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Research Article

The Orthogonal and Symplectic Schur Functions, Vertex Operators and Integrable Hierarchies

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ABSTRACT

In this paper, we first construct an integrable system whose solutions include the orthogonal Schur functions and the symplectic Schur functions. We find that the orthogonal Schur functions and the symplectic Schur functions can be obtained by one kind of Boson-Fermion correspondence which is slightly different from the classical one. Then, we construct a universal character which satisfies the bilinear equation of a new infinite-dimensional integrable orthogonal UC hierarchy.

1. INTRODUCTION

Boson-Fermion correspondence is well-known in mathematical physics [1,11]. Young diagrams and symmetric functions are of interest to many researchers and have many applications in mathematics including combinatorics and representation theory [3,10]. There are many relations between Boson-Fermion correspondence and symmetric functions.

The KP hierarchy [1] is one of the most important integrable hierarchies and it arises in many different fields of mathematics and physics such as enumerative algebraic geometry, topological field and string theory. Schur functions have close relations with the τ-functions of KP hierarchy. Schur functions give the characters of finite-dimensional irreducible representations of the general linear groups, see [3,10]. Schur functions can be realized from vertex operators as in Equation (6) of this paper, and these vertex operators can be used to construct Fermions which act on Bosonic Fock space, see [7,11]. By replacing nx_n by power sum, we find that the character of Young diagram in [11] is the same with the Schur function obtained from the Jacobi-Trudi formula, which tells us that the Schur functions are solutions of differential equations in the KP hierarchy, and the linear combinations of Schur functions with coefficients satisfying some relations (plücker relations) are also τ-functions of the KP hierarchy. In [12,13], the author generalized the KP hierarchy to the UC (universal character) hierarchy, whose τ-functions include universal characters [8].

The orthogonal and symplectic Schur functions are upgraded from Schur functions in the same setting [2]. Symplectic Schur functions are equal to orthogonal Schur functions with the conjugate Young diagrams. Like Schur functions, the symplectic and orthogonal Schur functions can also be realized from vertex operators as in Equation (19), and these vertex operators can also be used to construct Fermions. Then there certainly exists an integrable system. In this paper, we will construct this integrable system, and show that the symplectic and orthogonal Schur functions are its solutions.

This paper is arranged as follows. In Section 2, we will recall the definition of Schur function, its vertex operator realization, and the relations between Schur functions and KP hierarchy. In Section 3, we will recall the definitions of orthogonal and symplectic Schur functions, their respective vertex operator realization, then we will define an integrable system whose τ-function can be obtained from orthogonal and symplectic Schur function. In Section 4, we will construct a method to calculate orthogonal and symplectic Schur functions from a different kind of Boson-Fermion correspondence. In Section 5, we will construct the modified type of the integrable system which is constructed in Section 3. In Section 6, we will consider the universal character and the corresponding UC hierarchy.

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2. SCHUR FUNCTIONS, VERTEX OPERATOR AND THE KP HIERARCHY

Let $\mathbf{x} = (x_1, x_2, \cdots)$. The operators $h_n(\mathbf{x})$ are determined by the generating function:

$$e^{\xi(\mathbf{x},k)} := \sum_{n=0}^{\infty} h_n(\mathbf{x})k^n, \quad \text{where } \xi(\mathbf{x},k) = \sum_{n=1}^{\infty} x_n k^n \tag{1}$$

and set $h_n(\mathbf{x}) = 0$ for $n < 0$. Note that if we replace ix_i with the power sum $p_i = \sum_n x_n^i$, $h_n(\mathbf{x})$ is the complete homogeneous symmetric function [10]

$$\sum_{i_1 \le i_2 \le \cdots \le i_n} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{2}$$

For Young diagrams $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, the Schur function $S_\lambda = S_\lambda(\mathbf{x})$ is a polynomial in $\mathbb{C}[\mathbf{x}]$ defined by the Jacobi-Trudi formula [8]:

$$S_\lambda(\mathbf{x}) = \det\left(h_{\lambda_i - i + j}(\mathbf{x})\right)_{1 \le i,j \le l}. \tag{3}$$

Introduce the following vertex operators

$$V^+(k) = \sum_{n \in \mathbb{Z}} V_n^+ k^n = e^{\xi(\mathbf{x},k)} e^{-\xi(\tilde{\partial}_{\mathbf{x}}, z^{-1})}, \tag{4}$$

$$V^-(k) = \sum_{n \in \mathbb{Z}} V_n^- k^n = e^{-\xi(\mathbf{x},k)} e^{\xi(\tilde{\partial}_{\mathbf{x}}, k^{-1})}. \tag{5}$$

where $\tilde{\partial}_{\mathbf{x}} = (\partial_{x_1}, \frac{1}{2}\partial_{x_2}, \cdots, \frac{1}{n}\partial_{x_n}, \cdots)$. The operators V_i^+ are raising operators for the Schur functions

$$S_\lambda(\mathbf{x}) := V_{\lambda_1}^+ \cdots V_{\lambda_l}^+ \cdot 1 \tag{6}$$

where λ is a Young diagram $(\lambda_1, \lambda_2, \cdots, \lambda_l)$, and we denote $S_\lambda(\mathbf{x})$ by S_λ for short.

Introduce Fermions ψ_j^* and ψ_j for any $j \in \mathbb{Z} + \frac{1}{2}$ as operators satisfying the relations

$$\{\psi_j, \psi_k\} = 0, \ \{\psi_j^*, \psi_k^*\} = 0, \ \{\psi_j^*, \psi_k\} = \delta_{j+k,0} \tag{7}$$

where $\{A, B\} = AB + BA$. The generating functions of Fermions are

$$\psi(k) := \sum_{j \in \mathbb{Z}+1/2} \psi_j k^{-j-1/2}, \ \psi^*(k) := \sum_{j \in \mathbb{Z}+1/2} \psi_j^* k^{-j-1/2}.$$

The Fock representation space of Fermions is the space of Maya diagrams. A Maya diagram is made up of black and white stones lined up along the real line with the convention that all the stones are black far away to the right, whereas all the stones are white far away to the left. For example, the following is a Maya diagram

$$\begin{array}{ccccccccccc} \cdots & \circ & \circ & \bullet & \bullet & \circ & \bullet & \circ & \bullet & \bullet & \cdots \\ & -\frac{7}{2} & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \end{array} \tag{8}$$

By writing half integers $u_1, u_2, \cdots$ for the positions of the black stones, a Maya diagram is described as an increasing sequence of half integers

$$\mathbf{u} = \{u_n\}_{n \ge 1} \quad \text{with} \quad u_1 < u_2 < u_3 < \cdots.$$

For example, the Maya diagram in (8) is denoted by

$$-\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \frac{9}{2}, \cdots$$

Define the charge p of a Maya diagram as the number of white stones on the right half line minus the number of black stones on the left half line. For example, the charge of Maya diagram in (8) is zero.

Let $\mathcal{F}$ be the vector space based by the set of Maya diagrams, which is called Fermionic Fock space. The basis vector is written as $|\mathbf{u}\rangle$. In particular,

$$|0\rangle = |\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots\rangle.$$

The action of Fermions ψ_j and ψ_j^* for any $j \in \frac{1}{2} + \mathbb{Z}$ on Maya diagrams $|\mathbf{u}\rangle$ is determined by the formulas

$$\psi_j|\mathbf{u}\rangle = \begin{cases} (-1)^{i-1}|\cdots, u_{i-1}, u_{i+1}, \cdots\rangle & \text{if } u_i = -j \text{ for some } i, \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

$$\psi_j^*|\mathbf{u}\rangle = \begin{cases} (-1)^i|\cdots, u_i, j, u_{i+1}, \cdots\rangle & \text{if } u_i < j < u_{i+1} \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

There are three vector spaces which are isomorphic to each other [11]: the polynomial ring $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, x_2, \cdots]$ of infinitely many variables $\mathbf{x} = (x_1, x_2, \cdots)$ which is called the Bosonic Fock space, the charge zero part of the Fermionic Fock space $\mathcal{F}$, and the vector space Y spanned by Young diagrams. Therefore, the Maya diagram $|\mathbf{u}\rangle$ can be written as

$$|\mathbf{u}\rangle = |\lambda, n\rangle = |S_\lambda, n\rangle,$$

where n is the charge of $|\mathbf{u}\rangle$. In the special case of $n = 0$, we also write the Maya diagram $|\mathbf{u}\rangle$ as $|\lambda\rangle$.

Let $f(z, x) \in C[z, z^{-1}, x_1, x_2, \cdots]$. Define operators

$$e^K f(z, \mathbf{x}) := zf(z, \mathbf{x}), \quad k^{H_0} f(z, \mathbf{x}) := f(kz, \mathbf{x}). \tag{11}$$

Define the generating functions [5,11]

$$\tilde{V}(k) := \sum_{j\in\mathbb{Z}+\frac{1}{2}} \tilde{V}_j k^{-j-\frac{1}{2}} = V^+(k) e^K k^{H_0}, \tag{12}$$

$$\tilde{V}^*(k) := \sum_{j\in\mathbb{Z}+\frac{1}{2}} \tilde{V}_j^* k^{-j-\frac{1}{2}} = V^-(k) e^K k^{H_0}. \tag{13}$$

It can be checked that

$$\{\tilde{V}_i, \tilde{V}_j\} = 0, \ \ \{\tilde{V}_i^*, \tilde{V}_j^*\} = 0, \ \ \{\tilde{V}_i, \tilde{V}_j^*\} = \delta_{i+j,0}, \tag{14}$$

that is, the operators $\tilde{V}_i$, $\tilde{V}_j^*$ determine a representation of the algebra spanned by Fermions, see equations in (7).

Definition 2.1. *For an unknown function $\tau = \tau(\mathbf{x})$, the bilinear equation*

$$\sum_{j\in\mathbb{Z}+\frac{1}{2}} \tilde{V}_j^* \tau \otimes \tilde{V}_{-j} \tau = 0 \tag{15}$$

is called the KP hierarchy, see [5,11].

3. THE ORTHOGONAL SCHUR FUNCTION, THE SYMPLECTIC SCHUR FUNCTION, VERTEX OPERATORS AND AN INTEGRABLE HIERARCHY

For a Young diagram $\lambda = (\lambda_1, \cdots, \lambda_l)$, the orthogonal Schur function[6,9] is defined to be

$$S_\lambda^O := \det(h_{\lambda_i - i + j} - h_{\lambda_i - i - j})_{1\le i,j\le l}, \tag{16}$$

where h_n is the nth complete symmetric function of the form in equation (2). Define vertex operators

$$V_O(z) := (1 - z^2) e^{\xi(\mathbf{x}, z)} e^{-\xi(\tilde{\partial}\mathbf{x}, z^{-1})} e^{-\xi(\tilde{\partial}\mathbf{x}, z)} \tag{17}$$

$$V_O^*(z) := e^{-\xi(\mathbf{x}, z)} e^{\xi(\tilde{\partial}\mathbf{x}, z^{-1})} e^{\xi(\tilde{\partial}\mathbf{x}, z)} \tag{18}$$

and let

$$V_O(z) = \sum_{n\in\mathbb{Z}} V_n^O z^n, \quad V_O^*(z) = \sum_{n\in\mathbb{Z}} V_n^{O*} z^n.$$

Observe that this vertex operator $V_O(z)$ is the same as $V_\pi(z)$ for $\pi = (2)$ in [2].

The operator V_n^O is a raising operator of the orthogonal Schur function, i.e.,

$$S_\lambda^O(\mathbf{x}) = V_{\lambda_1}^O V_{\lambda_2}^O \cdots V_{\lambda_l}^O \cdot 1 \tag{19}$$

for a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$.

Define the generating functions

$$X^O(k) = \sum_{j\in\mathbb{Z}+\frac{1}{2}} X_j^O k^{-j-\frac{1}{2}} = V_O(k) e^K k^{H_0}, \tag{20}$$

$$X^{O*}(k) = \sum_{j\in\mathbb{Z}+\frac{1}{2}} X_j^{O*} k^{-j-\frac{1}{2}} = V_O^*(k) e^K k^{H_0}. \tag{21}$$

It can be checked that

$$\{X_i^O, X_j^O\} = 0,\ \{X_i^{O*},\ X_j^{O*}\} = 0,\ \{X_i^O,\ X_j^{O*}\} = \delta_{i+j,0}. \tag{22}$$

Definition 3.1. *For an unknown function $\tau = \tau(\mathbf{x})$, the bilinear equation*

$$\sum_{j\in\mathbb{Z}+\frac{1}{2}} X_j^{O*}\tau \otimes X_{-j}^O\tau = 0 \tag{23}$$

is called the orthogonal/symmplectic KP hierarchy, and denoted by OSKP hierarchy for short.

Equation (23) is equivalent to

$$\sum_{n+m=-1} V_n^{O*}\tau \otimes V_m^O\tau = 0. \tag{24}$$

It is obvious that equation (24) can be rewritten as

$$\frac{1}{2\pi i}\oint (1-z^2)e^{\xi(\mathbf{x}-\mathbf{x}',z)}dz\tau(\mathbf{x}'+[z^{-1}]+[z])\tau(\mathbf{x}-[z^{-1}]-[z]) = 0 \tag{25}$$

with $\mathbf{x} = (x_1, x_2, \cdots)$ and $\mathbf{x}' = (x_1', x_2', \cdots)$ being arbitrary parameters. Here the symbol $[z]$ denotes $(z, \frac{z^2}{2}, \frac{z^3}{3}, \cdots)$ and the integration means taking the coefficient of $\frac{1}{z}$ of the integrand in the formal Laurent series expansion in z. Then the equation (25) is equivalent to

$$\mathrm{Res}(1-z^2)e^{\xi(\mathbf{x}-\mathbf{x}',z)}\tau(\mathbf{x}'+[z^{-1}]+[z])\tau(\mathbf{x}-[z^{-1}]-[z]) = 0. \tag{26}$$

Let us replace $(\mathbf{x}', \mathbf{x})$ with $(\mathbf{x}+\mathbf{u}, \mathbf{x}-\mathbf{u})$ and consider the Taylor series expansion at $x' = x$, i.e., expand with respect to $\mathbf{u} = (u_1, u_2, \cdots)$. Hence, we obtain

$$\sum_{i-j+k=-1} P_i(-2\mathbf{u})P_j(\tilde{\partial}_{\mathbf{u}})P_k(\tilde{\partial}_{\mathbf{u}})\tau(\mathbf{x}+\mathbf{u})\tau(\mathbf{x}-\mathbf{u}) - \sum_{i-j+k=-3} P_i(-2\mathbf{u})P_j(\tilde{\partial}_{\mathbf{u}})P_k(\tilde{\partial}_{\mathbf{u}})\tau(\mathbf{x}+\mathbf{u})\tau(\mathbf{x}-\mathbf{u}) = 0. \tag{27}$$

By taking the coefficient of $\mathbf{u}^{\mathbf{n}} = u_1^{n_1}u_2^{n_2}\cdots$, we get many bilinear equations. Taking the coefficient of $1 = \mathbf{u}^{\mathbf{0}}$, we get

$$\sum_{k=0}^{\infty} P_{k+1}(D_{\mathbf{x}})P_k(D_{\mathbf{x}})\tau(\mathbf{x})\cdot\tau(\mathbf{x}) - \sum_{k=0}^{\infty} P_{k+3}(D_{\mathbf{x}})P_k(D_{\mathbf{x}})\tau(\mathbf{x})\cdot\tau(\mathbf{x}) = 0, \tag{28}$$

where $D_{\mathbf{x}} = (D_{x_1}, \frac{1}{2}D_{x_2}, \frac{1}{3}D_{x_3}, \cdots)$. We see that every differential equation with respect to $\mathbf{x}$ contained in the orthogonal KP hierarchy is of infinite order. This reflects the fact that the integrand of (25) with $\mathbf{x}' = \mathbf{x}$ may be singular not only at $z = 0$, but also at $z = \infty$.

For a Young diagram $\lambda = (\lambda_1, \cdots, \lambda_l)$, the symplectic Schur function[6,9] is defined to be

$$S_\lambda^{Sp} = \frac{1}{2}\det(h_{\lambda_i-i+j} + h_{\lambda_i-i-j+2})_{1\le i,j\le l},$$

where h_n is the nth complete symmetric function. The Symplectic symmetric function can be obtained by vertex operators as follows. Define the vertex operators

$$V_{Sp}(z) = e^{\xi(\mathbf{x},z)}e^{-\xi(\partial\mathbf{x},z^{-1})}e^{-\xi(\partial\mathbf{x},z)} \tag{29}$$

$$V_{Sp}^*(z) = (1-z^2)e^{-\xi(\mathbf{x},z)}e^{\xi(\partial\mathbf{x},z^{-1})}e^{\xi(\partial\mathbf{x},z)} \tag{30}$$

and let

$$V_{Sp}(z) = \sum_{n\in\mathbb{Z}} V_n^{Sp}z^n,\quad V_{Sp}^*(z) = \sum_{n\in\mathbb{Z}} V_n^{Sp*}z^n,$$

here the vertex operator $V_{Sp}(z)$ is the same as $V_\pi(z)$ for $\pi = (1^2)$ in [2].

The operator V_n^{Sp} is a raising operator of the symplectic Schur function, i.e.,

$$S_\lambda^{Sp}(\mathbf{x}) = S_\lambda^{Sp}(\mathbf{x}) = V_{\lambda_1}^{Sp}V_{\lambda_2}^{Sp}\cdots V_{\lambda_l}^{Sp}\cdot 1 \tag{31}$$

for a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$.

For an unknown function $\tau = \tau(\mathbf{x})$, the bilinear equation

$$\sum_{n+m=-1} V_n^{Sp*}\tau \otimes V_m^{Sp}\tau = 0 \tag{32}$$

gives the same integrable system as the bilinear equation (23), that is why we call this integrable system OSKP hierarchy.

4. ORTHOGONAL TYPE BOSON-FERMION CORRESPONDENCE

For Maya diagrams $|\mathbf{u}\rangle$ and $|\mathbf{v}\rangle$, the pairing $\langle\mathbf{v}|\mathbf{u}\rangle$ is defined by the formula

$$\langle\mathbf{v}|\mathbf{u}\rangle = \delta_{v_1+u_1,0}\delta_{v_2+u_2,0}\cdots.$$

Define operators H_n by

$$H_n = \sum_{j\in\mathbb{Z}+1/2} :\psi_{-j}\psi^*_{j+n}:$$

and $H(x) = \sum_{n=1}^{\infty} x_n H_n$.

From the actions of Fermions on Maya diagrams, we get the action of H_1 on a Maya diagram is H_1 sending a Maya diagram $|\mathbf{u}\rangle$ to the sum over all Maya diagrams which can be obtained from $|\mathbf{u}\rangle$ by moving a black stone to the right. We define P_n and Q_n from equations

$$\exp\left(\sum_{m\geq 1}\frac{H_m}{m}k^m\right) = \sum_{n\geq 0} Q_{(n)}k^n, \quad \exp\left(\sum_{m\geq 1}\frac{H_{-m}}{m}k^m\right) = \sum_{n\geq 0} P_{(n)}k^n \tag{33}$$

The action of $Q_{(m)}$ on Maya diagram is defined by $Q_{(m)}$ sending the Maya diagram $|\mathbf{u}\rangle$ to the sum over all Maya diagrams which can be obtained from $|\mathbf{u}\rangle$ by moving black stones m times to the right and no one black stone is moved twice. Then, $Q_{(1^m)}$ sends Maya diagram $|\mathbf{u}\rangle$ to the sum over all Maya diagrams which can be obtained from $|\mathbf{u}\rangle$ by moving black stones m times to the right and no two adjacent black stones move at the same time.

Define

$$\psi_j^O = \sum_{n=1}^{\infty}(-1)^n\psi_{n+j}Q_{1^n} - \sum_{n=1}^{\infty}(-1)^n\psi_{n+j+2}Q_{1^n}, \tag{34}$$

$$\psi_j^{O*} = \sum_{n=1}^{\infty}\psi_{n+j}Q_n. \tag{35}$$

The actions of ψ_j^O, ψ_j^{O*}, where $j \in \frac{1}{2}+\mathbb{Z}$, on Maya diagram can be obtained from the actions of ψ_j, ψ_j^* and $Q_{(m)}, Q_{(1^m)}$ on Maya diagram according to (34-35).

Let λ be a Young diagram, and λ' be its conjugate. The Frobenius notation $\lambda = (n_1, \cdots, n_l|m_1, \cdots, m_l)$ describes the Young diagram λ by $n_i = \lambda_i - i$, $m_i = \lambda'_i - i$, where l is the number of the boxes in the NW-SE diagonal line of λ.

Under the Boson-Fermion correspondence, the basis vector

$$\psi_{n_1}\cdots\psi_{n_l}\psi^*_{m_1}\cdots\psi^*_{m_l}|\mathrm{vac}\rangle \text{ for } n_1 < \cdots < n_l < 0 \text{ and } m_1 < \cdots < m_l < 0$$

of Fermionic Fock space of charge zero goes over into the Schur function S_λ multiplied by $a_\lambda = (-1)^{\sum_{i=1}^{l}(m_i+\frac{1}{2})+\frac{l(l-1)}{2}}$, where $\lambda = (-n_1 - \frac{1}{2}, \cdots, -n_l - \frac{1}{2}| - m_1 - \frac{1}{2}, \cdots, -m_l - \frac{1}{2})$, i.e.,

$$S_\lambda = a_\lambda\langle\mathrm{vac}|e^{H(x)}\psi_{n_1}\cdots\psi_{n_l}\psi^*_{m_1}\cdots\psi^*_{m_l}|\mathrm{vac}\rangle, \tag{36}$$

then we have

Proposition 4.1. *For $\lambda = (-n_1 - \frac{1}{2}, \cdots, -n_l - \frac{1}{2}| - m_1 - \frac{1}{2}, \cdots, -m_l - \frac{1}{2})$, the orthogonal Schur function S_λ^O is obtained from*

$$S_\lambda^O = (-1)^{\sum_{i=1}^{l}(m_i+\frac{1}{2})+\frac{l(l-1)}{2}}\langle vac|e^{H(x)}\psi^O_{n_1}\cdots\psi^O_{n_l}\psi^{O*}_{m_1}\cdots\psi^{O*}_{m_l}|vac\rangle. \tag{37}$$

Using the Fermions ψ_j and ψ_j^*, we can also get the orthogonal Schur function by the following formula.

Proposition 4.2. *For $\lambda = (-n_1 - \frac{1}{2}, \cdots, -n_l - \frac{1}{2}| - m_1 - \frac{1}{2}, \cdots, -m_l - \frac{1}{2})$, the orthogonal Schur function S_λ^O is obtained from*

$$S_\lambda^O = (-1)^{\sum_{i=1}^{l}(m_i+\frac{1}{2})+\frac{l(l-1)}{2}}\langle vac|e^{H(x)}e^{-\sum_{n=1}^{\infty}\frac{1}{2n}(H_n^2+H_{2n})}\psi_{n_1}\cdots\psi_{n_l}\psi^*_{m_1}\cdots\psi^*_{m_l}|vac\rangle. \tag{38}$$

We can get the symplectic Schur function similarly.

Proposition 4.3. *For $\lambda = (-n_1 - \frac{1}{2}, \cdots, -n_l - \frac{1}{2}| - m_1 - \frac{1}{2}, \cdots, -m_l - \frac{1}{2})$, the symplectic Schur function S_λ^{Sp} is obtained from*

$$S_\lambda^{Sp} = a_\lambda\langle vac|e^{H(x)}\psi^{Sp}_{n_1}\cdots\psi^{Sp}_{n_l}\psi^{Sp*}_{m_1}\cdots\psi^{Sp*}_{m_l}|vac\rangle \tag{39}$$

$$= a_\lambda\langle vac|e^{H(x)}e^{-\sum_{n=1}^{\infty}\frac{1}{2n}(H_n^2-H_{2n})}\psi_{n_1}\cdots\psi_{n_l}\psi^*_{m_1}\cdots\psi^*_{m_l}|vac\rangle, \tag{40}$$

where $a_\lambda = (-1)^{\sum_{i=1}^{l}(m_i+\frac{1}{2})+\frac{l(l-1)}{2}}$.

For example, we can calculate $S^{O}_{\Box\Box}$ in the following two ways. The first way is

$$\psi^{O}_{-\frac{3}{2}} = \psi_{-\frac{3}{2}} - \psi_{\frac{1}{2}} - \psi_{-\frac{1}{2}}Q + \psi_{\frac{3}{2}}Q + \cdots,$$
$$\psi^{O*}_{-\frac{1}{2}} = \psi^{*}_{-\frac{1}{2}} + \psi^{*}_{\frac{1}{2}}Q + \psi^{*}_{\frac{3}{2}}Q_2 + \cdots,$$

Then,

$$\begin{aligned} S^{O}_{\Box\Box} &= \langle \mathrm{vac}|e^{H(x)}\psi^{O}_{-\frac{3}{2}}\psi^{O*}_{-\frac{1}{2}}|\mathrm{vac}\rangle \\ &= \langle \mathrm{vac}|e^{H(x)}(\psi_{-\frac{3}{2}} - \psi_{\frac{1}{2}})\psi^{*}_{-\frac{1}{2}}|\mathrm{vac}\rangle \\ &= S_{\Box\Box} - S_0. \end{aligned}$$

In the second way, we know that for the Maya diagram

$$|\gamma\rangle = \begin{array}{ccccccccccc} \cdots & \circ & \circ & \circ & \bullet & \bullet & \circ & \bullet & \bullet & \bullet & \cdots \\ & -\frac{7}{2} & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \end{array}$$

we have $H_m\gamma = 0$ when $m > 2$. Then

$$\begin{aligned} S^{O}_{\Box\Box} &= \langle \mathrm{vac}|e^{H(x)}e^{-\sum_{n=1}^{\infty}\frac{1}{2n}(H_n^2 + H_{2n})}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|\mathrm{vac}\rangle \\ &= \langle \mathrm{vac}|e^{H(x)}(1 - \frac{1}{2}(H_1^2 + H_2))|\gamma\rangle \\ &= \langle \mathrm{vac}|e^{H(x)}(1 - Q_2)|\gamma\rangle \\ &= S_{\Box\Box} - S_0. \end{aligned}$$

Then, we obtain the orthogonal type Boson-Fermion correspondence.

Proposition 4.4. *The Fermions ψ^{O}_{j}, ψ^{O*}_{j} are realized in the Bosonic Fock space by X^{O}_{j}, X^{O*}_{j}, i.e., for any $|u\rangle \in \mathcal{F}$, we have*

$$\langle l|e^{H(x)}\psi^{O}_{j}|u\rangle = X^{O}_{j}\langle l|e^{H(x)}|u\rangle, \quad \langle l|e^{H(x)}\psi^{O*}_{j}|u\rangle = X^{O*}_{j}\langle l|e^{H(x)}|u\rangle, \tag{41}$$

where $\langle l| = \langle \cdots, l - \frac{5}{2}, l - \frac{3}{2}, l - \frac{1}{2}|$.

5. THE MODIFIED ORTHOGONAL KP HIERARCHY

Now, we consider the functional relations for a sequence of τ-functions connected by successive application of vertex operators. Let $\tau_0 := \tau(\mathbf{x})$ be a solution of the orthogonal KP hierarchy (23). Let $\tau_1 := V_O(\alpha)\tau$ and $\tau_1' = V_{Sp}(\alpha)\tau$ with an arbitrary constant $\alpha \in \mathbb{C}^{\times}$. Then τ_1 and τ_1' are also solutions of (23) . Moreover, we can deduce the bilinear equation

$$\sum_{n+m=-2} V^{O*}_{n}\tau_n \otimes V^{O}_{n}\tau_{n+1} = 0$$

from (23) multiplied by $1 \otimes V_O(\alpha)$ or

$$\sum_{n+m=-2} V^{Sp*}_{n}\tau_n \otimes V^{Sp}_{n}\tau_{n+1} = 0$$

from (32) multiplied by $1 \otimes V_{Sp}(\alpha)$. The two equations above can be equivalently rewritten into the equation

$$\frac{1}{2\pi i}\oint z(1 - z^2)e^{\xi(\mathbf{x}-\mathbf{x}',z)}dz\tau_n(\mathbf{x}' + [z^{-1}] + [z])\tau_{n+1}(\mathbf{x} - [z^{-1}] - [z]) = 0. \tag{42}$$

Replace $(\mathbf{x}', \mathbf{x})$ with $(\mathbf{x} + \mathbf{u}, \mathbf{x} - \mathbf{u})$ and consider the Taylor series expansion at $x' = x$, we obtain

$$\sum_{i-j+k=-2} P_i(-2\mathbf{u})P_j(\tilde{\partial}_{\mathbf{u}})P_k(\tilde{\partial}_{\mathbf{u}})\tau_n(\mathbf{x} + \mathbf{u})\tau_{n+1}(\mathbf{x} - \mathbf{u}) + \sum_{i-j+k=-4} P_i(-2\mathbf{u})P_j(\tilde{\partial}_{\mathbf{u}})P_k(\tilde{\partial}_{\mathbf{u}})\tau_n(\mathbf{x} + \mathbf{u})\tau_{n+1}(\mathbf{x} - \mathbf{u}) = 0. \tag{43}$$

By taking the coefficient of $\mathbf{u}^{\mathbf{n}} = u_1^{n_1}u_2^{n_2}\cdots$ for variety n, we will get many bilinear equations. Taking the coefficient of $1 = \mathbf{u}^{\mathbf{0}}$, we get

$$\sum_{k=0}^{\infty} P_{k+2}(D_{\mathbf{x}})P_k(D_{\mathbf{x}})\tau_n(\mathbf{x}) \cdot \tau_{n+1}(\mathbf{x}) - \sum_{k=0}^{\infty} P_{k+3}(D_{\mathbf{x}})P_k(D_{\mathbf{x}})\tau_n(\mathbf{x}) \cdot \tau_{n+1}(\mathbf{x}) = 0. \tag{44}$$

6. UNIVERSAL CHARACTER AND UC HIERARCHY

For a pair of Young diagrams $\lambda = (\lambda_1, \cdots, \lambda_l)$ and $\mu = (\mu_1, \cdots, \mu_{l'})$, we define the universal character as a polynomial in $\mathbf{x} = (x_1, x_2, \cdots)$ and $\mathbf{y} = (y_1, y_2, \cdots)$:

$$S^O_{[\lambda,\mu]}(\mathbf{x},\mathbf{y}) = (-1)^{ll'+\frac{l'(l'+1)}{2}} \times \det\begin{pmatrix} h_{\mu_{l'-i+1}-l-l'+i-j-1}(\mathbf{y}) - h_{\mu_{l'-i+1}-l-l'+i+j-1}(\mathbf{y}), & 1 \le i \le l' \\ h_{\lambda_{i-l'}+l'-i+j}(\mathbf{x}) - h_{\lambda_{i-l'}+l'-i-j}(\mathbf{x}), & l'+1 \le i \le l+l' \end{pmatrix}. \tag{45}$$

We can see that the orthogonal Schur function $S^O_\lambda(\mathbf{x})$ is a special case of the universal character: $S^O_\lambda(\mathbf{x}) = S^O_{[\lambda,\varnothing]}(\mathbf{x},\mathbf{y})$.

Let us introduce the vertex operators

$$X^+(k) = (1-k^2)e^{\xi(\mathbf{x},k)}e^{-\xi(\tilde{\partial}_\mathbf{y},k^{-1})}e^{-\xi(\tilde{\partial}_\mathbf{y},k)}e^{-\xi(\tilde{\partial}_\mathbf{x},k^{-1})}e^{-\xi(\tilde{\partial}_\mathbf{x},k)}, \tag{46}$$

$$X^-(k) = e^{-\xi(\mathbf{x},k)}e^{\xi(\tilde{\partial}_\mathbf{y},k^{-1})}e^{\xi(\tilde{\partial}_\mathbf{y},k)}e^{\xi(\tilde{\partial}_\mathbf{x},k^{-1})}e^{\xi(\tilde{\partial}_\mathbf{x},k)}, \tag{47}$$

$$Y^+(k) = (1-k^2)e^{\xi(\mathbf{y},k)}e^{-\xi(\tilde{\partial}_\mathbf{x},k^{-1})}e^{-\xi(\tilde{\partial}_\mathbf{x},k)}e^{-\xi(\tilde{\partial}_\mathbf{y},k)}e^{-\xi(\tilde{\partial}_\mathbf{y},k^{-1})}, \tag{48}$$

$$Y^-(k) = e^{-\xi(\mathbf{y},k^{-1})}e^{\xi(\tilde{\partial}_\mathbf{x},k^{-1})}e^{\xi(\tilde{\partial}_\mathbf{x},k)}e^{\xi(\tilde{\partial}_\mathbf{y},k)}e^{\xi(\tilde{\partial}_\mathbf{y},k^{-1})}, \tag{49}$$

and let $X^\pm(k) = \sum_{n\in\mathbb{Z}} X^\pm_n k^n$, $Y^\pm(k) = \sum_{n\in\mathbb{Z}} Y^\pm_n k^n$.

It can be checked that the $X^\pm_n$ satisfy the formionic relations: $X^\pm_n X^\pm_{m+1} + X^\pm_{n+1}X^\pm_n = 0$ and $X^+_{n+1}X^-_m + X^-_{m+1}X^+_n = \delta_{n+m+1,0}$. The same relations hold also for $Y^\pm_n$. Moreover, $X^\pm_n$ and $Y^\pm_n$ mutually commute.

Proposition 6.1. *The universal character $S^O_{[\lambda,\mu]}(\mathbf{x},\mathbf{y})$ can be obtained by means of these operators:*

$$S^O_{[\lambda,\mu]}(\mathbf{x},\mathbf{y}) = X^+_{\lambda_1}\cdots X^+_{\lambda_l}Y^+_{\mu_1}\cdots Y^+_{\mu_l}\cdot 1. \tag{50}$$

Proof. We will use the Vandermonde-like identity,

$$\det(k_i^{l-j} - k_i^{l+j}) = \prod_{1\le i<j\le l}(k_i-k_j)(1-k_ik_j).$$

Then,

$$\begin{aligned}
&X^+(k_1)\cdots X^+(k_l)Y^+(w_1^{-1})\cdots Y^+(w_l'^{-1})\cdot 1\\
&= \prod_{i=1}^{l}(1-k_i^2)\prod_{j=1}^{l'}(1-\frac{1}{w_i^2})\prod_{i<j}(1-k_ik_j)(1-\frac{k_j}{k_i})\prod_{i,k}(1-\frac{1}{k_iw_j})(1-\frac{k_i}{w_j})\\
&\quad\times\prod_{a<b}(1-\frac{w_b}{w_a})(1-\frac{1}{w_aw_b})e^{\xi(\mathbf{x},k_1)}\cdots e^{\xi(\mathbf{x},k_l)}e^{\xi(\mathbf{y},w_1^{-1})}\cdots e^{\xi(\mathbf{y},w_{l'}^{-1})}\\
&= (-1)^{ll'+\frac{l'(l'+1)}{2}}\prod_i k_{l-i}^{-(l+2l'-i)}\prod_j w_{l'-i+1}^{-(2l+2l'-i+1)}\det\begin{pmatrix} w_{l'-i+1}^{l+l'-j} - w_{l'-i+1}^{l+l'+j}, & 1\le i\le l'\\ k_{i-l'}^{l+l'-j} - k_{i-l'}^{l+l'+j}, & l'+1\le i\le l+l'\end{pmatrix}\\
&\quad\times e^{\xi(\mathbf{x},k_1)}\cdots e^{\xi(\mathbf{x},k_l)}e^{\xi(\mathbf{y},w_1^{-1})}\cdots e^{\xi(\mathbf{y},w_{l'}^{-1})}\\
&= (-1)^{ll'+\frac{l'(l'+1)}{2}}\det\begin{pmatrix} w_{l'-i+1}^{-l-l'+i-j-1} - w_{l'-i+1}^{-l-l'+i+j-1}, & 1\le i\le l'\\ k_{i-l'}^{-l'+i-j} - k_{i-l'}^{-l'+i+j}, & l'+1\le i\le l+l'\end{pmatrix}\\
&\quad\times e^{\xi(\mathbf{x},k_1)}\cdots e^{\xi(\mathbf{x},k_l)}e^{\xi(\mathbf{y},w_1^{-1})}\cdots e^{\xi(\mathbf{y},w_{l'}^{-1})}.
\end{aligned}$$

Taking the coefficient of $k_1^{\lambda_1}\cdots k_l^{\lambda_l}w_1^{-\mu_1}\cdots w_{l'}^{-\mu_l}$, we will get (50). □

We give a remark here to explain the difference between the universal characters $S^O_{[\lambda,\mu]}(\mathbf{x},\mathbf{y})$ here and that in our paper [4]. The vertex operators which realize $S^O_{[\lambda,\mu]}(\mathbf{x},\mathbf{y})$ in this paper are more complex than that in [4], but in this paper, the universal characters $S^O_{[\lambda,\mu]}(\mathbf{x},\mathbf{y})$ can be described by the determinant, that in [4] can not described by determinant.

Now we can define a UC hierarchy where UC is the abbreviation of universal character.

Definition 6.2. *For an unknown function $\tau = \tau(\mathbf{x},\mathbf{y})$, the system of bilinear relations*

$$\sum_{n+m=-1} X^-_n\tau\otimes X^+_m\tau = \sum_{n+m=-1} Y^-_n\tau\otimes Y^+_m\tau = 0 \tag{51}$$

is called the orthogonal UC hierarchy.

If $\tau = \tau(\mathbf{x}, \mathbf{y})$ does not depend on $\mathbf{y} = (y_1, y_2, \cdots)$, the second equality of (51) trivially holds and the first one is reduced to the bilinear expression (23) of the OSKP hierarchy. From this aspect, we treat the orthogonal UC hierarchy as an extension of the OSKP hierarchy.

It is obvious that (51) can be rewritten into the form

$$\frac{1}{2\pi i}\oint(1-z^2)e^{\xi(\mathbf{x}'-\mathbf{x}'',z)}dz \quad \tau(\mathbf{x}''+[z^{-1}]+[z],\mathbf{y}''+[z])\cdot\tau(\mathbf{x}'-[z^{-1}]-[z],\mathbf{y}'-[z])=0, \tag{52}$$

$$\frac{1}{2\pi i}\oint(1-w^2)e^{\xi(\mathbf{y}'-\mathbf{y}'',w)}dw \quad \tau(\mathbf{x}''+[w],\mathbf{y}''+[w^{-1}]+[w])\cdot\tau(\mathbf{x}'-[w],\mathbf{y}'-[w^{-1}]-[w]) \ =0 \tag{53}$$

for arbitrary $\mathbf{x}, \mathbf{x}', \mathbf{y}$ and $\mathbf{y}'$. Consider their Taylor expansions at $(\mathbf{x} = \mathbf{x}', \mathbf{y} = \mathbf{y}')$, that is, replacing $(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}')$ with $(\mathbf{x}-\mathbf{u}, \mathbf{x}+\mathbf{u}, \mathbf{y}-\mathbf{v}, \mathbf{y}+\mathbf{v})$ and expand with respect to $(\mathbf{u}, \mathbf{v}) = (u_1, u_2, \cdots, v_1, v_2, \cdots)$, then we get

$$\begin{aligned}&\sum_{i-j+k-m+n=-1} P_i(-2\mathbf{u})P_j(\tilde{\partial}_{\mathbf{u}})P_k(\tilde{\partial}_{\mathbf{u}})P_m(\tilde{\partial}_{\mathbf{u}})P_n(\tilde{\partial}_{\mathbf{u}})\tau(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})\tau(\mathbf{x}-\mathbf{u},\mathbf{y}-\mathbf{v})\\ &-\sum_{i-j+k-m+n=-3} P_i(-2\mathbf{u})P_j(\tilde{\partial}_{\mathbf{u}})P_k(\tilde{\partial}_{\mathbf{u}})P_m(\tilde{\partial}_{\mathbf{u}})P_n(\tilde{\partial}_{\mathbf{u}})\tau(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})\tau(\mathbf{x}-\mathbf{u},\mathbf{y}-\mathbf{v})=0.\end{aligned}$$

and

$$\begin{aligned}&\sum_{i-j+k-m+n=-1} P_i(-2\mathbf{v})P_j(\tilde{\partial}_{\mathbf{u}})P_k(\tilde{\partial}_{\mathbf{u}})P_m(\tilde{\partial}_{\mathbf{u}})P_n(\tilde{\partial}_{\mathbf{u}})\tau(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})\tau(\mathbf{x}-\mathbf{u},\mathbf{y}-\mathbf{v})\\ &-\sum_{i-j+k-m+n=-3} P_i(-2\mathbf{v})P_j(\tilde{\partial}_{\mathbf{u}})P_k(\tilde{\partial}_{\mathbf{u}})P_m(\tilde{\partial}_{\mathbf{u}})P_n(\tilde{\partial}_{\mathbf{u}})\tau(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})\tau(\mathbf{x}-\mathbf{u},\mathbf{y}-\mathbf{v})=0.\end{aligned}$$

Taking the coefficient of $\mathbf{u}^{\mathbf{n}}\mathbf{v}^{\mathbf{m}}$ leads to many differential equation with respect to $\mathbf{x}, \mathbf{y}$, these differential equations are all of infinite order. This reflects that the integrands above with $(\mathbf{x} = \mathbf{x}', \mathbf{y} = \mathbf{y}')$ may be singular not only at $z = 0, w = 0$ but also at $z = \infty, w = \infty$.

In the follows, we give a class of polynomial solutions of the orthogonal UC hierarchy. From the relations between $X_n^{\pm}$, $Y_n^{\pm}$, we obtain

$$\left(\sum_{n+m=-1} X_n^-\otimes X_m^+\right)(X_t^+\otimes X_t^+) = (X_{t+1}^+\otimes X_{t-1}^+)\left(\sum_{n+m=-1} X_n^-\otimes X_m^+\right) \tag{54}$$

$$\left(\sum_{n+m=-1} Y_n^-\otimes Y_m^+\right)(X_t^+\otimes X_t^+) = (X_t^+\otimes X_t^+)\left(\sum_{n+m=-1} Y_n^-\otimes Y_m^+\right) \tag{55}$$

that is, if $\tau = \tau(\mathbf{x}, \mathbf{y})$ is a solution of (51), so is $X_t^+\tau$, we can verify in the same way that $Y_t^+\tau$ is also a solution of (51). By equation (50), we obtain

Proposition 6.3. *All the universal characters $S^O_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y})$ are solutions of the orthogonal UC hierarchy.*

It is known that if $\tau = \tau(\mathbf{x}, \mathbf{y})$ is a solution of (51), so are $X^+(\alpha)\tau$ and $Y^+(\beta)\tau$ for arbitrary constants $\alpha, \beta \in \mathbb{C}^{\times}$. Then we will consider the bilinear relations among the solutions connected by the vertex operators. The modified orthogonal UC hierarchy is introduced as follows.

Definition 6.4. *Suppose $\tau_{m,n} = \tau_{m,n}(\mathbf{x}, \mathbf{y})$ is a solution of the orthogonal UC hierarchy (51). Let*

$$\tau_{m+1,n} = X^+(\alpha_m)\tau_{m,n}, \tau_{m,n+1} = Y^+(\beta_n)\tau_{m,n},$$
$$\tau_{m+1,n+1} = X^+(\alpha_m)Y^+(\beta_n)\tau_{m,n} = Y^+(\beta_n)X^+(\alpha_m)\tau_{m,n}$$

for arbitrary constants $\alpha_m, \beta_n \in \mathbb{C}^{\times}$. From equation (51), we can get the equations satisfied by $\tau_{m,n}$'s, which are called the modified orthogonal UC hierarchy.

For τ- function $\tau_{m,n} = \tau_{m,n}(\mathbf{x}, \mathbf{y})$, the modified orthogonal UC hierarchy includes the following bilinear equations:

$$\sum_{i+j=-2} X_i^-\tau_{m,n}\otimes X_j^+\tau_{m+1,n} = \sum_{i+j=-1} Y_i^-\tau_{m,n}\otimes Y_j^+\tau_{m+1,n} = 0, \tag{56}$$

$$\tau_{m,n}\otimes\tau_{m+1,n+1} - \sum_{i+j=0} X_i^-\tau_{m+1,n}\otimes X_j^+\tau_{m,n+1} = \sum_{i+j=-2} Y_i^-\tau_{m+1,n}\otimes Y_j^+\tau_{m,n+1} = 0. \tag{57}$$

Here the first equation and the second equation can be deduced from (51) by applying $1\otimes X^+(\alpha_m)$ and $X^+(\alpha_m)\otimes Y^+(\beta_n)$, respectively.

From the definition of $\tau_{m,n}$, for a solution $\tau_{0,0}$ of the orthogonal UC hierarchy, we have

$$\tau_{m,n} = \prod_{i=0}^{m-1} X^+(\alpha_i) \prod_{j=0}^{n-1} Y^+(\beta_j)\tau_{0,0}, \tag{58}$$

where

$$\prod_{i=0}^{m-1} X^+(\alpha_i) = X^+(\alpha_{m-1})\cdots X^+(\alpha_1)X^+(\alpha_0),$$

then we have the following bilinear equations.

Proposition 6.5. *For integers $m, n \geq 0$, it holds that*

$$\sum_{i+j=-m-1} X_i^-\tau_{0,0}\otimes X_j^+\tau_{m,n} = \sum_{i+j=-n-1} Y_i^-\tau_{0,0}\otimes Y_j^+\tau_{m,n} = 0, \tag{59}$$

$$\tau_{0,0}\otimes\tau_{1,n} - \sum_{i+j=0} X_i^-\tau_{1,0}\otimes X_j^+\tau_{0,n} = \sum_{i+j=-n-1} Y_i^-\tau_{1,0}\otimes Y_j^+\tau_{0,n} = 0, \tag{60}$$

$$\sum_{i+j=-m-1} X_i^-\tau_{0,1}\otimes X_j^+\tau_{m,0} = \tau_{0,0}\otimes\tau_{m,1} - \sum_{i+j=0} Y_i^-\tau_{0,1}\otimes Y_j^+\tau_{m,0} = 0. \tag{61}$$

The results above are obtained by applying $1\otimes\prod_{i=0}^{m-1}X^+(\alpha_i)\prod_{j=0}^{n-1}Y^+(\beta_j)$, $X^+(\alpha_0)\otimes\prod_{j=0}^{n-1}Y^+(\beta_j)$ and $Y^+(\beta_0)\otimes\prod_{i=0}^{m-1}X^+(\alpha_i)$ to (51).

Let us look closely at (59), which corresponds to the orthogonal UC hierarchy (51) when $m = n = 0$. It can be equivalently rewritten into

$$\frac{1}{2\pi i}\oint z^m(1-z^2)e^{\xi(\mathbf{x}-\mathbf{x}',z)}dz \quad \tau_{0,0}(\mathbf{x}'+[z^{-1}]+[z],\mathbf{y}'+[z^{-1}]+[z])\cdot\tau_{m,n}(\mathbf{x}-[z^{-1}]-[z],\mathbf{y}-[z^{-1}]-[z]) = 0, \tag{62}$$

$$\frac{1}{2\pi i}\oint w^n(1-w^2)e^{\xi(\mathbf{y}-\mathbf{y}',w)}dw \quad \tau_{0,0}(\mathbf{x}'+[w^{-1}]+[w],\mathbf{y}'+[w^{-1}]+[w])\cdot\tau_{m,n}(\mathbf{x}-[w^{-1}]-[w],\mathbf{y}-[w^{-1}]-[w]) = 0. \tag{63}$$

Let $I, J \subset \mathbb{Z}$ be a disjoint pair of finite indexing sets. By specializing the parameters in (62) and (63) as

$$\mathbf{x}' = \mathbf{x} - \sum_{j\in I}[t_i] + \sum_{j\in J}[t_j],\ \mathbf{y}' = \mathbf{y} - \sum_{j\in I}[t_i^{-1}] + \sum_{j\in J}[t_j^{-1}],$$

we get

$$\Omega_1 := z^m(1-z^2)e^{\xi(\mathbf{x}-\mathbf{x}',z)}dz = z^m(1-z^2)\frac{\prod_{j\in J}(1-t_jz)}{\prod_{j\in I}(1-t_jz)}dz,$$

$$\Omega_2 := w^n(1-w^2)e^{\xi(\mathbf{x}-\mathbf{x}',w)}dw = w^n(1-w^2)\frac{\prod_{j\in J}(1-w/t_j)}{\prod_{j\in I}(1-w/t_j)}dw.$$

Let $z = 1/w$, we find that

$$\Omega_2 = z^{|I|-|J|-m-n-4}\frac{\prod_{j\in I}(-t_j)}{\prod_{j\in J}(-t_j)}\Omega_1$$

Consequently, the integrands of (62) and (63) coincide up to constant functor if the condition $|I|-|J| = m+n+4$ holds. Let

$$F(z) = z^m(1-z^2)\frac{\prod_{j\in J}(1-t_jz)}{\prod_{j\in I}(1-t_jz)}\cdot\tau(\mathbf{x}'+[z^{-1}]+[z],\mathbf{y}'+[z^{-1}]+[z])\cdot\tau(\mathbf{x}-[z^{-1}]-[z],\mathbf{y}-[z^{-1}]-[z])$$

in the integrand of (62), hence, we get

$$\int_{C_1}F(z)dz = \int_{C_2}F(z)dz = 0,$$

where C_1 and C_2 are a positively oriented small circle around $z = 0$ and $z = \infty$ respectively such that all the other singularities are out of it. Then, we obtain

$$\sum_{i\in I}\mathrm{Res}_{z=1/t_i}F(z)dz = 0. \tag{64}$$

This means that the residue calculus at possible essential singularities $z = 0, \infty$ is avoided for the presence of two bilinear equations (62) and (63).

For a function $f = f(\mathbf{x}, \mathbf{y})$, we define a shift operator T_i by

$$T_i(f) = f(\mathbf{x} - [t_i], y - [t_i^{-1}])$$

and $T_{\{i_1,\cdots,i_r\}} := T_{i_1} \cdots T_{i_r}(f)$ for short. Then (64) gives

$$\sum_{i \in I} t_i^n (1 - \frac{1}{t_i^2}) \frac{\prod_{j \in J}(t_i - t_j)}{\prod_{j \in I/\{i\}}(t_i - t_j)} T_{I/\{i\}} \tau_{0,0}(\mathbf{x} + [t_i^{-1}], \mathbf{y} + [t_i]) T_{J \cup \{i\}} \tau_{m,n}(\mathbf{x} - [t_i^{-1}], \mathbf{y} - [t_i]) = 0,$$

which can be regarded as a difference equation with each t_i being the difference interval. Then, we have

Proposition 6.6. *The following equations hold:*

1. *If $|I| - |J| = m + n + 4$ and $m, n \geq 0$, then*

$$\sum_{i \in I} t_i^n (t_i^2 - 1) \frac{\prod_{j \in J}(t_i - t_j)}{\prod_{j \in I/\{i\}}(t_i - t_j)} T_{I/\{i\}} \tau_{0,0}(\mathbf{x} + [t_i^{-1}], \mathbf{y} + [t_i]) T_{J \cup \{i\}} \tau_{m,n}(\mathbf{x} - [t_i^{-1}], \mathbf{y} - [t_i]) = 0.$$

2. *If $|I| - |J| = n + 3$ and $n \geq 0$, then*

$$T_I(\tau_{0,0}) T_J(\tau_{1,n}) = \sum_{i \in I} (1 - \frac{1}{t_i^2}) \frac{\prod_{j \in J}(1 - t_j/t_i)}{\prod_{j \in I/\{i\}}(1 - t_j/t_i)} T_{I/\{i\}} \tau_{1,0}(\mathbf{x} + [t_i^{-1}], \mathbf{y} + [t_i]) \cdot T_{J \cup \{i\}} \tau_{0,n}(\mathbf{x} - [t_i^{-1}], \mathbf{y} - [t_i]) = 0.$$

3. *If $|I| - |J| = m + 3$ and $m \geq 0$, then*

$$T_I(\tau_{0,0}) T_J(\tau_{m,1}) = \sum_{i \in I} (1 - \frac{1}{t_i^2}) \frac{\prod_{j \in J}(1 - t_i/t_j)}{\prod_{j \in I/\{i\}}(1 - t_i/t_j)} T_{I/\{i\}} \tau_{1,0}(\mathbf{x} + [t_i^{-1}], \mathbf{y} + [t_i]) \cdot T_{J \cup \{i\}} \tau_{0,n}(\mathbf{x} - [t_i^{-1}], \mathbf{y} - [t_i]) = 0.$$

Let $m = 1, n = 0, I = 1, 2, 3, 4, J = \emptyset$, and let $t_3 = t_1^{-1},\ t_4 = t_2^{-1}$, the first equation in Proposition 6.6 reduces to

$$(1 - t_1 t_2)(t_2 - t_1) \tilde{T}_{12}(\tau_{0,0}) \tau_{1,1} = t_2(t_1^2 + 1) \tilde{T}_2(\tau_{1,0}) \tilde{T}_1(\tau_{0,1}) - t_1(t_2^2 + 1) \tilde{T}_1(\tau_{1,0}) \tilde{T}_2(\tau_{0,1}), \tag{65}$$

where the notation $\tilde{T}_i$ is a shift operator defined by

$$\tilde{T}_i f(\mathbf{x}, \mathbf{y}) = f(\mathbf{x} - [t_i] - [t_i^{-1}], \mathbf{y} - [t_i] - [t_i^{-1}]).$$

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

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