553)	0.0648706(0.324577)	0.0645641(0.324537)
100	15	[31]	0.0003732(0.322194)	0.0003785(0.322194)	0.0003493(0.322194)
		[32]	0.0004736(0.349041)	0.0004647(0.349041)	0.0005050(0.349041)
		[33]	0.0138877(0.300106)	0.0139039(0.300106)	0.0138432(0.300105)
		[34]	0.0322061(0.284619)	0.0321953(0.284618)	0.0322364(0.284621)
		[35]	0.0390078(0.288319)	0.0389973(0.288319)	0.0390370(0.288322)
		[36]	0.0073213(0.341275)	0.00734502(0.341275)	0.0072675(0.341274)
100	20	[37]	0.0123209(0.296028)	0.0122654(0.296027)	0.0123992(0.296030)
		[38]	0.0273074(0.271238)	0.0272604(0.271235)	0.0273765(0.271241)
		[39]	0.0186184(0.311621)	0.0184501(0.311615)	0.0188311(0.311629)
		[40]	0.0037491(0.355722)	0.0035876(0.355721)	0.0039160(0.355723)
		[41]	0.0386721(0.294793)	0.0386447(0.294791)	0.0387159(0.294796)
		[42]	0.0168828(0.269872)	0.0168550(0.269871)	0.0169318(0.269874)

MLE, maximum likelihood estimation.

Table 7 Remission times of bladder cancer patients data.

0.08	0.2	0.4	0.5	0.51	0.81	0.9	1.05	1.19	1.26	1.35	1.4
1.46	1.76	2.02	2.02	2.07	2.09	2.23	2.26	2.46	2.54	2.62	2.64
2.69	2.69	2.75	2.83	2.87	3.02	3.25	3.31	3.36	3.36	3.48	3.52
3.57	3.64	3.7	3.82	3.88	4.18	4.23	4.26	4.33	4.34	4.4	4.5
4.51	4.87	4.98	5.06	5.09	5.17	5.32	5.32	5.34	5.41	5.41	5.49
5.62	5.71	5.85	6.25	6.54	6.76	6.93	6.94	6.97	7.09	7.26	7.28
7.32	7.39	7.59	7.62	7.63	7.66	7.87	7.93	8.26	8.37	8.53	8.65
8.66	9.02	9.22	9.47	9.74	10.06	10.34	10.66	10.75	11.25	11.64	11.79
11.98	12.02	12.03	12.07	12.63	13.11	13.29	13.8	14.24	14.76	14.77	14.83
15.96	16.62	17.12	17.14	17.36	18.1	19.13	20.28	21.73	22.69	23.63	25.74
25.82	26.31	32.15	34.26	36.66	43.01	46.12	79.05				

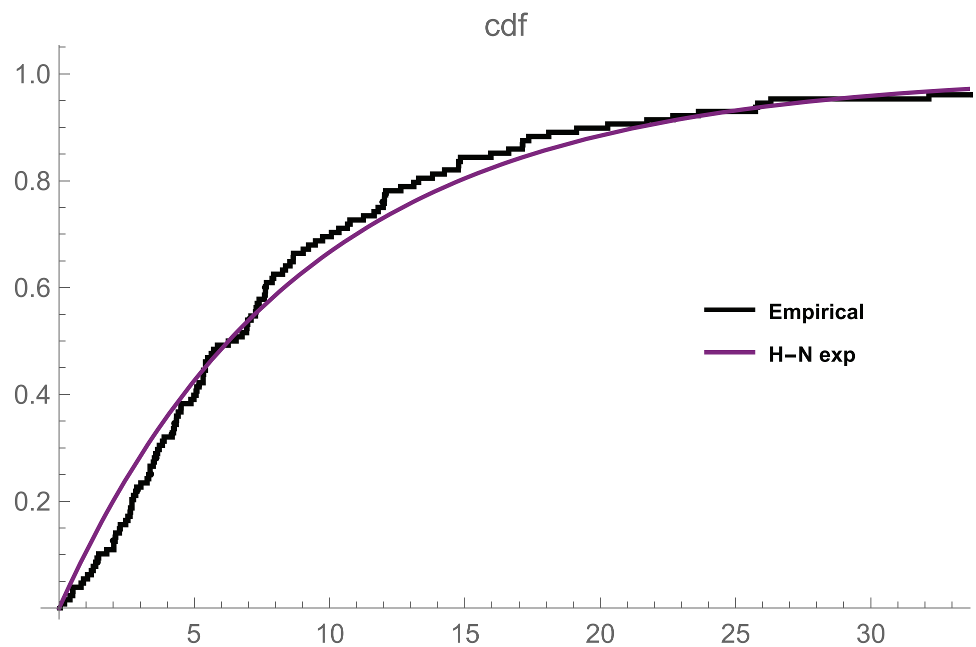

Figure 1 Emperical and fitted cdf plot of the remission times of bladder cancer patients data.

Table 8 The mle (CI in parenthesis) and Bayes estimates using different censoring schemes for the data set.

n	m	Scheme	$\hat{\alpha}_{mle}$	$\hat{\lambda}_{mle}$	$\hat{\alpha}_{\text{Lindely}}$	$\hat{\lambda}_{\text{Lindely}}$
128	128	(0 * 128)[a]	0.846349	0.127828	0.873219	0.132073
			(0.648884, 1.04381)	(0.0774782, 0.178178)		
128	53	(0*53, 75)	0.473106	0.244122	0.347462	0.291060
			(0.210417, 0.735742)	(0.0521221, 0.436122)		
128	59	(0*59, 1*69)	1.771420	0.157970	5.94787	0.171523
			(0.99333, 2.54951)	(0.0706334, 0.245307)		

The mle (CI in parenthesis) and Bayes estimates using different censoring schemes for a This 0*3(say) denotes that 0, 0, 0.

posterior summaries of interest can be derived and all the calculations are performed using the WinBUGS software. Table 9 lists the posterior descriptive summaries of interest. This table consists of MC error which considered as One way to assess the accuracy of the posterior estimates is by calculating the Monte Carlo error (MC error) for each parameter which estimates of the difference between the mean of sampled values and the true posterior mean. The simulation should be run until the MC error for each parameter of interest is less than about 5% of the sample standard deviation and this achieved in our example.

Table 9 The approximate Bayes estimates (sd in parenthesis and credible interval under different censoring schemes for the data set.

n	m	Scheme	Posterior Summaries	$\tilde{\alpha}_{\text{Bayes}}$	$\tilde{\lambda}_{\text{Bayes}}$
128	128	(0 * 128)[a]	Mean (sd)	2.856(0.1253)	0.02215(0.001777)
			credible interval	(2.722, 3.180)	(0.0188, 0.0256)
			MC error	0.001682	2.163E-5
128	53	(0*53,75)	mean (sd)	2.692(0.3328)	0.04565(0.003972)
			credible interval	(2.220, 3.454)	(0.0353, 0.04967)
			MC error	0.006645	7.798E-5

(a) This 0*3 (say) denotes that 0, 0, 0.

CONFLICTS OF INTEREST

The authors declare that there is no conflict of interest.

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APPENDIX

Lindley [9] developed an asymptotic expansion to evaluate the ratio of the following integral

$$u^* = E[u(\alpha, \lambda)] = \frac{\int_0^\infty \int_0^\infty u(\alpha, \lambda)\, exp[l(\alpha, \lambda \mid x) + \rho(\alpha, \lambda)] d\alpha d\lambda}{\int_0^\infty \int_0^\infty exp[l(\alpha, \lambda \mid x) + \rho(\alpha, \lambda)] d\alpha d\lambda}$$

where $u(\alpha, \lambda)$ is a function of α and λ only, $l(\alpha, \lambda \mid x)$ is the log-likelihood function (given by Equation (7)) and $\rho(\alpha, \lambda)$ is the logarithm of prior distribution (defined in Equation (10)).

Using Lindely's method, u^* can be approximated as

$$\begin{aligned} u^* = u + \tfrac{1}{2}[&(u_{\alpha\alpha} + 2u_\alpha \rho_\alpha)\sigma_{\alpha\alpha} + (u_{\alpha\lambda} + 2u_\alpha \rho_\lambda)\sigma_{\alpha\lambda} + (u_{\lambda\alpha} + 2u_\lambda \rho_\alpha)\sigma_{\lambda\alpha} + (u_{\lambda\lambda} + 2u_\lambda \rho_\lambda)(\sigma_{\lambda\lambda}) \\ &+ (u_\alpha \sigma_{\alpha\alpha} + u_\lambda \sigma_{\alpha\lambda})(L_{\alpha\alpha\alpha}\sigma_{\alpha\alpha} + L_{\alpha\lambda\alpha}\sigma_{\alpha\lambda} + L_{\lambda\alpha\alpha}\sigma_{\lambda\alpha} + L_{\lambda\lambda\alpha}\sigma_{\lambda\lambda}) \\ &+ (u_\alpha \sigma_{\lambda\alpha} + u_\lambda \sigma_{\lambda\lambda})(L_{\alpha\alpha\alpha}\sigma_{\alpha\alpha} + L_{\alpha\lambda\lambda}\sigma_{\alpha\lambda} + L_{\lambda\lambda\lambda}\sigma_{\lambda\lambda})] \end{aligned} \tag{A1}$$

The right-hand side of above equation are evaluated at the mle $\hat{u}$. We have the joint prior distribution $g(\alpha, \lambda) \propto \alpha^{a-1} e^{-b\alpha} \lambda^{c-1} e^{-d\lambda}$ and $\rho(\alpha, \lambda) = \ln[g(\alpha, \lambda)]$. Also, $\rho_\alpha = \frac{\partial \rho}{\partial \alpha}$ and $\rho_\lambda = \frac{\partial \rho}{\partial \lambda}$. When $u = \alpha$, we have $u_\alpha = \frac{\partial u}{\partial \alpha} = 1$ and $u_\alpha \alpha = 0 = u_\lambda = u_{\lambda\lambda} = u_{\lambda\alpha} = u_{\alpha\lambda}$ When $u = \lambda$, we have $u_\lambda = \frac{\partial u}{\partial \lambda} = 1$ and $u_\alpha = 0 = u_{\alpha\alpha} = u_{\lambda\lambda} = u_{\lambda\alpha} = u_{\alpha\lambda}$