

Research Article

A New Case of Separability in a Quartic Hénon-Heiles System

Nicola Sottocornola*

Department of Mathematics and Statistics, Zayed University, Abu Dhabi, UAE

ARTICLE INFO

Article History

Received 17 November 2020
 Accepted 09 April 2021

Keywords

Integrable systems
 separation of coordinates
 integration in quadratures

2020 Mathematics Subject Classification

37J35
 34M55

ABSTRACT

There are four quartic integrable Hénon-Heiles systems. Only one of them has been separated in the generic form while the other three have been solved only for particular values of the constants. We consider two of them, related by a canonical transformation, and we give their separation coordinates in a new case.

© 2021 The Author. Published by Atlantis Press B.V.

This is an open access article distributed under the CC BY-NC 4.0 license (<http://creativecommons.org/licenses/by-nc/4.0/>).

1. INTRODUCTION

Hénon-Heiles (HH) systems are Hamiltonian systems in \mathbb{R}^4 endowed with the standard symplectic form $dP_1 \wedge dQ_1 + dP_2 \wedge dQ_2$. The Hamiltonian function has the form

$$H = \frac{1}{2}(P_1^2 + P_2^2) + V(Q_1, Q_2)$$

where V is a polynomial function. There are four nontrivial integral cases with quartic potential whose name in the literature is HH4 followed by three numbers giving the ratios of the coefficients of the quartic monomials: 1:2:1, 1:6:1, 1:6:8 and 1:12:16. The generalized HH systems are obtained adding inverse terms to the potential V , without destroying the integrability of the system.

The problem of the integration in quadratures of these systems has been extensively studied in the last decades. The most efficient and elegant method for this purpose, is to find canonical coordinates that separate the Hamilton-Jacobi equation. In this paper we will deal with the delicate task of characterizing such coordinates. The difficulty of the task is well known so that, despite decades of efforts, only one of these four systems has been separated in the generic form: HH4 1:2:1. For the other three systems, the separation coordinates are known only in some degenerate cases. For HH4 1:12:16, the best available results can be found here [7]. In this paper we deal with HH4 1:6:1 and HH4 1:6:8 only.

Let's now introduce them.

2. THE LINK BETWEEN HH4 1:6:1 AND HH4 1:6:8

The generalized Hamiltonian function has the form:

$$H_{161} = \frac{1}{2}(P_1^2 + P_2^2) - \frac{1}{2}\omega(Q_1^2 + Q_2^2) - \frac{Q_1^4}{32} - \frac{3Q_1^2Q_2^2}{16} - \frac{Q_2^4}{32} - \frac{k_1^2}{2Q_1^2} - \frac{k_2^2}{2Q_2^2} \quad (1)$$

and depends on three arbitrary constants, ω , k_1 and k_2 . The last two terms are the inverse terms and the ratios of the coefficients of the quartic terms are 1:6:1 as expected. This Hamiltonian system possesses an integral of motion that we call K :

$$K_{161} = \left(P_1P_2 - Q_1Q_2 \left(\frac{Q_1^2}{8} + \frac{Q_2^2}{8} + \omega \right) \right)^2 - k_1^2 \left(\frac{P_2^2}{Q_1^2} - \frac{Q_2^2}{4} \right) - k_2^2 \left(\frac{P_1^2}{Q_2^2} - \frac{Q_1^2}{4} \right) + \frac{k_1^2k_2^2}{Q_1^2Q_2^2} \quad (2)$$

*Email: nicola.spinelli@zu.ac.ae.

The reader can easily check that these two functions are in involution with respect to the standard Poisson bracket hence the system is Liouville integrable. The separation coordinates for this system are unknown.

The canonical change of coordinates [1]:

$$\begin{aligned} Q_1 &= \frac{R_+}{2} \\ Q_2 &= \frac{R_-}{2} \\ P_1 &= \frac{R_+}{2} \left(-\frac{p_2}{q_2} - \frac{q_1}{2} - \frac{k_1 - k_2}{q_2^2} \right) + 2 \frac{k_1}{R_+} \\ P_2 &= \frac{R_-}{2} \left(-\frac{p_2}{q_2} + \frac{q_1}{2} - \frac{k_1 - k_2}{q_2^2} \right) - 2 \frac{k_2}{R_-} \end{aligned} \tag{3}$$

where

$$R_{\pm}^2 = -4q_1^2 \pm 8p_1 - 2q_2^2 + 16 \frac{p_2^2}{q_2^2} \mp 16 \frac{q_1 p_2}{q_2} + 32 \frac{(k_1 - k_2) p_2}{q_2^3} \mp 16 \frac{(k_1 - k_2) q_1}{q_2^2} + 16 \frac{(k_1 - k_2)^2}{q_2^4} + 16 \omega$$

changes HH4 1:6:1 into HH4 1:6:8:

$$\begin{aligned} h_{168} &= \frac{1}{2}(p_1^2 + p_2^2) + \frac{\omega(4q_1^2 + q_2^2)}{2} - \frac{q_1^4}{2} - \frac{3q_1^2 q_2^2}{8} - \frac{q_2^4}{16} - \gamma q_1 + \frac{\beta}{2q_2^2} \\ k_{168} &= \frac{1}{4} \left(p_2^2 - \frac{q_2^2(2q_1^2 + q_2^2 - 8\omega)}{8} + \frac{\beta}{q_2^2} \right)^2 - \frac{q_2^2(q_2 p_1 - 2q_1 p_2)^2}{16} \\ &\quad - \frac{\gamma}{4} \left(2\gamma q_2^2 - 4q_2 p_1 p_2 + \frac{q_1 q_2^4}{2} + q_1^3 q_2^2 + 4p_2^2 q_1 - 4\omega q_1 q_2^2 + 4 \frac{q_1 \beta}{q_2^2} \right). \end{aligned} \tag{4}$$

These functions are usually written in a slightly different form in the literature. It's easy to pass from one form to the other with a simple change of coordinates. The relationships between the coefficients of the two systems are

$$\gamma = \frac{1}{2}(k_1 + k_2) \quad \beta = -(k_1 - k_2)^2. \tag{5}$$

The separation coordinates of (4), in the case $\gamma = \omega = 0$, were found using Painlevé analysis in 1994 [6]:

$$2q_1^2 + q_2^2 - \frac{8p_2^2 \pm 8\sqrt{R}}{q_2^2} \tag{6}$$

where R is the polynomial obtained replacing $\beta = 0$ in k_{168} . The case $\omega \neq 0$ is treated in [8].

Inverting the change of coordinates (3), they provide the separation coordinates for HH4 1:6:1 in the symmetric case $k_1^2 = k_2^2$. As far as we know, no other cases have been separated to this day. In this paper we solve the case $k_1 k_2 = 0$. Before that, let's turn our attention to an alternative method to see the process of separation of coordinates.

3. THE KOWALEWSKI CONDITIONS

In 2005 F. Magri published a paper [3] revisiting the famous problem solved by S. Kowalewski in 1888 [2]: the so called Kowalewski top. The method adopted in the paper is general and can be applied even in the non-Hamiltonian case, provided that a convenient number of commuting vector fields and first integrals are present. It was subsequently refined in several publications over the years and finally presented in a complete form in [4,5], where the reader will find all the proofs that are omitted here.

Let's now summarize the key ideas in the case of a symplectic system in \mathbb{R}^4 with Hamiltonian functions H and K .

The method assumes the presence of a second Poisson tensor P_2 compatible with the tensor P_1 associated to the symplectic structure:

$$[P_1, P_2] = [P_2, P_2] = 0$$

where $[\dots]$ is the Schouten bracket. We also assume that the two Hamiltonian functions H and K are in involution with respect to the Poisson bracket associated to P_2 :

$$P_2(dH, dK) = 0.$$

At this stage one can build the torsionless, recursive operator $N = P_2 P_1^{-1}$. If N has maximal rank, the two distinct eigenvalues provide (half of) the separation coordinates. The explicit determination of the compatible Poisson tensor P_2 , that requires the calculation of six unknown

functions, can result quite cumbersome even in relatively simple cases. The number of unknown functions can be reduced to four, the components of a vector field X , looking for tensors $P_2 = L_X(P_1)$. Using this method the bi-Hamiltonian structure of cubic Hénon-Heiles systems can be calculated directly [9].

However, in the present case, the explicit determination of a bi-Hamiltonian structure in natural coordinates seems definitely too complicated. The good news is that one does not have to build up N (and, before this, P_2) in order to calculate its eigenvalues. To understand this point, we start observing that N acts on the vector fields tangent to the Lagrange foliation defined by the level surfaces of H and K [9]. This bi-dimensional foliation is spanned by the Hamiltonian vector fields X_H and X_K so that

$$\begin{aligned} NX_H &= m_1X_H + m_2X_K \\ NX_K &= m_3X_H + m_4X_K \end{aligned}$$

for some functions m_1, \dots, m_4 . It is now clear that the so called **control matrix** $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$ is nothing but the restriction of N to the leaves of the foliation, written in the basis associated with X_H and X_K . Furthermore the tensor M is also torsionless since it is the restriction of a torsionless tensor to an invariant surface. This is the first of the two properties that characterize M . The second one is that the vector fields X_H and X_K must commute with respect to the modified commutator

$$[X, Y]_M := [MX, Y] + [X, MY] - M[X, Y]$$

defined on the vector fields tangent to the Lagrangian foliation.

F. Magri proved that these two properties are necessary and sufficient conditions for the system to be separable and for the eigenvalues of M to be separation coordinates [5]. The point of interest in all this discussion is that these two conditions, $T(N) = 0$ and $[X, Y]_M = 0$, can be reduced to four differential constraints, on the entries of M , called Kowalewski Conditions (KC):

$$\begin{aligned} \{m_3, H\} &= \{m_1, K\} \\ \{m_4, H\} &= \{m_2, K\} \\ \{m_1m_3 + m_3m_4, H\} &= \{m_1^2 + m_2m_3, K\} \\ \{m_2m_3 + m_4^2, H\} &= \{m_1m_2 + m_2m_4, K\} \end{aligned} \tag{7}$$

We also need an extra condition for the new coordinates to be canonical: the trace and the determinant of M must be in involution

$$\{m_1 + m_4, m_1m_4 - m_2m_3\} = 0. \tag{8}$$

In order to solve the KC one has to solve four differential equations in four unknown functions m_1, \dots, m_4 ; this can be quite challenging. A first step could be to select a particular class of solutions that is easier to calculate but general enough to include most of the significant examples in the literature. The experience suggests this form for M :

$$\begin{aligned} m_1 &= aF^2 + bFG + cG^2 + dF + eG + f \\ m_2 &= gF + v \\ m_3 &= pF^2 + qFG + rG^2 + sF + tG + u \\ m_4 &= gG + w \end{aligned} \tag{9}$$

where all the coefficients a, b, c, \dots are constants of the motion while F and G are unknown functions.

If we agree to denote by

$$\dot{f} = X_H(f) \quad f' = X_K(f)$$

the derivatives of a function f along the given Hamiltonian fields, and replacing (9) into (7), we obtain the following

Proposition 3.1. *If the functions F and G are solutions of the equations*

$$\begin{aligned} F' &= \dot{G} \\ G' &= (\mu F + \tau)\dot{G} - (\mu G + v)\dot{F} \end{aligned} \tag{10}$$

where μ, τ and v are constants of the motion, then the functions

$$\begin{aligned} m_1 &:= -e(\mu F^2 + \tau F - G) - v(\mu F + \tau) + w \\ m_2 &:= eF + v \\ m_3 &:= -e(\mu FG + vF) - v(\mu G + v) \\ m_4 &:= eG + w \end{aligned} \tag{11}$$

verify the KC.

Therefore, if we limit our search to solutions of the form (9), the problem reduces to two differential equations (10) in two unknown functions F and G .

In [8] we suggested the method of the vector field Z , in order to reduce the task to the search of one single function V (the potential function) and a few constants.

Let's outline the method in the case of HH4 1:6:1. The idea is to extend the phase space including the constants of the problem as new coordinates, so turning our system into a Poisson one. In our example, we can extend the phase space to \mathbb{R}^6 with coordinates $(P_1, P_2, Q_1, Q_2, k_1, k_2)$. The Poisson tensor is obtained adding two extra columns and two extra lines of zeros to P_1 . We now consider the vector field Z so defined:

$$Z = X_V + w_1 \frac{\partial}{\partial k_1} + w_2 \frac{\partial}{\partial k_2}, \quad w_1, w_2 \in \mathbb{R}.$$

V, w_1, w_2 are unknown and X_V is the Hamiltonian vector field associated to V . Sometimes it may be useful to look for a potential function of the form $V = \ln f$ (some examples are given in [8]). The next step is to define the "Fundamental Functions" F and G of Proposition 3.1 in this way:

$$F = Z(H_{161}) \quad G = Z(K_{161}). \tag{12}$$

A simple calculation proves that the first of equations (10) is automatically verified with any choice of the potential function [8]. This means that the problem is finally reduced to the determination of a single function V (plus, eventually, the constants w_1, w_2) verifying the second equation in (10). It should be stressed that the involutivity condition (8) has to be checked independently from the KC.

It's time now to see how the method of the vector field Z can provide the separation coordinates for HH4 1:6:1 in the case $k_1 k_2 = 0$.

4. HH4 1:6:1 IN THE CASE $k_1 k_2 = 0$

The system (1)-(2) is invariant under the symmetry

$$(P_1, P_2, Q_1, Q_2, k_1, k_2) \longrightarrow (P_2, P_1, Q_2, Q_1, k_2, k_1)$$

so it's enough to solve the case $k_2 = 0$:

$$H = \frac{1}{2}(P_1^2 + P_2^2) - \frac{1}{2}\omega(Q_1^2 + Q_2^2) - \frac{Q_1^4}{32} - \frac{3Q_1^2 Q_2^2}{16} - \frac{Q_2^4}{32} - \frac{k^2}{2Q_1^2}$$

$$K = \left(P_1 P_2 - Q_1 Q_2 \left(\frac{Q_1^2}{8} + \frac{Q_2^2}{8} + \omega \right) \right)^2 - k^2 \left(\frac{P_2^2}{Q_1^2} - \frac{Q_2^2}{4} \right) \tag{13}$$

In order to apply the method of the field Z we extend the phase space to \mathbb{R}^5 with coordinates (P_1, P_2, Q_1, Q_2, k) . A first remark is that the system is homogeneous with respect to the following gradation:

$$P_1, P_2, \omega \sim 2 \quad Q_1, Q_2 \sim 1 \quad k \sim 3. \tag{14}$$

We detail now the steps of the algorithm.

1. We look for a vector field of the form $Z = X_V + w \partial/\partial k$ with $V = \ln f$ as suggested in Section 3. Our problem is now to find the unknown function f and the constant w .
2. Because the system is homogenous we look for homogeneous Fundamental Functions F and G . For that purpose, the presence of the term $\partial/\partial k$ forces $f \sim 4$.
3. Replacing f with the general homogeneous polynomial of degree 4 with respect to the gradation (14) and adding inverse terms like kP_2/Q_1 suggested by the form of the Hamiltonian functions, we are now able to calculate F and G with (12).
4. We can choose the coefficients of f and w in such a way that the second equation in (10) is verified:

$$V = \frac{1}{k} \ln \left(P_1 P_2 - Q_1 Q_2 \left(\frac{Q_1^2}{8} + \frac{Q_2^2}{8} + \omega \right) + \frac{kP_2}{Q_1} + \frac{kQ_1}{2} \right) \tag{15}$$

and $Z = X_V + \frac{1}{k} \frac{\partial}{\partial k}$.

5. We choose the integrals of motion in M in such a way that the involutivity condition (8) is verified (see (16)).

The results of this discussion can be summarized in the following

Theorem 4.1. Consider

- the vector field $Z = X_V + \frac{1}{k} \frac{\partial}{\partial k}$, where V is the function in (15);
- the functions F and G defined in (12);
- the Control Matrix

$$M = \begin{pmatrix} -2k^2F + 2H & 1 \\ -2k^2G + 4K + 8k^2\omega & 2H \end{pmatrix}. \tag{16}$$

Then the eigenvalues of M are separation (canonical) coordinates for (13).

Proof. A straightforward calculation gives

$$F = \frac{2Q_1^2Q_2 + 4Q_1P_1 + 4k}{Q_1^4Q_2 + (Q_2^3 + 8\omega Q_2 - 4k)Q_1^2 - 8P_1P_2Q_1 - 8kP_2}$$

$$G = 2\omega - \frac{Q_1^2}{4} - \frac{Q_2^2}{4} - P_2 - \frac{k(Q_1^4 + (3y^2 + 8\omega + 4P_2)Q_1^2 + 4Q_1Q_2P_1 + 4kQ_2)}{Q_1^4Q_2 + (Q_2^3 + 8\omega Q_2 - 4k)Q_1^2 - 8P_1P_2Q_1 - 8kP_2}$$

and (16) provides the four entries m_1, \dots, m_4 of the Control Matrix. With these functions (7) and (8) are verified so that the eigenvalues of M are separation coordinates. □

5. FINAL REMARKS

- If we replace K with $K - H^2$ the Control Matrix can be written in the simplified form:

$$M = \begin{pmatrix} -2kF + 4H & 1 \\ -2kG + 4K & 0 \end{pmatrix}.$$

F. Magri already pointed out a similar behavior of the Kowalewski top [5].

- Another set of separation coordinates can be obtained using quadratic functions in F and G :

$$M = \begin{pmatrix} -2k^2F^2 + G & F \\ -2k^2FG + 4KF & G \end{pmatrix} \tag{17}$$

- HH4 1:6:8 has been solved only in the particular cases $\beta\gamma = 0$ [10]. The eigenvalues of (16) or (17), through the change of coordinates (3), provide the separation coordinates for the case $\beta = -4\gamma^2$.
- HH4 1:6:1 with $k_1 = k_2 = k$ has already been solved using the method of the field Z and a potential function $V = \ln f$ [8]. The functions f for the cases $k_1 = k_2 = k$ and $(k_1, k_2) = (k, 0)$ are, respectively,

$$P_1P_2 - Q_1Q_2 \left(\frac{Q_1^2}{8} + \frac{Q_2^2}{8} + \omega \right) + \frac{kP_2}{Q_1} + \frac{kP_1}{Q_2} + \frac{k^2}{Q_1Q_2}$$

and

$$P_1P_2 - Q_1Q_2 \left(\frac{Q_1^2}{8} + \frac{Q_2^2}{8} + \omega \right) + \frac{kP_2}{Q_1} + \frac{kQ_1}{2}.$$

The idea is to guess, from these particular examples, the form of f for the generic case. The first part of the function is independent from the constants so it is reasonable to expect that it remains unchanged but for the last terms the situation is uncertain. For instance it is not clear why the term $kQ_1/2$ does not appear in the case $k_2 = k_1 = k$.

The generic case remains unsolved.

CONFLICTS OF INTEREST

The author declares no conflicts of interest.

ACKNOWLEDGEMENT

This work was funded by Zayed University, Abu Dhabi, through the RIF Grant R18050.

REFERENCES

- [1] S. Baker, V.Z. Enolskii, A.P. Fordy, Integrable quartic potentials and coupled KdV equations, *Phys. Lett. A* 201 (1995), 167–174.
- [2] S. Kowalewski, Sur le problème de la rotation d'un corps solide autour d'un point fixe, *Acta Math.* 12 (1889), 177–232.
- [3] F. Magri, The Kowalewski's top and the method of Syzygies, *Ann. Inst. Fourier, Grenoble* 55 (2005), 2147–2159.
- [4] F. Magri, The Kowalewski's top revisited, 2018. Available from: <https://arxiv.org/abs/1809.01879v1>.
- [5] F. Magri, The Kowalewski separability conditions. in: I. Krichever, S. Novikov, O. Ogievetsky, S. Shlosman (Eds.), *Volume Integrability, Quantization, and Geometry. Proceedings of Symposia in Pure Mathematics*, AMS, 2021.
- [6] V. Ravoson, A. Ramani, B. Grammaticos, Generalized separability for a Hamiltonian with nonseparable quartic potential, *Phys. Lett. A* 191 (1994), 91–95.
- [7] N. Sottocornola, Explicit integration of a generic Hénon-Heiles system with quartic potential, *J. Nonlinear Math. Phys.* 24 (2017), 346–355.
- [8] N. Sottocornola, Separation coordinates in Hénon-Heiles systems, *Phys. Lett. A* 383 (2019), 126027.
- [9] N. Sottocornola, Direct construction of a bi-Hamiltonian structure for cubic Hénon-Heiles systems, *J. Geom. Symmetry Phys.* 57 (2020), 99–109.
- [10] C. Verhoeven, M. Musette, R. Conte, General solution for Hamiltonians with extended cubic and quartic potentials, *Theor. Math. Phys.* 134 (2003), 128–138.