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Investigating Students' Errors in Graphing Polynomial Functions

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ABSTRACT

The transition from school to college mathematics courses involving advanced mathematics is a challenging task for many students. Undergraduate students in their first year are very often to experience difficulties in understanding mathematical concepts in a setting that emphasizes symbolic algebraic skills and graphing skills. Graphing skills are very important ability to grasp for students especially for whom mastering in mathematics. However, many references outlined that many students lack graphical skills. This study aims to investigate various students' errors in graphing polynomial functions which are considered as basic types of functions. To gather data on students' errors in graphing polynomial functions, two kinds of test are administered to first year students majoring in mathematics. The first test is directed to investigate students' prior knowledge on graphical skills learnt in school. On the other hand, the second one is aimed to gather information on students' graphical skills in higher order polynomial functions which have not been taught in school. The result reveals that many students make errors in graphing higher polynomial functions the same way as graphing linear and quadratic ones. Students' prior knowledge in graphing linear and quadratic functions essentially influences graphing skills in polynomial functions of higher degree.

Keywords: *Students' errors, Graphs of Polynomial Functions, Graphical Skills.*

1. INTRODUCTION

First year of college life is a transition phase for students to adapt with new academic environment and higher difficulty level of courses. In this period, mathematics courses involving advanced mathematics are found to be quite challenging for many students. Many students graduated from high school were found underprepared for college-level coursework in mathematics [1]. Barnett, Chavarin & Griffin [2] stated that many high school graduates begin college without the knowledge and skills needed to excel in college-level math courses. Poor alignment between high school and college-level expectations and coursework is considered as the main problem.

High school mathematics mainly emphasizes on knowing how to calculate and how to obtain the "correct answer". While college mathematics more emphasizes on conceptual understandings than calculation, hence more abstract, a higher demand for precise statements, and logical deductions. Concepts on polynomial functions are taught stepwise based on the degrees of the polynomials. In high school, linear and quadratic polynomials are discussed thoroughly covering the properties of functions, equations, inequalities, and graphing. Meanwhile, polynomials oh higher degree are taught in college in the first year. In addition to functions, equations, inequalities, and graphing, other important properties of polynomial functions are covered in college. Although similar concepts with lower degrees had been discussed before in high school, many college students still experienced difficulties particularly in graphing polynomial functions as stated in Curran [3].

Polynomial functions are the simplest of all functions in mathematics in part because they only involve multiplication and addition. In applied mathematics, phenomena involving arithmetic operations is simply modelled by polynomial functions. To develop a model function that represents a physical situation, the first step that could be carried out is by drawing one or more diagrams of the situation and then introduce one or more variables to represent quantities that are changing. Relationships that are present are investigated to express one of the quantities in terms of the other(s). One of many examples modelling using polynomial functions is conducted by Meek et al.[4]. E. coli and Bromide breakthrough is modelled using rational functions which are composed from polynomial functions. Meek et al. [4] reported that the models clearly tracked E. coli and Bromide distribution better than the lognormal model.

Cansiz, Küçük, & İşleyen [5] conducted a research on identifying students' misconceptions about functions. The research result pointed out that it was most commonly observed that students made mistakes to understand whether a graphic is a function graphic or not, and about the demonstration of the table and function especially on the function information test. Although students are certain to have experienced plots and graphs in high school, most students often struggle with the basics of graphing. Similar to Cansiz, Küçük, & İşleyen, Bardini et al. [6] conducted research on undergraduate mathematics students' understanding of the concept of functions. It reveals that many students were unable to give an appropriate definition or recognize whether a given graph or rule represents a function and could not make correct connections between function graphs and table of values. Walde [7] reported that many secondary scholl students in Ethiopia had difficulties in verbal and graphical representation of functions. Meisadewi, Anggraeni, & Supriatno [8] also reported that students have difficulties about using scientific language in graphing skills.

Graphs and plots are key in introductory courses in which quantitative skills are emphasized because they are the essence of giving students multiple representations of mathematical concepts. Graphs have been appraised as basic part of mathematics as it is very useful representation for summarizing sets of data and interpreting new information from complex data. Thus graphing skills are very important for students (mainly for mathematics students) to master the skills.

It is highly necessary to identify and wipe out mistakes made by students about mathematical concepts. When lecturers/teachers know about students' prior knowledge and their cognitive features that come along with them when educating them, the information regarding mistake identification can surely assist the lecturers in knowing students more easily and having information regarding where students may make mistakes and what kind of mistakes they may make and how they may think. Rahardi and Hasanah [9] analyzed students' skills on derivative of a function to have an insight about students' prior knowledge in derivative to order to well organize a course in ordinary differential equations. The resulting data can be used as a reference and feedback for lecturers to apply suitable learning model or media to assist them in teaching. On the other hand, Veloo, Krishnasamy & Abdullah [10] identified the errors made by senior high school students in Malaysia. It is also reported that the study has an implication for the student learning process and understanding graphs. In this paper, students' mistakes are investigated to obtain graphing skills of first-year students majoring in mathematics in Universitas Negeri Malang. This research also identifies errors in graphing linear and quadratic functions which have been studied in high school. The result of this study can be used by lecturers to organize preliminary courses involving graphing skills by understanding common mistakes in graphing a function.

2. METHODS

Introduction to Algebra is one of preliminary courses that must be taken by first-semester mathematics students in Universitas Negeri Malang. Concepts on polynomial functions are part of the content in the course which connect school mathematics and college mathematics. As the first semester is a transition phase for students in shifting school mathematics and college mathematics, it is very important to design the class of Introduction to Algebra course to assist first year students to succeed in the course by investigating students' errors. Concepts on polynomial functions cannot be away from graphing skills. Therefore, in this study, 70 first-year mathematics students are selected as participants of the study.

Two tests are administered to the participants to observe graphing skills on polynomial functions. The first test, which is given at the beginning of the semester, is aimed to check students' prior knowledge in graphing polynomial functions. The participants were given two polynomial functions (linear and quadratic functions) and were asked to sketch both graphs. The second test is given in the mid semester to examine students' errors in graphing polynomial function of degree 3. The later test is intended to discover whether prior knowledge in graphing lower degree polynomial functions plays an important role in graphing higher degree polynomials.

The result of both tests is analyzed to reveal students' mistakes in graphing polynomial functions which are categorized as the simplest kind of functions to graph. From 70 participants in this study, it is found five types of errors in graphing linear and quadratic functions and four types of errors in graphing a polynomial of degree 3. Eight participants are selected as research subjects to be investigated deeper the mistakes in graphing polynomials. This research result is very beneficial for lecturers in designing the course of Introduction to Algebra.

3. RESULTS AND DISCUSSIONS

The first test consists of three polynomial functions which are linear and quadratic functions. The two quadratic functions cover a quadratic function with real roots and a function with no real roots. The test is intended to uncover students' mistakes in graphing polynomial functions which have been taught in secondary levels. Types of errors that occur in students work are listed in Table 1 given in the following.

Table 1. Types of errors in graphing linear and quadratic functions

No	Type of error	Percentage
1	Parabola does not have a clearly defined max or min point	11%
2	Coordinate pair is either calculated and potted incorrectly	30%
3	Drawing a line segment between each coordinate pair (for quadratic function) or drawing a curve between two points (for linear function)	11%
4	Using wrong formula to obtain max or min value	14%
5	Calculate square root of a negative number which should be undefined	16%

Based on Table 1, many students made careless errors in either doing wrong calculations or plotting a coordinate pair incorrectly. This mistake surely has an impact on the graph of the function such as having wrong maximum or minimum values. Having wrong maximum or minimum value is also caused by using wrong discriminant formula to calculate. Based on students' work, a graph of quadratic function does not have a clearly defined maximum or minimum point due to students only calculate some points randomly (only for integer arguments). After plotting some points which they want, students only connecting the points without finding important features of linear or quadratic functions. The following are some descriptions of distinctive solution to some research subjects related to responding the first test.

Subject 1 draw the graph of a quadratic function $f(x) = x^2 + x - 6$ by first calculating the intercepts of both axes and determining the extreme point. However, as shown in Figure 1, Subject 1 use the wrong discriminant formula in determining the extreme point resulting having the wrong minimum value for the function.

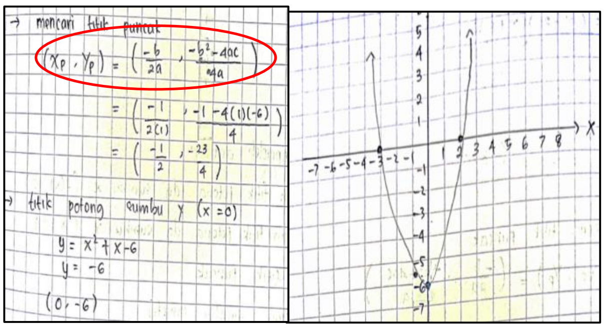

Figure 1 Error in using discriminant formula.

The discriminant of a quadratic function is defined by $D = b^2 - 4ac$. The discriminant identifies what type of roots that the equation has. The discriminant can also be used to find the maximum or minimum value of a quadratic function. The graph of function $f(x) = ax^2 + bx + c$ attains its maximum or minimum at $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$. However, Subject 1 used the wrong formula in computing the extreme point.

Similar to Subject 1, **Subject 2** also first computed some important points such as axes intercepts and extreme point. Although Subject 2 succeeded in determining the points, but Subject 2 failed to plot the extreme point $\left(-\frac{1}{2}, -\frac{25}{4}\right)$ correctly. In addition to this mistake, Subject 2 also neglected y-intercept that has been calculated properly. As indicated in Figure 2, the mistakes have implied the graph has the wrong minimal value and wrong y-intercept.

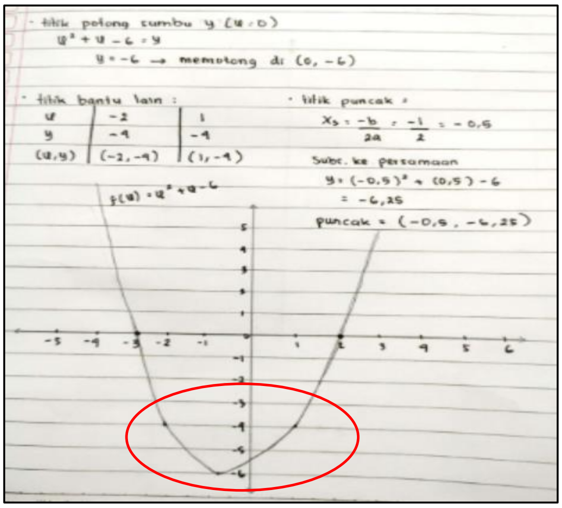

Figure 2 Mistake in plotting some points.

The test 1 asked students to sketch the graphs of three functions. The first is linear function and the other two are quadratic functions. Graphing two different kind of graphs simultaneously in a time possibly causes confusion to students. Interestingly, there are two types of errors that occur in students' work. Some students connect two points with a line segment in graphing quadratic functions, while other students do the contrary, i.e. some students connect two points with a curve when graphing a linear function. The followings are examples of unique solutions provided by some research subjects.

Subject 3 provided an answer to the instruction to sketch the graph for the quadratic function $f(x) = x^2 + 4$. The equation $x^2 + 4 = 0$ does not have real roots so some students may have problems in graphing as they would want to find the x-intercepts which do not exist. In graphing the function, Subject 3 only determined three coordinate pairs by choosing integer arguments for convenience and plotting them without bothering to find important features of quadratic function. In addition, Subject 3 connected each pair of two points with a line segment as graphing linear function as shown in Figure 3.

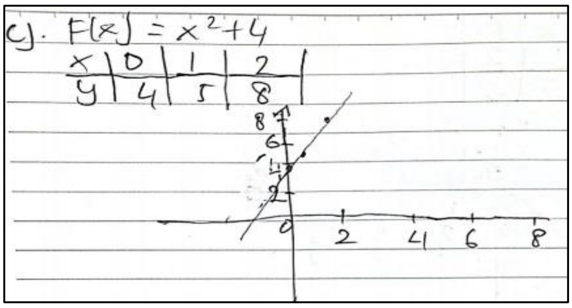

Figure 3 Connecting two points with a line segment in graphing quadratic function.

On the other hand, **Subject 4** did the opposite way as Subject 3. When graphing the linear function $f(x) = 2x + 5$, Subject 4 had determined important points as axes intercepts and other points without mistake. However, instead of plotting $(-1,3)$, Subject 4 plotted $(-1,-3)$ in the plane. Moreover, Subject 4 connected two points with a curve as connecting two points in graphing quadratic function as indicated in Figure 4. Subject 3 and Subject 4 underwent confusion between the graphs of linear and quadratic function. The confusion possibly comes because of the instruction of graphing linear and quadratic functions simultaneously at the same time. This indicates that Subject 3 and Subject 4 are lacking in conceptual understanding in polynomial functions particularly in linear and quadratic functions.

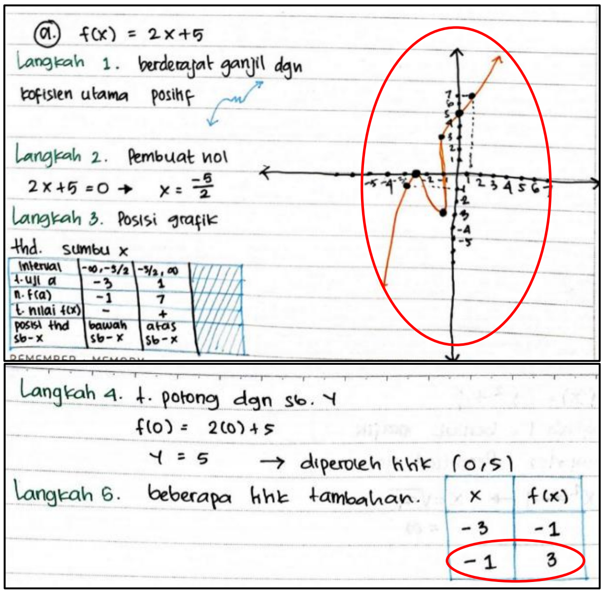

Figure 4 Connecting two points with a curve in graphing a linear function.

Other type of mistakes occurs when dealing with a quadratic function with no real roots. Some students followed the procedure in graphing a quadratic function by first identifying zeros of function. However, since the function $f(x) = x^2 + 4$ does not have real roots, some students had troubles in finding the roots. Some students including **Subject 5** even calculate the square root of negative number that supposed to be undefined as presented in Figure 5.

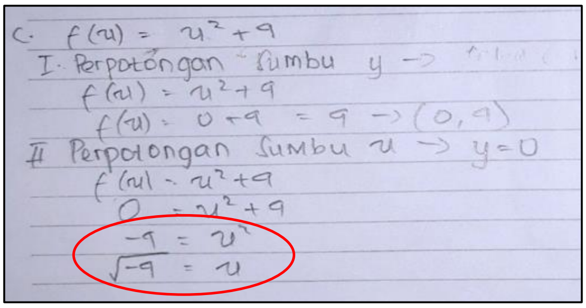

Figure 5 Example of misconception in radicals.

In this case, students failed to recognize that the equation does not have real solutions. Students also had a misconception in radicals as they were not aware that the domain of radical function is nonnegative real numbers. Based on similar finding by Özkan [11], to get of the misconception of being negative in the square root, ones should stress out the domain of radical functions.

In the second test, students were asked to sketch the graph of a polynomial function of degree 3. Almost all students are able to draw the graph of the function, but no more than 50% of them succeed in graphing the function perfectly. Some mistakes are still found in the students' work with various types of mistakes. The following Table 2 reveals the various types of errors that mathematics students made and the percentage of each type of error.

Table 2. Types of errors in graphing polynomial function of degree 3

No	Type of error	Percentage
1	The curve does not have a clearly defined max or min point	20%
2	Coordinate pair is either calculated and potted incorrectly	6%
3	Connecting two points with a line segment	7%
4	Lack of conceptual knowledge in polynomial functions and equations	6%

The most common type of mistakes occurred in the students' work based on Table 2 is unclear features in the polynomial graph particularly in maximum or minimum values. Similar to the mistakes appearing in the first test, many students only worked out for integer arguments for their own convenience. Having this simplicity, the graph is most likely unable to display important features of polynomial functions with higher degree. Despite plotting only a few points, some students connected each pair of points with a line segment. Some examples of students' answers are given in the following figures.

To sketch the polynomial function $f(x) = x^3 - 6x^2 + 11x - 6$, **Subject 6** computed some coordinate pairs by choosing small integer numbers as the arguments of the function. Subject 6 also connected two points with a line segment as displayed in Figure 6. The function should have a local maximum in the interval $(1,2)$ and a local minimum in the interval $(2,3)$, but it cannot be seen clearly since Subject 6 neglecting those important points and only joining two points with a line segment as graphing a linear function. Subject 6, in this case, has done mistakes of type 1 and type 2 in graphing a polynomial function of degree 3.

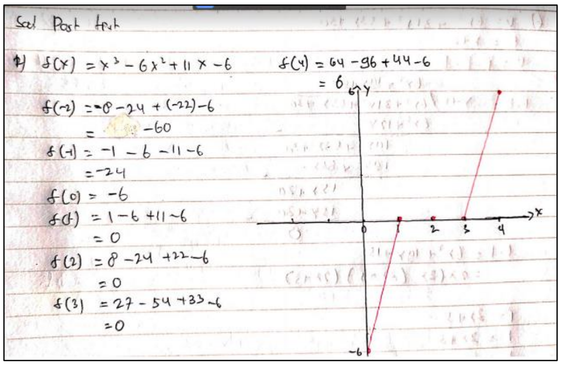

Figure 6 An example of unclear max or min point.

Similar error occurs in the solution provided by **Subject 7**. Subject 7 calculated the x-intercepts and the y-intercept of the graph. The function $f(x) = x^3 - 6x^2 + 11x - 6$ has three x-intercepts and exactly one y-intercept. After obtaining the four points and plotting them to the plane, Subject 7 ignored other points and only joining those four points as presented in Figure 7. This negligence results the losing important characteristics of polynomial functions such as maximum or minimum points and the behavior of the graph throughout the domain.

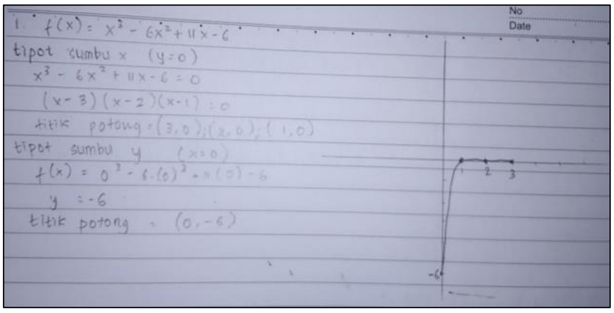

Figure 7 An example of a graph losing important features.

Careless errors which are similar to the ones in graphing polynomial functions with lower degree (linear or quadratic functions) also appear in graphing the polynomials with higher degree. Incorrect plotting of points or miscalculations still happen in the second test. Mistakes in connecting two points with a line segment also still occur as it indicates that many students are still lacking in conceptual understanding related to the characteristics of polynomial functions.

Misconceptions on polynomial functions and equations interestingly appear in the second test in the form of unable to differentiate between zeros of function, roots of an equation, and a factor of a polynomial function. Although these three terms are closely related, they have different meaning. Some students view the three terms as exactly the same meaning. For example, **Subject 8** intended to find the zeros of the function $f(x) = x^3 - 6x^2 + 11x - 6$, but he (or she) wrote: finding the root zeros of $f(x) = x^3 - 6x^2 + 11x - 6$ as shown in Figure 8. Subject 8 also made another mistake in ignoring mathematical symbols "=" in solving equation. He intended to solve the equation $x^2 - 4x + 3 = 0$, but he only wrote "$x^3 - 4x + 3$" without having the right side and the equation symbol as shown in Figure 8. Subject 8 did not aware that expression he wrote was not an equation. After factoring the algebraic expression, Subject 8 concluded that "the root zeros are $(x = -2)(x = -1)(x = -3)$". Despite errors in the terms and equation symbol, Subject 8 also commit some mistakes in ignoring mathematical symbols and careless errors in calculating simple equation $x - 2 = 0$ becoming $x = -2$. This strongly indicates that Subject 8 has very poor mathematical communication skills.

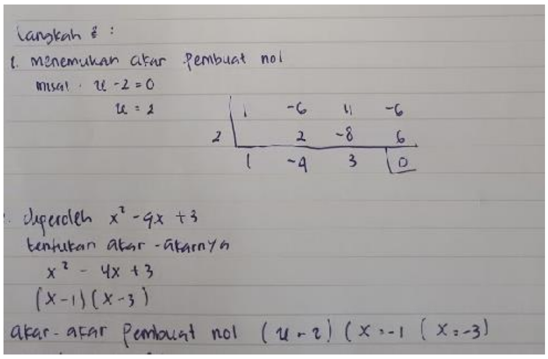

Figure 8 An example of misconception on polynomial functions.

Lacking in conceptual understanding in polynomial functions in the form of misused of the terms roots, zeros, and factors occurs in many students. The term "root" should be used when discussing about solution of an equation, while the term "zero" is employed when talking about any replacement of the variable that makes the function equals to zero. Finding the zeros of a function $f(x)$ is equivalent to find the roots of the equation $f(x) = 0$. If a is the zero of a function $f(x)$, then $x = a$ is a solution to the equation $f(x) = 0$. When a is the zero of a function $f(x)$, then the expression $(x - a)$ is a factor of $f(x)$. When a is a real zero of $f(x)$, the the point $(a, f(a))$ is an x-intercept of the graph $f(x)$. Hence, the terms "roots", "zeros", "factors", and "x-intercepts" are tightly related to each other. However, many students are unable to differentiate these terms and unable to use them properly.

4. FURTHER STUDY

In attempting the first test regarding graphing linear and quadratic functions, mistakes were found in various types. The most common mistakes is careless errors in calculation or plotting a point. The type of careless errors also found in [10] in solving the graph items in mathematics. In avoiding such kind of errors, Veloo et al. [10] suggested that students should avoid copying numbers or information incorrectly. The research result also suggested that students should not ignore basic mathematical skills involving addition, subtraction, multiplication and division.

In this study, students made mistakes in using quadratic formula involving discriminant formula. This finding is in line with previous research in [12] that students encountered difficulties in using quadratic formula. Didis & Erbas [12] reveal that when students used the quadratic formula to find the roots, students' incorrect answers were mainly based on either the incorrect calculation of the discriminant or incorrect use of the quadratic formula.

Fascinating mistakes found in the answers of test 1 when attempting to sketch a linear and quadratic function in the same worksheet. Some students joined two points with a line segment in trying to graph a quadratic function, while some others joined two points with a curve in graphing a linear function. The confusion between graphing linear functions and quadratic functions indicates that those students were lacking in conceptual understanding on graphing a function. This finding is supported by Veloo et al. [10] that lack of understanding becomes the major reason for making a mistake. Gultepe [13] also found that students with poor conceptual understanding were unsuccessful with graphs related to chemistry.

Similar types of errors also appear in students' work in graphing a polynomial function of degree 3. The types of mistakes in the second test consist of unclear important features of polynomial functions such as max or min point and smooth curve, careless errors in calculation and algebraic operation, connecting two points with a line segment, and lack of conceptual knowledge particularly in the terms roots, zeros, and factors of polynomials. Based on the results of identifying mistakes of both tests, it suggests that students' prior knowledge in graphing linear and quadratic functions plays an important role in succeeding graphing a polynomial function oh degree 3. This finding is in line with [14] which stated that prior knowledge, particularly procedural knowledge, from previous courses remarkably influenced student achievement. Rach and Ufer [15] also support the connection between prior knowledge and study success in the university study entrance phase (first year). Rach and Ufer [15] stated that well-connected knowledge about concepts of school mathematics is a crucial learning prerequisite besides procedural skills but not necessarily knowledge about formal symbolic representations.

5. CONCLUSION

Identifying students' errors and misconceptions are very essential aspects in teaching and learning process. Investigating and analyzing some students' mistakes may help them and other students to have a better understanding in mathematical concepts. Malahlela [16] prompted that students gained some conceptual knowledge of functions from the teaching which was pointed at the errors picked up as well as the misconceptions the errors emerged from. The finding from this study may have direct implications for classroom practice in teaching functions, specifically in polynomial functions. It surely is beneficial for lecturers teaching basic courses related to functions in shaping their class in order to enhance teaching and learning process and enable a better understanding in functions and the graphs.

The finding from this study suggests that many mathematics students in their first year have poor mathematical communication skills. Communication skills in mathematics includes the skill to use mathematical language to express mathematical ideas correctly. As a function has more than one representation including algebraic and graphical representation, mathematical communication is also the skills that must be mastered by mathematics students. This encourages to do future research in enhancing mathematical communication skills.

AUTHORS' CONTRIBUTIONS

All authors equally contributed to design the study and conduct the experiment. Dahliatul Hasanah analyzed the data and wrote the paper. All authors contributed to manuscript revisions. All authors approved the final version of the manuscript and agree to be held accountable for the content therein.

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