Research Article

The Commutator of Fuzzy Congruences in Universal Algebras

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ABSTRACT

We develop the commutator theory for fuzzy congruence relations of general universal algebras. In particular, for algebras in modular varieties, we characterize the commutator of fuzzy congruences using the Day's terms.

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1. INTRODUCTION

In group theory, the commutator is a binary operation on the lattice of normal subgroups of a group which has an important role in the study of solvable, Abelian and nilpotent groups. Given normal subgroups A and B of a group H , their commutator $[A, B]$ is defined to be the smallest normal subgroup of H containing all elements of the form $a^{-1}b^{-1}ab$ for $a \in A$ and $b \in B$. In other words, $[A, B]$ is the largest normal subgroup K of H such that in the quotient group H/K every element of A/K commutes with every element of B/K . Thus we have a binary operation in the lattice of normal subgroups. This binary operation, together with the lattice operations, carries much of the information about how a group is put together. The operation is also interesting in its own right. It is a commutative, monotone operation, completely distributive with respect to joins in the lattice.

There is also an operation naturally defined on the lattice of ideals of a ring, which has these properties, namely the product of ideals. For ideals I and J of a ring $[R]$ be the ideal of R generated by all the products ij and ji , with $i \in I$ and $j \in J$. The congruency between these two contexts extends to the following fact: $[I, J]$ is the smallest ideal K of R for which every element of the ring I/K commutes multiplicatively with all elements of the ring J/K .

The structural properties of groups and rings were extended to the variety of algebras with permuting congruences by J.D.H. Smith in [1]. His work has laid the foundation for generalizing the commutator theory from groups and rings to an abstract operation on the lattice of congruences of an algebra in permutable varieties. This operation has the same useful properties that the commutator for groups (which is a special case of it) possesses. The resulting theory has also many general applications.

The concept of fuzzy sets was first introduced by L. A. Zadeh [2] in 1965. This idea brings a new approach to model problems relating uncertain and imprecise conditions. In 1971, A. Rosenfeld [3] applied this theory and has formulated the concept of a fuzzy subgroup of a group. Since then, many researchers have been studying fuzzy subalgebras of several algebraic structures such as rings (see [4–6]), modules (see [7–9]), vector spaces (see [10,11]), lattices (see [12–17]), pseudo-complemented semi-lattice (see [18]), posets (see [19,20]), MS-algebras (see [21–24]), universal algebras (see [25–28]), etc.

Fuzzy congruences have been studied in different algebraic structures; in semigroups (see [29,30]), in groups (see [31–33]), in rings (see [34,35]), in modules and vector spaces (see [36,37]), in lattice structures (see [15, 17, 38]), etc. More generally, L. Filep and G.I. Maurer [39], B. Šešelja and A. Tepavčević [40], A. Di Nola and G. Gerla [41] have studied fuzzy congruences in some general context in universal algebras. U.M. Swamy and D.V. Raju [42] showed that

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the set $FCon(A)$ of all L -fuzzy congruences of an algebra A forms an algebraic closure fuzzy set system, where L is a complete lattice satisfying the infinite meet distributive law. In particular, if L is the unit interval $[0, 1]$, then $FCon(A)$ is an algebraic lattice and this result coincides with that of Murali [43].

In all the abovementioned articles, a binary operation c on the lattice of fuzzy congruences on an algebra A having the following properties:

- (1) $c(\Theta, \Phi) \leq \Theta \cap \Phi$
- (2) $c(\Theta, \Phi) = c(\Phi, \Theta)$
- (3) $\pi(c(\Theta, \Phi)) = c(\pi(\Theta), \pi(\Phi))$
- (4) $c(\bigvee_{i \in I} \Theta_i, \Phi) = \bigvee_{i \in I} c(\Theta_i, \Phi)$

for each $\Theta, \Phi, \Theta_i \in FCon(A)$ and any surjective homomorphism $\pi : A \rightarrow B$, was not obtained in any form. Taking this into account, in this paper, we consider a general universal algebra A of a fixed type $\mathfrak{F}$ and define a binary operation, which we call it the commutator, on the lattice of fuzzy congruence relations on A having the above properties. In the particular case of modular varieties, we characterize this commutator using the Day's terms. In this vein, we formulate and prove the fuzzy version of the shifting lemma. Moreover, we give another important characterization of the commutator in the sense of Hagemann and Hermann [44].

The paper is structured as follows: Section 2 presents some notations and basic results we use in what follows. Section 3 is mainly devoted to the development of the commutator theory for fuzzy congruences in general universal algebras with abstract finitary operations. The concept of centralizers is studied in the viewpoint of fuzzy logic and used to define the commutator.

In Section 4, we state and prove the fuzzy version of the well-known result in algebraic geometry called the shifting lemma and we give a detailed characterization for the commutator of fuzzy congruences in modular varieties. In Section 5, we give another description for the commutator of fuzzy congruences in modular varieties in the sense of Hagemann and Hermann [44].

2. PRELIMINARIES

For the standard concepts in universal algebras, we refer to [45,46]. Throughout this paper $\mathcal{V}$ is a class of algebras of a fixed type $\mathfrak{F}$ and $A \in \mathcal{V}$, i.e., A is an algebra of type $\mathfrak{F}$. $Z(Z^+, \text{ resp. } Z_0)$ denotes the set of all integers (positive integers, resp. non-negative integers). For $n \in Z_0$, we denote by $\mathfrak{F}_n$, the set of all n -ary fundamental operations of A . We use the notation $\mathbf{a}$ to denote the tuples $a_1, \dots, a_n$ of elements from A . For $n \in Z^+$, the n -ary terms of type $\mathfrak{F}$ are formal expressions obtained in finitely many steps by the following process:

- (1) The variables $x_1, \dots, x_n$ are n -ary terms of type $\mathfrak{F}$.
- (2) If $m \in Z_0$, $t_1, \dots, t_m$ are n -ary terms of type $\mathfrak{F}$ and $f \in \mathfrak{F}_m$, then $f(t_1, \dots, t_m)$ is also a term of type $\mathfrak{F}$.

T_n denotes the set of n -ary terms of type $\mathfrak{F}$. An equivalence relation θ on A is called a congruence on A if it is compatible with all fundamental operations f on A . The set of all congruence relations

on A denoted by $Con(A)$ is a complete lattice together with the usual inclusion order. For $\theta, \phi \in Con(A)$ their relational product $\theta \circ \phi$ is a relation on A defined by

$$\theta \circ \phi = \{(x, y) \in A \times A : \exists z \in A \text{ such that } (x, z) \in \phi \text{ and } (z, y) \in \theta\}$$

A is called congruence permutable (or shortly permutable) if $\theta \circ \phi = \phi \circ \theta$ for all $\theta, \phi \in Con(A)$, and $\mathcal{V}$ is permutable if all algebras in $\mathcal{V}$ are so. A.I. Mal'cev has proved that a variety $\mathcal{V}$ is permutable if and only if there is a term p of the variety so that $\mathcal{V}$ models the equations [47]:

$$p(x, z, z) \approx x \text{ and } p(x, x, z) \approx z$$

The following definition is summarized mainly from [48,49].

Definition 2.1. Let $\alpha, \beta, \delta \in Con(A)$.

- (1) $M(\alpha, \beta)$ is the set of all matrices of the form

$$\begin{bmatrix} t(\mathbf{a}^1, \mathbf{b}^1) & t(\mathbf{a}^1, \mathbf{b}^2) \\ t(\mathbf{a}^2, \mathbf{b}^1) & t(\mathbf{a}^2, \mathbf{b}^2) \end{bmatrix}$$
 where $\mathbf{a}^i$, $i = 1, 2$ is a sequence of n elements $\mathbf{b}^j$ is a sequence of m elements of A , $m, n \geq 0$ satisfying $a_k^1 \alpha a_k^2$ and $b_j^1 \beta b_j^2$ for $k \leq n$ and $j \leq m$ and $t \in T_{n+m}$.
- (2) We say α centralizes β modulo δ and write $C(\alpha, \beta; \delta)$ provided that for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(\alpha, \beta)$, $a\delta b$ implies $c\delta d$.
- (3) The commutator $[\alpha, \beta]$ is the smallest congruence δ on A such that $C(\alpha, \beta; \delta)$ holds.
- (4) The symmetric commutator $[\alpha, \beta]_s$ is the smallest congruence δ on A such that both $C(\alpha, \beta; \delta)$ and $C(\beta, \alpha; \delta)$ hold.

It was proposed by Gougen [50] that a complete residuated lattice (to which the unit interval $[0, 1]$ is a special case of it) is the best candidate to take truth values of fuzzy statements. Residuated lattices have been introduced by Ward and Dilworth [51]. From the view point of fuzzy logic, residuated lattices have been thoroughly investigated by H\u00f6hle (see e.g. [52–54]). More recently, Belohlavek [55] used residuated lattices to study algebras with fuzzy equality. However, in the study of fuzzy substructures (like fuzzy subalgebras, fuzzy ideals, fuzzy congruences, etc.) complete residuated lattices in general are not suitable to obtain a Rosenfeld type characterization for such structures using their level sets. For this reason, we choose special complete residuated lattices called complete Brouwerian lattices (complete lattices satisfying the infinite meet distributive property) to be the structure of truth degrees for fuzzy statements. In the sequel, $L = (L, \wedge, \vee, 0, 1)$ is a complete Brouwerian lattice; i.e., L is a complete lattice satisfying the infinite meet distributive law. An L -fuzzy subset of A is any mapping $\mu : A \rightarrow L$. For each $\alpha \in L$, the α -level set of μ denoted by μ_α is a subset of A given by

$$\mu_\alpha = \{x \in A : \alpha \leq \mu(x)\}$$

For L -fuzzy subsets μ and ν of A , we write $\mu \leq \nu$ to mean $\mu(x) \leq \nu(x) \forall x \in A$ in the ordering of L . μ will be called normalized if there is some $x \in A$ with $\mu(x) = 1$. By an L -fuzzy relation on A , we mean an L -fuzzy subset of $A \times A$. From now on, we drop the prefix

“L—” and simply say fuzzy subsets respectively fuzzy relations on A . Note also that we use lower case Greek letters like $\theta, \phi, \dots$ to denote crisp relations on A , whereas we use upper case Greek letters like $\Theta, \Phi, \dots$ to denote fuzzy relations on A . The following definition is due to [42].

Definition 2.2. A fuzzy relation Θ on A is said to be

- (1) reflexive if: $\Theta(x, x) = 1$ for all $x \in A$,
- (2) symmetric if: $\Theta(x, y) = \Theta(y, x)$ for all $x, y \in A$,
- (3) transitive if: $\Theta(x, z) \geq \Theta(x, y) \wedge \Theta(y, z)$ for all $x, y, z \in A$.

A reflexive, symmetric and transitive fuzzy relation on A is called a fuzzy equivalence relation on A .

Definition 2.3. A fuzzy relation Θ on A is called compatible, if

$$\Theta(f^A(x_1, x_2, \dots, x_n), f^A(y_1, y_2, \dots, y_n)) \geq \Theta(x_1, y_1) \wedge \dots \wedge \Theta(x_n, y_n)$$

for every $n \in \mathbb{Z}^+$, $f \in \mathfrak{F}_n$ and all $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in A$. A compatible fuzzy equivalence relation on A is called a fuzzy congruence relation on A . We denote by $FCon(A)$ the set of all fuzzy congruence relations on A .

For $\Theta, \Phi \in FCon(A)$, their composition $\Theta \circ \Phi$ is a fuzzy relation on A given by

$$\Theta \circ \Phi(x, y) = \bigvee \left\{ \Theta(x, z) \wedge \Phi(z, y) : z \in A \right\}$$

for all $x, y \in A$. For a positive integer n , by Θ^n , we mean $\Theta \circ \Theta \circ \dots \circ \Theta$ (n copies). If $t(x_1, \dots, x_m)$ is an m -ary term operation on A and $\Theta \in FCon(A)$, then it holds that

$$\Theta(t(a_1, \dots, a_m), t(b_1, \dots, b_m)) \geq \Theta(a_1, b_1) \wedge \dots \wedge \Theta(a_m, b_m)$$

for all $a_1, \dots, a_m, b_1, \dots, b_m \in A$.

3. THE COMMUTATOR

The concept of the commutator of fuzzy subgroups was defined and used to study the notion of commutative fuzzy sets, nilpotent fuzzy subgroups and solvable fuzzy subgroups in [56, 57]. In this section, we define the commutator of fuzzy congruences in a more general setting in universal algebras.

Let $\Theta, \Phi \in FCon(A)$ and let $M(A)_{2 \times 2}$ be the set of all matrices of the form $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ where each $u_{ij} \in A$ for $i, j \in \{1, 2\}$. Define a fuzzy subset $M_L(\Theta, \Phi)$ of $M(A)_{2 \times 2}$ by

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) = \bigvee \left\{ \left[\bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \right] \wedge \left[\bigwedge_{r=1}^m \Phi(b_r^1, b_r^2) \right] : \begin{array}{l} \text{where } u_{ij} = t(a_1^i, \dots, a_n^i, b_1^j, \dots, b_m^j) \\ i, j \in \{1, 2\}, t \in T_{n+m} \end{array} \right\}$$

Lemma 3.1. Let $\Theta, \Theta', \Phi, \Phi' \in FCon(A)$. If $\Theta \leq \Theta'$ and $\Phi \leq \Phi'$, then

$$M(\Theta, \Phi) \leq M(\Theta', \Phi) \text{ and } M(\Theta, \Phi) \leq M(\Theta, \Phi')$$

Theorem 3.2. For any $\Theta, \Phi \in FCon(A)$, and each $u_{ij} \in A$:

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) = \bigvee \left\{ \alpha \in L : \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in M(\Theta_\alpha, \Phi_\alpha) \right\}$$

Proof. Let us define two sets G and H as follows:

$$G = \bigvee \left\{ \left[\bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \right] \wedge \left[\bigwedge_{r=1}^m \Phi(b_r^1, b_r^2) \right] : u_{ij} = t(a_1^i, \dots, a_n^i, b_1^j, \dots, b_m^j) \right. \\ \left. \text{where } i, j \in \{1, 2\}, n, m \geq 1 \text{ and } t \in T_{n+m} \right\}$$

$$H = \left\{ \alpha \in L : \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in M(\Theta_\alpha, \Phi_\alpha) \right\}$$

Then it is clear that both G and H are subsets of L . Our aim is to show that $\bigvee H = \bigvee G$. If $\alpha \in H$, then $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in M(\Theta_\alpha, \Phi_\alpha)$. i.e.,

$$u_{11} = t(a_1^1, \dots, a_n^1, b_1^1, \dots, b_m^1), \quad u_{12} = t(a_1^1, \dots, a_n^1, b_1^2, \dots, b_m^2), \\ u_{21} = t(a_1^2, \dots, a_n^2, b_1^1, \dots, b_m^1), \quad u_{22} = t(a_1^2, \dots, a_n^2, b_1^2, \dots, b_m^2).$$

for some $t \in T_{n+m}$, where $(a_k^1, a_k^2) \in \Theta_\alpha$, $(b_j^1, b_j^2) \in \Phi_\alpha$ for $k = 1, 2, \dots, n, j = 1, 2, \dots, m$, which implies that,

$$\bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \geq \alpha \text{ and } \bigwedge_{k=1}^n \Phi(b_1^1, b_1^2) \geq \alpha$$

If we put

$$\lambda = \left(\bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \right) \wedge \left(\bigwedge_{k=1}^n \Phi(b_1^1, b_1^2) \right),$$

then $\lambda \in G$ such that $\alpha \leq \lambda$. Since α is arbitrary in H , we can conclude that for each $\alpha \in H$ there is $\lambda \in G$ such that $\alpha \leq \lambda$. So that $\bigvee H \leq \bigvee G$. To prove the other inequality, let $\alpha \in G$. Then

$$\alpha = \left(\bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \right) \wedge \left(\bigwedge_{j=1}^m \Phi(b_j^1, b_j^2) \right)$$

for some $a_k^1, a_k^2, b_j^1, b_j^2 \in A$, where $k = 1, 2, \dots, n, j = 1, 2, \dots, m$, and

$$u_{ij} = t(a_1^i, a_2^i, \dots, a_n^i, b_1^j, b_2^j, \dots, b_m^j)$$

for $i, j \in \{1, 2\}$. So we have

$$\bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \geq \alpha \text{ and } \bigwedge_{j=1}^m \Phi(b_j^1, b_j^2) \geq \alpha$$

which gives that $(a_k^1, a_k^2) \in \Theta_\alpha$ for all $k = 1, 2, \dots, n$ and $(b_r^1, b_r^2) \in \Phi_\alpha$ for all $r = 1, 2, \dots, m$. Thus $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in M(\Theta_\alpha, \Phi_\alpha)$. So that $\alpha \in H$, i.e., $G \subseteq H$ and hence $\vee G \leq \vee H$. Therefore the equality holds and this completes the proof. $\square$

Remark. For each $a, b, c, d \in A$, it holds that

$$M_L(\Phi, \Theta) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{tr} \right)$$

Lemma 3.3. Let $\Theta, \Phi \in FCon(A)$. For each $\alpha \in L$, we have

$$M_L(\Theta, \Phi)_\alpha = \bigcup \left\{ \bigcap_{\beta \in H} M(\Theta_\beta, \Phi_\beta) : H \subseteq L, \alpha \leq \sup H \right\}$$

Proof. For each $\alpha \in L$, let us define a set B_α as follows:

$$B_\alpha = \bigcup \left\{ \bigcap_{\beta \in H} M(\Theta_\beta, \Phi_\beta) : H \subseteq L, \alpha \leq \sup H \right\}$$

We show that $M_L(\Theta, \Phi)_\alpha = B_\alpha$. Let $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in M_L(\Theta, \Phi)_\alpha$, then $M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) \geq \alpha$. Let us take the set H as given in Theorem 3.2. Then $\alpha \leq \sup H$ and $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in M(\Theta_\gamma, \Phi_\gamma)$ for all $\gamma \in H$, i.e.,

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in \bigcap_{\gamma \in H} M(\Theta_\gamma, \Phi_\gamma)$$

which implies that $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in B_\alpha$ and hence $M_L(\Theta, \Phi)_\alpha \subseteq B_\alpha$.

To prove the other inclusion let $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in B_\alpha$. Then there exists $H \subseteq L$ such that $\alpha \leq \sup H$ and $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in M(\Theta_\gamma, \Phi_\gamma)$ for all $\gamma \in H$. From Theorem 3.2, we have the following:

$$\begin{aligned} M(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) &= \bigvee \left\{ \lambda \in L : \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right. \\ &\quad \left. \in M(\Theta_\lambda, \Phi_\lambda) \right\} \\ &\geq \vee H \\ &\geq \alpha, \end{aligned}$$

So that $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in M_L(\Theta, \Phi)_\alpha$ which gives $B_\alpha \subseteq M_L(\Theta, \Phi)_\alpha$ and hence the equality holds.

Remember that a fuzzy subset μ of A is a fuzzy subalgebra of A if the following conditions are satisfied:

- (1) If $f \in \mathfrak{F}$ is nullary, then $\mu(f^A) = 1$
- (2) If $f \in \mathfrak{F}$ is n -ary, $n > 0$ and $x_1, \dots, x_n \in A$, then

$$\mu(f^A(x_1, \dots, x_n)) \geq \mu(x_1) \bigwedge \dots \bigwedge \mu(x_n)$$

For a fuzzy subset λ of A , always there is the smallest fuzzy subalgebra of A containing λ , namely a fuzzy subalgebra generated by λ and is denoted by $FS_g(\lambda)$. As proved in [42], $FS_g(\lambda)$ can be characterized as follows.

Theorem 3.4. For any fuzzy subset λ of A , $FS_g(\lambda)$ is characterized as follows: iff f is a nullary operation symbol, then $FS_g(\lambda)(f^A) = 1$ and for any $x \in A$:

$$FS_g(\lambda)(x) = \bigvee \left\{ \bigwedge_{i=1}^n \lambda(x_i) : x = t(x_1, \dots, x_n), n > 0 \text{ and } t \in T_n \right\}.$$

Theorem 3.5. For any $\Theta, \Phi \in FCon(A)$, $M_L(\Theta, \Phi)$ is the fuzzy subalgebra of $M(A)_{2 \times 2}$ generated by the fuzzy subset η of $M(A)_{2 \times 2}$ defined in the following way: for each $a, b, c, d \in A$

$$\eta \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{cases} \Theta(a, c) & \text{if } a = b, c = d \\ \Phi(a, b) & \text{if } a = c, b = d \\ 0 & \text{otherwise} \end{cases}$$

Proof. For any $a, b, c, d \in A$, consider the following:

$$\begin{aligned} &M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \\ &= \bigvee \left\{ \left[\bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \right] \bigwedge \left[\bigwedge_{r=1}^m \Phi(b_r^1, b_r^2) \right] : \right. \\ &\quad a = t(a_1^1, \dots, a_n^1, b_1^1, \dots, b_m^1), \\ &\quad b = t(a_1^1, \dots, a_n^1, b_1^2, \dots, b_m^2), \\ &\quad c = t(a_1^2, \dots, a_n^2, b_1^1, \dots, b_m^1), \\ &\quad d = t(a_1^2, \dots, a_n^2, b_1^2, \dots, b_m^2) \text{ where } t \in T_{n+m} \} \\ &= \bigvee \left\{ \left[\bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \right] \bigwedge \left[\bigwedge_{r=1}^m \Phi(b_r^1, b_r^2) \right] : \right. \\ &\quad \left[\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right] = \left[\begin{bmatrix} t(a_1^1, \dots, a_n^1, b_1^1, \dots, b_m^1) & t(a_1^1, \dots, a_n^1, b_1^2, \dots, b_m^2) \\ t(a_1^2, \dots, a_n^2, b_1^1, \dots, b_m^1) & t(a_1^2, \dots, a_n^2, b_1^2, \dots, b_m^2) \end{bmatrix} \right] \\ &\quad \left. \text{for some term } t \in T_{n+m} \right\} \end{aligned}$$

$$= \bigvee \left\{ \left[\bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \right] \wedge \left[\bigwedge_{r=1}^m \Phi(b_r^1, b_r^2) \right] : \begin{bmatrix} a & b \\ c & d \end{bmatrix} = t(Q) \right. \\ \left. \text{for some term } t \in T_{n+m} \right\}$$

$$= \bigvee \left\{ \left[\bigwedge_{k=1}^n \eta \left(\begin{bmatrix} a_k^1 & a_k^1 \\ a_k^2 & a_k^2 \end{bmatrix} \right) \right] \wedge \left[\bigwedge_{r=1}^m \eta \left(\begin{bmatrix} b_r^1 & b_r^2 \\ b_r^1 & b_r^2 \end{bmatrix} \right) \right] : \right. \\ \left. \begin{bmatrix} a & b \\ c & d \end{bmatrix} = t(Q) \text{ for some term } t \in T_{n+m} \right\} \\ = FS_g(\eta) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \text{ (by Theorem 3.4)}$$

where

$$Q = \left[\begin{bmatrix} a_1^1 & a_1^1 \\ a_1^2 & a_1^2 \end{bmatrix}, \dots, \left[\begin{bmatrix} a_n^1 & a_n^1 \\ a_n^2 & a_n^2 \end{bmatrix}, \left[\begin{bmatrix} b_1^1 & b_1^2 \\ b_1^1 & b_1^2 \end{bmatrix}, \dots, \left[\begin{bmatrix} b_m^1 & b_m^2 \\ b_m^1 & b_m^2 \end{bmatrix} \right. \right.$$

Definition 3.6. Let $\Theta, \Phi, \Psi \in FCon(A)$. We say that Θ centralizes Φ modulo Ψ if

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) \wedge \Psi(u_{11}, u_{12}) \\ = M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) \wedge \Psi(u_{21}, u_{22})$$

for each $u_{ij} \in A$. In this case, we write $FC(\Theta, \Phi; \Psi)$ holds for short to say that Θ centralizes Φ modulo Ψ . In this notation the letter F —stands to indicate that Θ, Φ and Ψ are all fuzzy congruences.

Theorem 3.7. Θ centralizes Φ modulo Ψ if and only if Θ_α centralizes Φ_α modulo Ψ_α for all $\alpha \in L$.

Proof. Suppose that $FC(\Theta, \Phi; \Psi)$ holds. Then

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) \wedge \Psi(u_{11}, u_{12}) \\ = M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) \wedge \Psi(u_{21}, u_{22})$$

for all $u_{ij} \in A$. Let $\alpha \in L$, $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in M(\Theta_\alpha, \Phi_\alpha)$ and $(u_{11}, u_{12}) \in \Psi_\alpha$. Then

$$u_{11} = t(a_1^1, \dots, a_n^1, b_1^1, \dots, b_m^1), \quad u_{12} = t(a_1^1, \dots, a_n^1, b_1^2, \dots, b_m^2) \\ u_{21} = t(a_1^2, \dots, a_n^2, b_1^1, \dots, b_m^1), \quad u_{22} = t(a_1^2, \dots, a_n^2, b_1^2, \dots, b_m^2)$$

where $(a_k^1, a_k^2) \in \Theta_\alpha$, $(b_j^1, b_j^2) \in \Phi_\alpha$ for all $k = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, i.e.,

$$\Theta(a_1^1, a_1^2) \wedge \dots \wedge \Theta(a_n^1, a_n^2) \geq \alpha, \quad \Phi(b_1^1, b_1^2) \wedge \dots \wedge \Phi(b_m^1, b_m^2) \\ \geq \alpha \text{ and } \Psi(u_{11}, u_{12}) \geq \alpha$$

Now we have the following:

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) \geq \bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \wedge \bigwedge_{j=1}^m \Phi(b_j^1, b_j^2) \\ \geq \alpha$$

so that

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) \wedge \Psi(u_{11}, u_{12}) \geq \alpha$$

Since $FC(\Theta, \Phi; \Psi)$ holds, it follows that

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) \wedge \Psi(u_{21}, u_{22}) \\ = M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) \\ \wedge \Psi(u_{11}, u_{12}) \geq \alpha$$

Therefore $\Psi(u_{21}, u_{22}) \geq \alpha$. Mean that $(u_{21}, u_{22}) \in \Psi_\alpha$. Thus $C(\Theta_\alpha, \Phi_\alpha; \Psi_\alpha)$ holds. Conversely suppose that $C(\Theta_\alpha, \Phi_\alpha; \Psi_\alpha)$ holds for all $\alpha \in L$. Let $u_{ij} \in A$ for $i, j \in \{1, 2\}$. Put

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) \wedge \Psi(u_{11}, u_{12}) = \alpha$$

Then $\alpha \in L$ such that

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in M_L(\Theta, \Phi)_\alpha \text{ and } (u_{11}, u_{12}) \in \Psi_\alpha$$

By Lemma 3.3 there exists some $H \subseteq L$ such that $\alpha \leq \sup H$ and

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in M(\Theta_\beta, \Phi_\beta) \text{ for all } \beta \in H. \text{ Let us define a set}$$

$$G = \left\{ \alpha \wedge \beta : \beta \in H \right\}$$

Then G is a subset of L such that $\sup G = \alpha$ and for each $\gamma \in G$, there is $\beta \in H$ such that $\gamma \leq \beta$. Moreover, it can be verified that

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in M(\Theta_\gamma, \Phi_\gamma) \text{ and } (u_{11}, u_{12}) \in \Psi_\gamma \text{ for all } \gamma \in G$$

Since by our assumption, $C(\Theta_\gamma, \Phi_\gamma; \Psi_\gamma)$ holds, it follows that $(u_{21}, u_{22}) \in \Psi_\gamma$ for all $\gamma \in G$, which gives that $\Psi(u_{21}, u_{22}) \geq \alpha$. Therefore

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) \wedge \Psi(u_{21}, u_{22}) \geq \alpha = \\ M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) \wedge \Psi(u_{11}, u_{12})$$

Similarly, we can show that

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) \wedge \Psi(u_{11}, u_{12}) \geq \\ M_L(\Theta, \Phi) \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) \wedge \Psi(u_{21}, u_{22})$$

and hence the equality holds. Thus Θ centralizes Φ modulo Ψ . Hence proved. $\square$

Lemma 3.8. $FC(\Theta, \Phi; \Theta \wedge \Phi)$ holds for each $\Theta, \Phi \in FCon(A)$

Proof. Let $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in M(A)_{2 \times 2}$. Note first that we may write $[u_{ij}]$ instead of $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ for simplicity. Now consider the following:

$$M_L(\Theta, \Phi) ([u_{ij}]) \wedge (\Theta \wedge \Phi) (u_{11}, u_{12}) \\ = \bigvee \left\{ \left[\bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \right] \wedge \left[\bigwedge_{r=1}^m \Phi(b_r^1, b_r^2) \right] \right. \\ \left. \wedge \Theta(u_{11}, u_{12}) \wedge \Phi(u_{11}, u_{12}) : \right. \\ \left. u_{ij} = t(a_1^i, \dots, a_n^i, b_1^j, \dots, b_m^j) \right. \\ \left. \text{where } i, j \in \{1, 2\}, \text{ and } t \in T_{n+m} \right\}$$

Now let

$$u_{11} = t(a_1^1, \dots, a_n^1, b_1^1, \dots, b_m^1), \quad u_{12} = t(a_1^1, \dots, a_n^1, b_1^2, \dots, b_m^2) \\ u_{21} = t(a_1^2, \dots, a_n^2, b_1^1, \dots, b_m^1), \quad u_{22} = t(a_1^2, \dots, a_n^2, b_1^2, \dots, b_m^2)$$

be any expression of u_{ij} 's using an arbitrary term operation t on A . By the transitive property of θ we have the following:

$$\Theta(u_{21}, u_{22}) \geq \Theta(u_{21}, u_{11}) \wedge \Theta(u_{11}, u_{12}) \wedge \Theta(u_{12}, u_{22}) \\ \geq \bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \wedge \Theta(u_{21}, u_{11})$$

Again using the compatibility property of Φ , we get

$$\Phi(u_{21}, u_{22}) \geq \bigwedge_{r=1}^m \Phi(b_r^1, b_r^2)$$

Using the above two inequalities we get the following:

$$M_L(\Theta, \Phi) ([u_{ij}]) \wedge (\Theta \wedge \Phi) (u_{21}, u_{22}) \\ \geq \left[\bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \right] \wedge \left[\bigwedge_{r=1}^m \Phi(b_r^1, b_r^2) \right] \wedge \Theta(u_{21}, u_{22}) \\ \wedge \Phi(u_{21}, u_{22}) \\ \geq \left[\bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \right] \wedge \left[\bigwedge_{r=1}^m \Phi(b_r^1, b_r^2) \right] \wedge \Theta(u_{11}, u_{12}) \\ \geq \left[\bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \right] \wedge \left[\bigwedge_{r=1}^m \Phi(b_r^1, b_r^2) \right] \wedge (\Theta \wedge \Phi) (u_{11}, u_{12})$$

Since the expressions of u_{ij} 's are arbitrary, it follows that

$$M_L(\Theta, \Phi) ([u_{ij}]) \wedge (\Theta \wedge \Phi) (u_{21}, u_{22}) \\ = \bigvee \left\{ \left[\bigwedge_{k=1}^n \Theta(a_k^1, a_k^2) \right] \wedge \left[\bigwedge_{r=1}^m \Phi(b_r^1, b_r^2) \right] \right. \\ \left. \wedge \Theta(u_{11}, u_{12}) \wedge \Phi(u_{11}, u_{12}) : \right. \\ \left. u_{ij} = t(a_1^i, \dots, a_n^i, b_1^j, \dots, b_m^j), \text{ where } i, j \in \{1, 2\}, t \in T_{n+m} \right\} \\ = M_L(\Theta, \Phi) ([u_{ij}]) \wedge (\Theta \wedge \Phi) (u_{11}, u_{12})$$

That is

$$M_L(\Theta, \Phi) ([u_{ij}]) \wedge (\Theta \wedge \Phi) (u_{21}, u_{22}) \geq M_L(\Theta, \Phi) ([u_{ij}]) \\ \wedge (\Theta \wedge \Phi) (u_{11}, u_{12})$$

The other inequality can be verified in a similar way. So that the equality holds and therefore $FC(\Theta, \Phi; \Theta \wedge \Phi)$ holds. $\square$

Corollary 3.9. Let $\Theta, \Phi, \Psi \in FCon(A)$. If $\Theta \wedge \Phi \leq \Psi \leq \theta$, then $FC(\Theta, \Phi; \Psi)$ holds.

Proof. Let $a, b, c, d \in A$. Clearly,

$$\Phi(a, b) \geq M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

and by our assumption, $\Theta(a, b) \geq \Psi(a, b)$, together imply

$$(\Theta \wedge \Phi)(a, b) \wedge M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \geq \\ \Psi(a, b) \wedge M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

By the above lemma, $FC(\Theta, \Phi; \Theta \wedge \Phi)$ holds and using this, we got the following:

$$\psi(c, d) \geq (\Theta \wedge \Phi)(c, d) \\ \geq (\Theta \wedge \Phi)(a, b) \wedge M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \\ \geq \Psi(a, b) \wedge M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

Therefore $FC(\Theta, \Phi; \Psi)$ holds. $\square$

Lemma 3.10. If $FC(\Theta, \Phi; \Psi_i)$ holds for each $i \in I$, then $FC(\Theta, \Phi; \bigwedge_{i \in I} \Psi_i)$ holds.

Proof. Suppose that $FC(\Theta, \Phi; \Psi_i)$ holds for each $i \in I$. For each $i, j \in \{1, 2\}$ and each $u_{ij} \in A$, consider the following:

$$M_L(\Theta, \Phi) ([u_{ij}]) \wedge (\bigwedge_{i \in I} \Psi_i (u_{11}, u_{12})) \\ = \bigwedge_{i \in I} (M_L(\Theta, \Phi) ([u_{ij}]) \wedge \Psi_i (u_{11}, u_{12})) \\ = \bigwedge_{i \in I} (M_L(\Theta, \Phi) ([u_{ij}]) \wedge \Psi_i (u_{21}, u_{22})) \\ = M_L(\Theta, \Phi) ([u_{ij}]) \wedge (\bigwedge_{i \in I} \Psi_i (u_{21}, u_{22}))$$

Therefore $FC(\Theta, \Phi; \bigwedge_{i \in I} \Psi_i)$ holds. $\square$

Lemma 3.11. If $FC(\Theta_i, \Phi; \Psi)$ holds for each $i \in I$, then $FC(\bigvee_{i \in I} \Theta_i, \Phi; \Psi)$ holds.

Proof. Suppose that $FC(\Theta_i, \Phi; \Psi)$ holds for all $i \in I$. Put $\Gamma = \bigvee_{i \in I} \Theta_i$. Then for any $a, b, c, d \in A$

$$M_L(\Gamma, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \bigvee \left\{ \left[\bigwedge_{i=1}^n \Gamma(a_i, b_i) \right] \wedge \left[\bigwedge_{j=1}^m \Phi(c_j, d_j) \right] : \right. \\ \left. \begin{array}{l} a = t(\mathbf{a}, \mathbf{c}), b = t(\mathbf{a}, \mathbf{d}) \\ c = t(\mathbf{b}, \mathbf{c}), d = t(\mathbf{b}, \mathbf{d}) \text{ where } t \in T_{n+m} \end{array} \right\}$$

Let $\mathbf{a}, \mathbf{b} \in A^n, \mathbf{c}, \mathbf{d} \in A^m$ and $t(\mathbf{x}, \mathbf{y})$ be an $(m + n)$ -ary term operation A such that $a = t(\mathbf{a}, \mathbf{c}), b = t(\mathbf{a}, \mathbf{d}), c = t(\mathbf{b}, \mathbf{c}), d = t(\mathbf{b}, \mathbf{d})$.

Claim. $\Psi(c, d) \geq \Psi(a, b) \wedge \left[\bigwedge_{i=1}^n \Gamma(a_i, b_i) \right] \wedge \left[\bigwedge_{j=1}^m \Phi(c_j, d_j) \right]$

Remember also that for each $x, y \in A$:

$$\Gamma(x, y) = \bigvee \left\{ \bigwedge_{j=1}^n \Theta_{ij}(x_{j-1}, x_j) : n \in \mathbb{Z}^+, x_0, x_1, \dots, x_n \in A, x = x_0, y = x_n \right\}$$

For each $1 \leq j \leq n$, let $u_{j0}, u_{j1}, \dots, u_{jm} \in A$ such that $a_j = u_{j0}$ and $b_j = u_{jm}$. Let $i_1, \dots, i_l \in I$ be arbitrary. What we need to show is that

$$\Psi(c, d) \geq \Psi(a, b) \wedge \left[\bigwedge_{j=1}^n \left[\bigwedge_{k=0}^{l-1} \Theta_{i_{k+1}}(u_{j_k}, u_{j_{k+1}}) \right] \right] \\ \wedge \left[\bigwedge_{j=1}^m \Phi(c_j, d_j) \right]$$

For each $k = 0, 1, \dots, l$, let us define vectors $\mathbf{u}_k$ as $\mathbf{u}_k = (u_{1k}, u_{2k}, \dots, u_{nk})$. Then it is clear that $\mathbf{u}_0 = (a_1, a_2, \dots, a_n) = \mathbf{a}$ and $\mathbf{u}_l = (b_1, b_2, \dots, b_n) = \mathbf{b}$. For each $k = 0, 1, \dots, l$ we show that

$$\Psi(t(\mathbf{u}_k, \mathbf{c}), t(\mathbf{u}_k, \mathbf{d})) \geq \Psi(a, b) \wedge \left[\bigwedge_{j=1}^n \left[\bigwedge_{k=0}^{l-1} \Theta_{i_{k+1}}(u_{j_k}, u_{j_{k+1}}) \right] \right] \\ \wedge \left[\bigwedge_{j=1}^m \Phi(c_j, d_j) \right]$$

Here we use induction on k . If $k = 0$, then $\mathbf{u}_k = \mathbf{a}$ and hence the result holds trivially. Let $k > 0$ and assume the result to be true for all $k < l - 1$. Using the fact $FC(\Theta_i, \Phi; \Psi)$ holds for all $i \in I$ and the induction hypothesis we get the following:

$$\Psi(t(\mathbf{u}_{k+1}, \mathbf{c}), t(\mathbf{u}_{k+1}, \mathbf{d})) \\ \geq \Psi(t(\mathbf{u}_k, \mathbf{c}), t(\mathbf{u}_k, \mathbf{d})) \wedge \\ M_L(\Theta_{i_{k+1}}, \Phi) \left(\begin{bmatrix} t(\mathbf{u}_k, \mathbf{c}) & t(\mathbf{u}_k, \mathbf{d}) \\ t(\mathbf{u}_{k+1}, \mathbf{c}) & t(\mathbf{u}_{k+1}, \mathbf{d}) \end{bmatrix} \right) \\ \geq \Psi(t(\mathbf{u}_k, \mathbf{c}), t(\mathbf{u}_k, \mathbf{d})) \wedge \left[\bigwedge_{j=1}^n \Theta_{i_{k+1}}(u_{j_k}, u_{j_{k+1}}) \right] \\ \wedge \left[\bigwedge_{j=1}^m \Phi(c_j, d_j) \right] \\ \geq \Psi(a, b) \wedge \left[\bigwedge_{j=1}^n \left[\bigwedge_{k=0}^{l-1} \Theta_{i_{k+1}}(u_{j_k}, u_{j_{k+1}}) \right] \right] \wedge \left[\bigwedge_{j=1}^m \Phi(c_j, d_j) \right]$$

In particular if $k = l$, then $k = l$, then $\mathbf{u}_k = \mathbf{b}$ and hence $t(\mathbf{u}_k, \mathbf{c}) = c$ and $t(\mathbf{u}_k, \mathbf{d}) = d$. Thus

$$\Psi(c, d) \geq \Psi(a, b) \wedge \left[\bigwedge_{j=1}^n \left[\bigwedge_{k=0}^{l-1} \Theta_{i_{k+1}}(u_{j_k}, u_{j_{k+1}}) \right] \right] \\ \wedge \left[\bigwedge_{j=1}^m \Phi(c_j, d_j) \right]$$

Since each u_{j_k} is arbitrary with $u_{j_0} = a_j$ and $u_{j_l} = b_j$, it follows that

$$\Psi(c, d) \geq \Psi(a, b) \wedge \left[\bigwedge_{i=1}^n \Gamma(a_i, b_i) \right] \wedge \left[\bigwedge_{j=1}^m \Phi(c_j, d_j) \right]$$

Hence proved. $\square$

Definition 3.12. Let $\Theta, \Phi \in FCon(A)$. The commutator $[\Theta, \Phi]$ of Θ and Φ is defined to be the smallest fuzzy congruence Ψ on A for which $FC(\Theta, \Phi; \Psi)$ holds. The symmetric commutator $[\Theta, \Phi]_s$ is defined as the smallest fuzzy congruence Ψ on A for which both $FC(\Theta, \Phi; \Psi)$ and $FC(\Phi, \Theta; \Psi)$ hold.

Lemma 3.13. The following conditions hold for all $\Theta, \Phi \in FCon(A)$:

- (1) $[\Theta, \Phi]_s = [\Phi, \Theta]_s$
- (2) $[\Theta, \Phi] \leq [\Theta, \Phi]_s \leq \Theta \wedge \Phi$
- (3) $[\Theta, \Phi]$ and $[\Theta, \Phi]_s$ are monotone in both Θ and Φ .

Theorem 3.14. Let $\Theta, \Phi \in FCon(A)$. Then

$$[\Theta, \Phi](x, y) = \bigvee \{ \alpha \in L : (x, y) \in [\Theta_\alpha, \Phi_\alpha] \}$$

Proof. For each $x, y \in A$, let

$$\Psi(x, y) = \bigvee \{ \alpha \in L : (x, y) \in [\Theta_\alpha, \Phi_\alpha] \}$$

We first show that Ψ is a fuzzy congruence relation on A . Clearly, it is reflexive and symmetric. We show that it is transitive. Let $x, y, z \in A$.

$$\Psi(x, y) \wedge \Psi(y, z) = \bigvee \{ \alpha \in L : (x, y) \in [\Theta_\alpha, \Phi_\alpha] \} \\ \wedge \bigvee \{ \beta \in L : (y, z) \in [\Theta_\beta, \Phi_\beta] \} \\ = \bigvee \left\{ \alpha \wedge \beta : \alpha, \beta \in L, (x, y) \in [\Theta_\alpha, \Phi_\alpha], (y, z) \in [\Theta_\beta, \Phi_\beta] \right\}$$

Let $\alpha, \beta \in L$ such that $(x, y) \in [\Theta_\alpha, \Phi_\alpha]$ and $(y, z) \in [\Theta_\beta, \Phi_\beta]$. If we put $\lambda = \alpha \wedge \beta$, then $\lambda \leq \alpha, \beta$ which gives $\Theta_\alpha \cup \Theta_\beta \subseteq \Theta_\lambda$ and $\Phi_\alpha \cup \Phi_\beta \subseteq \Phi_\lambda$. Since the commutator of congruences is monotone, we get

$$[\Theta_\alpha, \Phi_\alpha] \subseteq [\Theta_\lambda, \Phi_\lambda] \text{ and } [\Theta_\beta, \Phi_\beta] \subseteq [\Theta_\lambda, \Phi_\lambda]$$

So that $(x, y), (y, z) \in [\Theta_\lambda, \Phi_\lambda]$. Using the transitive property of the congruence $[\Theta_\lambda, \Phi_\lambda]$, it holds that $(x, z) \in [\Theta_\lambda, \Phi_\lambda]$. So we have

the following:

$$\begin{aligned}\Psi(x, y) \wedge \Psi(y, z) &= \bigvee \left\{ \alpha \wedge \beta : \alpha, \beta \in L, (x, y) \in [\Theta_\alpha, \Phi_\alpha], (y, z) \in [\Theta_\beta, \Phi_\beta] \right\} \\ &\leq \bigvee \left\{ \lambda \in L : (x, z) \in [\Theta_\lambda, \Phi_\lambda] \right\} \\ &= \Psi(x, z)\end{aligned}$$

Thus Ψ is transitive and hence a fuzzy equivalence relation on A . Let $f \in \mathfrak{F}$ be n -ary, $n > 0$ and $x_1, \dots, x_n, y_1, \dots, y_n \in A$.

$$\begin{aligned}\Psi(x_1, y_1) \wedge \dots \wedge \Psi(x_n, y_n) &= \bigwedge_{i=1}^n \bigvee \left\{ \alpha_i \in L : (x_i, y_i) \in [\Theta_{\alpha_i}, \Phi_{\alpha_i}] \right\} \\ &= \bigvee \left\{ \bigwedge_{i=1}^n \alpha_i : \alpha_i \in L, (x_i, y_i) \in [\Theta_{\alpha_i}, \Phi_{\alpha_i}] \right\}\end{aligned}$$

Let $\alpha_1, \dots, \alpha_n$ be a sequence in $(x_i, y_i) \in [\Theta_{\alpha_i}, \Phi_{\alpha_i}]$ for each i . If we put $\lambda = \bigwedge_{i=1}^n \alpha_i$ then we get that

$$[\Theta_{\alpha_i}, \Phi_{\alpha_i}] \subseteq [\Theta_\lambda, \Phi_\lambda]$$

for all $i = 1, 2, \dots, n$. So that $(x_i, y_i) \in [\Theta_\lambda, \Phi_\lambda]$

for each i . Using the compatibility property of the congruence $[\Theta_\lambda, \Phi_\lambda]$, it follows that

$$(f^A(x_1, \dots, x_n), f^A(y_1, \dots, y_n)) \in [\Theta_\lambda, \Phi_\lambda]$$

So we have the following:

$$\begin{aligned}\Psi(x_1, y_1) \wedge \dots \wedge \Psi(x_n, y_n) &= \bigvee \left\{ \bigwedge_{i=1}^n \alpha_i : \alpha_i \in L, (x_i, y_i) \in [\Theta_{\alpha_i}, \Phi_{\alpha_i}] \right\} \\ &\leq \bigvee \left\{ \lambda \in L : (f^A(x_1, \dots, x_n), f^A(y_1, \dots, y_n)) \in [\Theta_\lambda, \Phi_\lambda] \right\} \\ &= \Psi(f^A(x_1, \dots, x_n), f^A(y_1, \dots, y_n))\end{aligned}$$

Therefore Ψ is a fuzzy congruence on A . Next we show that $FC(\Theta, \Phi; \Psi)$ holds. For this it is enough to show that

$$\Psi(a, b) \geq \Psi(c, d) \wedge M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

for all $a, b, c, d \in A$.

$$\begin{aligned}\Psi(c, d) \wedge M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) &= \bigvee \left\{ \alpha \in L : (c, d) \in [\Theta_\alpha, \Phi_\alpha] \right\} \wedge \bigvee \left\{ \beta \in L : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(\Theta_\beta, \Phi_\beta) \right\} \\ &= \bigvee \left\{ \alpha \wedge \beta : (c, d) \in [\Theta_\alpha, \Phi_\alpha] \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(\Theta_\beta, \Phi_\beta) \right\}\end{aligned}$$

for any $\alpha, \beta \in L$ with $(c, d) \in [\Theta_\alpha, \Phi_\alpha]$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(\Theta_\beta, \Phi_\beta)$, if we put $\lambda = \alpha \wedge \beta$, then $\lambda \in L$ such that $(c, d) \in [\Theta_\lambda, \Phi_\lambda]$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(\Theta_\lambda, \Phi_\lambda)$. Since Plot the current gain centralizes Φ_λ modulo $[\Theta_\lambda, \Phi_\lambda]$, $(a, b) \in [\Theta_\lambda, \Phi_\lambda]$. The following follows from the above inequality:

$$\begin{aligned}\Psi(c, d) \wedge M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) &= \bigvee \left\{ \alpha \wedge \beta : (c, d) \in [\Theta_\alpha, \Phi_\alpha] \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(\Theta_\beta, \Phi_\beta) \right\} \\ &\leq \bigvee \left\{ \lambda \in L : (a, b) \in [\Theta_\lambda, \Phi_\lambda] \right\} \\ &= \Psi(a, b)\end{aligned}$$

Therefore $FC(\Theta, \Phi; \Psi)$ holds. Let Γ be any other fuzzy congruence on A for which $FC(\Theta, \Phi; \Gamma)$ holds. By Theorem 4.6 $C(\Theta_\alpha, \Phi_\alpha; \Gamma_\alpha)$ holds for all $\alpha \in L$. Since $[\Theta_\alpha, \Phi_\alpha]$ is the smallest congruence ρ on A for which Θ_α centralizes Φ_α modulo ρ , we get $[\Theta_\alpha, \Phi_\alpha] \leq \Gamma_\alpha$ for all $\alpha \in L$. Now for each $a, b \in L$ consider the following:

$$\begin{aligned}\Psi(a, b) &= \bigvee \left\{ \alpha \in L : (a, b) \in [\Theta_\alpha, \Phi_\alpha] \right\} \\ &\leq \bigvee \left\{ \alpha \in L : (a, b) \in \Gamma_\alpha \right\} \\ &= \Gamma(a, b)\end{aligned}$$

This completes the proof. $\square$

Corollary 3.15. For each $\alpha \in L$ and $\Theta, \Phi \in FCon(A)$

$$[\Theta, \Phi]_\alpha = \bigcup \left\{ \bigcap_{\beta \in H} [\Theta_\beta, \Phi_\beta] : H \subseteq L, \alpha \leq \sup H \right\}$$

4. THE COMMUTATOR IN MODULAR VARIETIES

In this section we give a detailed characterization for the commutator of fuzzy congruences in modular varieties. It is proved that the commutator and the symmetric commutator of fuzzy congruences defined in the previous section are identical in modular varieties.

Remember that A is said to be modular if the lattice $Con(A)$ is modular, and $\mathcal{V}$ is modular if all algebras in $\mathcal{V}$ are so. The following theorem gives internal characterization for modular varieties and it is taken from [58].

Theorem 4.1. A variety $\mathcal{V}$ is modular if and only if for some n there are terms $m_0(x, y, z, u), \dots, m_n(x, y, z, u)$ such that $\mathcal{V}$ satisfies

- (1) $m_0(x, y, z, u) \approx x, m_n(x, y, z, u) \approx u$
- (2) $m_i(x, y, y, x) \approx x$ for all $i \leq n$
- (3) $m_i(x, x, y, y) \approx m_{i+1}(x, x, y, y)$ for all even $i < n$
- (4) $m_i(x, y, y, z) \approx m_{i+1}(x, y, y, z)$ for all odd $i < n$

The terms $m_0(x, y, z, u), \dots, m_n(x, y, z, u)$ are called Day's terms.

Lemma 4.2. Let $\mathcal{V}$ be a modular variety with Day terms $m_0, \dots, m_n$ and $A \in \mathcal{V}$. Let $\Theta \in FCon(A)$. Then for any $a, b, c, d \in A$ it holds

that

$$\Theta(a, c) \wedge \Theta(b, d) = \Theta(b, d) \wedge \left[\bigwedge_{i=1}^n \Theta(m_i(a, a, c, c), m_i(a, b, d, c)) \right]$$

Proof. For each consider $i \in \{0, 1, \dots, n\}$ the following:

$$\begin{aligned} & \Theta(m_i(a, a, c, c), m_i(a, b, d, c)) \\ & \geq \Theta(m_i(a, a, c, c), m_i(a, a, a, a)) \wedge \Theta(m_i(a, a, a, a), m_i(a, b, d, c)) \\ & \geq \Theta(a, c) \wedge \Theta(m_i(a, a, a, a), m_i(a, b, d, c)) \\ & = \Theta(a, c) \wedge \Theta(a, m_i(a, b, d, c)) \\ & = \Theta(a, c) \wedge \Theta(m_i(a, b, b, a), m_i(a, b, d, c)) \\ & \geq \Theta(a, c) \wedge \Theta(b, d) \end{aligned}$$

which implies that

$$\bigwedge_{i=1}^n \Theta(m_i(a, a, c, c), m_i(a, b, d, c)) \geq \Theta(a, c) \wedge \Theta(b, d)$$

Computing $\Theta(b, d)$ on both side of this inequality with the binary operation “ $\wedge$ ” provides that

$$\Theta(b, d) \wedge \left[\bigwedge_{i=1}^n \Theta(m_i(a, a, c, c), m_i(a, b, d, c)) \right] \geq \Theta(a, c) \wedge \Theta(b, d)$$

To prove the other side of the inequality, we first show that

$$\begin{aligned} & \Theta(b, d) \wedge \Theta(m_i(a, a, c, c), a) \geq \Theta(b, d) \\ & \wedge \left[\bigwedge_{j=1}^n \Theta(m_j(a, a, c, c), m_j(a, b, d, c)) \right] \end{aligned}$$

for all $i \in \{0, 1, \dots, n\}$. We use induction on i . If $i = 0$, then it is straight forward. Now Assume the result to be true for all $0 \leq i < n$. We need to show that the result holds for $i + 1$.

Case 1. i is odd

$$\begin{aligned} \Theta(m_{i+1}(a, b, d, c), a) & \geq \Theta(m_{i+1}(a, b, d, c), m_{i+1}(a, b, b, c)) \\ & \quad \wedge \Theta(m_{i+1}(a, b, b, c), a) \\ & \geq \Theta(b, d) \wedge \Theta(m_{i+1}(a, b, b, c), a) \\ & = \Theta(b, d) \wedge \Theta(m_i(a, b, b, c), a) \\ & \geq \Theta(b, d) \wedge \Theta(m_i(a, b, b, c), m_i(a, b, d, c)) \\ & \quad \wedge \Theta(m_i(a, b, b, c), a) \\ & \geq \Theta(b, d) \wedge \Theta(m_i(a, b, b, c), a) \\ & \geq \Theta(b, d) \wedge \left[\bigwedge_{j=1}^n \Theta(m_j(a, b, d, c), m_j(a, a, c, c)) \right] \end{aligned}$$

Then it follows that

$$\begin{aligned} & \Theta(b, d) \wedge \Theta(m_{i+1}(a, b, d, c), a) \geq \Theta(b, d) \\ & \wedge \left[\bigwedge_{j=1}^n \Theta(m_j(a, b, d, c), m_j(a, a, c, c)) \right] \end{aligned}$$

Case 2. i is even. For simplicity, let us put $\alpha = \Theta(m_{i+1}(a, b, d, c), m_{i+1}(a, a, c, c))$ and $\beta = \Theta(m_i(a, a, c, c), m_i(a, b, d, c))$

Now consider the following:

$$\begin{aligned} & \Theta(b, d) \wedge \Theta(m_{i+1}(a, b, d, c), a) \\ & \geq \Theta(b, d) \wedge \Theta(m_{i+1}(a, b, d, c), m_{i+1}(a, a, c, c)) \\ & \quad \wedge \Theta(m_{i+1}(a, a, c, c), a) \\ & = \Theta(b, d) \wedge \alpha \wedge \Theta(m_{i+1}(a, a, c, c), a) \\ & = \Theta(b, d) \wedge \alpha \wedge \Theta(m_i(a, a, c, c), a) \\ & \geq \Theta(b, d) \wedge \alpha \wedge \Theta(m_i(a, a, c, c), m_i(a, b, d, c)) \\ & \quad \wedge \Theta(m_i(a, b, d, c), a) \\ & = \Theta(b, d) \wedge \alpha \wedge \beta \wedge \Theta(m_i(a, b, d, c), a) \\ & \geq \Theta(b, d) \wedge \alpha \wedge \beta \wedge \left[\bigwedge_{j=1}^n \Theta(m_j(a, b, d, c), m_j(a, a, c, c)) \right] \\ & = \Theta(b, d) \wedge \left[\bigwedge_{j=1}^n \Theta(m_j(a, b, d, c), m_j(a, a, c, c)) \right] \end{aligned}$$

Thus the result holds for all $i \in \{0, 1, \dots, n\}$. In particular, it works for $i = n$, i.e.,

$$\begin{aligned} & \Theta(b, d) \wedge \Theta(m_n(a, b, d, c), a) \geq \Theta(b, d) \\ & \quad \wedge \left[\bigwedge_{j=1}^n \Theta(m_j(a, b, d, c), m_j(a, a, c, c)) \right] \end{aligned}$$

which is equivalent to that

$$\Theta(b, d) \wedge \Theta(a, c) \geq \Theta(b, d) \wedge \left[\bigwedge_{j=1}^n \Theta(m_j(a, b, d, c), m_j(a, a, c, c)) \right]$$

This completes the proof. $\square$

In the following lemma, we state and prove the fuzzy version of the Shifting Lemma proved by P.H. Gumm [59].

Lemma 4.3. (Shifting Lemma: The Fuzzy Version) Let $\mathcal{V}$ be a modular variety with Day terms $m_0, \dots, m_n$ and $A \in \mathcal{V}$. Let $\Theta, \Phi, \Psi \in FCon(A)$ with $\Theta \wedge \Phi \leq \Psi$. Then for any $a, b, c, d \in A$ it holds that

$$\begin{aligned} & \Theta(a, b) \wedge \Theta(c, d) \wedge \Phi(a, c) \wedge \Phi(b, d) \wedge \Psi(a, c) = \Theta(a, b) \\ & \quad \wedge \Theta(c, d) \wedge \Phi(a, c) \wedge \Phi(b, d) \wedge \Psi(b, d) \end{aligned}$$

Proof. We first show that

$$\Psi(a, c) \geq \Theta(a, b) \wedge \Theta(c, d) \wedge \Phi(a, c) \wedge \Phi(b, d) \wedge \Psi(b, d)$$

For each $i \in \{0, 1, \dots, n\}$, it is clear that

$$\Theta(m_i(a, a, c, c), m_i(a, b, d, c)) \geq \Theta(a, b) \wedge \Theta(c, d)$$

Also, by the above lemma one can show that

$$\Phi(m_i(a, a, c, c), m_i(a, b, d, c)) \geq \Phi(a, c) \wedge \Phi(b, d)$$

Since $\Theta \wedge \phi \leq \Psi$, it follows that

$$\Psi(m_i(a, a, c, c), m_i(a, b, d, c)) \geq \Theta(a, b) \wedge \Theta(c, d) \wedge \Phi(a, c) \wedge \Phi(b, d)$$

which implies that

$$\bigwedge_{i=0}^n \Psi(m_i(a, a, c, c), m_i(a, b, d, c)) \geq \Theta(a, b) \wedge \Theta(c, d) \wedge \Phi(a, c) \wedge \Phi(b, d)$$

Computing $\Psi(b, d)$ on both side of this inequality using the binary operation $\bigwedge$ gives the following:

$$\Psi(b, d) \wedge \left[\bigwedge_{i=0}^n \Psi(m_i(a, a, c, c), m_i(a, b, d, c)) \right] \geq \Theta(a, b) \wedge \Theta(c, d) \wedge \Phi(a, c) \wedge \Phi(b, d) \wedge \Psi(b, d)$$

Again from the above lemma, we have

$$\Psi(a, c) \geq \Psi(b, d) \wedge \left[\bigwedge_{i=0}^n \Psi(m_i(a, a, c, c), m_i(a, b, d, c)) \right]$$

Thus

$$\Psi(a, c) \geq \Theta(a, b) \wedge \Theta(c, d) \wedge \Phi(a, c) \wedge \Phi(b, d) \wedge \Psi(b, d)$$

By interchanging a and b (respectively c and d) each other we can also show that

$$\Psi(b, d) \geq \Theta(a, b) \wedge \Theta(c, d) \wedge \Phi(a, c) \wedge \Phi(b, d) \wedge \Psi(a, c)$$

This completes the proof. $\square$

Definition 4.4. For $\Theta, \Phi \in FCon(A)$, $X_L(\Theta, \Phi)$ be a fuzzy relation on A defined as follows: For each $y, z \in A$, if y and z can be expressed as $y = m_i(a, b, c, d)$ and $z = m_i(a, a, c, c)$, then

$$X_L(\Theta, \Phi)(y, z) = \bigvee \left\{ M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) : y = m_i(a, b, d, c), z = m_i(a, a, c, c) \right\}$$

Otherwise, $X_L(\Theta, \Phi)(y, z) = 0$.

Theorem 4.5. Let $\Theta, \Phi \in FCon(A)$. Then

$$X_L(\Theta, \Phi)(y, z) = \bigvee \{ \alpha \in L : (y, z) \in X(\Theta_\alpha, \Phi_\alpha) \}.$$

Proof. If there are no $a, b, c, d \in A$ such that $y = m_i(a, b, d, c)$ and $z = m_i(a, a, c, c)$ then $(y, z) \notin X(\Theta_\alpha, \Phi_\alpha)$ for all $\alpha \in L$, so that

$$\bigvee \{ \alpha \in L : (y, z) \in X(\Theta_\alpha, \Phi_\alpha) \} = \bigvee \emptyset = 0 = X_L(\Theta, \Phi)(y, z)$$

Let us define two sets G and H as follows:

$$G = \{ \alpha \in L : (y, z) \in X(\Theta_\alpha, \Phi_\alpha) \},$$

$$H = \left\{ M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) : y = m_i(a, b, d, c), z = m_i(a, a, c, c) \right\}$$

We need to show that $\sup G = \sup H$. Let $\alpha \in G$. Then $(y, z) \in X(\Theta_\alpha, \Phi_\alpha)$, which implies that, $y = m_i(a, b, d, c)$ and $z = m_i(a, a, c, c)$ for some $a, b, c, d \in A$ with $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(\Theta_\alpha, \Phi_\alpha)$. By Theorem 4.1 we get

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \geq \alpha$$

If we put $\beta = M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$, then $\beta \in H$ such that $\alpha \leq \beta$ so that $\alpha \leq \sup H$. Since α is arbitrary in G it follows that $\sup G \leq \sup H$. To prove the other side of the inequality, let $\beta \in H$. Then $\beta = M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$ for some $a, b, c, d \in A$ with $y = m_i(a, b, d, c)$ and $z = m_i(a, a, c, c)$, i.e., $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_L(\Theta, \Phi)_\beta$. By Theorem 4.2, there exists $K \subseteq L$ such that $\beta \leq \sup K$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(\Theta_\alpha, \Phi_\alpha)$ for all $\alpha \in K$. This is equivalent to that $(y, z) \in X(\Theta_\alpha, \Phi_\alpha)$ for all $\alpha \in K$. Now consider the following:

$$\begin{aligned} \sup G &= \sup \{ \alpha \in L : (y, z) \in X(\Theta_\alpha, \Phi_\alpha) \} \\ &\geq \sup \{ \alpha \in L : \alpha \in K \} \\ &\geq \beta \end{aligned}$$

Since β is arbitrary in $\sup G \geq \sup H$ and hence the equality holds.

Theorem 4.6. Let $\Theta, \Phi, \Psi \in FCon(A)$, $A \in \mathcal{V}$ and $\mathcal{V}$ be modular. Then the following are equivalent:

1. $FC(\Theta, \Phi; \Psi)$ holds.
2. $X(\Theta, \Phi) \leq \Psi$
3. $FC(\Phi, \Theta; \Psi)$ holds.
4. $X(\Phi, \Theta) \leq \Psi$

Proof. (1) $\Rightarrow$ (2). Suppose that $FC(\Theta, \Phi; \Psi)$ holds. Let $y, z \in A$. If there are no $a, b, c, d \in A$ such that

$$y = m_i(a, a, c, c) \text{ and } z = m_i(a, b, d, c)$$

then $X_L(\Theta, \Phi)(y, z) = 0 \leq \Psi(y, z)$. Suppose that

$$y = m_i(a, a, c, c) \text{ (respectively } z = m_i(a, b, d, c))$$

is an arbitrary expression of y (respectively z) using the Day's terms. Let $\mathbf{a}, \mathbf{b} \in A^n$, $\mathbf{c}, \mathbf{d} \in A^m$ and $t(x, y)$ be an $(n + m)$ -ary term operation on A such that

$$\begin{aligned} a &= t(\mathbf{a}, \mathbf{c}) & b &= t(\mathbf{a}, \mathbf{d}) \\ c &= t(\mathbf{b}, \mathbf{c}) & d &= t(\mathbf{b}, \mathbf{d}) \end{aligned}$$

If we define a term s as follows:

$$\begin{aligned} s(x^1, x^2, y^1, y^2, y^3, y^4, y^5, y^6) &= m_i \left(t(y^1, y^2), t(x^1, y^3), \right. \\ &\quad \left. t(y^4, y^5), t(x^2, y^6) \right) \end{aligned}$$

Then we have

$$\begin{aligned} y &= m_i(a, a, c, c) \\ &= m_i(t(\mathbf{a}, \mathbf{c}), t(\mathbf{a}, \mathbf{c}), t(\mathbf{b}, \mathbf{c}), t(\mathbf{b}, \mathbf{c})) \\ &= s(\mathbf{a}, \mathbf{b}, \mathbf{a}, \mathbf{c}, \mathbf{c}, \mathbf{b}, \mathbf{c}, \mathbf{c}) \end{aligned}$$

Similarly

$$z = s(\mathbf{a}, \mathbf{b}, \mathbf{a}, \mathbf{c}, \mathbf{d}, \mathbf{b}, \mathbf{d}, \mathbf{c})$$

Again from

$$a = m_i(a, c, c, a)$$

we get

$$a = s(\mathbf{a}, \mathbf{b}, \mathbf{a}, \mathbf{c}, \mathbf{c}, \mathbf{b}, \mathbf{c}, \mathbf{c})$$

Thus it can be verified that

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} a & a \\ y & z \end{bmatrix} \right) \geq \left[\bigwedge_{i=1}^n \Theta(a_i, b_i) \right] \bigwedge \left[\bigwedge_{j=1}^n \Phi(c_j, d_j) \right]$$

Since $\mathbf{a}, \mathbf{b} \in A^n$, $\mathbf{c}, \mathbf{d} \in A^m$ and the term $t(\mathbf{x}, \mathbf{y})$ are arbitrary, it follows that

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} a & a \\ y & z \end{bmatrix} \right) \geq M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

Again since a, b, c and d are arbitrary elements of A with

$$y = m_i(a, a, c, c) \text{ (respectively } z = m_i(a, b, d, c) \text{)}$$

it holds that

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} a & a \\ y & z \end{bmatrix} \right) \geq X_L(\Theta, \Phi)(y, z)$$

Now consider the following assertion:

$$\begin{aligned} \Psi(y, z) &\geq \Psi(y, z) \bigwedge M_L(\Theta, \Phi) \left(\begin{bmatrix} a & a \\ y & z \end{bmatrix} \right) \\ &= \Psi(a, a) \bigwedge M_L(\Theta, \Phi) \left(\begin{bmatrix} a & a \\ y & z \end{bmatrix} \right) \\ &= M_L(\Theta, \Phi) \left(\begin{bmatrix} a & a \\ y & z \end{bmatrix} \right) \\ &\geq X_L(\Theta, \Phi)(y, z) \end{aligned}$$

Therefore $X_L(\Theta, \Phi) \leq \Psi$ and hence the result holds.

(2) $\Rightarrow$ (3). Suppose that $X_L(\Theta, \Phi) \leq \Psi$. For any $a, b, c, d \in A$ we show that

$$\Psi(a, c) \geq \Psi(b, d) \bigwedge M_L(\Phi, \Theta) \left(\begin{bmatrix} b & d \\ a & c \end{bmatrix} \right)$$

For this consider the following:

$$\begin{aligned} M_L(\Phi, \Theta) \left(\begin{bmatrix} b & d \\ a & c \end{bmatrix} \right) &= M_L(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \\ &\leq X_L(\Theta, \Phi) (m_i(a, a, c, c), m_i(a, b, d, c)) \\ &\leq \Psi(m_i(a, a, c, c), m_i(a, b, d, c)) \end{aligned}$$

Since i is arbitrary, it follows that

$$M_L(\Phi, \Theta) \left(\begin{bmatrix} b & d \\ a & c \end{bmatrix} \right) \leq \bigwedge_{i=0}^n \Psi(m_i(a, a, c, c), m_i(a, b, d, c))$$

computing $\Psi(a, c)$ on both side of the above inequality using the binary operation " $\bigwedge$ " provides

$$\begin{aligned} \Psi(b, d) \bigwedge M_L(\Phi, \Theta) \left(\begin{bmatrix} b & d \\ a & c \end{bmatrix} \right) \\ \leq \Psi(b, d) \bigwedge \left[\bigwedge_{i=0}^n \Psi(m_i(a, a, c, c), m_i(a, b, d, c)) \right] \\ \leq \Psi(a, c) \quad (\text{by Lemma 5.2}). \end{aligned}$$

By symmetry, it can also be proved that

$$\Psi(a, c) \bigwedge M_L(\Phi, \Theta) \left(\begin{bmatrix} b & d \\ a & c \end{bmatrix} \right) \leq \Psi(b, d)$$

which implies

$$\Psi(a, c) \bigwedge M_L(\Phi, \Theta) \left(\begin{bmatrix} b & d \\ a & c \end{bmatrix} \right) = \Psi(b, d) \bigwedge M_L(\Phi, \Theta) \left(\begin{bmatrix} b & d \\ a & c \end{bmatrix} \right).$$

Therefore $FC(\Phi, \Theta; \Psi)$ holds. $\square$

Corollary 4.7. Let $\Theta, \Phi \in FCon(A)$, $A \in \mathcal{V}$ and $\mathcal{V}$ be modular. Then

$$[\Theta, \Phi] = [\Phi, \Theta] = [\Theta, \Phi]_s$$

Theorem 4.8. Let $\Theta, \Phi, \Psi \in FCon(A)$, $A \in \mathcal{V}$ and $\mathcal{V}$ be modular. Then $FC(\Theta, \Phi; \Psi)$ holds if and only if $[\Theta, \Phi] \leq \Psi$.

Proof. If $FC(\Theta, \Phi; \Psi)$ holds, then $[\Theta, \Phi]$ is the smallest fuzzy congruence Γ on A such that $FC(\Theta, \Phi; \Gamma)$ holds, we get $[\Theta, \Phi] \leq \Psi$. Conversely, suppose that $[\Theta, \Phi] \leq \Psi$. If we put $\Gamma = [\Theta, \Phi]$, then $FC(\Theta, \Phi; \Gamma)$ holds and by the above theorem, $X_L(\Theta, \Phi) \leq \Gamma \leq \Psi$. Again by the equivalency in the above theorem, we get $FC(\Phi, \Theta; \Psi)$ holds. $\square$

Corollary 4.9. Let $\Theta, \Phi \in FCon(A)$, $A \in \mathcal{V}$ and $\mathcal{V}$ be modular. Then

$$[\Theta, \Phi] = FS_g(X_L(\Theta, \Phi))$$

where by $FS_g(X_L(\Theta, \Phi))$, we mean a fuzzy subalgebra of $A \times A$ generated by the fuzzy set $X_L(\Theta, \Phi)$.

Theorem 4.10. Let $\phi \in FCon(A)$, $A \in \mathcal{V}$ and $\mathcal{V}$ be modular. If $\{\theta_i\}_{i \in I}$ is an indexed family of fuzzy congruence relations on A , then

$$\left[\bigvee_{i \in I} \theta_i, \phi \right] = \bigvee_{i \in I} [\theta_i, \phi]$$

Proof. By monotonicity, it follows that

$$[\theta_i, \phi] \leq \left[\bigvee_{i \in I} \theta_i, \phi \right]$$

To prove the other side of the inequality let us $\psi_i = [\theta_i, \phi]$ for all $i \in I$ and $\psi = \bigvee_{i \in I} [\theta_i, \phi]$. By definition $FC(\theta_i, \phi; \psi_i)$ holds for all $i \in I$. It follows from Theorem 4.6 that

$$X_L(\theta_i, \phi) \leq \psi_i \leq \psi$$

Again by Theorem 4.6 it holds that $FC(\theta_i, \phi; \psi)$ for all $i \in I$. By Lemma 3.11, $FC\left(\bigvee_{i \in I} \theta_i, \phi; \psi\right)$ holds so that $\left[\bigvee_{i \in I} \theta_i, \phi\right] \leq \psi$. Hence proved. $\square$

5. CHARACTERIZING THE COMMUTATOR OF FUZZY CONGRUENCES IN THE SENSE OF HAGEMANN AND HERRMANN

In this section, we give another characterization for the commutator of fuzzy congruences in modular varieties in the sense of Hagemann and Herrmann [44]. For each $\Theta, \Phi \in FCon(A)$, let $A(\Theta, \Phi)$ be a fuzzy subset of $M(A)_{2 \times 2}$ given by

$$A(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \Theta(a, c) \wedge \Theta(b, d) \wedge \Phi(a, b) \wedge \Phi(c, d).$$

Let us define $\delta_{\Theta, \Phi}$ to be a fuzzy subset of $M(A)_{2 \times 2}$ such that

$$\delta_{\Theta, \Phi} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{cases} A(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) & \text{if } a = c \text{ and } b = d \\ 0 & \text{otherwise} \end{cases}$$

for all $a, b, c, d \in A$, i.e.,

$$\delta_{\Theta, \Phi} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{cases} \Phi(a, b) & \text{if } a = c \text{ and } b = d \\ 0 & \text{otherwise} \end{cases}$$

Put

$$\Delta_{\Theta, \Phi} = \bigcap \{ \Psi \in FCon(A \times A) : \delta_{\Theta, \Phi} \leq \Psi \leq \Theta \times \Theta \}$$

Also, let us define $\delta^{\Theta, \Phi}$ and $\Delta^{\Theta, \Phi}$ dual to $\delta_{\Theta, \Phi}$ and $\Delta_{\Theta, \Phi}$ respectively, i.e.,

$$\delta^{\Theta, \Phi} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{cases} A(\Theta, \Phi) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) & \text{if } a = b \text{ and } c = d \\ 0 & \text{otherwise} \end{cases}$$

for all $a, b, c, d \in A$, i.e.,

$$\delta^{\Theta, \Phi} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{cases} \Theta(a, c) & \text{if } a = b \text{ and } c = d \\ 0 & \text{otherwise} \end{cases}$$

Put

$$\Delta^{\Theta, \Phi} = \bigcap \{ \Psi \in FCo(A \times A) : \delta^{\Theta, \Phi} \leq \Psi \leq \Phi \times \Phi \}$$

Lemma 5.1. If $\mathcal{V}$ is modular, then $\Delta_{\Theta, \Phi}$ is the least transitive fuzzy relation on $A \times A$ such that

$$M_L(\Theta, \Phi) \leq \Delta_{\Theta, \Phi} \leq \Theta \times \Theta.$$

Proof. Clearly $M_L(\Theta, \Phi)$ is reflexive and symmetric as a fuzzy relation on $A \times A$ such that

$$M_L(\Theta, \Phi) \leq \Delta_{\Theta, \Phi} \leq \Theta \times \Theta.$$

Moreover, it follows from Theorem 3.5 that $M_L(\Theta, \Phi)$ is compatible with all fundamental operations of $A \times A$ so if Γ is the transitive closure of $M_L(\Theta, \Phi)$, then Γ is just the fuzzy congruence on $A \times A$ generated by $M_L(\Theta, \Phi)$. Clearly, $\delta_{\Theta, \Phi} \leq M_L(\Theta, \Phi)$. So that $\Delta_{\Theta, \Phi} \leq \Gamma$. On the other hand, it follows from the modularity of $\mathcal{V}$ that $M_L(\Theta, \Phi) \leq \Delta_{\Theta, \Phi}$ and hence $\Gamma \leq \Delta_{\Theta, \Phi}$. Hence proved. $\square$

Lemma 5.2. If $\mathcal{V}$ is modular, then $\Delta^{\Theta, \Phi} = \Delta_{\Theta, \Phi}$.

Proof. One can easily show that

$$\Delta^{\Theta, \Phi} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \Delta_{\Theta, \Phi} \left(\begin{bmatrix} a & c \\ b & d \end{bmatrix} \right)$$

for all $a, b, c, d \in A$. Since $\mathcal{V}$ is modular, we have $M_L(\Theta, \Phi) = M_L(\Phi, \Theta)$. Then it follows from Lemma 5.1 that $\Delta_{\Theta, \Phi} = \Delta^{\Theta, \Phi}$. $\square$

Theorem 5.3. Let $\mathcal{V}$ be modular. Then for each $x, y \in A$;

$$\Delta_{\Theta, \Phi} \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right) = \bigvee \left\{ \Delta_{\Theta, \Phi} \left(\begin{bmatrix} x & a \\ y & a \end{bmatrix} \right) : a \in A \right\}$$

Proof. The inequality

$$\Delta_{\Theta, \Phi} \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right) \bigvee \left\{ \Delta_{\Theta, \Phi} \left(\begin{bmatrix} x & a \\ y & a \end{bmatrix} \right) : a \in A \right\}$$

holds trivially. To prove the other side of the inequality, let $a \in A$. By Lemma 5.1, $\Delta_{\Theta, \Phi}$ is the transitive closure of $M_L(\Theta, \Phi)$. So that we have

$$\Delta_{\Theta, \Phi} \left(\begin{bmatrix} x & a \\ y & a \end{bmatrix} \right) = \bigvee \left\{ \bigwedge_{i=1}^n M_L(\Theta, \Phi) \left(\begin{bmatrix} x_{i-1} & x_i \\ y_{i-1} & y_i \end{bmatrix} \right) : \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \dots, \begin{bmatrix} x_n \\ y_n \end{bmatrix} \in A \times A \text{ such that } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \text{ and } \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} \right\}$$

Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \dots, \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ be arbitrary elements of $A \times A$ with $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix}$. Then it can be verified that

$$M_L(\Theta, \Phi) \left(\begin{bmatrix} x_{i-1} & x_i \\ y_{i-1} & y_i \end{bmatrix} \right) \leq \Phi(y_{i-1}, y_i)$$

for all $i = 1, 2, \dots, n$. So that

$$\bigwedge_{i=1}^n M_L(\Theta, \Phi) \left(\begin{bmatrix} x_{i-1} & x_i \\ y_{i-1} & y_i \end{bmatrix} \right) \leq \bigwedge_{i=1}^n \Phi(y_{i-1}, y_i)$$

Again by the transitive property of Φ we get the following:

$$\begin{aligned} \bigwedge_{i=1}^n M_L(\Theta, \Phi) \left(\begin{bmatrix} x_{i-1} & x_i \\ y_{i-1} & y_i \end{bmatrix} \right) &\leq \bigwedge_{i=1}^n \Phi(y_{i-1}, y_i) \\ &\leq \Phi(y_0, y_n) \\ &= \Phi(y, a) \end{aligned}$$

It follows that

$$\begin{aligned} \Delta_{\theta, \phi} \left(\begin{bmatrix} x & a \\ y & a \end{bmatrix} \right) &\leq \Phi(a, y) \\ &= \delta_{\theta, \phi} \left(\begin{bmatrix} a & y \\ a & y \end{bmatrix} \right) \\ &\leq \Delta_{\theta, \phi} \left(\begin{bmatrix} a & y \\ a & y \end{bmatrix} \right) \end{aligned}$$

This implies that

$$\begin{aligned} \Delta_{\theta, \phi} \left(\begin{bmatrix} x & a \\ y & a \end{bmatrix} \right) &= \Delta_{\theta, \phi} \left(\begin{bmatrix} x & a \\ y & a \end{bmatrix} \right) \wedge \Delta_{\theta, \phi} \left(\begin{bmatrix} a & y \\ a & y \end{bmatrix} \right) \\ &\leq \Delta_{\theta, \phi} \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right) \end{aligned}$$

Since $a \in A$ is arbitrary, it follows that

$$\bigvee \left\{ \Delta_{\theta, \phi} \left(\begin{bmatrix} x & a \\ y & a \end{bmatrix} \right) : a \in A \right\} \leq \Delta_{\theta, \phi} \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right)$$

Hence the equality holds. $\square$

Theorem 5.4. Let $\mathcal{V}$ be modular. Then for each $x, y \in A$;

$$\Delta_{\theta, \phi} \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right) = \bigvee \left\{ \Delta_{\theta, \phi} \left(\begin{bmatrix} x & y \\ b & b \end{bmatrix} \right) : b \in A \right\}$$

Proof. The inequality

$$\Delta_{\theta, \phi} \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right) \leq \bigvee \left\{ \Delta_{\theta, \phi} \left(\begin{bmatrix} x & y \\ b & b \end{bmatrix} \right) : b \in A \right\}$$

holds trivially. To prove the other side of the inequality, let $b \in A$ be arbitrary. Define fuzzy subsets η_1 and η_2 as follows: for each $x, y, z, u \in A$

$$\eta_1 \left(\begin{bmatrix} x & z \\ y & u \end{bmatrix} \right) = \begin{cases} 1 & \text{if } x = z \\ 0 & \text{otherwise} \end{cases}$$

and

$$\eta_2 \left(\begin{bmatrix} x & z \\ y & u \end{bmatrix} \right) = \begin{cases} 1 & \text{if } y = u \\ 0 & \text{otherwise} \end{cases}$$

Then it is clear that $\eta_1 \wedge \eta_2 \leq \Delta_{\Theta, \Phi}$. Therefore by the fuzzy version of the shifting lemma it holds that

$$\begin{aligned} &\Delta_{\theta, \phi} \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right) \\ &\geq \eta_1 \left(\begin{bmatrix} x & x \\ y & b \end{bmatrix} \right) \wedge \eta_1 \left(\begin{bmatrix} y & y \\ y & b \end{bmatrix} \right) \wedge \eta_2 \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right) \wedge \eta_2 \left(\begin{bmatrix} x & y \\ b & b \end{bmatrix} \right) \\ &\quad \wedge \Delta_{\theta, \phi} \left(\begin{bmatrix} x & y \\ b & b \end{bmatrix} \right) \\ &= \Delta_{\theta, \phi} \left(\begin{bmatrix} x & y \\ b & b \end{bmatrix} \right). \end{aligned}$$

Since $b \in A$ is arbitrary, it follows that

$$\bigvee \left\{ \Delta_{\theta, \phi} \left(\begin{bmatrix} x & y \\ b & b \end{bmatrix} \right) : b \in A \right\} \leq \Delta_{\theta, \phi} \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right).$$

Hence the equality holds. $\square$

Theorem 5.5. Let $\mathcal{V}$ be modular. Then the commutator $[\theta, \phi]$ can be characterized as

$$[\Theta, \Phi](x, y) = \Delta_{\Theta, \Phi} \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right)$$

for all $x, y \in A$.

Proof. We first show that

$$[\Theta, \Phi](a, c) \geq \Delta_{\Theta, \Phi} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \wedge [\Theta, \Phi](b, d)$$

for all $a, b, c, d \in A$. Since $\Delta_{\Theta, \Phi}$ is the transitive closure of $M_L(\Theta, \Phi)$

$$\begin{aligned} &\Delta_{\theta, \phi} \left(\begin{bmatrix} a & c \\ b & d \end{bmatrix} \right) \\ &= \bigvee \left\{ \bigwedge_{i=1}^n M_L(\Theta, \Phi) \left(\begin{bmatrix} x_{i-1} & x_i \\ y_{i-1} & y_i \end{bmatrix} \right) : \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \dots, \begin{bmatrix} x_n \\ y_n \end{bmatrix} \in A^2 \right. \\ &\quad \left. \text{such that } \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \text{ and } \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} \right\} \end{aligned}$$

Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \dots, \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ be arbitrary elements of $A \times A$ with $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$. Then, since $FC(\Theta, \Phi; [\Theta, \Phi])$ holds it can be verified that

$$[\Theta, \Phi](a, c) \geq \bigwedge_{i=1}^n M_L(\Theta, \Phi) \left(\begin{bmatrix} x_{i-1} & x_i \\ y_{i-1} & y_i \end{bmatrix} \right) \wedge [\Theta, \Phi](b, d)$$

So that

$$[\Theta, \Phi](a, c) \geq \Delta_{\Theta, \Phi} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \wedge [\Theta, \Phi](b, d)$$

By symmetry it is also true that

$$[\Theta, \Phi](b, d) \geq \Delta_{\Theta, \Phi} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \wedge [\Theta, \Phi](a, c)$$

So that

$$[\Theta, \Phi](a, c) \wedge \Delta_{\Theta, \Phi} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \Delta_{\Theta, \Phi} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \wedge [\Theta, \Phi](b, d)$$

In particular,

$$\begin{aligned} [\Theta, \Phi](x, y) \wedge \Delta_{\Theta, \Phi} \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right) &= \Delta_{\Theta, \Phi} \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right) \wedge [\Theta, \Phi](y, y) \\ &= \Delta_{\Theta, \Phi} \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right) \end{aligned}$$

Implying that

$$\Delta_{\Theta, \Phi} \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right) \leq [\Theta, \Phi](x, y)$$

Moreover, the above equality is equivalent to saying that $[\Theta, \Phi]$ is a fuzzy class of the fuzzy congruence $\Delta_{\Theta, \Phi}$ on $M(A)_{2 \times 2}$. On the other hand, let η be a fuzzy class of $\Delta_{\Theta, \Phi}$. Without loss of generality we can assume that $\eta = \left[\begin{bmatrix} a \\ a \end{bmatrix} \right]_{\Delta_{\Theta, \Phi}}$ for some $a \in A$. Clearly η is a fuzzy congruence on A . Furthermore, one can easily observe that $FC(\Theta, \Phi; \eta)$ holds. So that $[\Theta, \Phi] \leq \eta$. Now consider the following:

$$\begin{aligned} [\Theta, \Phi](x, y) &\leq \eta(x, y) \\ &= \left[\begin{bmatrix} a \\ a \end{bmatrix} \right]_{\Delta_{\Theta, \Phi}}(x, y) \\ &= \Delta_{\Theta, \Phi} \left(\begin{bmatrix} x & a \\ y & a \end{bmatrix} \right) \\ &\leq \bigvee \left\{ \Delta_{\Theta, \Phi} \left(\begin{bmatrix} x & a \\ y & a \end{bmatrix} \right) : a \in A \right\} \\ &= \Delta_{\Theta, \Phi} \left(\begin{bmatrix} x & y \\ y & y \end{bmatrix} \right) \quad (\text{by Theorem 5.3}) \end{aligned}$$

Hence the equality holds and the proof ends. $\square$

Corollary 5.6. Let $\mathcal{V}$ be modular. Then for each $x, y \in A$;

$$[\theta, \phi](x, y) = \bigvee \left\{ \Delta_{\theta, \phi} \left(\begin{bmatrix} x & a \\ y & a \end{bmatrix} \right) : a \in A \right\}$$

Corollary 5.7. Let $\mathcal{V}$ be modular. Then for each $x, y \in A$;

$$[\theta, \phi](x, y) = \bigvee \left\{ \Delta_{\theta, \phi} \left(\begin{bmatrix} x & y \\ b & b \end{bmatrix} \right) : b \in A \right\}$$

6. CONCLUSION

For an algebra A of a given type $\mathfrak{F}$ we obtain a binary operation $[\cdot, \cdot]$ (rep. $[\cdot, \cdot]_s$) called the commutator (resp. the symmetric commutator) on the lattice of fuzzy congruence relations on A having the following properties:

- (1) $[\Theta, \Phi]_s = [\Phi, \Theta]_s$
- (2) $[\Theta, \Phi] \leq [\Theta, \Phi]_s \leq \Theta \wedge \Phi$
- (3) $[\Theta, \Phi]$ and $[\Theta, \Phi]_s$ are monotone in both Θ and Φ .

In particular, in modular varieties, the commutator of fuzzy congruences coincides with that symmetric commutator and an algebraic characterization is obtained for it using the Day's terms. The Day's terms are also used in the paper to prove the fuzzy version of the shifting lemma. Moreover, in modular varieties, it is shown that the commutator is distributive over arbitrary join of fuzzy congruences. Another characterization is obtained for this commutator in the sense of Hagemann and Herman [44].

CONFLICTS OF INTEREST

The authors declare of no conflicts of interest.

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