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Inference in Simple Step-Stress Accelerated Life Tests for Type-II Censoring Lomax Data

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ABSTRACT

In this paper, step-stress accelerated life test is considered to obtain the failure time data of highly reliable units in specified conditions. It is assumed that the lifetime data of such units follow Lomax distribution with a scale parameter depends on the stress level and the shape parameter remains constant. It is also assumed that failure times occur according to a cumulative exposure model (CEM). Using this model, the maximum likelihood estimators (MLEs) and the respective confidence intervals (CIs) based on the asymptotic normality theory as well as the ones based on parametric bootstrap method are considered. In the context of prediction, point and interval predictions are also addressed. A simulation study has been performed to assess the estimation and prediction methods and a real dataset is analyzed for illustrative purposes.

1. INTRODUCTION

Accelerated Life Tests (ALT) are commonly used to evaluate the lifetime of highly reliable products or components within a reasonable testing time. In ALT the products or components are run at higher than usual levels of stress (including temperature, voltage, pressure, etc.) to obtain failures quickly. The data obtained from such an accelerated test are then transformed to estimate the distribution of failures under specified conditions. The model of ALT is chosen according to the relationship between the parameters of the failure distribution and accelerated stress conditions. If a constant stress level is used and some selected stress levels are very low, there are many nonfailed products or components during the testing time, which reduces the effectiveness of accelerated tests. To overcome this problem, step-stress accelerated life test (SSALT) are used. Detailed discussions on ALT tests can be found in Nelson [1], Lawless [2], Kundu and Ganguly [3].

In the SSALT, the stress level in the model will be changed in steps at various intermediate stages of experiment. Specifically, a test unit is subjected to a specified level of stress for a prefixed period of time, if it does not fail during that period of time, then the stress level is increased for future prefixed time. This process continues until the test units fail or some termination conditions are met. The SSALT with two levels of stress is known as simple SSALT. Miller and Nelson [4] described an optimal simple step-stress plan for (ALT) when failure times are exponentially distributed and not censored. Bai *et al.* [5] extended the results of Miller and Nelson [4] to the case of censoring. Bai and Kim [6] presented an optimum simple step-stress testing for the Weibull distribution under Type I censoring. Xiong [7] assumed that the mean life of an experimental unit is a log-linear function of the stress level. Based on this assumption, he developed inference for the parameters of the log-linear link function.

The censoring data is of natural interest in survival, reliabilityand medical studies due to cost or time considerations. The Type-II censoring scheme is one of the popular mechanisms of collecting data in lifetime analysis. It is often used in testing of equipment where all items are put on test at the same time and the test is terminated when the predetermined r number of the items have failed. Balakrishnan *et al.* [8] presented point and interval estimation for a simple step-stress model with Type-II censoring for exponential distribution. Kateri and Balakrishnan [9] estimated the original parameters of a simple step-stress model for the Weibull distribution under Type-II censoring. Watkins [10] argued that it is preferable to work with the original parameters in step-stress model, see also Kundu and Ganguly [3].

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For Pareto distribution, Kamal *et al.* [11] presented estimates of the parameters for simple step-stress model of Pareto distribution for uncensored data. Chandra and Khan [12] considered the simple step-stress model, and obtained estimators of the parameters for Lomax with Type-I censoring. Hassan *et al.* [13] considered the SSALTs based on an adaptive Type-II progressive hybrid censoring with product's lifetime following Lomax distribution where the scale parameter of the lifetime distribution at any stress level is assumed to be log-linear function of the stress level.

The prediction of future censored observations based on the information available is a fundamental problem in statistics. It is widely used in survival, medical and engineering studies. For detailed discussions on point and interval prediction, one may refer to Kaminsky and Rhodin [14], Asgharzadeh *et al.* [15] and Saadati Nik *et al.* [16]. In the context of ALT, Basak and Balakrishnan [17,18] considered the problem of predicting the failure times of censored items for a simple step-stress model from exponential distribution with progressive Type-II censoring and Type-II right censoring, respectively.

The main aim of this paper is to estimate the original parameters of the model and then predict future order statistics based on Type-II censored Lomax data under simple step-stress with cumulative exposure model (CEM). It may worth mentioning that no attention has been paid for the problem of prediction of future lifetimes of Lomax distribution under CEM. Details of the model are given in Section 2. Based on the proposed model, maximum likelihood estimation procedure and its asymptotic normality are discussed in Section 3. Confidence intervals (CIs) based on the asymptotic normality of the maximum likelihood estimators (MLEs) and bootstrap methods are established in Section 4. We tackle the problem of predicting the future failure times in Section 5. A simulation study for assessing the estimation and prediction methods discussed in the previous sections are performed in Section 6. Finally, the paper is concluded in Section 7.

2. MODEL DESCRIPTION AND RELATED ASSUMPTIONS

Here, we give a description of the Lomax CEM for simple step-stress model, terminologies and its related assumptions.

Notations:

n: Number of units placed on the test
τ: Stress change time, at which the stress level is changed
n_1: Number of units failed before τ
n_2: Number of units failed after τ, until a specified number of failures r will be observed, hence:
$r = n_1 + n_2$
t_i: Failure time of the test unit $i, i = 1, 2, \dots, n$
α_j: Scale parameters of stress level $j, j = 1, 2$
β: Shape parameter
$F(.)$: The cumulative distribution function (cdf)
$f(.)$: The probability distribution function (pdf)
$F_j(.)$: The cdf at stress level $j, j = 1, 2$
$f_j(.)$: The pdf at stress level $j, j = 1, 2$

Basic assumptions:

1. Units are tested at two stress levels $S_1 < S_2$;
2. The failure times of the units for any stress level follow Lomax distribution;
3. The scale parameters for the life distribution are $\alpha_j, j = 1, 2$, corresponding to stress level $S_j, j = 1, 2$;
4. β is independent of the stress level S_j;
5. Failures follow the CEM.

In step-stress model, a CEM is assumed in which the lifetime distribution of the units at one stress level is related to the lifetime distribution of the units at the next level. The model assumes that the remaining lifetime of the experiment units depends only the cumulative exposure the units have experienced, with no memory on how this exposure was accumulated, see Kundu and Ganguly [3].

Lomax distribution, is a special case of the second kind of Pareto distribution, it was proposed by Lomax [19]. It has been shifted from Pareto distribution so that its support begins at zero. It has been used in business, economics, insurance, queueing theory and engineering. Its pdf is given by

$$f(t, \alpha, \beta) = \frac{\beta}{\alpha}\left(1 + \frac{t}{\alpha}\right)^{-\beta-1}, t > 0, \beta > 0, \alpha > 0, \tag{1}$$

with cdf

$$F(t,\alpha,\beta)=1-\left(1+\frac{t}{\alpha}\right)^{-\beta}, t>0, \beta>0, \alpha>0, \tag{2}$$

where β is the shape parameter, and α is the scale parameter. The hazard rate function of Lomax distribution is decreasing in t and given by

$$h(t)=\frac{\beta}{\alpha+t},$$

So Lomax distribution may describe the lifetime of a decreasing failure rate items. Bryson [20] recommended Lomax distribution as an alternative to the exponential distribution when the data are heavy tailed.

The test is conducted as follows. All n units are initially put on the lower stress S_1 and run until time τ. Then, the stress is changed to high level S_2, and the test continues until a prespecified number of failures r are observed. Let n_1 denotes the random number of failures before τ, and $n_2=r-n_1$, denotes the number of failures after τ. If $n_1=r$, then the test is terminated at the first step. Otherwise, the stress level is increased to the next step, and the test continues until the required r failures. The ordered failure times that are observed are denoted by:

$$t_{1:n}<\ldots<t_{n_1:n}<\tau\leq t_{n_1+1:n}<\ldots<t_{r:n} \tag{3}$$

The CEM for simple step-stress test is given by

$$F(t)=\begin{cases} F_1(t), & 0\leq t<\tau \\ F_2(t-\tau+h), & \tau\leq t<\infty, \end{cases} \tag{4}$$

where the equivalent starting time, h is a solution of the equation $F_1(\tau)=F_2(h)$. Solving this equation for h, we get $h=\frac{\alpha_2}{\alpha_1}\tau$.

As a consequence of that, the Lomax CEM for simple step-stress model is:

$$F(t)=\begin{cases} 1-\left(1+\frac{t}{\alpha_1}\right)^{-\beta}, & 0\leq t<\tau \\ 1-\left(1+\frac{\tau}{\alpha_1}+\frac{t-\tau}{\alpha_2}\right)^{-\beta}, & \tau\leq t<\infty, \end{cases} \tag{5}$$

with the corresponding pdf

$$f(t)=\begin{cases} \frac{\beta}{\alpha_1}\left(1+\frac{t}{\alpha_1}\right)^{-\beta-1}, & 0\leq t<\tau \\ \frac{\beta}{\alpha_2}\left(1+\frac{\tau}{\alpha_1}+\frac{t-\tau}{\alpha_2}\right)^{-\beta-1}, & \tau\leq t<\infty. \end{cases} \tag{6}$$

3. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we consider the MLEs of the parameters β, α_1, and α_2 involved in the CEM Lomax model based on the observed Type-II censored data:

$$\boldsymbol{t}=\left(t_{1:n},\ldots,t_{n_1:n},t_{n_1+1:n},\ldots,t_{r:n}\right).$$

The likelihood function based on the censored data, $\boldsymbol{t}$ is given by

$$L\left(\beta,\alpha_1,\alpha_2|\boldsymbol{t}\right)=\begin{cases} \frac{n!}{r!}\prod_{i=1}^{n_1} f_1\left(t_{i:n}\right)\left[1-F_1\left(t_{r:n}\right)\right]^{n-r}, \; n_1=r & \text{(7a)} \\ \frac{n!}{r!}\prod_{i=1}^{r} f_2\left(t_{i:n}\right)\left[1-F_2\left(t_{r:n}\right)\right]^{n-r}, \; n_1=0 & \text{(7b)} \\ \frac{n!}{r!}\prod_{i=1}^{r} f\left(t_{i:n}\right)\left[1-F\left(t_{r:n}\right)\right]^{n-r}, \quad 1\leq n_1\leq r-1, & \text{(7c)} \end{cases} \tag{7}$$

From the likelihood function given in (7a), (7b) and (7c), it is observed that the MLEs of the parameters exist only if $1 \leq n_1 \leq r-1$. Therefore we get the likelihood function of β, α_1 and α_2 for (7c) based on the observed Type-II censored sample as follows:

$$L\left(\beta, \alpha_1, \alpha_2 | \boldsymbol{t}\right) = \frac{n!}{n_1!(r-n_1)!} \prod_{i=1}^{n_1} f_1\left(t_{i:n}\right) \prod_{i=n_1+1}^{r} f_2\left(t_{i:n}\right) \left[1 - F_2\left(t_{r:n}\right)\right]^{n-r}, \tag{8}$$

Using (5) and (6), we immediately have

$$\begin{aligned} L\left(\beta, \alpha_1, \alpha_2 | \boldsymbol{t}\right) \propto & \prod_{i=1}^{n_1} \left\{ \frac{\beta}{\alpha_1} \left(1 + \frac{t_{i:n}}{\alpha_1}\right)^{-\beta-1} \right\} \prod_{i=n_1+1}^{r} \left\{ \frac{\beta}{\alpha_2} \left(1 + \frac{\tau}{\alpha_1} + \frac{t_{i:n} - \tau}{\alpha_2}\right)^{-\beta-1} \right\} \\ & \times \left[\left(1 + \frac{\tau}{\alpha_1} + \frac{t_{r:n} - \tau}{\alpha_2}\right)^{-\beta}\right]^{n-r}. \end{aligned} \tag{9}$$

Consequently, the log-likelihood function $L^* = \log(L)$ is given by

$$\begin{aligned} L^*\left(\beta, \alpha_1, \alpha_2 | \boldsymbol{t}\right) \propto & -(\beta+1) \left[\sum_{i=1}^{n_1} \log\left(1 + \frac{t_{i:n}}{\alpha_1}\right) + \sum_{i=n_1+1}^{r} \log\left(1 + \frac{\tau}{\alpha_1} + \frac{t_{i:n} - \tau}{\alpha_2}\right)\right] + r \log(\beta) \\ & -n_1 \log\left(\alpha_1\right) - \left(r - n_1\right) \log\left(\alpha_2\right) - (n-r)\, \beta \log\left(1 + \frac{\tau}{\alpha_1} + \frac{t_{r:n} - \tau}{\alpha_2}\right), \end{aligned} \tag{10}$$

So, the likelihood equations are given by

$$\frac{\partial L^*}{\partial \alpha_1} = -(\beta+1) \left[\sum_{i=1}^{n_1} \frac{-t_{i:n}}{\alpha_1^2 \left(1 + \frac{t_{i:n}}{\alpha_1}\right)} + \sum_{i=n_1+1}^{r} \frac{-\tau}{\alpha_1^2 \left(1 + \frac{\tau}{\alpha_1} + \frac{t_{i:n} - \tau}{\alpha_2}\right)}\right] \frac{-n_1}{\alpha_1} + \frac{(n-r)\beta\tau}{\alpha_1^2 \left(1 + \frac{\tau}{\alpha_1} + \frac{t_{r:n} - \tau}{\alpha_2}\right)} = 0 \tag{11}$$

$$\frac{\partial L^*}{\partial \alpha_2} = -(\beta+1) \sum_{i=n_1+1}^{r} \frac{-(t_{i:n} - \tau)}{\alpha_2^2 \left(1 + \frac{\tau}{\alpha_1} + \frac{t_{i:n} - \tau}{\alpha_2}\right)} - \frac{(r - n_1)}{\alpha_2} + \frac{(n-r)\, \beta \left(t_{r:n} - \tau\right)}{\alpha_2^2 \left(1 + \frac{\tau}{\alpha_1} + \frac{t_{r:n} - \tau}{\alpha_2}\right)} = 0 \tag{12}$$

$$\frac{\partial L^*}{\partial \beta} = -\left[\sum_{i=1}^{n_1} \log\left(1 + \frac{t_{i:n}}{\alpha_1}\right) + \sum_{i=n_1+1}^{r} \log\left(1 + \frac{\tau}{\alpha_1} + \frac{t_{i:n} - \tau}{\alpha_2}\right)\right] + \frac{r}{\beta} - (n-r) \log\left(1 + \frac{\tau}{\alpha_1} + \frac{t_{i:n} - \tau}{\alpha_2}\right) = 0 \tag{13}$$

From (13), The MLE $\hat{\beta}$ of β is obtained analytically and it is given by

$$\hat{\beta} = \frac{r}{\sum_{i=1}^{n_1} \log\left(1 + \frac{t_{i:n}}{\alpha_1}\right) + \sum_{i=n_1+1}^{r} \log\left(1 + \frac{\tau}{\alpha_1} + \frac{t_{i:n} - \tau}{\alpha_2}\right) + (n-r) \log\left(1 + \frac{\tau}{\alpha_1} + \frac{t_{i:n} - \tau}{\alpha_2}\right)} \tag{14}$$

The estimation procedure, through Eqs. (11)–(13), does not result in closed form. Therefore, Eqs. (11) and (12) can be solved numerically using Newton–Raphson method and the resulting estimates are denoted by $\hat{\alpha}_1$ and $\hat{\alpha}_2$.

4. CIs FOR THE MODEL PARAMETERS

Here, we present two different methods for constructing CIs for the unknown parameters β, α_1 and α_2. The first one is based on the asymptotic distributions of the MLEs and the other one uses parametric bootstrap technique.

4.1. Approximate CIs

It is known that for large sample sizes and under some regularity conditions, the MLEs are consistent and normally distributed

Let $FI\left(\beta, \alpha_1, \alpha_2\right) = \left[FI_{ij}\left(\beta, \alpha_1, \alpha_2\right)\right]$, for $i, j = 1, 2, 3$, denotes the observed Fisher information (FI) matrix of β, α_1 and α_2, where

$$FI_{ij}\left(\beta, \alpha_1, \alpha_2\right) = -\left(\nabla^2 L^*\left(\beta, \alpha_1, \alpha_2\right)\right). \tag{15}$$

So, the observed FI matrix is given by

$$FI = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}, \tag{16}$$

where the elements of FI are obtained as follows:

$$I_{11} = \frac{\partial^2 L^*}{\partial \beta^2} = \frac{-r}{\beta^2}, \tag{17}$$

$$I_{12} = I_{21} = \frac{\partial^2 L^*}{\partial \beta \partial \alpha_1} = \left[\sum_{i=1}^{n_1} \frac{t_{i:n}}{\alpha_1^2 + t_{i:n}\alpha_1} + \sum_{i=n_1+1}^{r} \frac{\tau}{\alpha_1^2 + \tau\alpha_1 + \frac{\alpha_1^2}{\alpha_2}\left(t_{i:n} - \tau\right)} \right] + \frac{(n-r)\,\tau}{\alpha_1^2 + \tau\alpha_1 + \frac{\alpha_1^2}{\alpha_2}\left(t_{r:n} - \tau\right)}, \tag{18}$$

$$I_{13} = I_{31} = \frac{\partial^2 L^*}{\partial \alpha_2 \partial \beta} = -(\beta+1) \sum_{i=n_1+1}^{r} \frac{\left(t_{i:n} - \tau\right)}{\alpha_2^2 + \tau\frac{\alpha_2^2}{\alpha_1} + \alpha_2\left(t_{i:n} - \tau\right)} + \frac{(n-r)\,\beta\left(t_{r:n} - \tau\right)}{\alpha_2^2 + \tau\frac{\alpha_2^2}{\alpha_1} + \alpha_2\left(t_{r:n} - \tau\right)} \tag{19}$$

$$\begin{aligned} I_{22} = \frac{\partial^2 L^*}{\partial {\alpha_1}^2} = &-(\beta+1)\left[\sum_{i=1}^{n_1} \frac{t_{i:n}\left(2\alpha_1 + t_{i:n}\right)}{\left(\alpha_1^2 + t_{i:n}\alpha_1\right)^2} + \sum_{i=n_1+1}^{r} \frac{\tau\left[2\alpha_1 + \tau + \frac{2\alpha_1}{\alpha_2}\left(t_{i:n} - \tau\right)\right]}{\left[\alpha_1^2 + \tau\alpha_1 + \frac{\alpha_1^2}{\alpha_2}\left(t_{i:n} - \tau\right)\right]^2} \right] \\ &+ \frac{n_1}{\alpha_1^2} - \frac{(n-r)\beta\tau\left[2\alpha_1 + \tau + \frac{2\alpha_1}{\alpha_2}\left(t_{r:n} - \tau\right)\right]}{\left[\alpha_1^2 + \tau\alpha_1 + \frac{\alpha_1^2}{\alpha_2}\left(t_{r:n} - \tau\right)\right]^2}, \end{aligned} \tag{20}$$

$$I_{23} = \frac{\partial^2 L^*}{\partial \alpha_1 \partial \alpha_2} - (\beta+1) \sum_{i=n_1+1}^{r} \frac{-\tau({\alpha_1}^2\left(t_{i:n} - \tau\right)}{\alpha_2^2\left[\alpha_1^2 + \tau\alpha_1 + \frac{\alpha_1^2}{\alpha_2}\left(t_{i:n} - \tau\right)\right]^2} + \frac{(n-r)\,\beta\tau{\alpha_1}^2\left(t_{r:n} - \tau\right)}{\alpha_2^2\left[\alpha_1^2 + \tau\alpha_1 + \frac{\alpha_1^2}{\alpha_2}\left(t_{r:n} - \tau\right)\right]^2}, \tag{21}$$

$$\begin{aligned} I_{33} = \frac{\partial^2 L^*}{\partial {\alpha_2}^2} = &-(\beta+1) \sum_{i=n_1+1}^{r} \frac{\left(t_{i:n} - \tau\right)\left[2\alpha_2 + \frac{2\tau\alpha_2}{\alpha_1} + \left(t_{i:n} - \tau\right)\right]}{\left[\alpha_2^2 + \tau\frac{\alpha_2^2}{\alpha_1} + \alpha_2\left(t_{i:n} - \tau\right)\right]^2} \\ &+ \frac{\left(r - n_1\right)}{\alpha_2^2} - \frac{(n-r)\beta\left(t_{r:n} - \tau\right)\left[2\alpha_2 + \frac{2\tau\alpha_2}{\alpha_1} + \left(t_{r:n} - \tau\right)\right]}{\left[\alpha_2^2 + \tau\frac{\alpha_2^2}{\alpha_1} + \alpha_2\left(t_{r:n} - \tau\right)\right]^2}. \end{aligned} \tag{22}$$

The inverse of FI is the asymptotic variance-covariance matrix of the MLEs $\hat{\beta}, \hat{\alpha}_1$ and $\hat{\alpha}_2$ of the parameters β, α_1 and α_2. Therefore, the asymptotic variance of these estimates is

$$Var\begin{pmatrix} \hat{\beta} \\ \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{pmatrix} = (FI)^{-1} = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \tag{23}$$

So, the $100\left(1 - \frac{\gamma}{2}\right)$ % asymptotic confidence intervals (ACIs) of the parameters β, α_1 and α_2 are, respectively,

$$\hat{\beta} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{11}}, \tag{24}$$

$$\hat{\alpha}_1 \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{22}}, \tag{25}$$

$$\hat{\alpha}_2 \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{33}}, \tag{26}$$

where z_p is the p-th upper percentile of the standard normal distribution.

4.2. Bootstrap CIs

The parametric bootstrap sampling is used to construct CIs for the unknown parameters β, α_1 and α_2. The following describe the steps for generating these CIs.

Algorithm for computing bootstrap CIs:

Step 1: Compute the MLEs of β, α_1 and α_2 based on the algorithm described in Section 3, say $\hat{\beta}, \hat{\alpha}_1$ and $\hat{\alpha}_2$;

Step 2: Simulate the first r order statistics $U_{1:n}, U_{2:n}, \dots, U_{r:n}$ from a sample from uniform distribution $U(0,1)$;

Step 3: For given value of stress change time τ, Find n_1 such that:

$$U_{n_1} < 1 - \left(1 + \frac{\tau}{\hat{\alpha}_1}\right)^{-\hat{\beta}} \leq U_{n_1+1:n;}$$

Step 4: The ordered observations $t_{1:n} < t_{2:n} < \dots < t_{n_1:n} < t_{n_{1+1}:n} < \dots < t_{r:n}$ are calculated as follows:

$$t_{i:n} = \begin{cases} \hat{\alpha}_1 \left[\left(1 - U_{i:n}\right)^{\frac{-1}{\hat{\beta}}} - 1 \right], & i = 1, 2, \dots, n_1 \\ \hat{\alpha}_2 \left[\left(1 - U_{i:n}\right)^{\frac{-1}{\hat{\beta}}} - 1 \right] + \tau \left(1 - \frac{\hat{\alpha}_2}{\hat{\alpha}_1}\right), & i = n_1 + 1, \dots, r \end{cases}$$

Step 5: Compute the MLEs of β, α_1 and α_2 based on $t_{1:n}, t_{2:n}, \dots, t_{n_1:n}, t_{n_{1+1}:n}, \dots, t_{r:n}$, say, $\hat{\beta}^{(1)}, \hat{\alpha}_1^{(1)}$, and $\hat{\alpha}_{21}^{(1)}$.

Step 6: Repeat steps 2–5 M times to obtain M sets of MLEs of β, α_1 and α_2. Then a two-sided $100\left(1 - \frac{\gamma}{2}\right)$% bootstrap confidence intervals (BCIs) of the parameters β, α_1 and α_2 are respectively,

$$\hat{\beta} \pm z_{1-\frac{\gamma}{2}} \sqrt{MSE\left(\hat{\beta}\right)} \tag{27}$$

$$\hat{\alpha}_1 \pm z_{1-\frac{\gamma}{2}} \sqrt{MSE\left(\hat{\alpha}_1\right)} \tag{28}$$

$$\hat{\alpha}_2 \pm z_{1-\frac{\gamma}{2}} \sqrt{MSE\left(\hat{\alpha}_2\right)} \tag{29}$$

where $MSE\left(\hat{\theta}\right) = Var\left(\hat{\theta}\right) + \left(\hat{\theta} - \bar{\hat{\theta}}\right)^2$, with $\bar{\hat{\theta}}$ is the average of the simulated estimates $\hat{\theta}$.

5. PREDICTION OF FUTURE ORDER STATISTICS

In this section, we consider the problem of prediction of future failure time based on some observed failure times under the current simple step-stress model. The problem can be expressed as follows. Let $T_{1:n} < T_{2:n} < \dots < T_{r:n}$ be the observed ordered lifetime units known as informative sample, and let $T_{s:n}, s = r+1, \dots, n$, be the unobserved future order statistics from the same sample. The prediction problem involves the prediction of the future order statistics $T_{s:n}$, given the first r observed ordered statistics $T_{i:n}, 0 < i \leq r$.

Using the Markovian property of censored order statistics, it is well-known that the conditional distribution of $Y = T_{s:n}$ given $\boldsymbol{T} = \boldsymbol{t} = (t_{1:n}, \dots, t_{n_1:n}, t_{n_1+1:n}, \dots, t_{r:n})$ is just the distribution of $Y = T_{s:n}$ given $T_{r:n} = t_{r:n}$. This implies that the density of Y given $\boldsymbol{T} = \boldsymbol{t}$ is the same as the density of the $(s-r)$-th order statistic out of $(n-r)$ units from the population with density $\varphi\left(y\right) = \frac{f(y)}{1-F\left(t_{r:n}\right)}, y > t_{r:n}$ (left truncated density at $t_{r:n}$), where $F\left(y\right)$ is given in Eq. (5). Hence, we obtain

$$\begin{aligned} f_{T_{s:n}|\mathrm{T}}\left(y|\boldsymbol{\theta}, data\right) = c \times \frac{\beta}{\alpha_2}\left(1 + \frac{\tau}{\alpha_1} + \frac{y-\tau}{\alpha_2}\right)^{-\beta(n-s+1)-1} \left(1 + \frac{\tau}{\alpha_1} + \frac{t_{r:n}-\tau}{\alpha_2}\right)^{-\beta(n-r)} \\ \times\left[\left(1 + \frac{\tau}{\alpha_1} + \frac{t_{r:n}-\tau}{\alpha_2}\right)^{-\beta} - \left(1 + \frac{\tau}{\alpha_1} + \frac{y-\tau}{\alpha_2}\right)^{-\beta}\right]^{s-r-1}, y > t_{r:n}, \end{aligned} \tag{30}$$

where $\boldsymbol{\theta} = \left(\beta, \alpha_1, \alpha_2\right)$, $c = \frac{(n-r)!}{(s-r-1)!(n-s)!}$.

5.1. Point Prediction

In this subsection, we present two techniques for obtaining point predictors of future lifetimes, $T_{s:n}, s = r+1, \ldots, n$.

5.1.1. Maximum likelihood predictor (MLP)

The MLP was proposed by Kaminsky and Rhodin [14]. This method involves prediction of future order statistics and also estimation of the parameters in the model. The predictive likelihood function (PLF) of $Y = T_{s:n}$ is given by

$$\begin{aligned} L\left(y, \boldsymbol{\theta} | \boldsymbol{t}\right) = L &= f_{T_{s:n}|\boldsymbol{T}}\left(y | \boldsymbol{t}, \boldsymbol{\theta}\right) .f_{\boldsymbol{T}}\left(\boldsymbol{t}, \boldsymbol{\theta}\right) \\ &= f_{T_{s:n}|T_{r:n}}\left(y | t_{r:n}, \boldsymbol{\theta}\right) .f_{\boldsymbol{T}}\left(\boldsymbol{t}, \boldsymbol{\theta}\right), \end{aligned} \tag{31}$$

where $f_{T_{s:n}|T_{r:n}}\left(y | t_{r:n}, \boldsymbol{\theta}\right)$ is the conditional density of $Y = T_{s:n}$ given the observed value of $\boldsymbol{T} = \boldsymbol{t}$, as in Eq. (30), and $f_{\boldsymbol{T}}\left(\boldsymbol{t}, \boldsymbol{\theta}\right)$ is the density of $\boldsymbol{T}$. Therefore, Eq. (31) can be expressed as

$$\begin{aligned} &L \propto \prod_{i=1}^{n_1} f_1\left(t_{i:n}\right) \prod_{i=n_1+1}^{r} f_2\left(t_{i:n}\right) \left[F_2\left(y\right) - F_2\left(t_{r:n}\right)\right]^{s-r-1} f_2\left(y\right) \left[1 - F_2\left(y\right)\right]^{n-s}, \\ &0 \le n_1 \le r, r+1 \le s \le n. \end{aligned} \tag{32}$$

Considering the case when $1 \le n_1 < r \le n$, we obtain

$$L \propto \frac{\beta^{r+1}}{\alpha_1^{n_1} \alpha_2^{n_2+1}} \prod_{i=1}^{n_1} \left(1 + \frac{t_{i:n}}{\alpha_1}\right)^{-\beta-1} \prod_{i=n_1+1}^{r} \left[h\left(t_{i:n}\right)\right]^{-\beta-1} \times \left[h\left(y\right)\right]^{-\beta(n-s+1)-1} \times \left[\left[h\left(t_{r:n}\right)\right]^{-\beta} - \left[h\left(y\right)\right]^{-\beta}\right]^{s-r-1}, \tag{33}$$

where

$$h\left(t\right) = 1 + \frac{\tau}{\alpha_1} + \frac{t-\tau}{\alpha_2}.$$

Consequently, the log PLF can be written as

$$\begin{aligned} \log L \propto\ & (r+1) \log \beta - n_1 \log\left(\alpha_1\right) - \left(n_2+1\right) \log\left(\alpha_2\right) \\ & - (\beta+1) \left[\sum_{i=1}^{n_1} \log\left(1 + \frac{t_{i:n}}{\alpha_1}\right) + \sum_{i=n_1+1}^{r} \log h\left(t_{i:n}\right)\right] - \left[\beta\left(n-s+1\right) - 1\right] \log h\left(y\right) \\ & + (s-r-1) \log\left(\left[h\left(t_{r:n}\right)\right]^{-\beta} - \left[h\left(y\right)\right]^{-\beta}\right). \end{aligned} \tag{34}$$

By (34), the predictive likelihood equations (PLEs) for $\beta, \alpha_1, \alpha_2$ and y are obtained and presented as follows:

$$\begin{aligned} \frac{\partial \log L}{\partial \beta} &= \frac{r+1}{\beta} - \left[\sum_{i=1}^{n_1} \log\left(1 + \frac{t_{i:n}}{\alpha_1}\right) + \sum_{i=n_1+1}^{r} \log h\left(t_{i:n}\right)\right] - (n-s+1) \log h\left(y\right) \\ &\quad + (s-r-1) \frac{\left[h\left(t_{r:n}\right)\right]^{-\beta} \log h\left(t_{r:n}\right) - \left[h\left(y\right)\right]^{-\beta} \log h\left(y\right)}{\left[h\left(t_{r:n}\right)\right]^{-\beta} - \left[h\left(y\right)\right]^{-\beta}} \\ &= 0. \end{aligned} \tag{35}$$

$$\begin{aligned} \frac{\partial \log L}{\partial \alpha_1} &= -\frac{n_1}{\alpha_1} - (\beta+1) \left[\sum_{i=1}^{n_1} \frac{-t_{i:n}}{\alpha_1^2 \left(1 + \frac{t_{i:n}}{\alpha_1}\right)} + \sum_{i=n_1+1}^{r} \frac{-\tau}{\alpha_1^2 h\left(t_{i:n}\right)}\right] \\ &\quad + \frac{\left[\beta(n-s+1)+1\right]\tau}{\alpha_1^2 h(y)} + (s-r-1) \frac{\beta\tau}{\alpha_1^2} \times \frac{\left\{\left[h\left(t_{r:n}\right)\right]^{-\beta-1} - \left[h\left(y\right)\right]^{-\beta-1}\right\}}{\left[h\left(t_{r:n}\right)\right]^{-\beta} - \left[h\left(y\right)\right]^{-\beta}} \\ &= 0. \end{aligned} \tag{36}$$

$$\frac{\partial \log L}{\partial \alpha_2} = -\frac{n_2+1}{\alpha_2} + (\beta+1)\sum_{i=n_1+1}^{r}\left\{\frac{\left(t_{i:n}-\tau\right)}{\alpha_2^2 h\left(t_{i:n}\right)}\right\} + \frac{\left[\beta\left(n-s+1\right)+1\right]\left[y-\tau\right]}{\alpha_2^2 h\left(y\right)}$$

$$(s-r-1)\,\frac{\beta}{\alpha_2^2}\times\frac{\left\{\left[t_{r:n}-\tau\right]\left[h(t_{r:n})\right]^{-\beta-1}-\left[y-\tau\right]\left[h(\mathrm{y})\right]^{-\beta-1}\right\}}{\left[h(t_{r:n})\right]^{-\beta}-\left[h(\mathrm{y})\right]^{-\beta}} \tag{37}$$

$$= 0.$$

$$\frac{\partial \log L}{\partial y} = -\frac{[\beta(n-s+1)]}{\alpha_2 h(y)} + (s-r-1)\frac{\beta}{\alpha_2}\times\frac{[h(\mathrm{y})]^{-\beta-1}\}}{[h(t_{r:n})]^{-\beta}-[h(\mathrm{y})]^{-\beta}} = 0. \tag{38}$$

Since Eqs. (35)–(38) cannot be solved analytically, numerical methods will be used to solve them simultaneously, which leads to find the MLP of Y and the predictivess maximum likelihood estimators (PMLEs) of β, α_1 and α_2. The resulting MLP of Y is denoted by $\hat{Y}_M$.

5.1.2. Conditional median predictor (CMP)

The CMP was first suggested by Raqab and Nagaraja [21]. A predictor $\hat{Y}$ is called the CMP of Y, if it is the median of the conditional distribution of Y given $\boldsymbol{T}=\boldsymbol{t}$, that is

$$P_\theta\left(Y \le \hat{Y}|\boldsymbol{T}=\boldsymbol{t}\right) = P_\theta\left(Y \ge \hat{Y}|\boldsymbol{T}=\boldsymbol{t}\right). \tag{39}$$

Based on the conditional distribution of Y given $\boldsymbol{T}=\boldsymbol{t}$, we can obtain

$$P_\theta\left(Y \le \hat{Y}|\boldsymbol{T}=\boldsymbol{t}\right) = P_\theta\left(1-\left[\frac{\alpha_1\alpha_2+\alpha_2\tau+\alpha_1\left(Y-\tau\right)}{\alpha_1\alpha_2+\alpha_2\tau+\alpha_1\left(t_{r:n}-\tau\right)}\right]^{-\beta} \ge 1-\left[\frac{\alpha_1\alpha_2+\alpha_2\tau+\alpha_1\left(\hat{Y}-\tau\right)}{\alpha_1\alpha_2+\alpha_2\tau+\alpha_1\left(t_{r:n}-\tau\right)}\right]^{-\beta}\middle|\boldsymbol{T}=\boldsymbol{t}\right). \tag{40}$$

It can be shown that, given $\boldsymbol{T}=\boldsymbol{t}$, the distribution of

$$Z = 1-\left[\frac{\alpha_1\alpha_2+\alpha_2\tau+\alpha_1\left(Y-\tau\right)}{\alpha_1\alpha_2+\alpha_2\tau+\alpha_1\left(t_{r:n}-\tau\right)}\right]^{-\beta},$$

is a Beta distribution with parameters $s-r$ and $n-s+1$, denoted by $Beta\,(s-r, n-s+1)$. So, we can obtain the CMP of Y as

$$\hat{Y}_{CMP} = \left[\alpha_2+\frac{\alpha_2}{\alpha_1}\tau+\left(t_{r:n}-\tau\right)\right]\left[1-M_B\right]^{\frac{-1}{\beta}}-\alpha_2-\frac{\alpha_2}{\alpha_1}\tau+\tau, \tag{41}$$

where B is a $Beta\,(s-r, n-s+1)$ variate and M_B represents the median of B. We compute an approximate CMP of Y by replacing β, α_1 and α_2 by their corresponding MLEs.

5.2. Prediction Intervals (PIs)

Another aspect of prediction problem is to predict the future unobserved order statistics by constructing PIs for $Y=T_{s:n}, s=r+1,\ldots,n$ based on the Type-II censored sample $\boldsymbol{T}=\left(T_{1:n}, T_{2:n},\ldots,T_{r:n}\right)$. The pivot and highest conditional density (HCD) methods are used to construct PIs.

5.2.1. Pivotal-based PIs

Let us consider the random variable

$$Z = 1-\left[\frac{\alpha_1\alpha_2+\alpha_2\tau+\alpha_1\left(Y-\tau\right)}{\alpha_1\alpha_2+\alpha_2\tau+\alpha_1\left(t_{r:n}-\tau\right)}\right]^{-\beta}, \quad Y > t_r. \tag{42}$$

It can be easily shown that the conditional density in (30) is a unimodal function of Z. Since the distribution of Z given $\boldsymbol{T} = \boldsymbol{t}$, is a Beta distribution with parameters $s-r$ and $n-s+1$, then Z can be considered as a pivotal quantity for obtaining the PI of Y. By considering $(1-\gamma), 0<\gamma<1$, as prediction coefficient and using (42), we obtain

$$P\left(B_{\frac{\gamma}{2}} < Z < B_{1-\frac{\gamma}{2}}\right) = 1-\gamma,$$

where B_γ is the $100\gamma - th$ percentile of the distribution $Beta\,(s-r, n-s+1)$. Therefore, a $(1-\gamma)\,100\%$ PI of Y is $(L(\boldsymbol{T}), U(\boldsymbol{T}))$, where

$$\begin{aligned} L(\boldsymbol{T}) &= \left[\alpha_2 + \tfrac{\alpha_2}{\alpha_1}\tau + \left(t_{r:n} - \tau\right)\right]\left[1 - B_{\frac{\gamma}{2}}\right]^{\frac{-1}{\beta}} - \alpha_2 - \tfrac{\alpha_2}{\alpha_1}\tau + \tau, \\ U(\boldsymbol{T}) &= \left[\alpha_2 + \tfrac{\alpha_2}{\alpha_1}\tau + \left(t_{r:n} - \tau\right)\right]\left[1 - B_{1-\frac{\gamma}{2}}\right]^{\frac{-1}{\beta}} - \alpha_2 - \tfrac{\alpha_2}{\alpha_1}\tau + \tau. \end{aligned} \tag{43}$$

Since β, α_1 and α_2 are unknown, the MLEs of the parameters can be plugged in (43) to deduce approximate prediction limits, $L(\boldsymbol{T})$ and $U(\boldsymbol{T})$.

5.2.2. Highest conditional density PIs

Now we consider the conditional distribution of

$$Z = 1 - \left[\frac{\alpha_1\alpha_2 + \alpha_2\tau + \alpha_1(Y-\tau)}{\alpha_1\alpha_2 + \alpha_2\tau + \alpha_1\left(t_{r:n} - \tau\right)}\right]^{-\beta},$$

given $\boldsymbol{T} = \boldsymbol{t}$. Its density is given by

$$g(z) = \frac{(n-r)!}{(s-r-1)!\,(n-s)!} z^{s-r-1}(1-z)^{n-s}, 0<z<1. \tag{44}$$

The density in (44) is unimodal function. An interval $\left(d_1, d_2\right)$ is called HCD PI of content $1-\gamma$ if $\left(d_1, d_2\right) = \left\{d : d \in [0,1], f(d) \geq k\right\} \subseteq [0,1]$, where

$$\int_{d_1}^{d_2} g(z)\,dz = 1-\gamma,$$

for some $k > 0$. Now, if $r+1 < s < n$, then $g(z)$ is a unimodal function in z, and it attains its maximum value at $\delta = \frac{s-r-1}{n-r-1} \in (0,1)$. Using Theorem 9.3.2 of Casella and Berger [22], the HCD PI can be obtained by finding two points $d_1 = 100\left(\frac{\gamma}{2}\right) - th$ percentile, and $d_2 = 100\left(1-\frac{\gamma}{2}\right) - th$ percentile, with $d_1 \leq \delta \leq d_2$, satisfying

$$\int_{d_1}^{d_2} g(z)\,dz = 1-\gamma, \tag{45}$$

and,

$$g\left(d_1\right) = g\left(d_2\right). \tag{46}$$

Eqs. (45) and (46) can be simplified as

$$B_{d_2}(s-r, n-s+1) - B_{d_1}(s-r, n-s+1) = 1-\gamma, \tag{47}$$

and

$$\left(\frac{1-d_2}{1-d_1}\right)^{n-s} = \left(\frac{d_1}{d_2}\right)^{s-r-1}, \tag{48}$$

where

$$B_v(a,b)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^v u^{a-1}(1-u)^{b-1}du,$$

is incomplete beta function with $\Gamma(a+1)=a\Gamma(a)$ for a real number a and $\Gamma(a+1)=a!$ for an integer a. Consequently, a $(1-\gamma)\,100\%$ HCD PI of Y is given by $\left(L_2(\boldsymbol{T}),U_2(\boldsymbol{T})\right)$, with

$$\begin{aligned}L_2(\boldsymbol{T})&=\left[\alpha_2+\tfrac{\alpha_2}{\alpha_1}\tau+\left(t_{r:n}-\tau\right)\right]\left[1-d_1\right]^{\frac{-1}{\beta}}-\alpha_2-\tfrac{\alpha_2}{\alpha_1}\tau+\tau,\\ U_2(\boldsymbol{T})&=\left[\alpha_2+\tfrac{\alpha_2}{\alpha_1}\tau+\left(t_{r:n}-\tau\right)\right]\left[1-d_2\right]^{\frac{-1}{\beta}}-\alpha_2-\tfrac{\alpha_2}{\alpha_1}\tau+\tau.\end{aligned}\tag{49}$$

For the special case when $s=r+1$ and $s<n$, $g(z)$ is decreasing function starting from $(n-r)$ at $z=0$ to 0 at $z=1$. In this case, the PI for Y is of the form $(0,d_2)$ such that $d_2=1-\gamma^{1/(n-r)}$. This in turns implies that

$$L_2(\boldsymbol{T})=t_{r:n},\,U_2(\boldsymbol{T})=\left[\alpha_2+\frac{\alpha_2}{\alpha_1}\tau+\left(t_{r:n}-\tau\right)\right](\gamma)^{\frac{-1}{\beta(n-r)}}-\alpha_2-\frac{\alpha_2}{\alpha_1}\tau+\tau.$$

When $s=r+1$ and $s=n$, $g(z)$ is uniform $U(0,1)$. Here d_1 and d_2 are taken such that $d_1=\gamma/2$ and $d_2=1-\gamma/2$. Therefore,

$$L_2(\boldsymbol{T})=\left[\alpha_2+\frac{\alpha_2}{\alpha_1}\tau+\left(t_{r:n}-\tau\right)\right]\left[1-\gamma/2\right]^{\frac{-1}{\beta}}-\alpha_2-\frac{\alpha_2}{\alpha_1}\tau+\tau,$$

and

$$U_2(\boldsymbol{T})=\left[\alpha_2+\frac{\alpha_2}{\alpha_1}\tau+\left(t_{r:n}-\tau\right)\right]\left[\gamma/2\right]^{\frac{-1}{\beta}}-\alpha_2-\frac{\alpha_2}{\alpha_1}\tau+\tau.$$

Finally, when $s=n$ and $s>r+1$, the density $g(z)$ is increasing function with $g(0)=0$ and $g(1)=(n-r)$. Therefore, we choose the PI for Y of the form $(d_1,1)$ such that

$$\int_{d_1}^1 g(z)\,dz=1-\gamma,$$

which implies that

$$d_1=\gamma^{\frac{1}{n-r}}.$$

So, a $(1-\gamma)\,100\%$ HCD PI of Y is given by

$$L_2(\boldsymbol{T})=\left[\alpha_2+\frac{\alpha_2}{\alpha_1}\tau+\left(t_{r:n}-\tau\right)\right]\left[1-\gamma^{\frac{1}{n-r}}\right]^{\frac{-1}{\beta}}-\alpha_2-\frac{\alpha_2}{\alpha_1}\tau+\tau,\text{ and }U_2(\boldsymbol{T})=\infty.$$

6. SIMULATION STUDY AND DATA ANALYSIS

Here in this section, we conduct a simulation study to compute the MLEs of the parameters under the simple step-stress model, as well the corresponding CIs. Computations of the prediction methods are described in another simulation experiment. A real dataset is considered to illustrate the different techniques proposed in this paper.

6.1. Simulation Study

In this section, we perform an intensive Monte Carlo (MC) simulation study for computing the MLEs $\hat{\beta}, \hat{\alpha}_1$ and $\hat{\alpha}_2$ of the parameters β, α_1 and α_2 as well as the prediction of future order statistics $Y = T_{s:n}\ (s \geq r+1)$. The algorithm used for generating the data and computing the MLEs of the parameters β, α_1 and α_2 is performed according to the following steps:

Algorithm for generating the data and finding the MLEs:

Step 1: Generate a random sample of size n from uniform distribution U (0, 1), and obtain the order statistics: $U_{1:n} < U_{2:n} < \ldots < U_{n:n}$;

Step 2: Find the random variable n_1 such that

$U_{n_1} < P(T \leq \tau) = F_1(\tau) \leq U_{n_1+1:n}$, where T represents the failure time such that

$$U_{n_1} < 1 - \left(1 + \frac{\tau}{\alpha_1}\right)^{-\beta} \leq U_{n_1+1:n} \tag{50}$$

Step 3: Generate a sample based on observed failure times data as follows:

$$t_{i:n} = \begin{cases} \alpha_1 \left[\left(1 - U_{i:n}\right)^{\frac{-1}{\beta}} - 1 \right], & i = 1, 2, \ldots, n_1 \\ \alpha_2 \left[\left(1 - U_{i:n}\right)^{\frac{-1}{\beta}} - 1 \right] + \tau\left(1 - \frac{\alpha_2}{\alpha_1}\right), & i = n_1 + 1, \ldots, r \end{cases} \tag{51}$$

Step 4: Compute the MLEs of β, α_1 and α_2 using Eqs. (11), (12) and (14) based on $t_{i:n}, t_{2:n}, \ldots, t_{n_1:n}, t_{n_{1+1}:n}, \ldots, t_{r:n}$ as in Step 3.

Also, the CIs of the parameters and PIs of the censored lifetimes are computed. The simulation process is repeated $M = 1000$ times. The simulated biases and mean square errors (MSEs) of these estimates are also computed. Moreover, the CIs using the asymptotic and bootstrap arguments are obtained. The bias and MSE of an estimate of θ, say, $\hat{\theta}$ and its MSE is defined as

$$bias\left(\hat{\theta}\right) = \frac{1}{M}\sum_{k=1}^{M}\left(\hat{\theta}_k - \theta\right),$$

and

$$MSE\left(\hat{\theta}\right) = \frac{1}{M}\sum_{k=1}^{M}\left(\hat{\theta}_k - \theta\right)^2.$$

The respective expressions for prediction bias and mean square prediction error (MSPE) can be also defined. The respective ACIs and BCIs are obtained using Eqs. (24–29) and the PIs of $Y = T_{s:n}\ (s \geq r+1)$ are computed using (43) and (49).

The MLEs of β, α_1 and α_2, and respective CIs are computed for sample sizes $n = 80$ and 100, with censoring schemes: $\tau = 9, 10, 11, 12$. The true values of the parameters used for simulating CEM Lomax model are $\beta = 0.8, 0.9, 1, \alpha_1 = 2$, and $\alpha_2 = 1.25$. In the context of prediction, we consider particular values of *n, r* and *s,* and then generate Type-II censored samples from this model according to the following cases: $\beta = 0.75, \alpha_1 = 1, \alpha_2 = 0.5, \tau = 2$ and $\beta = 0.9, \alpha_1 = 2, \alpha_2 = 1, \tau = 3$. Using these random samples, prediction biases and MSPEs of the predictors are computed. The so obtained results involving biases, MSEs, MSPEs as well the average lengths (ALs), and coverage probabilities (CPs) of the CIs and PIs are presented in Tables 1–7. The following remarks can be concluded follows from these tables:

1. It is clear that the MSEs and biases of the estimates decrease as n or r increases. This indicates that the method of estimation provides asymptotically consistent estimators of the CEM Lomax parameters.
2. As τ increases, more failures will occur before τ, which means more information about α_1, and then better performance of $\hat{\alpha}_1$. Though, it is observed that the bias and MSE of $\hat{\alpha}_2$ decrease as τ increases, the bias and MSE of $\hat{\beta}$ depend on the value of β.
3. It can be noticed from the values presented in the tables that the MSEs of β have small values when compared to the respective estimates of α_1 and α_2. It is evident that the MSEs of α_1 are smaller than that of α_2.
4. For the interval estimation, it is easily checked that the CIs obtained by bootstrap method outperform the ones based on the asymptotic normality approach. As a result of that, BCIs of β, α_1 and α_2 are recommended to be used.
5. For prediction problem, we notice that the prediction biases of the CMP are smaller than those of the MLP for all the considered cases. By considering the MSPE as an optimality criterion, it is checked that, the CMP performs better than the MLP. However, the ratio of MSPEs of MLPs to the MSPEs of CMPs becomes closer to 1 in many cases, especially when s is close to r. This observation can be explained by observing that the MLEs of the parameters are close to the corresponding PMLEs in most of the considered cases.

Table 1 MLEs, biases, MSEs and CPs of the 95% asymptotic CIs of $\beta, \alpha_1, \alpha_2$, with (n, r) = (80, 60).

$\beta = 0.8, \alpha_1 = 2, \alpha_2 = 1.25$						
	τ	**MLE**	**Bias**	**MSE**	**95% ACI Coverage**	**95% BCI Coverage**
β	9	0.7852	−0.0148	0.0785	0.948	0.960
	10	0.7227	−0.0773	0.0516	0.972	0.970
	11	0.6876	−0.1124	0.0465	0.992	0.986
	12	0.6599	−0.1401	0.0526	0.988	0.990
α_1	9	2.3482	0.3482	2.5051	0.882	0.957
	10	2.1212	0.1212	1.4445	0.923	0.957
	11	2.0301	0.0301	1.0995	0.954	0.959
	12	1.9257	−0.0743	0.9000	0.946	0.960
α_2	9	1.9915	0.7415	6.4717	0.874	0.956
	10	1.6670	0.4170	3.7032	0.893	0.946
	11	1.4333	0.1833	2.329	0.910	0.958
	12	1.2856	0.0356	1.9889	0.904	0.956

Table 2 MLEs, biases, MSEs and CPs of the 95% asymptotic CIs of $\beta, \alpha_1, \alpha_2$, with (n, r) = (100, 80).

$\beta = 0.8, \alpha_1 = 2, \alpha_2 = 1.25$						
	τ	**MLE**	**Bias**	**MSE**	**95% ACI Coverage**	**95% BCI Coverage**
β	9	0.8575	0.0575	0.0723	0.921	0.947
	10	0.8227	0.0227	0.0501	0.936	0.948
	11	0.7754	−0.0246	0.0394	0.963	0.963
	12	0.7469	−0.0531	0.0319	0.980	0.969
α_1	9	2.4178	0.4178	1.9530	0.876	0.951
	10	2.3054	0.3054	1.3815	0.880	0.940
	11	2.1744	0.1744	1.1948	0.911	0.954
	12	2.0859	0.0859	0.8862	0.930	0.958
α_2	9	2.0469	0.7969	5.3077	0.887	0.950
	10	1.9652	0.7152	4.9554	0.879	0.957
	11	1.7746	0.5246	3.3664	0.906	0.951
	12	1.6604	0.4140	2.6056	0.909	0.945

Table 3 MLEs, biases, MSEs, and CPs of the 95% asymptotic CIs of $\beta, \alpha_1, \alpha_2$, with (n, r) = (80, 60).

$\beta = 0.9, \alpha_1 = 2, \alpha_2 = 1.25$						
	τ	**MLE**	**Bias**	**MSE**	**95% ACI Coverage**	**95% BSCI Coverage**
β	9	0.7701	−0.1299	0.0718	0.988	0.991
	10	0.7442	−0.1558	0.0710	0.990	0.991
	11	0.7201	−0.1799	0.0782	0.982	0.998
	12	0.7047	−0.1953	0.0817	0.997	0.997
α_1	9	2.0643	0.0643	1.3899	0.967	0.967
	10	2.0279	0.0279	1.1223	0.944	0.965
	11	1.9282	0.0728	0.8448	0.936	0.979
	12	1.9133	−0.0669	0.7297	0.982	0.979
α_2	9	1.4948	0.2448	3.0729	0.900	0.963
	10	1.3131	0.0631	2.7393	0.874	0.965
	11	1.1537	−0.0963	1.1017	0.892	0.964
	12	1.0113	−0.2387	1.0616	0.850	0.950

6. As expected, for fixed values of n and r, the MSPEs of MLP and CMP increase as s increases, which is due to the fluctuation of the variable to be predicted as s gets large.

7. The PIs obtained using the HCD method competes the PIs obtained by the pivot method for all the considered cases. It can be also observed that, for fixed values of n and r, the ALs increase as s increases for the PIs obtained by both methods.

Table 4 | MLEs, biases, MSEs and CPs of the 95% asymptotic CIs of $\beta, \alpha_1, \alpha_2$, with (n, r) = (100, 80).

	$\boldsymbol{\beta = 0.9, \alpha_1 = 2, \alpha_2 = 1.25}$					
	τ	**MLE**	**Bias**	**MSE**	**95% ACI Coverage**	**95% BSCI Coverage**
β	9	0.8865	−0.0135	0.0559	0.952	0.950
	10	0.8160	−0.0840	0.0458	0.986	0.976
	11	0.7850	−0.1150	0.0417	0.988	0.984
	12	0.7623	−0.1377	0.0507	0.994	0.986
α_1	9	2.2341	0.2341	1.1963	0.888	0.942
	10	2.0222	0.0222	0.8070	0.940	0.964
	11	1.9864	−0.0136	0.6912	0.942	0.970
	12	1.8781	−0.1219	0.5280	0.970	0.966
α_2	9	1.7801	0.5301	2.7761	0.908	0.950
	10	1.5348	0.2848	2.1824	0.916	0.952
	11	1.4878	0.2378	1.7031	0.924	0.956
	12	1.2898	0.0398	1.5214	0.914	0.956

Table 5 | MLEs, biases, MSEs and CPs of the 95% asymptotic CIs of $\beta, \alpha_1, \alpha_2$, with (n, r) = (80, 60).

	$\boldsymbol{\beta = 1.0, \alpha_1 = 2, \alpha_2 = 1.25}$					
	τ	**MLE**	**Bias**	**MSE**	**95% ACI Coverage**	**95% BSCI Coverage**
β	9	0.8034	−0.1966	0.0872	1	1
	10	0.7883	−0.2117	0.1040	0.996	0.996
	11	0.8257	−0.1843	0.1466	0.982	0.994
	12	0.8369	−0.1631	0.1837	0.978	0.980
α_1	9	2.0240	0.0240	0.7260	0.968	0.96
	10	1.9995	−0.0005	0.7393	0.984	0.978
	11	2.1093	0.1093	0.6523	0.998	0.994
	12	2.1457	0.1457	0.7084	0.998	0.994
α_2	9	1.0685	−0.1815	1.0900	0.811	0.958
	10	0.8593	−0.3907	0.9046	0.790	0.970
	11	0.7611	−0.4889	0.7521	0.712	0.978
	12	0.6584	−0.5916	0.7753	0.720	0.988

Table 6 | MLEs, biases, MSEs and CPs of the 95% asymptotic CIs of $\beta, \alpha_1, \alpha_2$, with (n, r) = (100, 80).

	$\boldsymbol{\beta = 1.0, \alpha_1 = 2, \alpha_2 = 1.25}$.					
	τ	**MLE**	**Bias**	**MSE**	**95% ACI Coverage**	**95% BCI Coverage**
β	9	0.8823	−0.1177	0.0668	0.988	0.986
	10	0.8089	−0.1911	0.0677	0.998	0.996
	11	0.7950	−0.2050	0.0813	0.998	0.998
	12	0.7518	−0.2482	0.0920	1	1
α_1	9	2.0115	0.0115	0.9732	0.926	0.958
	10	1.8298	−0.1702	0.5595	0.966	0.964
	11	1.8188	−0.1812	0.5712	0.978	0.980
	12	1.7385	−0.2615	0.6001	0.984	0.986
α_2	9	1.5083	0.2583	2.4100	0.908	0.948
	10	1.2295	−0.0205	0.9654	0.936	0.950
	11	1.1304	−0.1196	1.2285	0.878	0.962
	12	1.0946	−0.1554	1.3214	0.880	0.956

6.2. Data Analysis

To illustrate the inference methods developed in this paper, we analyze a real data, which has been considered by Liu [23]. It represents the lifetimes (in seconds) of nanocrystalline embedded high-k device put under a specific test. Forty devices are put into a step-stress experiment with stress change time $\tau = 600$ seconds. Thirty-eight failures have been observed before the termination of the experiment. These data have been used previously by Amleh and Raqab [24]. The data are recorded as follows:

Table 7 Biases and MSPEs of point predictors and ALs of PIs.

		$\beta = 0.75, \alpha_1 = 1, \alpha_2 = 0.5, \tau = 2$					
(n, r)		**MLP**		**CMP**		**AL of 95% PI**	
	s	**Bias**	**MSPE**	**Bias**	**MSPE**	**Pivotal**	**HCD**
	46	−0.3649	1.1128	−0.1718	1.1506	1.2568	0.1480
	47	−0.4863	1.5260	−0.2544	1.3474	2.2454	1.8864
(60, 45)	48	−0.6471	2.3881	−0.3253	1.9904	3.2060	2.7991
	49	−0.6652	2.9289	−0.1780	2.6681	4.6869	4.1938
	50	−1.1352	4.3791	−0.4449	3.5368	6.2673	5.7453
	61	−0.1605	0.7793	−0.0201	0.8329	0.9126	0.1829
	62	−0.2297	0.8835	−0.0721	0.8433	1.4573	1.2443
(80, 60)	63	−0.2774	1.3145	−0.0611	1.2003	2.0844	1.8477
	64	−0.3710	1.5837	−0.1036	1.3801	2.6724	2.4234
	65	−0.5871	2.0118	−0.2542	1.6426	3.4052	3.1404
	76	−0.1718	0.5268	−0.0621	0.5345	0.6802	0.1516
	77	−0.2344	0.7573	−0.1035	0.7114	1.0803	0.9306
(100, 75)	78	−0.2611	0.9498	−0.1027	0.8503	1.5309	1.3679
	79	−0.2640	1.0555	−0.0701	0.9299	1.9418	1.7748
	80	−0.3032	1.3059	−0.0714	1.0778	2.3822	2.2087
		$\beta = 0.9, \alpha_1 = 2, \alpha_2 = 1, \tau = 3$					
(n, r)		**MLP**		**CMP**		**AL of 95% PI**	
	s	**Bias**	**MSPE**	**Bias**	**MSPE**	**Pivotal**	**HCD**
	46	−0.3741	1.6559	−0.1399	1.5706	1.5316	0.2866
	47	−0.3883	1.9687	−0.1320	1.7814	2.6049	2.2036
(80, 60)	48	−0.4973	2.5723	−0.1505	2.2118	3.8176	3.3597
	49	−0.6573	4.1886	−0.1659	3.5741	5.3672	4.8425
	50	−0.9158	5.4664	−0.2428	4.2851	7.1562	6.6011
	61	−0.3307	1.1359	−0.1711	1.1350	1.0854	0.1581
	62	−0.3293	1.4811	−0.1564	1.3253	1.6908	1.4511
(80, 60)	63	−0.3326	1.8630	−0.1091	1.6063	2.4758	2.2058
	64	−0.4379	2.0766	−0.1424	1.7844	3.0917	2.8215
	65	−0.4344	2.7592	−0.0541	2.2375	4.0470	3.7475
	76	−0.2012	0.9318	−0.0542	0.9263	0.8429	−3.631
	77	−0.1543	1.1033	−0.0131	1.0822	1.3402	1.1578
(100, 75)	78	−0.2474	1.3081	−0.0806	1.1830	1.7540	1.5775
	79	−0.2841	1.4982	−0.0770	1.2583	2.2571	2.0723
	80	−0.4999	1.7211	−0.2498	1.3557	2.6765	2.4929

Data on the lifetimes of nanocrystalline embedded high-k device

Stress Level					**Recorded Data**						
1	8	38	72	97	122	140	163	170	188	198	223
	256	257	265	448							
2	608	611	614	615	616	620	623	623	624	624	631
	636	646	654	660	673	675	680	684	692	693	730
	745										

For computational ease we divide all of the values by 100 and this will not affect the statistical inference. To visualize the accuracy of the Lomax distribution under the CEM, the true CDF of the lifetimes is plotted in Figure 1, along with the corresponding empirical CDF. To check the goodness-of-fit of the data to the Lomax distribution, Kolmogorov–Smirnov (K–S) test is applied. The K––S statistic of the distance between the fitted and the empirical distribution function is K–S = 0.1772 and the corresponding p-value = 0.7741. Therefore, it is reasonable to use the Lomax distribution under CEM as an appropriate model for fitting these data.

Suppose the life test ended when the 30-th lifetime is observed, i.e., we observe a Type-II censored sample with $n = 40, r = 30$. The problem is to obtain the MLEs of the parameters, point predictors of the unobserved lifetimes $Y = T_{s:n}, s = 31, 32, 33, 34, 35$ and the associated PIs.

First we compute the MLEs of β, α_1 and α_2 by solving (11), (12) and (14) simultaneously, it is found that $\hat{\beta} = 1.7517$, $\hat{\alpha}_1 = 17.2768$ and $\hat{\alpha}_2 = 0.7364$. For predicting the future censored failures, point predictors as well as PIs are displayed in Table 8. It can be observed that the values of the MLPs and CMPs are close to the true values. Moreover, the point predictors obtained are lying within all the considered PIs

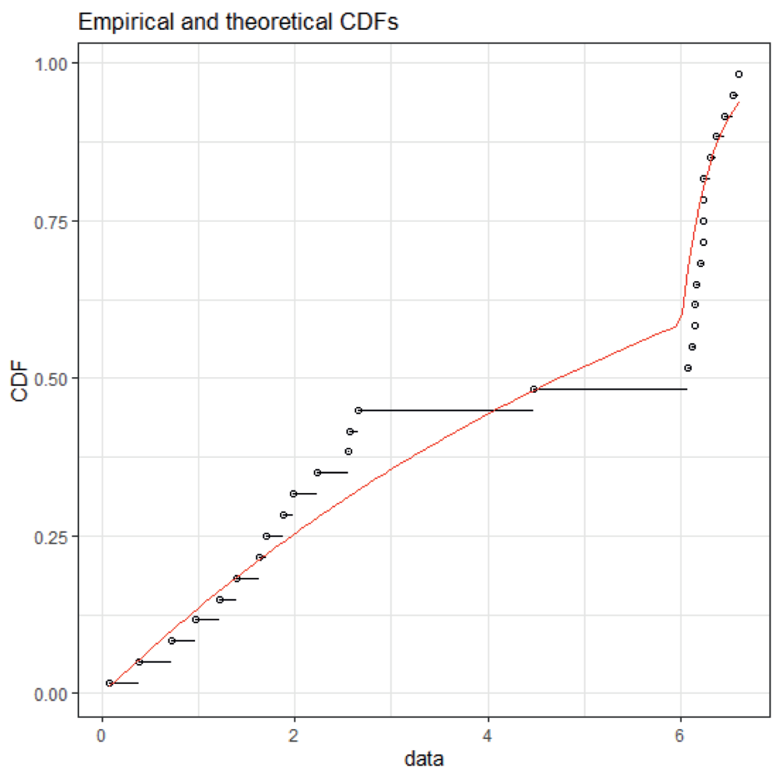

Figure 1 The empirical CDF (dots); and the estimated CDF based on MLE (solid line).

Table 8 Point and interval prediction for future lifetimes of $Y = T_{s:n}$.

s	True Value	MLP	CMP	95% Pivotal PI	95% HCD PI
31	6.73	6.600	6.664	(6.602, 6.973)	(6.600, 6.862)
32	6.75	6.669	6.769	(6.623, 7.236)	(6.607, 7.134)
33	6.80	6.743	6.896	(6.664, 7.539)	(6.644, 7.435)
34	6.84	6.825	7.053	(6.722, 7.918)	(6.706, 7.830)
35	6.92	6.922	7.251	(6.800, 8.424)	(6.793, 8.381)

except when $s = r + 1$. It can be observed that the CMP has a clear advantage if s is close to r, while the MLP is closer to the true values when s is close to n. It can be observed also that all PIs obtained contain the true values of the future order statistics. The PIs become wider when s gets large, the reason is that the variation of $Y = T_{s:n}$ tends to be high as Y moves away from the observed lifetimes. Although all PIs are close in the sense of AL criterion but the PIs obtained by HCD method have shortest lengths.

7. CONCLUSION

In this paper, we have addressed the ALTs under CEM Type-II censoring Lomax data to produce the failure time data of highly reliable units in specified conditions. Under this setup, we have considered the estimation problem of the model parameters using the maximized likelihood and bootstrap methods. Further, point prediction by using the maximum likelihood prediction and conditional median prediction methods are also addressed. The PIs including pivot and conditional highest density arguments are considered. It is observed via MC simulation studies that the CIs based on the bootstrap method are more valid than the CIs based on the asymptotic normality method. In the prediction front, the CMP as a point predictor and highest conditional density prediction interval outperform the MLP and pivot prediction interval, respectively.

CONFLICTS OF INTEREST

The authors declare that there is no conflict of interest.

AUTHORS' CONTRIBUTIONS

The first author performed the writing up of the material, checking and conducting the numerical simulation while the second author developed the models, methodology and editing of the paper.

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