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Research Article

The Arithmetic Operator of Fuzzy Regular Prismoid Numbers and Its Application to Fuzzy Risk Analysis

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ABSTRACT

We devote to study the arithmetic operator of fuzzy regular prismoid numbers as well as the degree of similarity between fuzzy regular prismoid numbers, and then the arithmetic operator and the degree of similarity are applied in risk analysis. Firstly, the arithmetic operator of fuzzy regular prismoid numbers are researched, and some properties of the arithmetic operators are discussed. At the same time, the fuzzy regular prismoid numbers approximation of 2-dimensional fuzzy numbers is discussed as an indispensable part to study the fuzzy risk analysis. Then, the degree of similarity between fuzzy regular prismoid numbers are deeply researched, which are the basic work of studying the fuzzy risk analysis. Finally, the proposed results are applied in risk analysis of a company.

1. INTRODUCTION

Fuzzy numbers are a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models. Since Zadeh [1] put forward the concept of fuzzy numbers to describe the properties of probability functions in 1975, many researchers have been involved in the development of various aspects of the theory and applications of fuzzy numbers. With further study of the theory and applications of fuzzy numbers, the following forms of fuzzy numbers are produced respectively. To study the operations of fuzzy numbers, Dubois and Prade [2] proposed the conception of L-R fuzzy number in 1978. L-R fuzzy numbers as the most general form of fuzzy numbers have been used extensively [3]. With the extensive use of fuzzy numbers, Kaufmann and Gupta [4] introduced the conception of triangular fuzzy number to solve the economic problems. In order to present a comparison method for fuzzy numbers Cheng [5] presented the notion of trapezoid fuzzy number in 1998. As a natural generalization of L-R fuzzy numbers from one-dimension to n-dimension, Wang and Wu [6] generated the concept of fuzzy n-cell numbers in 2002. Later, Wang *et al.* [7] extended to a fuzzy n-ellipsoid numbers to solve the problem of uncertainty multichannel digital information. For reasons of simplifying the calculation of 2-dimensional fuzzy numbers, Hai *et al.* [8] put forward the concept of fuzzy 2-cell prismoid numbers in 2020. With the wide application of fuzzy numbers, a large number of outstanding results are constantly emerging. For instance, in view of the fuzzy sets and fuzzy numbers may have some degree of uncertainty and error when available data either come from unreliable sources or refer to events in the future, Seiti *et al.* [9] presented a novel concept R-numbers for a better modeling of the risks and errors associated with fuzzy numbers. In R-numbers, the membership function has not been taken into account in risk modeling of the fuzzy sets, then Seiti *et al.* [10] suggested a concept called R-sets in 2021.

The arithmetic operators as the root in the theory and application for fuzzy numbers were discussed by enormous researchers [11,12]. In 1975, Zadeh [13] first proposed the arithmetic operations on fuzzy numbers by the use of the extension principle. Dubois and Prade [14] made used of discretized fuzzy numbers to successfully give the arithmetical operations of fuzzy numbers which is the exact analytical fuzzy operations. Later, Dubois and Prade [15] discussed the nonlinear operations of fuzzy multiplication and fuzzy division on triangular fuzzy numbers by the interval arithmetic. The arithmetic operations of fuzzy numbers are usually approached either by use of the interval arithmetic or the extension principle. The interval fuzzy arithmetic is the most common approach in different applications, due to its simplicity and the availability of computational methods [16]. Based on Archimedean t-norms, Wagenknech *et al.* [17] derived some formulas for the inclusion of LR-number arithmetic operations. In 2005, Guerra and Stefanin [18] studied the arithmetic operations on fuzzy numbers by α-cut. In the category-theoretic point of view Bica [19] obtained a construction of additive and multiplicative structures for fuzzy numbers in 2007. Considering that the Hukuhara difference [20] is not invertible operations and appears to have several limitations and restrictive,

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Stefanini [21] proposed the difference and division of fuzzy numbers as inverse operations to addition and multiplication. In 2014, Cano *et al.* [22] proposed a single- level constrained fuzzy arithmetic which is interval arithmetic on α-levels. Based on the summation and multiplication for random variables, Stupnanova introduced the arithmetics of fuzzy numbers in [23]. The interval fuzzy arithmetic is criticized for accumulation of fuzziness, a phenomenon that causes overestimation of uncertainties in the resulting [24]. This problem can be reduced by the extension principle fuzzy arithmetic [25]. Existing computational methods extension principle fuzzy arithmetic are mainly divided into three methods: min t-norm, drastic product t-norm as well as product and Lukasiewicz t-norms. Klir [26] introduced an extended fuzzy arithmetic using the min t-norm on trapezoidal fuzzy numbers. Mesiar [27] as well as Hong and Do [28] developed a computational method for implementing extended fuzzy arithmetic using the drastic product t-norm on triangular fuzzy numbers. In order to reduce the uncertainty overestimation problem, Seresht and Fayek [25] introduced an extended fuzzy arithmetic using product and Lukasiewicz t-norms on triangular fuzzy numbers. In view of the arithmetical operations based on Zadeh's extension principle may no longer be preserve the basic operational properties of fuzzy numbers, Holcapek and Stepnicka [29] presented a novel framework for arithmetics of extensional fuzzy numbers that preserved the important algebraic properties of the arithmetic of real numbers.

As part of applications for fuzzy numbers, researchers began to study fuzzy risk analysis [30]. With the development of theory and applications of fuzzy risk analysis, there are more and more researchers have devoted to study it [31–34]. Just as important, the similarity measure between fuzzy numbers was studied by many researchers to measure similarity [35–39]. All of these researches are concentrated on 1-dimensional fuzzy numbers. In many instances, it becomes more reasonable that the practical problem is described by n-dimensional fuzzy numbers [40,41]. For instance, success of a company results in part from its working efficiency of employee which based on their production speed and the quality of their products. If the production speed and the quality of products are denoted by x and y, respectively, then the working efficiency of the employee can be characterized by two-dimension quantity (x, y). If the quantity is an estimated quantity, then using a two-dimension fuzzy number to express the business management is more suitable than a crisp two-dimension quantity. For example, if the production speed and the quality of products are about 0.85 and 0.985 respectively, then the person's working state can be expressed by the 2-dimension fuzzy number as following:

$$\tilde{u}(x,y)=\begin{cases}\dfrac{x-0.72}{0.06}, \ 0.72\le x\le 0.78, \ \dfrac{0.05x+0.0198}{0.06}\le y\le \dfrac{-0.01x+0.0672}{0.06},\\[2ex] \dfrac{y-0.93}{0.05}, \ 0.93\le y\le 0.98, \ \dfrac{0.06y-0.0198}{0.05}\le x\le \dfrac{-0.05y+0.095}{0.05},\\[2ex] 1, \ 0.78<x<0.92, \ 0.98<y<0.99,\\[2ex] \dfrac{0.97-x}{0.05}, \ 0.92\le x\le 0.97, \ \dfrac{0.05x-0.095}{-0.05}\le y\le \dfrac{0.01x+0.0403}{0.05},\\[2ex] \dfrac{1-y}{0.01}, \quad 0.99\le y\le 1, \quad \dfrac{0.06y-0.06}{-0.01}\le x\le \dfrac{-0.05y+0.0403}{-0.01},\\[2ex] 0, \quad \text{otherwise}.\end{cases}$$

Then the working efficiency of employee can be expressed by the 2-dimensional fuzzy number which is named fuzzy regular prismoid number due to its spatial graph in this paper.

The main tasks of dealing with fuzzy risk analysis are two aspects. One is arithmetic operators of fuzzy numbers. Usually the arithmetic operator of fuzzy regular prismoid numbers is not a fuzzy regular prismoid numbers. How to solve this problem, it becomes particularly important and urgent. The other aspect is the similarity of the fuzzy numbers. The purpose of this paper is to study the arithmetic operators and the similarity of fuzzy regular prismoid numbers, which are used to handle fuzzy risk analysis problems. From this perspective, the presented results should be useful for fuzzy risk analysis. The paper is organized as following. Firstly, arithmetic operators of fuzzy regular prismoid numbers are defined, which are the basic work of studying fuzzy risk analysis. Meanwhile, the fuzzy regular prismoid numbers approximation of 2-dimensional fuzzy numbers is discussed. In Section 3, we start by considering the radius of gyration for fuzzy regular prismoid numbers, and then based on distance and the radius of gyration for fuzzy regular prismoid numbers a similarity measure for fuzzy regular prismoid numbers is being proposed. Finally, we use the regular prismoid numbers to deal with fuzzy risk analysis problems in Section 4.

2. THE ARITHMETIC OPERATION AND FUZZY REGULAR PRISMOID NUMBERS APPROXIMATION OF 2-DIMENSIONAL FUZZY NUMBERS

Throughout this study, $F(R^n)$ denotes the set of all fuzzy subsets on the n-dimensional Euclidean space R^n. If $\tilde{u}\in F(R^n)$, $r\in(0,1]$, then we write the r-level sets of $\tilde{u}$ as $[\tilde{u}]^r=\{x\in R^n : \tilde{u}(x)\ge r\}$. Suppose that $\tilde{u}\in F(R^n)$, satisfies the following conditions:

(1) $\tilde{u}$ is a normal fuzzy set, i.e., an $x_0 \in R^n$ exists such that $\tilde{u}(x_0) = 1$,

(2) $\tilde{u}$ is a convex fuzzy set, i.e., $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min\{\tilde{u}(x), \tilde{u}(y)\}$ for any $x, y \in R^n$ and $\lambda \in [0,1]$,

(3) $\tilde{u}$ is upper semicontinuous,

(4) $[\tilde{u}]^0 = \overline{\{x \in R^n : \tilde{u}(x) > 0\}} = \overline{\bigcup_{r\in(0,1]}[\tilde{u}]^r}$ is compact, where $\overline{A}$ denotes the closure of A.

Then $\tilde{u}$ is called a fuzzy number. We use E^n to denote the fuzzy number space [42].

It is clear that each $u \in R^n$ can be considered as a fuzzy number $\tilde{u}$ defined by

$$\tilde{u}(x) = \begin{cases} 1, & x = u \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $\tilde{u}, \tilde{v} \in E^2$, and the membership function of the projective for $\tilde{u}, \tilde{v}$ on the xoz and yoz plane satisfy

$$\tilde{p}_u(x) = \begin{cases} L_u(x), & a_x \leq x < b_x, \\ 1, & b_x \leq x \leq c_x, \\ R_u(x), & c_x < x \leq d_x, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tilde{p}_u(y) = \begin{cases} L_u(y), & a_y \leq y < b_y, \\ 1, & b_y \leq y \leq c_y, \\ R_u(y), & c_y < y \leq d_y, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tilde{p}_v(x) = \begin{cases} L_v(x), & a_x \leq x < b_x, \\ 1, & b_x \leq x \leq c_x, \\ R_v(x), & c_x < x \leq d_x, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tilde{p}_v(y) = \begin{cases} L_v(y), & a_y \leq y < b_y, \\ 1, & b_y \leq y \leq c_y, \\ R_v(y), & c_y < y \leq d_y, \\ 0, & \text{otherwise.} \end{cases}$$

Let $R^{-1}_{\tilde{p}_u(x)}(r)$, $R^{-1}_{\tilde{p}_v(x)}(r)$, $L^{-1}_{\tilde{p}_u(x)}(r)$, $L^{-1}_{\tilde{p}_v(x)}(r)$, $R^{-1}_{\tilde{p}_u(y)}(r)$, $R^{-1}_{\tilde{p}_v(y)}(r)$, $L^{-1}_{\tilde{p}_u(y)}(r)$ and $L^{-1}_{\tilde{p}_v(y)}(r)$ be the inverse function of $R_u(x)$, $R_v(x)$, $L_u(x)$, $L_v(x)$, $R_u(y)$, $R_v(y)$, $L_u(y)$ and $L_v(y)$, respectively. The weighted pseudometric $D : E^2 \times E^2 \to [0, +\infty)$ between $\tilde{u}$ and $\tilde{v}$ is defined by

$$D(\tilde{u}, \tilde{v}) = \frac{1}{2}\left\{\left[\int_0^1 f(r)(R^{-1}_{\tilde{p}_u(x)}(r) - R^{-1}_{\tilde{p}_v(x)}(r))^2 dr + \int_0^1 f(r)(R^{-1}_{\tilde{p}_u(y)}(r) - R^{-1}_{\tilde{p}_v(y)}(r))^2 dr\right]^{\frac{1}{2}} + \left[\int_0^1 f(r)(L^{-1}_{\tilde{p}_u(x)}(r) - L^{-1}_{\tilde{p}_v(x)}(r))^2 dr + \int_0^1 f(r)(L^{-1}_{\tilde{p}_u(y)}(r) - L^{-1}_{\tilde{p}_v(y)}(r))^2 dr\right]^{\frac{1}{2}}\right\},$$

where the function $f(r)$ is increasing on $[0, 1]$, $f(r) > 0$ for all $r \in (0, 1]$ with $f(0) = 0$ and $\int_0^1 f(r)dr = \frac{1}{2}$.

The trapezoidal fuzzy number as a special fuzzy number is widely used in fuzzy nonlinear regression, fuzzy decision-making, fuzzy optimization and other fields [43–45]. However, it becomes more reasonable that the practical problem is described by n-dimensional fuzzy numbers. We generalize the trapezoidal fuzzy number to 2-dimension and propose the concept of fuzzy regular prismoid number as follows.

Definition 1. Let $\tilde{u} \in E^2$, $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in R$, $a_1 \le a_2 \le a_3 \le a_4$ and $b_1 \le b_2 \le b_3 \le b_4$. If the membership function of $\tilde{u}$ can be defined as

$$\tilde{u}(x, y) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, \; a_1 \le x \le a_2, \; \dfrac{(b_2-b_1)x + a_2b_1 - a_1b_2}{a_2-a_1} \le y \le \dfrac{(b_3-b_4)x + a_2b_4 - a_1b_3}{a_2-a_1}, \\ \dfrac{y-b_1}{b_2-b_1}, \; b_1 \le y \le b_2, \; \dfrac{(a_2-a_1)y + a_1b_2 - a_2b_1}{b_2-b_1} \le x \le \dfrac{(a_3-a_4)y + a_4b_2 - a_3b_1}{b_2-b_1}, \\ \quad 1, \; a_2 \le x \le a_3, \; b_2 \le y \le b_3, \\ \dfrac{a_4-x}{a_4-a_3}, \; a_3 \le x \le a_4, \; \dfrac{(b_2-b_1)x + a_3b_1 - a_4b_2}{a_3-a_4} \le y \le \dfrac{(b_3-b_4)x + a_3b_4 - a_4b_3}{a_3-a_4}, \\ \dfrac{b_4-y}{b_4-b_3}, \; b_3 \le y \le b_4, \; \dfrac{(a_2-a_1)y + a_1b_3 - a_2b_4}{b_3-b_4} \le x \le \dfrac{(a_3-a_4)y + a_4b_3 - a_3b_4}{b_3-b_4}, \\ \quad 0, \; \text{otherwise}. \end{cases}$$

Then $\tilde{u}$ is called a fuzzy regular prismoid number. For brevity, the fuzzy regular prismoid number are denoted as $\tilde{u} = (a_1, a_2, a_3, a_4;\; b_1, b_2, b_3, b_4)$. We use $P(E^2)$ to denote the fuzzy regular prismoid number space.

Let $\tilde{u} \in P(E^2)$. For any $r \in (0, 1]$,

$$[\tilde{u}]^r = \left[(a_2-a_1)r + a_1,\; a_4 - (a_4-a_3)r\right] \times \left[(b_2-b_1)r + b_1,\; b_4 - (b_4-b_3)r\right].$$

Given $\tilde{u} = (a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4)$ and $\tilde{v} = (c_1, c_2, c_3, c_4;\; d_1, d_2, d_3, d_4)$ are both fuzzy regular prismoid numbers, the weighted pseudo-metric $D : P(E^2) \times P(E^2) \to [0, +\infty)$ between $\tilde{u}$ and $\tilde{v}$ is

$$\begin{aligned} D(\tilde{u}, \tilde{v}) = \frac{1}{2}\Bigg\{ & \left[\int_0^1 f(r)(R^{-1}_{\tilde{p}_u(x)}(r) - R^{-1}_{\tilde{p}_v(x)}(r))^2 dr + \int_0^1 f(r)(R^{-1}_{\tilde{p}_u(y)}(r) - R^{-1}_{\tilde{p}_v(y)}(r))^2 dr\right]^{\frac{1}{2}} \\ & + \left[\int_0^1 f(r)(L^{-1}_{\tilde{p}_u(x)}(r) - L^{-1}_{\tilde{p}_v(x)}(r))^2 dr + \int_0^1 f(r)(L^{-1}_{\tilde{p}_u(y)}(r) - L^{-1}_{\tilde{p}_v(y)}(r))^2 dr\right]^{\frac{1}{2}} \Bigg\} \\ = \frac{1}{2}\Bigg\{ & \left[\int_0^1 f(r)(a_4 - c_4 - (a_4 - c_4 - a_3 + c_3)r)^2 dr + \int_0^1 f(r)(b_4 - d_4 - (b_4 - d_4 - b_3 + d_3)r)^2 dr\right]^{\frac{1}{2}} \\ & + \left[\int_0^1 f(r)((a_2 - a_1 - c_2 + c_1)r + a_1 - c_1)^2 dr + \int_0^1 f(r)((b_2 - b_1 - d_2 + d_1)r + b_1 - d_1)^2 dr\right]^{\frac{1}{2}} \Bigg\}. \end{aligned}$$

2.1. The Arithmetic Operation of 2-Dimensional Fuzzy Numbers

The arithmetic operation play an important role in fuzzy risk analysis, hereafter the arithmetic operation for the 2-dimensional fuzzy numbers will be researched.

Definition 2. Let $\tilde{u}, \tilde{v} \in E^2$. If $[\tilde{u}]^r = [u_1^-(r), u_1^+(r)] \times [u_2^-(r), u_2^+(r)]$ and $[\tilde{v}]^r = [v_1^-(r), v_1^+(r)] \times [v_2^-(r), v_2^+(r)]$ for any $r \in [0, 1]$, then the arithmetic operation for $\tilde{u}$ and $\tilde{v}$ are defined as follows:

(1) Addition $\oplus$ of the fuzzy regular prismoid numbers satisfies

$$\begin{aligned} [\tilde{u} \oplus \tilde{v}]^r = & \left[\inf_{\beta \ge r} \min_{\lambda \in [0,1]} \left\{(\lambda u_1^-(\beta) + (1-\lambda)u_1^+(\beta)) + (\lambda v_1^-(\beta) + (1-\lambda)v_1^+(\beta))\right\}, \sup_{\beta \ge r} \right. \\ & \left. \max_{\lambda \in [0,1]} \left\{(\lambda u_1^-(\beta) + (1-\lambda)u_1^+(\beta)) + (\lambda v_1^-(\beta) + (1-\lambda)v_1^+(\beta))\right\}\right] \\ & \times \left[\inf_{\beta \ge r} \min_{\lambda \in [0,1]} \left\{(\lambda u_2^-(\beta) + (1-\lambda)u_2^+(\beta)) + (\lambda v_2^-(\beta) + (1-\lambda)v_2^+(\beta))\right\}, \sup_{\beta \ge r} \right. \\ & \left. \max_{\lambda \in [0,1]} \left\{(\lambda u_2^-(\beta) + (1-\lambda)u_2^+(\beta)) + (\lambda v_2^-(\beta) + (1-\lambda)v_2^+(\beta))\right\}\right], \end{aligned}$$

for any $r \in [0, 1]$.

(2) Subtraction $\ominus$ of the fuzzy regular prismoid numbers satisfies

$$[\tilde{u} \ominus \tilde{v}]^r = \Big[\inf_{\beta \ge r} \min_{\lambda \in [0,1]} \left\{(\lambda u_1^-(\beta) + (1-\lambda)u_1^+(\beta)) - (\lambda v_1^-(\beta) + (1-\lambda)v_1^+(\beta))\right\}, \\ \sup_{\beta \ge r} \max_{\lambda \in [0,1]} \left\{(\lambda u_1^-(\beta) + (1-\lambda)u_1^+(\beta)) - (\lambda v_1^-(\beta) + (1-\lambda)v_1^+(\beta))\right\}\Big] \\ \times \Big[\inf_{\beta \ge r} \min_{\lambda \in [0,1]} \left\{(\lambda u_2^-(\beta) + (1-\lambda)u_2^+(\beta)) - (\lambda v_2^-(\beta) + (1-\lambda)v_2^+(\beta))\right\}, \\ \sup_{\beta \ge r} \max_{\lambda \in [0,1]} \left\{(\lambda u_2^-(\beta) + (1-\lambda)u_2^+(\beta)) - (\lambda v_2^-(\beta) + (1-\lambda)v_2^+(\beta))\right\}\Big],$$

for any $r \in [0, 1]$.

(3) Multiplication $\otimes$ of the fuzzy regular prismoid numbers satisfies

$$[\tilde{u} \otimes \tilde{v}]^r = \Big[\inf_{\beta \ge r} \min_{\lambda \in [0,1]} \left\{(\lambda u_1^-(\beta) + (1-\lambda)u_1^+(\beta)) \cdot (\lambda v_1^-(\beta) + (1-\lambda)v_1^+(\beta))\right\}, \\ \sup_{\beta \ge r} \max_{\lambda \in [0,1]} \left\{(\lambda u_1^-(\beta) + (1-\lambda)u_1^+(\beta)) \cdot (\lambda v_1^-(\beta) + (1-\lambda)v_1^+(\beta))\right\}\Big] \\ \times \Big[\inf_{\beta \ge r} \min_{\lambda \in [0,1]} \left\{(\lambda u_2^-(\beta) + (1-\lambda)u_2^+(\beta)) \cdot (\lambda v_2^-(\beta) + (1-\lambda)v_2^+(\beta))\right\}, \\ \sup_{\beta \ge r} \max_{\lambda \in [0,1]} \left\{(\lambda u_2^-(\beta) + (1-\lambda)u_2^+(\beta)) \cdot (\lambda v_2^-(\beta) + (1-\lambda)v_2^+(\beta))\right\}\Big],$$

for any $r \in [0, 1]$.

(4) Division $\oslash$ of the fuzzy regular prismoid numbers satisfies

$$[\tilde{u} \oslash \tilde{v}]^r = \Big[\inf_{\beta \ge r} \min_{\lambda \in [0,1]} \left\{(\lambda u_1^-(\beta) + (1-\lambda)u_1^+(\beta)) \div (\lambda v_1^-(\beta) + (1-\lambda)v_1^+(\beta))\right\}, \\ \sup_{\beta \ge r} \max_{\lambda \in [0,1]} \left\{(\lambda u_1^-(\beta) + (1-\lambda)u_1^+(\beta)) \div (\lambda v_1^-(\beta) + (1-\lambda)v_1^+(\beta))\right\}\Big] \\ \times \Big[\inf_{\beta \ge r} \min_{\lambda \in [0,1]} \left\{(\lambda u_2^-(\beta) + (1-\lambda)u_2^+(\beta)) \div (\lambda v_2^-(\beta) + (1-\lambda)v_2^+(\beta))\right\}, \\ \sup_{\beta \ge r} \max_{\lambda \in [0,1]} \left\{(\lambda u_2^-(\beta) + (1-\lambda)u_2^+(\beta)) \div (\lambda v_2^-(\beta) + (1-\lambda)v_2^+(\beta))\right\}\Big],$$

for any $r \in [0, 1]$.

Proposition 1. Let $\tilde{u} = (a_1, a_2, a_3, a_4;\ b_1, b_2, b_3, b_4)$, $\tilde{v} = (c_1, c_2, c_3, c_4;\ d_1, d_2, d_3, d_4)$ and $\tilde{w} = (e_1, e_2, e_3, e_4;\ f_1, f_2, f_3, f_4)$ be all fuzzy regular prismoid numbers. Then

(1) For any $r \in [0, 1]$,

$$[\tilde{u} \oplus \tilde{v}]^r = [(a_2 - a_1 + c_2 - c_1)r + a_1 + c_1, a_4 + c_4 - (a_4 - a_3 + c_4 - c_3)r] \times [(b_2 - b_1 + d_2 - d_1)r + b_1 + d_1, b_4 + d_4 - (b_4 - b_3 + d_4 - d_3)r];$$

(2) For any $r \in [0, 1]$,

$$[\tilde{u} \ominus \tilde{v}]^r = \Big[\inf_{\beta \ge r} \min \left\{(a_2 - a_1 - c_2 + c_1)\beta + a_1 - c_1, a_4 - c_4 - (a_4 - a_3 - c_4 + c_3)\beta\right\}, \\ \sup_{\beta \ge r} \max \left\{(a_2 - a_1 - c_2 + c_1)\beta + a_1 - c_1, a_4 - c_4 - (a_4 - a_3 - c_4 + c_3)\beta\right\}\Big] \\ \times \Big[\inf_{\beta \ge r} \min \left\{(b_2 - b_1 - d_2 + d_1)\beta + b_1 - d_1, b_4 - d_4 - (b_4 - b_3 - d_4 + d_3)\beta\right\}, \\ \sup_{\beta \ge r} \max \left\{(b_2 - b_1 - d_2 + d_1)\beta + b_1 - d_1, b_4 - d_4 - (b_4 - b_3 - d_4 + d_3)\beta\right\}\Big];$$

(3) $\tilde{u} \ominus \tilde{u} = \{(0, 0)\}$;

(4) $(\tilde{u} \oplus \tilde{v}) \ominus \tilde{v} = \tilde{u}$;

(5) $\tilde{u} \otimes (\tilde{v} \oplus \tilde{w}) = (\tilde{u} \otimes \tilde{v}) \oplus (\tilde{u} \otimes \tilde{w})$;

(6) $\tilde{u} \otimes (\tilde{v} \ominus \tilde{w}) = (\tilde{u} \otimes \tilde{v}) \ominus (\tilde{u} \otimes \tilde{w})$;

(7) $\tilde{u} \oslash \tilde{u} = \{(1, 1)\}$;

(8) $(\tilde{u} \otimes \tilde{v}) \oslash \tilde{v} = \tilde{u}$;

(9) $\tilde{u} \oslash (\tilde{v} \oplus \tilde{w}) = (\tilde{u} \oslash \tilde{v}) \oplus (\tilde{u} \oslash \tilde{w})$;

(10) $\tilde{u} \oslash (\tilde{v} \ominus \tilde{w}) = (\tilde{u} \oslash \tilde{v}) \ominus (\tilde{u} \oslash \tilde{w})$.

Proof. According to Definition 2, the proof of (1), (2), (3) and (7) are immediate.

(4) We have from (1) and (2) that

$$\begin{aligned}[(\tilde{u} \oplus \tilde{v}) \ominus \tilde{v}]^r &= \left[\inf_{\beta\geq r} \min\left\{(a_2-a_1)\beta+a_1,\ a_4-(a_4-a_3)\beta\right\}, \sup_{\beta\geq r}\max\left\{(a_2-a_1)\beta+a_1, a_4-(a_4-a_3)\beta\right\}\right] \\ &\quad\times\left[\inf_{\beta\geq r} \min\left\{(b_2-b_1)\beta+b_1,\ b_4-(b_4-b_3)\beta\right\}, \sup_{\beta\geq r}\max\left\{(b_2-b_1)\beta+b_1,\ b_4-(b_4-b_3)\beta\right\}\right] \\ &= \left[(a_2-a_1)r+a_1,\ \ a_4-(a_4-a_3)r\right]\times\left[(b_2-b_1)r+b_1,\ \ b_4-(b_4-b_3)r\right] \\ &= [\tilde{u}]^r,\end{aligned}$$

for any $r \in [0,1]$, which implies that $(\tilde{u} \oplus \tilde{v}) \ominus \tilde{v} = \tilde{u}$.

(5) It follows from Definition 2 and (1) that

$$\begin{aligned}[\tilde{u} \otimes (\tilde{v} \oplus \tilde{w})]^r = &\left[\inf_{\beta\geq r}\min_{\lambda\in[0,1]}\left\{\lambda\left[(a_2-a_1)\beta+a_1\right]+(1-\lambda)\left[a_4-(a_4-a_3)\beta\right]\right\}\right. \\ &\cdot\left\{\lambda\left[(c_2-c_1+e_2-e_1)\beta+c_1+e_1\right]+(1-\lambda)\left[c_4+e_4-(c_4-c_3+e_4-e_3)\beta\right]\right\}, \\ &\sup_{\beta\geq r}\max_{\lambda\in[0,1]}\left\{\lambda\left[(a_2-a_1)\beta+a_1\right]+(1-\lambda)\left[a_4-(a_4-a_3)\beta\right]\right\} \\ &\left.\cdot\left\{\lambda\left[(c_2-c_1+e_2-e_1)\beta+c_1+e_1\right]+(1-\lambda)\left[c_4+e_4-(c_4-c_3+e_4-e_3)\beta\right]\right\}\right] \\ &\times\left[\inf_{\beta\geq r}\min_{\lambda\in[0,1]}\left\{\lambda\left[(b_2-b_1)\beta+b_1\right]+(1-\lambda)\left[b_4-(b_4-b_3)\beta\right]\right\}\right. \\ &\cdot\left\{\lambda\left[(d_2-d_1+f_2-f_1)\beta+d_1+f_1\right]+(1-\lambda)\left[d_4+f_4-(d_4-d_3+f_4-f_3)\beta\right]\right\}, \\ &\sup_{\beta\geq r}\max_{\lambda\in[0,1]}\left\{\lambda\left[(b_2-b_1)\beta+b_1\right]+(1-\lambda)\left[b_4-(b_4-b_3)\beta\right]\right\} \\ &\cdot\left\{\lambda\left[(d_2-d_1+f_2-f_1)\beta+d_1+f_1\right]+(1-\lambda)\left[d_4+f_4-(d_4-d_3+f_4-f_3)\beta\right]\right\},\end{aligned}$$

for any $r \in [0,1]$. According to the law of the multiplication allocating rule, we obtain

$$\begin{aligned}[\tilde{u} \otimes (\tilde{v} \oplus \tilde{w})]^r = &[\inf_{\beta\geq r}\min_{\lambda\in[0,1]}\left\{\left(\lambda[(a_2-a_1)\beta+a_1]+(1-\lambda)[a_4-(a_4-a_3)\beta]\right)\cdot\left(\lambda[(c_2-c_1)\beta+c_1]+(1-\lambda)[c_4-(c_4-c_3)\beta]\right)\right. \\ &\left.+\left(\lambda[(a_2-a_1)\beta+a_1]+(1-\lambda)[a_4-(a_4-a_3)\beta]\right)\cdot\left(\lambda[(e_2-e_1)\beta+e_1]+(1-\lambda)[e_4-(e_4-e_3)\beta]\right)\right\}, \\ &\sup_{\beta\geq r}\max_{\lambda\in[0,1]}\left\{\left(\lambda[(a_2-a_1)\beta+a_1]+(1-\lambda)[a_4-(a_4-a_3)\beta]\right)\cdot\left(\lambda[(c_2-c_1)\beta+c_1]+(1-\lambda)[c_4-(c_4-c_3)\beta]\right)\right. \\ &\left.+\left(\lambda[(a_2-a_1)\beta+a_1]+(1-\lambda)[a_4-(a_4-a_3)\beta]\right)\cdot\left(\lambda[(e_2-e_1)\beta+e_1]+(1-\lambda)[e_4-(e_4-e_3)\beta]\right)\right\}] \\ &\times[\inf_{\beta\geq r}\min_{\lambda\in[0,1]}\left\{\left(\lambda[(b_2-b_1)\beta+b_1]+(1-\lambda)[b_4-(b_4-b_3)\beta]\right)\cdot\left(\lambda[(d_2-d_1)\beta+d_1]+(1-\lambda)[d_4-(d_4-d_3)\beta]\right)\right\} \\ &\left.+\left(\lambda[(b_2-b_1)\beta+b_1]+(1-\lambda)[b_4-(b_4-b_3)\beta]\right)\cdot\left(\lambda[(f_2-f_1)\beta+f_1]+(1-\lambda)[f_4-(f_4-f_3)\beta]\right)\right\}, \\ &\sup_{\beta\geq r}\max_{\lambda\in[0,1]}\left\{\left(\lambda[(b_2-b_1)\beta+b_1]+(1-\lambda)[b_4-(b_4-b_3)\beta]\right)\cdot\left(\lambda[(d_2-d_1)\beta+d_1]+(1-\lambda)[d_4-(d_4-d_3)\beta]\right)\right. \\ &\left.+\left(\lambda[(b_2-b_1)\beta+b_1]+(1-\lambda)\left[b_4-(b_4-b_3)\beta\right]\right)\cdot\left(\lambda[(f_2-f_1)\beta+f_1]+(1-\lambda)[f_4-(f_4-f_3)\beta]\right)\right\}],\end{aligned}$$

for any $r \in [0,1]$. Based on the monotonicity of $f_1(\lambda) = \lambda[(a_2-a_1)\beta+a_1]+(1-\lambda)[a_4-(a_4-a_3)\beta], f_2(\lambda) = \lambda[(b_2-b_1)\beta+b_1]+(1-\lambda)[b_4-(b_4-b_3)\beta], f_3(\lambda) = \lambda[(c_2-c_1)\beta+c_1]+(1-\lambda)[c_4-(c_4-c_3)\beta], f_4(\lambda) = \lambda[(d_2-d_1)\beta+d_1]+(1-\lambda)[d_4-(d_4-d_3)\beta], f_5(\lambda) =$

$\lambda[(e_2 - e_1)\beta + e_1] + (1-\lambda)[e_4 - (e_4 - e_3)\beta]$ and $f_6(\lambda) = \lambda[(f_2 - f_1)\beta + f_1] + (1-\lambda)[f_4 - (f_4 - f_3)\beta]$, we have

$$
\begin{aligned}
[\tilde{u} \otimes (\tilde{v} \oplus \tilde{w})]^r = & \Big[\inf_{\beta \geq r} \min_{\lambda \in [0,1]} \{(\lambda[(a_2 - a_1)\beta + a_1] + (1-\lambda)[a_4 - (a_4 - a_3)\beta]) \cdot (\lambda[(c_2 - c_1)\beta + c_1] + (1-\lambda)[c_4 - (c_4 - c_3)\beta])\} \\
& + \inf_{\beta \geq r} \min_{\lambda \in [0,1]} \left\{\left(\lambda[(a_2 - a_1)\beta + a_1] + (1-\lambda)[a_4 - (a_4 - a_3)\beta]\right) \cdot \left(\lambda[(e_2 - e_1)\beta + e_1] + (1-\lambda)[+e_4 - (e_4 - e_3)\beta]\right)\right\}, \\
& \sup_{\beta \geq r} \max_{\lambda \in [0,1]} \left\{\left(\lambda[(a_2 - a_1)\beta + a_1] + (1-\lambda)[a_4 - (a_4 - a_3)\beta]\right) \cdot \left(\lambda[(c_2 - c_1)\beta + c_1] + (1-\lambda)[c_4 - (c_4 - c_3)\beta]\right)\right\} \\
& + \sup_{\beta \geq r} \max_{\lambda \in [0,1]} \left\{\left(\lambda[(a_2 - a_1)\beta + a_1] + (1-\lambda)[a_4 - (a_4 - a_3)\beta]\right) \cdot \left(\lambda[(e_2 - e_1)\beta + e_1] + (1-\lambda)[e_4 - (e_4 - e_3)\beta]\right)\right\}\Big] \\
& \times \Big[\inf_{\beta \geq r} \min_{\lambda \in [0,1]} \left\{\left(\lambda[(b_2 - b_1)\beta + b_1] + (1-\lambda)[b_4 - (b_4 - b_3)\beta]\right) \cdot \left(\lambda[(d_2 - d_1)\beta + d_1] + (1-\lambda)[d_4 - (d_4 - d_3)\beta]\right)\right\} \\
& + \inf_{\beta \geq r} \min_{\lambda \in [0,1]} \left\{\left(\lambda[(b_2 - b_1)\beta + b_1] + (1-\lambda)[b_4 - (b_4 - b_3)\beta]\right) \cdot \left(\lambda[(f_2 - f_1)\beta + f_1] + (1-\lambda)[f_4 - (f_4 - f_3)\beta]\right)\right\}, \\
& \sup_{\beta \geq r} \max_{\lambda \in [0,1]} \left\{\left(\lambda[(b_2 - b_1)\beta + b_1] + (1-\lambda)[b_4 - (b_4 - b_3)\beta]\right) \cdot \left(\lambda[(d_2 - d_1)\beta + d_1] + (1-\lambda)[d_4 - (d_4 - d_3)\beta]\right)\right. \\
& + \sup_{\beta \geq r} \max_{\lambda \in [0,1]} \left\{\left(\lambda[(b_2 - b_1)\beta + b_1] + (1-\lambda)[b_4 - (b_4 - b_3)\beta]\right) \cdot \left(\lambda[(f_2 - f_1)\beta + f_1] + (1-\lambda)[f_4 - (f_4 - f_3)\beta]\right)\right\}\Big],
\end{aligned}
$$

for any $r \in [0, 1]$. Then,

$$[\tilde{u} \otimes (\tilde{v} \oplus \tilde{w})]^r = [(\tilde{u} \otimes \tilde{v}) \oplus (\tilde{u} \otimes \tilde{w})]^r,$$

for any $r \in [0, 1]$, which implies that $\tilde{u} \otimes (\tilde{v} \oplus \tilde{w}) = (\tilde{u} \otimes \tilde{v}) \oplus (\tilde{u} \otimes \tilde{w})$.

The proof of (8) similar to (4) as well as the proof of (6), (9) and (10) are similar to (5).

2.2. Fuzzy Regular Prismoid Numbers Approximation of 2-Dimensional Fuzzy Numbers

The fuzzy regular prismoid numbers approximation of 2-dimensional fuzzy numbers will be discussed, which is an indispensable part to study the fuzzy risk analysis based on fuzzy regular prismoid numbers.

Lemma 1. [46] Let $f : R^n \to R$ be a strictly convex and differentiable function, $g_1, g_2, \cdots, g_m : R^n \to R$ be convex and differentiable functions. Then $\overline{x}$ solves the convex programing problem

$min\, f(x)$,

$s.t.\ g_i(x) \leq b_i\ (i = 1, 2, \cdots, m)$,

if and only if there exist $\mu_i\ (i = 1, 2, \cdots, m)$, such that

(1) $\nabla f(\overline{x}) + \Sigma_{i=1}^{m} \mu_i \nabla g_i(\overline{x}) = 0$,

(2) $g_i(\overline{x}) - b_i \leq 0$,

(3) $\mu_i \geq 0$,

(4) $\mu_i(b_i - g_i(\overline{x})) = 0$.

Let $\tilde{u} \in E^2$. We will try to find a fuzzy regular prismoid number $\tilde{v} \in P(E^2)$, which is the nearest fuzzy regular prismoid number of $\tilde{u}$ and preserves the core of $\tilde{u}$ with respect to the weighted pseudometric. Thus we have to find such real numbers $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 that minimize

$$
\begin{aligned}
D(\tilde{u}, \tilde{v}) = \frac{1}{2}\Bigg\{ & \left[\int_0^1 f(r)(R^{-1}_{\tilde{p}_u(x)}(r) - a_4 + (a_4 - a_3)r)^2 dr + \int_0^1 f(r)(R^{-1}_{\tilde{p}_u(y)}(r) - b_4 + (b_4 - b_3)r)^2 dr\right]^{\frac{1}{2}} \\
& + \left[\int_0^1 f(r)(L^{-1}_{\tilde{p}_u(x)}(r) - a_1 - (a_2 - a_1)r)^2 dr + \int_0^1 f(r)(L^{-1}_{\tilde{p}_u(y)}(r) - b_1 - (b_2 - b_1)r)^2 dr\right]^{\frac{1}{2}}\Bigg\},
\end{aligned}
$$

with respect to condition core$\tilde{u}$ = core$\tilde{v}$, i.e.,

$$a_2 = L^{-1}_{\tilde{p}_u(x)}(1),\ a_3 = R^{-1}_{\tilde{p}_u(x)}(1),\ b_2 = L^{-1}_{\tilde{p}_u(y)}(1),\ b_3 = R^{-1}_{\tilde{p}_u(y)}(1).$$

Therefore, we only need minimize the function

$$
\begin{aligned}
F(a_1, a_4, b_1, b_4) = & \int_0^1 f(r)\left[L^{-1}_{\tilde{p}_u(x)}(r) - a_1 - (L^{-1}_{\tilde{p}_u(x)}(1) - a_1)r\right]^2 dr + \int_0^1 f(r)\left[R^{-1}_{\tilde{p}_u(x)}(r) - a_4 + (a_4 - R^{-1}_{\tilde{p}_u(x)}(1))r\right]^2 dr \\
& + \int_0^1 f(r)\left[L^{-1}_{\tilde{p}_u(y)}(r) - b_1 - (L^{-1}_{\tilde{p}_u(y)}(1) - b_1)r\right]^2 dr + \int_0^1 f(r)\left[R^{-1}_{\tilde{p}_u(y)}(r) - b_4 + (b_4 - R^{-1}_{\tilde{p}_u(y)}(1))r\right]^2 dr \\
= & \left(a_1\right)^2 \int_0^1 f(r)(1-r)^2 dr + (a_4)^2 \int_0^1 f(r)(1-r)^2 dr + \left(b_1\right)^2 \int_0^1 f(r)(1-r)^2 dr + (b_4)^2 \int_0^1 f(r)(1-r)^2 dr \\
& + 2a_1 \int_0^1 f(r)(1-r)\left[rL^{-1}_{\tilde{p}_u(x)}(1) - L^{-1}_{\tilde{p}_u(x)}(r)\right] dr + 2a_4 \int_0^1 f(r)(1-r)\left[rR^{-1}_{\tilde{p}_u(x)}(1) - R^{-1}_{\tilde{p}_u(x)}(r)\right] dr \\
& + 2b_1 \int_0^1 f(r)(1-r)\left[rL^{-1}_{\tilde{p}_u(x)}(1) - L^{-1}_{\tilde{p}_u(x)}(r)\right] dr + 2b_4 \int_0^1 f(r)(1-r)\left[rR^{-1}_{\tilde{p}_u(x)}(1) - R^{-1}_{\tilde{p}_u(x)}(r)\right] dr \\
& + \int_0^1 f(r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))^2 dr + \int_0^1 f(r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))^2 dr \\
& + \int_0^1 f(r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))^2 dr + \int_0^1 f(r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))^2 dr,
\end{aligned} \tag{1}
$$

subject to

$$
\begin{aligned}
a_1 - a_4 \le 0, \\
b_1 - b_4 \le 0.
\end{aligned} \tag{2}
$$

Theorem 1. *Suppose that $\tilde{u} \in E^2$ and the membership function for the projective of $\tilde{u}$ on the xoz and yoz plane satisfy*

$$
\tilde{p}_u(x) = \begin{cases} L_x(x), & a_x \le x < b_x, \\ 1, & b_x \le x \le c_x, \\ R_x(x), & c_x < x \le d_x, \\ 0, & otherwise, \end{cases}
$$

$$
\tilde{p}_u(y) = \begin{cases} L_y(y), & a_y \le y < b_y, \\ 1, & b_y \le y \le c_y, \\ R_y(y), & c_y < y \le d_y, \\ 0, & otherwise, \end{cases}
$$

respectively. Furthermore, $\tilde{v} \in P(E^2)$ and

$$
\tilde{v}(x,y) = \begin{cases}
\dfrac{x-a_1}{a_2-a_1}, \ a_1 \le x \le a_2, \ \dfrac{(b_2-b_1)x + a_2b_1 - a_1b_2}{a_2-a_1} \le y \le \dfrac{(b_3-b_4)x + a_2b_4 - a_1b_3}{a_2-a_1}, \\
\dfrac{y-b_1}{b_2-b_1}, \ b_1 \le y \le b_2, \ \dfrac{(a_2-a_1)y + a_1b_2 - a_2b_1}{b_2-b_1} \le x \le \dfrac{(a_3-a_4)y + a_4b_2 - a_3b_1}{b_2-b_1}, \\
1, \quad a_2 \le x \le a_3, \ b_2 \le y \le b_3, \\
\dfrac{a_4-x}{a_4-a_3}, \ a_3 \le x \le a_4, \ \dfrac{(b_2-b_1)x + a_3b_1 - a_4b_2}{a_3-a_4} \le y \le \dfrac{(b_3-b_4)x + a_3b_4 - a_4b_3}{a_3-a_4}, \\
\dfrac{b_4-y}{b_4-b_3}, \ b_3 \le y \le b_4, \ \dfrac{(a_2-a_1)y + a_1b_3 - a_2b_4}{b_3-b_4} \le x \le \dfrac{(a_3-a_4)y + a_4b_3 - a_3b_4}{b_3-b_4}, \\
0, \quad otherwise.
\end{cases}
$$

$\tilde{v}$ is the nearest fuzzy regular prismoid number to $\tilde{u}$ preserves the core of $\tilde{u}$ with respect to the weighted pseudometric.

(1) If

$$\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr > \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr,$$
$$\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr > \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr,$$

we have

$$a_1 = a_4 = \frac{\int_0^1 f(r)(1-r)[(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1)) + (R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))]dr}{2\int_0^1 f(r)(1-r)^2 dr},$$

$$b_1 = b_4 = \frac{\int_0^1 f(r)(1-r)[(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1)) + (R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))]dr}{2\int_0^1 f(r)(1-r)^2 dr}.$$

(2) If

$$\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr > \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr,$$
$$\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr \le \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr,$$

we have

$$a_1 = a_4 = \frac{\int_0^1 f(r)(1-r)[(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1)) + (R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))]dr}{2\int_0^1 f(r)(1-r)^2 dr},$$

$$b_1 = \frac{\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr},$$

$$b_4 = \frac{\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr}.$$

(3) If

$$\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr \le \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr,$$
$$\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr > \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr,$$

we have

$$a_1 = \frac{\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr},$$

$$a_4 = \frac{\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr},$$

$$b_1 = b_4 = \frac{\int_0^1 f(r)(1-r)[(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1)) + (R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))]dr}{2\int_0^1 f(r)(1-r)^2 dr}.$$

(4) If

$$\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr \le \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr,$$

$$\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr \le \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr,$$

we have

$$a_1 = \frac{\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr},$$

$$a_4 = \frac{\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr},$$

$$b_1 = \frac{\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr},$$

$$b_4 = \frac{\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr}.$$

Where the function $f(r)$ is nonnegative and increasing on $[0, 1]$ *with* $f(0) = 0$, $\int_0^1 f(r)dr = \frac{1}{2}$.

Proof. Because the function F in (1) and the conditions (2) satisfy the hypothesis of convexity and differentiability in Lemma 1, the conditions (i)–(iv) in Lemma 1 with respect to the minimization problem (1) in the conditions (2) can be shown as following:

$$2a_1 \int_0^1 f(r)(1-r)^2 dr + 2\int_0^1 f(r)(1-r)\left[rL^{-1}_{\tilde{p}_u(x)}(1) - L^{-1}_{\tilde{p}_u(x)}(r)\right] dr + \mu_1 = 0, \tag{3}$$

$$2a_4 \int_0^1 f(r)(1-r)^2 dr + 2\int_0^1 f(r)(1-r)\left[rR^{-1}_{\tilde{p}_u(x)}(1) - R^{-1}_{\tilde{p}_u(x)}(r)\right] dr - \mu_1 = 0, \tag{4}$$

$$2b_1 \int_0^1 f(r)(1-r)^2 dr + 2\int_0^1 f(r)(1-r)\left[rL^{-1}_{\tilde{p}_u(y)}(1) - L^{-1}_{\tilde{p}_u(y)}(r)\right] dr + \mu_2 = 0, \tag{5}$$

$$2b_4 \int_0^1 f(r)(1-r)^2 dr + 2\int_0^1 f(r)(1-r)\left[rR^{-1}_{\tilde{p}_u(y)}(1) - R^{-1}_{\tilde{p}_u(y)}(r)\right] dr - \mu_2 = 0, \tag{6}$$

$$\mu_1(a_1 - a_4) = 0, \tag{7}$$

$$\mu_2(b_1 - b_4) = 0, \tag{8}$$

$$\mu_1 \ge 0, \tag{9}$$

$$\mu_2 \ge 0. \tag{10}$$

(1) In the case $\mu_1 > 0$ and $\mu_2 > 0$, the solution of the system (3)–(10) is

$$a_1 = a_4 = \frac{\int_0^1 f(r)(1-r)[(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1)) + (R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))]dr}{2\int_0^1 f(r)(1-r)^2 dr},$$

$$b_1 = b_4 = \frac{\int_0^1 f(r)(1-r)[(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1)) + (R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))]dr}{2\int_0^1 f(r)(1-r)^2 dr},$$

$$\mu_1 = \int_0^1 f(r)(1-r)[(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1)) - (R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))]dr,$$

$$\mu_2 = \int_0^1 f(r)(1-r)[(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1)) - (R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))]dr.$$

It is obvious that $\mu_1 > 0$ and $\mu_2 > 0$. Then the conditions (3)–(10) are verified. Furthermore, we can prove that

$$\begin{aligned}
a_4 - a_3 &= \frac{\int_0^1 f(r)(1-r)[(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1)) + (R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))]dr}{2\int_0^1 f(r)(1-r)^2 dr} - R^{-1}_{\tilde{p}_u(x)}(1) \\
&\geq \frac{\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr} - R^{-1}_{\tilde{p}_u(x)}(1) \\
&= \frac{\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - R^{-1}_{\tilde{p}_u(x)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr} \\
&\geq 0,
\end{aligned}$$

$$\begin{aligned}
a_2 - a_1 &= L^{-1}_{\tilde{p}_u(x)}(1) - \frac{\int_0^1 f(r)(1-r)[(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1)) + (R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))]dr}{2\int_0^1 f(r)(1-r)^2 dr} \\
&\geq L^{-1}_{\tilde{p}_u(x)}(1) - \frac{\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr} \\
&= \frac{\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(1) - L^{-1}_{\tilde{p}_u(x)}(r))dr}{\int_0^1 f(r)(1-r)^2 dr} \\
&\geq 0.
\end{aligned}$$

Which implies that $a_4 \geq a_3 \geq a_2 \geq a_1$. Similarly, we have $b_4 \geq b_3 \geq b_2 \geq b_1$. Then $\tilde{v} = (a_1, a_2, a_3, a_4;\ b_1, b_2, b_3, b_4)$ is the nearest fuzzy regular prismoid number to $\tilde{u}$ in this case.

(2) In this case $\mu_1 > 0$ and $\mu_2 = 0$, the solution of the system (3)–(10) is

$$a_1 = a_4 = \frac{\int_0^1 f(r)(1-r)[(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1)) + (R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))]dr}{2\int_0^1 f(r)(1-r)^2 dr},$$

$$b_1 = \frac{\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr},$$

$$b_4 = \frac{\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr},$$

$$\mu_1 = \int_0^1 f(r)(1-r)[(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1)) - (R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))]dr,$$

$$\mu_2 = 0.$$

It is obvious that $\mu_1 > 0$ and $\mu_2 = 0$. Then the conditions (3)–(10) are verified. Similar to (1) we have $a_4 \geq a_3 \geq a_2 \geq a_1$. Moreover, we can prove that

$$\begin{aligned} b_4 - b_3 &= \frac{\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr} - R^{-1}_{\tilde{p}_u(y)}(1) \\ &= \frac{\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - R^{-1}_{\tilde{p}_u(y)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr} \\ &\geq 0, \end{aligned}$$

$$\begin{aligned} b_2 - b_1 &= L^{-1}_{\tilde{p}_u(y)}(1) - \frac{\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr} \\ &= \frac{\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(1) - L^{-1}_{\tilde{p}_u(y)}(r))dr}{\int_0^1 f(r)(1-r)^2 dr} \\ &\geq 0. \end{aligned}$$

Which implies that $b_4 \geq b_3 \geq b_2 \geq b_1$. Then $\tilde{v} = (a_1, a_2, a_3, a_4;\ b_1, b_2, b_3, b_4)$ is the nearest fuzzy regular prismoid number to $\tilde{u}$ in this case.

(3) In the case $\mu_1 = 0$ and $\mu_2 > 0$, the solution of the system (3)–(10) is

$$a_1 = \frac{\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr},$$

$$a_4 = \frac{\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr},$$

$$b_1 = b_4 = \frac{\int_0^1 f(r)(1-r)[(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1)) + (R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))]dr}{2\int_0^1 f(r)(1-r)^2 dr},$$

$$\mu_1 = 0,$$

$$\mu_2 = \int_0^1 f(r)(1-r)[(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1)) - (R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))]dr.$$

It is obvious that $\mu_1 = 0$ and $\mu_2 > 0$. Then the conditions (3)–(10) are verified. Similar to (1) and (2) we have $a_4 \geq a_3 \geq a_2 \geq a_1$ and $b_4 \geq b_3 \geq b_2 \geq b_1$. Then $\tilde{v} = (a_1, a_2, a_3, a_4;\ b_1, b_2, b_3, b_4)$ is the nearest fuzzy regular prismoid number to $\tilde{u}$ in this case.

(4) In the case $\mu_1 = 0$ and $\mu_2 = 0$, the solution of the system (3)–(10) is

$$a_1 = \frac{\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr},$$

$$a_4 = \frac{\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr},$$

$$b_1 = \frac{\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr},$$

$$b_4 = \frac{\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr},$$

$$\mu_1 = 0,$$

$$\mu_2 = 0.$$

Then the conditions (3)–(10) are verified. Similar to (1) and (2) we have $a_4 \geq a_3 \geq a_2 \geq a_1$ and $b_4 \geq b_3 \geq b_2 \geq b_1$. Then $\tilde{v} = (a_1, a_2, a_3, a_4;\ b_1, b_2, b_3, b_4)$ is the nearest fuzzy regular prismoid number to $\tilde{u}$ in this case.

For any 2-dimensional fuzzy number, we can apply one and only one of the above situations (1)–(4) to calculate the fuzzy regular prismoid approximation of it. We denote

$$\begin{aligned}
\Omega_1 = \{\tilde{u} \in E^2 :\ & \int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr > \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr,\\
& \int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr > \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr\},\\
\Omega_2 = \{\tilde{u} \in E^2 :\ & \int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr > \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr,\\
& \int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr \leq \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr\},\\
\Omega_3 = \{\tilde{u} \in E^2 :\ & \int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr \leq \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr,\\
& \int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr > \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr\},\\
\Omega_4 = \{\tilde{u} \in E^2 :\ & \int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr \leq \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr,\\
& \int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr \leq \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr\}.
\end{aligned}$$

It is obvious that the cases (1)–(4) cover the set of all 2-dimensional fuzzy numbers and $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ are disjoint sets. So the approximation operator always gives a fuzzy regular prismoid number.

Example 1.

Let $\tilde{u} \in E^2$. If the r-level set of $\tilde{u}$ satisfies

$$[\tilde{u}]^r = \left[\frac{0.0115r^2 + 0.3479r + 2.5122}{0.26r + 3.04}, \frac{0.009r^2 - 0.4136r + 4.3326}{4.41 - 0.29r}\right] \times \left[\frac{0.0141r^2 + 0.4047r + 2.8062}{0.27r + 3.53}, \frac{0.0025r^2 - 0.327r + 4.4909}{4.66 - 0.22r}\right],$$

for any $r \in [0, 1]$, and $[u]^1 \approx [0.8702, 0.9534] \times [0.8487, 0.9384]$. It is obvious that the projective of $\tilde{u}$ on the xoz and yoz plane satisfy

$$[\tilde{p}_u(x)]^r = \left[\frac{0.0115r^2 + 0.3479r + 2.5122}{0.26r + 3.04}, \frac{0.009r^2 - 0.4136r + 4.3326}{4.41 - 0.29r}\right],$$

$$[\tilde{p}_u(y)]^r = \left[\frac{0.0141r^2 + 0.4047r + 2.8062}{0.27r + 3.53}, \frac{0.0025r^2 - 0.327r + 4.4909}{4.66 - 0.22r}\right],$$

for any $r \in [0, 1]$ respectively. Taking $f(r) = r$, we have

$$\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr \approx 0.0689,$$

$$\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr \approx 0.0819,$$

$$\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr \approx 0.0662,$$

$$\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr \approx 0.0803.$$

Then,

$$\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr < \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr,$$

$$\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr < \int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr.$$

It follows from Theorem 1 that

$$a_1 = \frac{\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(x)}(r) - rL^{-1}_{\tilde{p}_u(x)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr} \approx 0.8263,$$

$$a_4 = \frac{\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(x)}(r) - rR^{-1}_{\tilde{p}_u(x)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr} \approx 0.9824,$$

$$b_1 = \frac{\int_0^1 f(r)(1-r)(L^{-1}_{\tilde{p}_u(y)}(r) - rL^{-1}_{\tilde{p}_u(y)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr} \approx 0.7950,$$

$$b_4 = \frac{\int_0^1 f(r)(1-r)(R^{-1}_{\tilde{p}_u(y)}(r) - rR^{-1}_{\tilde{p}_u(y)}(1))dr}{\int_0^1 f(r)(1-r)^2 dr} \approx 0.9640.$$

Therefore, the fuzzy regular prismoid number $\tilde{v} = (0.8263, 0.8702, 0.9534, 0.9824;\ 0.7950, 0.8487, 0.9384, 0.9640)$ is the nearest fuzzy regular prismoid number to $\tilde{u}$ preserves the core of $\tilde{u}$ with respect to the weighted distance.

3. THE SIMILARITY MEASURE BETWEEN FUZZY REGULAR PRISMOID NUMBERS

3.1. The Radius of Gyration for Fuzzy Regular Prismoid Numbers

We will mention the moment of inertia and the radius of gyration about a region of space Ω. Consider a space Ω located in rectangular coordinate system *Oxyz*. The moment of inertia for the space Ω with respect to the *x*-axis, the moment of inertia for the space Ω with respect to the *y*-axis and the moment of inertia for the space Ω with respect to the *z*-axis are defined, respectively, as

$$I_x = \int\int\int_\Omega (y^2 + z^2)dV, \tag{11}$$

$$I_y = \int\int\int_\Omega (z^2 + x^2)dV, \tag{12}$$

$$I_z = \int\int\int_\Omega (x^2 + y^2)dV. \tag{13}$$

The radius of gyration for a space Ω with respect to *x*-axis, *y*-axis and *z*-axis are defined, respectively, as

$$r_x = \sqrt{\frac{I_x}{V}}, \tag{14}$$

$$r_y = \sqrt{\frac{I_y}{V}}, \tag{15}$$

$$r_z = \sqrt{\frac{I_z}{V}}, \tag{16}$$

where V is the volume of the space Ω.

Example 2.

For fuzzy regular prismoid number $\tilde{u} = (0.72, 0.78, 0.92, 0.97;\ 0.72, 0.78, 0.92, 0.97)$ with membership function as given in Definition 1.

Let $\Omega_1 = \{(x, y, z) : 0.72 \le x \le 0.78,\ x \le y \le -0.83x + 1.57,\ 0 \le z \le 16.67x - 12\}$.

$$\int\int\int_{\Omega_1}(y^2+z^2)dV = \int_{0.72}^{0.78} dx \int_{x}^{-0.83x+1.57} dy \int_0^{16.67x-12}(y^2+z^2)dz \approx 0.0046,$$

$$\int\int\int_{\Omega_1}(z^2+x^2)dV = \int_{0.72}^{0.78} dx \int_{x}^{-0.83x+1.57} dy \int_0^{16.67x-12}(z^2+x^2)dz \approx 0.0039,$$

$$\int\int\int_{\Omega_1}(x^2+y^2)dV = \int_{0.72}^{0.78} dx \int_{x}^{-0.83x+1.57} dy \int_0^{16.67x-12}(x^2+y^2)dz \approx 0.0069,$$

$$\int\int\int_{\Omega_1} 1dV = \int_{0.72}^{0.78} dx \int_{x}^{-0.83x+1.57} dy \int_0^{16.67x-12} 1dz \approx 0.0053.$$

Let $\Omega_2 = \{(x, y, z) : 0.72 \le y \le 0.78,\ y \le x \le -0.83y + 1.57,\ 0 \le z \le 16.67y - 12\}$.

$$\int\int\int_{\Omega_2}(y^2+z^2)dV = \int_{0.72}^{0.78} dy \int_{y}^{-0.83y+1.57} dx \int_0^{16.67y-12}(y^2+z^2)dz \approx 0.0039,$$

$$\int\int\int_{\Omega_2}(z^2+x^2)dV = \int_{0.72}^{0.78} dy \int_{y}^{-0.83y+1.57} dx \int_0^{16.67y-12}(z^2+x^2)dz \approx 0.0046,$$

$$\int\int\int_{\Omega_2}(x^2+y^2)dV = \int_{0.72}^{0.78} dy \int_{y}^{-0.83y+1.57} dx \int_0^{16.67y-12}(x^2+y^2)dz \approx 0.0069,$$

$$\int\int\int_{\Omega_2} 1dV = \int_{0.72}^{0.78} dy \int_{y}^{-0.83y+1.57} dx \int_0^{16.67y-12} 1dz \approx 0.0053.$$

Let $\Omega_3 = \{(x, y, z) : 0.92 \le x \le 0.97,\ -1.2x + 1.884 \le y \le x,\ 0 \le z \le -20x + 19.4\}$.

$$\int\int\int_{\Omega_3}(y^2+z^2)dV = \int_{0.92}^{0.97} dx \int_{-1.2x+1.884}^{x} dy \int_0^{-20x+19.4}(y^2+z^2)dz \approx 0.0039,$$

$$\int\int\int_{\Omega_3}(z^2+x^2) = \int_{0.92}^{0.97} dx \int_{-1.2x+1.884}^{x} dy \int_0^{-20x+19.4}(z^2+x^2)dz \approx 0.0046,$$

$$\int\int\int_{\Omega_3}(x^2+y^2) = \int_{0.92}^{0.97} dx \int_{-1.2x+1.884}^{x} dy \int_0^{-20x+19.4}(x^2+y^2)dz \approx 0.0071,$$

$$\int\int\int_{\Omega_3} 1dV = \int_{0.92}^{0.97} dx \int_{-1.2x+1.884}^{x} dy \int_0^{-20x+19.4} 1dz \approx 0.0044.$$

Let $\Omega_4 = \{(x, y, z) : 0.92 \le y \le 0.97,\ -1.2y + 1.884 \le x \le y,\ 0 \le z \le -20y + 19.4\}$.

$$\int\int\int_{\Omega_4}(y^2+z^2)dV = \int_{0.72}^{0.78} dy \int_{y}^{-0.83y+1.57} dx \int_0^{16.67y-12}(y^2+z^2)dz \approx 0.0046,$$

$$\int\int\int_{\Omega_4}(z^2+x^2)dV = \int_{0.72}^{0.78} dy \int_{y}^{-0.83y+1.57} dx \int_0^{16.67y-12}(z^2+x^2)dz \approx 0.0039,$$

$$\int\int\int_{\Omega_4}(x^2+y^2)dV = \int_{0.72}^{0.78} dy \int_{y}^{-0.83y+1.57} dx \int_0^{16.67y-12}(x^2+y^2)dz \approx 0.0071,$$

$$\int\int\int_{\Omega_4} 1dV = \int_{0.72}^{0.78} dy \int_{y}^{-0.83y+1.57} dx \int_0^{16.67y-12} 1dz \approx 0.0044.$$

Let $\Omega_5 = \{(x, y, z) : 0.78 \leq x \leq 0.92,\ 0.78 \leq y \leq 0.92,\ 0 \leq z \leq 1\}$.

$$\int\int\int_{\Omega_5}(y^2+z^2)dV = \int_{0.78}^{0.92} dx \int_{0.78}^{0.92} dy \int_0^1 (y^2+z^2)dz \approx 0.0207,$$

$$\int\int\int_{\Omega_5}(z^2+x^2)dV = \int_{0.78}^{0.92} dx \int_{0.78}^{0.92} dy \int_0^1 (z^2+x^2)dz \approx 0.0207,$$

$$\int\int\int_{\Omega_5}(x^2+y^2)dV = \int_{0.78}^{0.92} dx \int_{0.78}^{0.92} dy \int_0^1 (x^2+y^2)dz \approx 0.0284,$$

$$\int\int\int_{\Omega_5} 1dV = \int_{0.78}^{0.92} dx \int_{0.78}^{0.92} dy \int_0^1 1dz \approx 0.0196.$$

Let $\Omega = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5$. We have from (11), (12) and (13) that

$$\begin{aligned} I_x &= \int\int\int_{\Omega}(y^2+z^2)dV \\ &= \int\int\int_{\Omega_1}(y^2+z^2)dV + \int\int\int_{\Omega_2}(y^2+z^2)dV + \int\int\int_{\Omega_3}(y^2+z^2)dV + \int\int\int_{\Omega_4}(y^2+z^2)dV + \int\int\int_{\Omega_5}(y^2+z^2)dV \\ &\approx 0.0376, \end{aligned}$$

$$\begin{aligned} I_y &= \int\int\int_{\Omega}(z^2+x^2)dV \\ &= \int\int\int_{\Omega_1}(z^2+x^2)dV + \int\int\int_{\Omega_2}(z^2+x^2)dV + \int\int\int_{\Omega_3}(z^2+x^2)dV + \int\int\int_{\Omega_4}(z^2+x^2)dV + \int\int\int_{\Omega_5}(z^2+x^2)dV \\ &\approx 0.0376, \end{aligned}$$

$$\begin{aligned} I_z &= \int\int\int_{\Omega}(x^2+y^2)dV \\ &= \int\int\int_{\Omega_1}(x^2+y^2)dV + \int\int\int_{\Omega_2}(x^2+y^2)dV + \int\int\int_{\Omega_3}(x^2+y^2)dV + \int\int\int_{\Omega_4}(x^2+y^2)dV + \int\int\int_{\Omega_5}(x^2+y^2)dV \\ &\approx 0.0563. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} V &= \int\int\int_{\Omega} 1dV \\ &= \int\int\int_{\Omega_1} 1dV + \int\int\int_{\Omega_2} 1dV + \int\int\int_{\Omega_3} 1dV + \int\int\int_{\Omega_4} 1dV + \int\int\int_{\Omega_5} 1dV \\ &\approx 0.0390. \end{aligned}$$

According to (14), (15) and (16), we obtain the radius of gyration for fuzzy regular prismoid number $\tilde{u}$ with respect to x-axis, y-axis and z-axis are $r_x = \sqrt{\frac{I_x}{V}} \approx 0.9821$, $r_y = \sqrt{\frac{I_y}{V}} \approx 0.9821$, $r_z = \sqrt{\frac{I_z}{V}} \approx 1.2009$, respectively.

3.2. The Radius of Gyration Similarity Measure between Fuzzy Regular Prismoid Numbers

A similarity measure for fuzzy regular prismoid numbers is being proposed, which is based on distance and the radius of gyration for fuzzy regular prismoid numbers.

Definition 3. Let $\tilde{u}$ and $\tilde{v}$ be both fuzzy regular prismoid numbers. The degree of similarity between the fuzzy regular prismoid numbers $\tilde{u}$ and $\tilde{v}$, denoted as $S(\tilde{u}, \tilde{v})$, is defined as

$$S(\tilde{u},\tilde{v}) = \frac{1}{1+D(\tilde{u},\tilde{v})} \times \frac{\min\{r_x(\tilde{u}), r_x(\tilde{v})\} + \min\{r_y(\tilde{u}), r_y(\tilde{v})\} + \min\{r_z(\tilde{u}), r_z(\tilde{v})\}}{\max\{r_x(\tilde{u}), r_x(\tilde{v})\} + \max\{r_y(\tilde{u}), r_y(\tilde{v})\} + \max\{r_z(\tilde{u}), r_z(\tilde{v})\}},$$

where $r_x(\tilde{u})$, $r_y(\tilde{u})$ and $r_z(\tilde{u})$ are the radius of gyration for $\tilde{u}$ with respect to x-axis, y-axis and z-axis respectively, as well as $r_x(\tilde{v})$, $r_y(\tilde{v})$ and $r_z(\tilde{v})$ are the radius of gyration for $\tilde{v}$ with respect to x-axis, y-axis and z-axis respectively.

The larger value of $S(\tilde{u}, \tilde{v})$ gives the more similarity between the fuzzy regular prismoid numbers $\tilde{u}$ and $\tilde{v}$.

Some of the properties for the similarity measure of fuzzy regular prismoid numbers are researched as follows.

Property 2. Let $\tilde{u} = (a_1, a_2, a_3, a_4;\ b_1, b_2, b_3, b_4)$ and $\tilde{v} = (c_1, c_2, c_3, c_4;\ d_1, d_2, d_3, d_4)$ be both fuzzy regular prismoid numbers. The degree of similarity $S(\tilde{u}, \tilde{v}) = 1$ if and only if fuzzy regular prismoid numbers $\tilde{u}$ and $\tilde{v}$ are identical.

Proof. If fuzzy regular prismoid numbers $\tilde{u}$ and $\tilde{v}$ are identical, i.e., $a_1 = c_1, a_2 = c_2, a_3 = c_3, a_4 = c_4, b_1 = d_1, b_2 = d_2, b_3 = d_3, b_4 = d_4$. It is obvious that $D(\tilde{u},\tilde{v}) = 0$, $r_x(\tilde{u}) = r_x(\tilde{v})$, $r_y(\tilde{u}) = r_y(\tilde{v})$ and $r_z(\tilde{u}) = r_z(\tilde{v})$, thus

$$S(\tilde{u},\tilde{v}) = \frac{1}{1+D(\tilde{u},\tilde{v})} \times \frac{\min\{r_x(\tilde{u}),r_x(\tilde{v})\} + \min\{r_y(\tilde{u}),r_y(\tilde{v})\} + \min\{r_z(\tilde{u}),r_z(\tilde{v})\}}{\max\{r_x(\tilde{u}),r_x(\tilde{v})\} + \max\{r_y(\tilde{u}),r_y(\tilde{v})\} + \max\{r_z(\tilde{u}),r_z(\tilde{v})\}} = 1.$$

Only if: Assume that $S(\tilde{u},\tilde{v}) = 1$. According to

$$\frac{1}{1+D(\tilde{u},\tilde{v})} \le 1,$$

$$\frac{\min\{r_x(\tilde{u}),r_x(\tilde{v})\} + \min\{r_y(\tilde{u}),r_y(\tilde{v})\} + \min\{r_z(\tilde{u}),r_z(\tilde{v})\}}{\max\{r_x(\tilde{u}),r_x(\tilde{v})\} + \max\{r_y(\tilde{u}),r_y(\tilde{v})\} + \max\{r_z(\tilde{u}),r_z(\tilde{v})\}} \le 1,$$

we have $D(\tilde{u},\tilde{v}) = 0$ and

$$\frac{\min\{r_x(\tilde{u}),r_x(\tilde{v})\} + \min\{r_y(\tilde{u}),r_y(\tilde{v})\} + \min\{r_z(\tilde{u}),r_z(\tilde{v})\}}{\max\{r_x(\tilde{u}),r_x(\tilde{v})\} + \max\{r_y(\tilde{u}),r_y(\tilde{v})\} + \max\{r_z(\tilde{u}),r_z(\tilde{v})\}} = 1,$$

which implies that $\tilde{u} = \tilde{v}$.

Property 3. Let $\tilde{u},\tilde{v} \in P(E^2)$. Then $S(\tilde{u},\tilde{v}) = S(\tilde{v},\tilde{u})$.

Proof. We have from $D(\tilde{u},\tilde{v}) = D(\tilde{v},\tilde{u})$, $\min\{r_x(\tilde{u}),r_x(\tilde{v})\} = \min\{r_x(\tilde{v}),r_x(\tilde{u})\}$, $\min\{r_y(\tilde{u}),r_y(\tilde{v})\} = \min\{r_y(\tilde{v}),r_y(\tilde{u})\}$, $\min\{r_z(\tilde{u}),r_z(\tilde{v})\} = \min\{r_z(\tilde{v}),r_z(\tilde{u})\}$, $\max\{r_x(\tilde{u}),r_x(\tilde{v})\} = \max\{r_x(\tilde{v}),r_x(\tilde{u})\}$, $\max\{r_y(\tilde{u}),r_y(\tilde{v})\} = \max\{r_y(\tilde{v}),r_y(\tilde{u})\}$, $\max\{r_z(\tilde{u}),r_z(\tilde{v})\} = \max\{r_z(\tilde{v}),r_z(\tilde{u})\}$ that

$$\begin{aligned} S(\tilde{u},\tilde{v}) &= \frac{1}{1+D(\tilde{u},\tilde{v})} \times \frac{\min\{r_x(\tilde{u}),r_x(\tilde{v})\} + \min\{r_y(\tilde{u}),r_y(\tilde{v})\} + \min\{r_z(\tilde{u}),r_z(\tilde{v})\}}{\max\{r_x(\tilde{u}),r_x(\tilde{v})\} + \max\{r_y(\tilde{u}),r_y(\tilde{v})\} + \max\{r_z(\tilde{u}),r_z(\tilde{v})\}} \\ &= \frac{1}{1+D(\tilde{v},\tilde{u})} \times \frac{\min\{r_x(\tilde{v}),r_x(\tilde{u})\} + \min\{r_y(\tilde{v}),r_y(\tilde{u})\} + \min\{r_z(\tilde{v}),r_z(\tilde{u})\}}{\max\{r_x(\tilde{v}),r_x(\tilde{u})\} + \max\{r_y(\tilde{v}),r_y(\tilde{u})\} + \max\{r_z(\tilde{v}),r_z(\tilde{u})\}} \\ &= S(\tilde{v},\tilde{u}). \end{aligned}$$

Property 4. Let $\tilde{u},\tilde{v} \in P(E^2)$. Then $0 \le S(\tilde{u},\tilde{v}) \le 1$.

Proof. From $0 \le \frac{1}{1+D(\tilde{u},\tilde{v})} \le 1$ and $\min\{r_x(\tilde{u}),r_x(\tilde{v})\} + \min\{r_y(\tilde{u}),r_y(\tilde{v})\} + \min\{r_z(\tilde{u}),r_z(\tilde{v})\} \le \max\{r_x(\tilde{u}),r_x(\tilde{v})\} + \max\{r_y(\tilde{u}),r_y(\tilde{v})\} + \max\{r_z(\tilde{u}),r_z(\tilde{v})\}$, we have $0 \le S(\tilde{u},\tilde{v}) \le 1$.

Example 3.

For fuzzy regular prismoid numbers $\tilde{u} = (0.72, 0.78, 0.92, 0.97;\ 0.72, 0.78, 0.92, 0.97)$ and $\tilde{v} = (0.8263, 0.8702, 0.9534, 0.9824;\ 0.7950, 0.8487, 0.9384, 0.9640)$ with membership function as given in Definition 1.

According to Example 2, the radius of gyration for fuzzy regular prismoid number $\tilde{u}$ with respect to x-axis, y-axis and z-axis are $r_x(\tilde{u}) = \sqrt{\frac{I_x}{V}} \approx 0.9821$, $r_y(\tilde{u}) = \sqrt{\frac{I_y}{V}} \approx 0.9821$, $r_z(\tilde{u}) = \sqrt{\frac{I_z}{V}} \approx 1.2009$.

For $\tilde{v} = (0.8341, 0.8701, 0.9535, 0.9895;\ 0.8086, 0.8488, 0.9382, 0.9784)$, we have $I_x = \int\int\int_\Omega (y^2+z^2)dV \approx 0.0165$, $I_y = \int\int\int_\Omega (z^2+x^2)dV \approx 0.0171$, $I_z = \int\int\int_\Omega (x^2+y^2)dV \approx 0.0261$ and $V = \int\int\int_\Omega 1dV \approx 0.0160$. According to (14), (15) and (16), we obtain the radius of gyration for fuzzy regular prismoid number $\tilde{v}$ with respect to x-axis, y-axis and z-axis are $r_x(\tilde{v}) = \sqrt{\frac{I_x}{V}} \approx 1.0108$, $r_y(\tilde{v}) = \sqrt{\frac{I_y}{V}} \approx 1.0301$, $r_z(\tilde{v}) = \sqrt{\frac{I_z}{V}} \approx 1.2688$, respectively. From

$$\begin{aligned} D(\tilde{u},\tilde{v}) = \frac{1}{2}\Bigg\{ &\left[\int_0^1 f(r)(R^{-1}_{\tilde{p}_u(x)}(r) - R^{-1}_{\tilde{p}_v(x)}(r))^2 dr + \int_0^1 f(r)(R^{-1}_{\tilde{p}_u(y)}(r) - R^{-1}_{\tilde{p}_v(y)}(r))^2 dr\right]^{\frac{1}{2}} \\ &+ \left[\int_0^1 f(r)(L^{-1}_{\tilde{p}_u(x)}(r) - L^{-1}_{\tilde{p}_v(x)}(r))^2 dr + \int_0^1 f(r)(L^{-1}_{\tilde{p}_u(y)}(r) - L^{-1}_{\tilde{p}_v(y)}(r))^2 dr\right]^{\frac{1}{2}} \Bigg\} \end{aligned}$$

$$= \frac{1}{2}\left\{\left[\int_0^1 f(r)(a_4 - c_4 - (a_4 - c_4 - a_3 + c_3)r)^2 dr + \int_0^1 f(r)(b_4 - d_4 - (b_4 - d_4 - b_3 + d_3)r)^2 dr\right]^{\frac{1}{2}}\right.$$

$$\left. + \left[\int_0^1 f(r)((a_2 - a_1 - c_2 + c_1)r + a_1 - c_1)^2 dr + \int_0^1 f(r)((b_2 - b_1 - d_2 + d_1)r + b_1 - d_1)^2 dr\right]^{\frac{1}{2}}\right\}$$

$$\approx 0.0524,$$

we have

$$S(\tilde{u}, \tilde{v}) = \frac{1}{1 + D(\tilde{u}, \tilde{v})} \times \frac{\min\{r_x(\tilde{u}), r_x(\tilde{v})\} + \min\{r_y(\tilde{u}), r_y(\tilde{v})\} + \min\{r_z(\tilde{u}), r_z(\tilde{v})\}}{\max\{r_x(\tilde{u}), r_x(\tilde{v})\} + \max\{r_y(\tilde{u}), r_y(\tilde{v})\} + \max\{r_z(\tilde{u}), r_z(\tilde{v})\}}$$

$$\approx \frac{1}{1.0524} \times \frac{\min\{0.9821,\ 1.0108\} + \min\{0.9821,\ 1.0301\} + \min\{1.2009,\ 1.2688\}}{\max\{0.9821,\ 1.0108\} + \max\{0.9821,\ 1.0301\} + \max\{1.2009,\ 1.2688\}}$$

$$= \frac{1}{1.0524} \times \frac{0.9821 + 0.9821 + 1.2009}{1.0108 + 1.0301 + 1.2688}$$

$$\approx 0.9087.$$

4. FUZZY RISK ANALYSIS BASED ON FUZZY REGULAR PRISMOID NUMBERS

Assume that there is a component A consisting of n sub-components $A_1, A_2, \cdots, A_n$ and each sub-component is evaluated by two evaluating items probability of failure and severity of loss, as well as $\tilde{R}_i$ denotes the probability of failure and $\tilde{\omega}_i$ denotes the severity of loss of the sub-component A_i, respectively, where $\tilde{R}_i$ and $\tilde{\omega}_i$ are regular prismoid numbers ($1 \le i \le n$). The algorithm for dealing with fuzzy risk analysis is now presented as follows:

Step 1: Use the regular prismoid number arithmetic operations to calculate the probability of failure $\tilde{R}$ of component A,

$$\tilde{R} = \left(\sum_{i=1}^{n} \tilde{R}_i \otimes \tilde{\omega}_i\right) \oslash \sum_{i=1}^{n} \tilde{\omega}_i,$$

where $\tilde{R} \in E^2$.

Step 2: Use Theorem 1 to calculate the nearest fuzzy regular prismoid number $\tilde{R}^*$ to $\tilde{R}$ preserves the centroid of $[\tilde{R}]^1$ with respect to the weighted pseudometric.

Step 3: Calculate the radius of gyration for each linguistic term shown in Table 1 with respect to x-axis, y-axis and z-axis.

Step 4: According to the distance D of the fuzzy regular prismoid numbers, we calculate the distance of $\tilde{R}^*$ with each linguistic term.

Step 5: Use the similarity measure of regular prismoid numbers to calculate the similarity measure of $\tilde{R}^*$ with each linguistic term.

Step 6: Find the largest similarity of $\tilde{R}^*$ and it is considered as a risk value of the system in linguistic term. The largest the similarity measure, the highest the probability of failure of component A.

Table 1 A 9-member linguistic term set.

Linguistic Terms	Linguistic Valued Regular Prismoid Number
Absolutely low	(0, 0, 0, 0; 0, 0, 0, 0)
Very low	(0, 0, 0.02, 0.07; 0, 0, 0.02, 0.07)
Low	(0.04, 0.1, 0.18, 0.23; 0.04, 0.1, 0.18, 0.23)
Fairly low	(0.17, 0.22, 0.36, 0.42; 0.17, 0.22, 0.36, 0.42)
Medium	(0.32, 0.41, 0.58, 0.65; 0.32, 0.41, 0.58, 0.65)
Fairly high	(0.58, 0.63, 0.80, 0.86; 0.58, 0.63, 0.80, 0.86)
High	(0.72, 0.78, 0.92, 0.97; 0.72, 0.78, 0.92, 0.97)
Very high	(0.93, 0.98, 1.0, 1.0; 0.93, 0.98, 1.0, 1.0)
Absolutely high	(1.0, 1.0, 1.0, 1.0; 1.0, 1.0, 1.0, 1.0)

Example 4.

Every company plays a major role in creating employment opportunities and improving the standard of living of people. Therefore, it is necessary to study the fuzzy risk analysis on every company. The key to success of a company is to depend on the following sub-components. A_1: Business management, which based on the personal ability and management inexperience. A_2: Working efficiency of employee, which based on their production speed and the quality of their products. A_3: Technology, which based on the hardware investments and the technological level of employees. A_4: Product, which based on products popularity and products market. A_5: Geographical location, which based on transportation and peripheral economic. Obviously, these linguistic terms are expressed by fuzzy regular prismoid numbers more precise and more actual. Assume that the fuzzy regular prismoid numbers of the probability of failure and severity of loss of the sub-components are shown in Table 2.

Step 1: We have from Table 2 that $[\tilde{R}_1]^r = [0.05r+0.93, 1.0]\times[0.05r+0.93, 1.0]$, $[\tilde{\omega}_1]^r = [0.05r+0.58, 0.86-0.06r]\times[0.05r+0.58, 0.86-0.06r]$, $[\tilde{R}_2]^r = [0.06r+0.72, 0.97-0.05r]\times[0.05r+0.93, 1.0]$, $[\tilde{\omega}_2]^r = [0.05r+0.58, 0.86-0.06r]\times[0.05r+0.58, 0.86-0.06r]$, $[\tilde{R}_3]^r = [1.0, 1.0]\times[0.06r+0.72, 0.97-0.05r]$, $[\tilde{\omega}_3]^r = [0.06r+0.72, 0.97-0.05r]\times[0.06r+0.72, 0.97-0.05r]$, $[\tilde{R}_4]^r = [0.06r+0.72, 0.97-0.05r]\times[0.05r+0.58, 0.86-0.06r]$, $[\tilde{\omega}_4]^r = [0.05r+0.58, 0.86-0.06r]\times[0.05r+0.93, 1]$, $[\tilde{R}_5]^r = [0.06r+0.72, 0.97-0.05r]\times[0.05r+0.93, 1]$, $[\tilde{\omega}_5]^r = [0.05r+0.58, 0.86-0.06r]\times[0.06r+0.72, 0.97-0.05r]$. Then it follows from Definition 2 that

$$
\begin{aligned}
[\tilde{R}_1 \otimes \tilde{\omega}_1]^r = &\left[\inf_{\beta\geq r}\min_{\lambda\in[0,1]}\{\lambda(0.05\beta+0.93)+(1-\lambda)\cdot 1.0\}\cdot\{\lambda(0.05\beta+0.58)+(1-\lambda)(0.86-0.06\beta)\},\right.\\
&\left.\sup_{\beta\geq r}\max_{\lambda\in[0,1]}\{\lambda(0.05\beta+0.93)+(1-\lambda)\cdot 1.0\}\cdot\{\lambda(0.05\beta+0.58)+(1-\lambda)(0.86-0.06\beta)\}\right]\\
&\times\left[\inf_{\beta\geq r}\min_{\lambda\in[0,1]}\{\lambda(0.05\beta+0.93)+(1-\lambda)\cdot 1.0\}\cdot\{\lambda(0.05\beta+0.58)+(1-\lambda)(0.86-0.06\beta)\},\right.\\
&\left.\sup_{\beta\geq r}\max_{\lambda\in[0,1]}\{\lambda(0.05\beta+0.93)+(1-\lambda)\cdot 1.0\}\cdot\{\lambda(0.05\beta+0.58)+(1-\lambda)(0.86-0.06\beta)\}\right]\\
=&\left[\inf_{\beta\geq r}\left\{0.0025\beta^2+0.0755\beta+0.5394\right\},\sup_{\beta\geq r}\{0.86-0.06\beta\}\right]\\
&\times\left[\inf_{\beta\geq r}\left\{0.0025\beta^2+0.0755\beta+0.5394\right\},\sup_{\beta\geq r}\{0.86-0.06\beta\}\right]\\
=&\left[0.0025r^2+0.0755r+0.5394, 0.86-0.06r\right]\times\left[0.0025r^2+0.0755r+0.5394, 0.86-0.06r\right],
\end{aligned}
$$

for any $r \in (0, 1]$. Similarly, we obtain

$$
\begin{aligned}
&[\tilde{R}_2 \otimes \tilde{\omega}_2]^r = [0.003r^2+0.0708r+0.4176, 0.003r^2-0.1012r+0.8342]\times[0.0025r^2+0.0755r+0.5394, 0.86-0.06r],\\
&[\tilde{R}_3 \otimes \tilde{\omega}_3]^r = [0.003r^2+0.0708r+0.4176, 0.003r^2-0.1012r+0.8342]\times[0.0036r^2+0.0864r+0.5184, 0.0025r^2-0.097r+0.9409],\\
&[\tilde{R}_4 \otimes \tilde{\omega}_4]^r = [0.003r^2+0.0708r+0.4176, 0.003r^2-0.1012r+0.8342]\times[0.0025r^2+0.0755r+0.5394, 0.86-0.06r],\\
&[\tilde{R}_5 \otimes \tilde{\omega}_5]^r = [0.003r^2+0.0708r+0.4176, 0.003r^2-0.1012r+0.8342]\times[0.003r^2+0.0918r+0.6696, 0.97-0.05r],
\end{aligned}
$$

for any $r \in (0, 1]$. Thus,

$$
\left[\sum_{i=1}^{5}\tilde{R}_i\otimes\tilde{\omega}_i\right]^r = [0.0115r^2+0.3479r+2.5122, 0.009r^2-0.4136r+4.3326]\times[0.0141r^2+0.4047r+2.8062, 0.0025r^2-0.327r+4.4909]
$$

for any $r \in (0, 1]$. Furthermore, we have

$$
\left[\sum_{i=1}^{5}\tilde{\omega}_i\right]^r = [0.26r+3.04, 4.41-0.29r]\times[0.27r+3.53, 4.66-0.22r],
$$

Table 2 Fuzzy regular prismoid numbers of $\tilde{R}_i$ and $\tilde{\omega}_i$ for Ssb-components A_1; A_2, ..., A_5.

A_i	Probability of Failure $\tilde{R}_i$	Severity of Loss $\tilde{\omega}_i$
A_1	(0.93, 0.98, 1.0, 1.0; 0.93, 0.98, 1.0, 1.0)	(0.58, 0.63, 0.80, 0.86; 0.58, 0.63, 0.80, 0.86)
A_2	(0.72, 0.78, 0.92, 0.97; 0.93, 0.98, 1.0, 1.0)	(0.58, 0.63, 0.80, 0.86; 0.58, 0.63, 0.80, 0.86)
A_3	(1.0, 1.0, 1.0, 1.0; 0.72, 0.78, 0.92, 0.97)	(0.72, 0.78, 0.92, 0.97; 0.72, 0.78, 0.92, 0.97)
A_4	(0.72, 0.78, 0.92, 0.97; 0.58, 0.63, 0.80, 0.86)	(0.58, 0.63, 0.80, 0.86; 0.93, 0.98, 1.0, 1.0)
A_5	(0.72, 0.78, 0.92, 0.97; 0.93, 0.98, 1.0, 1.0)	(0.58, 0.63, 0.80, 0.86; 0.72, 0.78, 0.92, 0.97)

for any $r \in (0, 1]$. Then the probability of failure $\tilde{R}$ of component A is showing as follow.

$$[\tilde{R}]^r = \left[\left(\sum_{i=1}^{n} \tilde{R}_i \otimes \tilde{\omega}_i\right) \oslash \sum_{i=1}^{n} \tilde{\omega}_i\right]^r$$

$$= \left[\inf_{\beta \geq r} \min_{\lambda \in [0,1]} \left\{\lambda(0.0115\beta^2 + 0.3479\beta + 2.5122) + (1-\lambda)(0.009\beta^2 - 0.4136\beta + 4.3326)\right\}\right.$$

$$\div \{\lambda(0.26\beta + 3.04) + (1-\lambda)(4.41 - 0.29\beta)\}],$$

$$\sup_{\beta \geq r} \max_{\lambda \in [0,1]} \left\{\lambda(0.0115\beta^2 + 0.3479\beta + 2.5122) + (1-\lambda)(0.009\beta^2 - 0.4136\beta + 4.3326)\right\}$$

$$\div \{\lambda(0.26\beta + 3.04) + (1-\lambda)(4.41 - 0.29\beta)\}]$$

$$\times \left[\inf_{\beta \geq r} \min_{\lambda \in [0,1]} \left\{\lambda(0.0141\beta^2 + 0.4047\beta + 2.8062) + (1-\lambda)(0.0025\beta^2 - 0.327\beta + 4.4909)\right\}\right.$$

$$\div \{\lambda(0.27\beta + 3.53) + (1-\lambda)(4.66 - 0.22\beta)\}],$$

$$\sup_{\beta \geq r} \max_{\lambda \in [0,1]} \{\lambda(0.0141\beta^2 + 0.4047\beta + 2.8062) + (1-\lambda)(0.0025\beta^2 - 0.327\beta + 4.4909)\}$$

$$\div \{\lambda(0.27\beta + 3.53) + (1-\lambda)(4.66 - 0.22\beta)\}]$$

$$= \left[\inf_{\beta \geq r}\left\{\frac{0.0115\beta^2 + 0.3479\beta + 2.5122}{0.26\beta + 3.04}\right\}, \sup_{\beta \geq r}\left\{\frac{0.009\beta^2 - 0.4136\beta + 4.3326}{4.41 - 0.29\beta}\right\}\right]$$

$$\times \left[\inf_{\beta \geq r}\left\{\frac{0.0141\beta^2 + 0.4047\beta + 2.8062}{0.27\beta + 3.53}\right\}, \sup_{\beta \geq r}\left\{\frac{0.0025\beta^2 - 0.327\beta + 4.4909}{4.66 - 0.22\beta}\right\}\right]$$

$$= \left[\frac{0.0115r^2 + 0.3479r + 2.5122}{0.26r + 3.04}, \frac{0.009r^2 - 0.4136r + 4.3326}{4.41 - 0.29r}\right] \times \left[\frac{0.0141r^2 + 0.4047r + 2.8062}{0.27r + 3.53}, \frac{0.0025r^2 - 0.327r + 4.4909}{4.66 - 0.22r}\right],$$

for any $r \in [0, 1]$.

Step 2: According to Example 1 the nearest fuzzy regular prismoid number to $\tilde{R}$ preserves the centroid of $[\tilde{R}]^1$ with respect to the weighted pseudometric is $\tilde{R}^* = (0.8263, 0.8702, 0.9534, 0.9824;\ 0.7950, 0.8487, 0.9384, 0.9640.;$

Step 3: The radius of gyration for each linguistic term with respect to x-axis, y-axis and z-axis as shown in Table 3.

Step 4: The distance of $\tilde{R}^*$ with each linguistic term as shown in Table 4.

Step 5: Calculate the similarity measure of $\tilde{R}^*$ with each linguistic term, we have

$$S(\tilde{R}^*, Absolutely - low) = 0$$

$$S(\tilde{R}^*, very - low) \approx 0.1407$$

$$S(\tilde{R}^*, low) \approx 0.1990$$

Table 3 The radius of gyration for each linguistic term.

Linguistic Terms	r_x	r_y	r_z
Absolutelylow	$r_x(A-L) = 0$	$r_y(A-L) = 0$	$r_z(A-L) = 0$
Very-low	$r_x(V-L) \approx 0.4155$	$r_y(V-L) \approx 0.4155$	$r_z(V-L) \approx 0.0447$
Low	$r_x(L) \approx 0.4783$	$r_y(L) \approx 0.4783$	$r_z(A-L) \approx 0.2033$
Fairly-low	$r_x(F-L) \approx 0.5770$	$r_y(F-L) \approx 0.5770$	$r_z(F-L) \approx 0.4229$
Medium	$r_x(M) \approx 0.6911$	$r_y(M) \approx 0.6911$	$r_z(M) \approx 0.7003$
Fairly high	$r_x(F-H) \approx 0.8802$	$r_y(F-H) \approx 0.8802$	$r_z(F-H) \approx 1.0198$
High	$r_x(H) \approx 0.9821$	$r_y(H) \approx 0.9821$	$r_z(H) \approx 1.2009$
Very high	$r_x(V-H) \approx 1.0579$	$r_y(V-H) \approx 1.0579$	$r_z(V-H) \approx 1.3767$
Absolutely high	$r_x(A-H) \approx 1.4142$	$r_y(A-H) \approx 1.4142$	$r_z(A-H) \approx 1.4142$

Table 4 The distance of $\tilde{R}^*$ with each linguistic term.

Linguistic Terms	D
Absolutely low	$D(\tilde{R}^*, A - L) \approx 0.8992$
Very low	$D(\tilde{R}^*, V - L) \approx 0.8809$
Low	$D(\tilde{R}^*, L) \approx 0.7608$
Fairly low	$D(\tilde{R}^*, F - L) \approx 0.6075$
Medium	$D(\tilde{R}^*, M) \approx 0.4077$
Fairly high	$D(\tilde{R}^*, F - H) \approx 0.1828$
High	$D(\tilde{R}^*, H) \approx 0.0524$
Very high	$D(\tilde{R}^*, V - H) \approx 0.0835$
Absolutely high	$D(\tilde{R}^*, A - H) \approx 0.1019$

$$S(\tilde{R}^*, fairly - low) \approx 0.2964$$

$$S(\tilde{R}^*, medium) \approx 0.4470$$

$$S(\tilde{R}^*, fairly - high) \approx 0.7102$$

$$S(\tilde{R}^*, high) \approx 0.9087$$

$$S(\tilde{R}^*, very - high) \approx 0.8746$$

$$S(\tilde{R}^*, absolutely - high) \approx 0.7080$$

Step 6: It is obvious that 0.9087 is the largest value. Therefore, the the regular prismoid number $\tilde{R}^*$ is translated into the linguistic term *high*. That is, for this system the risk of failure to the company is *high*.

5. CONCLUSION

Taking into account that fuzzy regular prismoid number can provide more flexibility and tractability to represent the imprecise information, we will use the regular prismoid numbers to deal with fuzzy risk analysis problems. The arithmetic operator of fuzzy regular prismoid numbers and the degree of similarity between fuzzy regular prismoid numbers are introduced, which are the basic work of studying fuzzy risk analysis. Furthermore, we establish the fuzzy regular prismoid numbers approximation of 2-dimensional fuzzy numbers. Finally, the arithmetic operator and the degree of similarity for fuzzy regular prismoid numbers are applied in fuzzy risk analysis. In this sense the research is a first step toward solving fuzzy risk analysis. Our proposal for future work is to use fuzzy regular prismoid numbers to investigate fuzzy risk analysis and applications.

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest

AUTHORS' CONTRIBUTIONS

From the conception of the idea, writing the manuscript, interpretation of the results to data preparation and analysis are all completed by the author independently.

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REFERENCES

[1] L.A. Zadeh, The concept of linguistic variable and its application to approximate reasoning-I, Inf. Sci. 8 (1975), 199–249.
[2] D. Dubois, H. Prade, Operations on fuzzy numbers, Int. J. Syst. Sci. 9 (1978), 613–626.
[3] B. Asady, The revised method of ranking LR fuzzy number based on deviation degree, Expert Syst. Appl. 37 (2010), 5056–5060.
[4] A. Kaufmann, M.M. Gupta, Fuzzy Mathematical Models in Engineering and Managment Science, Elseiver Science Publishers B.V., Amsterdam, the Netherlands, 1988. http://refhub.elsevier.com/S0020-0255(16)30131-1/sbref0015
[5] C.H. Cheng, A new approach for ranking fuzzy numbers by distance method, Fuzzy Sets Syst. 95 (1998), 307–317.
[6] G.X. Wang, C.X. Wu, Fuzzy n-cell numbers and the differential of fuzzy n-cell number value mappings, Fuzzy Sets Syst. 130 (2002), 367–381.
[7] G. Wang, P. Shi, B. Wang, J. Zhang, Fuzzy n-ellipsoid numbers and representations of uncertain multichannel digital information, IEEE Trans. Fuzzy Syst. 22 (2014), 1113–1126.
[8] S.X. Hai, Z.T. Gong, Z.Z. Chen, Weighted pseudometric approximation of 2-dimensional fuzzy numbers by fuzzy 2-cell prismoid numbers preserving the centroid, Fuzzy Sets Syst. 387 (2020), 158–173.
[9] H. Seiti, A. Hafezalkotob, L. Martinez, R-numbers: a new risk modeling associated with fuzzy numbers and its application to decision making, Inf. Sci. 483 (2019), 206–231.
[10] H. Seiti, A. Hafezalkotob, L. Martinez, R-sets: comprehensive fuzzy sets risk modeling for risk-based information fusion and decision-making, IEEE Trans. Fuzzy Syst. 29 (2021), 385–399.
[11] R. Kangari, L.S. Riggs, Construction risk assessment by linguistics, IEEE Trans. Eng. Manag. 36 (1989), 126–131.
[12] Z.Y. Xu, S.C. Shang, W.B. Qian, W.H. Shu, A method for fuzzy risk analysis based on the new similarity of trapezoidal fuzzy numbers, Expert Syst. Appl. 37 (2010), 1920–1927.
[13] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-III, Inf. Sci. 9 (1975), 43–80.
[14] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, NY, USA, 1980. http://refhub.elsevier.com/S1568-4946(14)00547-X/sbref0015
[15] D. Dubois, H. Prade, Possibility Theory: an Approach to Computerized Processing of Uncertainty, Plenum Press, New York, NY, USA, 1988.
[16] M.H. Alavidoost, M. Tarimoradi, M.F. Zarandi, Fuzzy adaptive genetic algorithm for multi-objective assembly line balancing problems, Appl. Soft Comput. 34 (2015), 655–677.
[17] M. Wagenknecht, R. Hampel, V. Schneider, Computational aspects of fuzzy arithmetics based on Archimedean t-norms, Fuzzy Sets Syst. 123 (2001), 49–62.
[18] M.L. Guerra, L. Stefanini, Approximate fuzzy arithmetic operations using monotonic interpolations, Fuzzy Sets Syst. 150 (2005), 5–33.
[19] A.M. Bica, Algebraic structures for fuzzy numbers from categorial point of view, Soft Comput. 11 (2007), 1099–1105.
[20] M. Hukuhara, Integration des applications measurables dont la valeur est un compact convexe, Funkcialaj Ekvacioj. 10 (1967), 205–229. http://refhub.elsevier.com/S0165-0114(14)00155-9/bib68756Bs1
[21] L. Stefanini, A generalization of Hukuhara difference and division for interval and fuzzy arithmetic, Fuzzy Sets Syst. 161 (2010), 1564–1584.
[22] Y. Chalco-Cano, W.A. Lodwick, B. Bede, Single level constraint interval arithmetic, Fuzzy Sets Syst. 257 (2014), 146–168.
[23] A. Stupnanova, A probabilistic approach to the arithmetics of fuzzy numbers, Fuzzy Sets Syst. 264 (2015), 64–75.
[24] M. Kumar, Applying weakest t-norm based approximate intuitionistic fuzzy arithmetic operations on different types of intuitionistic fuzzy numbers to evaluate reliability of PCBA fault, Appl. Soft Comput. 23 (2014), 387–406.
[25] N.G. Seresht, A.R. Fayek, Computational method for fuzzy arithmetic operations on triangular fuzzy numbers by extension principle, Int. J. Approx. Reason. 106 (2019), 172–193.
[26] G.J. Klir, Fuzzy arithmetic with requisite constraints, Fuzzy Sets Syst. 91 (1997), 165–175.
[27] R. Mesiar, Shape preserving additions of fuzzy intervals, Fuzzy Sets Syst. 86 (1997), 73–78.
[28] D.H. Hong, H.Y. Do, Fuzzy system reliability analysis by the use of $T\omega$ (the weakest t-norm) on fuzzy number arithmetic operations, Fuzzy Sets Syst. 90 (1997), 307–316.
[29] M. Holcapek, M. Stepnicka, MI-algebras: anew framework for arithmetics of (extensional) fuzzy numbers, Fuzzy Sets Syst. 257 (2014), 102–131.
[30] R. Lelliott, Fuzzy sets, natural language computations, and risk analysis, Fuzzy Sets Syst. 27 (1988), 395–396.
[31] S.J. Chen, S.M. Chen, Fuzzy risk analysis based on measures of similarity between interval-valued fuzzy numbers, Comput. Appl. Math. 55 (2008), 1670–1685.
[32] S.J. Chen, S.M. Chen, Fuzzy risk analysis based on similarity measures of generalized fuzzy numbers, IEEE Trans. Fuzzy Syst. 11 (2003), 45–56.
[33] H.A. Khorshidi, S. Nikfalazar, An improved similarity measure for generalized fuzzy numbers and its application to fuzzy risk analysis, Appl. Soft Comput. 52 (2017), 478–486.
[34] Y.M. Wang, T.M.S. Elhag, Fuzzy TOPSIS method based on alpha level sets with an application to bridge risk assessment, Expert Syst. Appl. 31 (2006), 309–319.
[35] R. Chutia, M.K. Gogoi, Fuzzy risk analysis in poultry farming using a new similarity measure on generalized fuzzy numbers, Comput. Ind. Eng. 115 (2018), 543–558.

[36] R. Chutia, M.K. Gogoi, Fuzzy risk analysis in poultry farming based on a novel similarity measure of fuzzy numbers, Appl. Soft Comput. 66 (2018), 60–76.
[37] S.R. Hejazi, A. Doostparast, S.M. Hosseini, An improved fuzzy risk analysis based on a new similarity measures of generalized fuzzy numbers, Expert Syst. Appl. 38 (2011), 9179–9185.
[38] K. Patra, S.K. Mondal, Fuzzy risk analysis using area and height based similarity measure on generalized trapezoidal fuzzy numbers and its application, Appl. Soft Comput. 28 (2015), 276–284.
[39] S.H. Wei, S.M. Chen, A new approach for fuzzy risk analysis based on similarity measures of generalized fuzzy numbers, Expert Syst. Appl. 36 (2009), 589–598.
[40] Z.T. Gong, S.X. Hai, Convexity of n-dimensional fuzzy number-valued functions and its applications, Fuzzy Sets Syst. 295 (2016), 19–36.
[41] S.X. Hai, Z.T. Gong, H.X. Li, Generalized differentiability for n-dimensional fuzzy-number-valued functions and fuzzy optimization, Inf. Sci. 374 (2016), 151–163.
[42] P. Diamond, P. Kloeden, Characterization of compact subsets of fuzzy sets, Fuzzy Sets Syst. 29 (1989), 341–348.
[43] Y.L. He, C.H. Wei, H. Long, R.A.R. Ashfaq, J.Z. Huang, Random weight network-based fuzzy nonlinear regression for trapezoidal fuzzy number data, Appl. Soft Comput. 70 (2018), 959–979.
[44] P. Ziemba, NEAT F-PROMETHEE-a new fuzzy multiple criteria decision making method based on the adjustment of mapping trapezoidal fuzzy numbers, Expert Syst. Appl. 110 (2018), 363–380.
[45] J. Sadeghi, S.T.A. Niaki, Two parameter tuned multi-objective evolutionary algorithms for a bi-objective vendor managed inventory model with trapezoidal fuzzy demand, Appl. Soft Comput. 30 (2015), 567–576.
[46] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, NY, USA, 1970. http://refhub.elsevier.com/S0165 -0114(18)31080-7/bib5254526F636B6166 656C6C617232303034s1