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The Girth of the Total Graph of $\mathbb{Z}_n$

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ABSTRACT
Let R be a commutative ring with a non-zero identity, and $Z(R)$ is a set of zero-divisors of R. The total graph of R, denoted $T_\Gamma(R)$, is an (undirected) graph with all elements R as vertices of $T_\Gamma(R)$ and for distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$. The girth of $T_\Gamma(R)$ is the length of the shortest cycle in $T_\Gamma(R)$, its denoted by $gr\big(T_\Gamma(R)\big)$. In this paper, we discuss the characterization of the total graph of $\mathbb{Z}_n$, $T_\Gamma(\mathbb{Z}_n)$and $gr\big(T_\Gamma(\mathbb{Z}_n)\big)$.
Keywords: *Total graph, Commutative ring, Zero divisors, Girth.*

1. INTRODUCTION

Graphs are a very interesting topic to be discussed because they are general, have images, and have many benefits. One of the beneficial application of a graph in the health sector is how a total graph forms a polypeptide chain in the genetic code [1]. The total graph in that paper construct graph from algebraic structure.

Let R be a commutative ring with non-zero identity elements and $Z(R)$ is the set of all zero-divisors in R, whereas $Z(R)^* = Z(R) - \{0\}$ and set of regular elements in R is $Reg(R) = R - Z(R)$ [2]. A ring R is called an integral domain if and only if $Z(R) = \{0\}$ [3]. Anderson & Badawi [2] introduced the concept of a total graph of R, denoted by $T_\Gamma(R)$, is an (undirected) graph where the vertices are all elements of R and for each two different vertices $x, y \in R$ is adjacent if and only if $x + y \in Z(R)$. The subgraphs of $T_\Gamma(R)$ which are induced by $Z(R)$ and $Reg(R)$ is denoted by $Z_\Gamma(R)$ and $Reg_\Gamma(R)$ respectively. $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the ring of integers and the ring of integers modulo n respectively.

Example 1.1 Let $R = \mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ be a ring of integer modulo 4. Then $Z(\mathbb{Z}_4) = \{\bar{0}, \bar{2}\} = \langle\bar{2}\rangle$. The graph $T_\Gamma(\mathbb{Z}_4)$ have $V\big(T_\Gamma(\mathbb{Z}_4)\big) = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ and $E\big(T_\Gamma(\mathbb{Z}_4)\big) = \{(\bar{0}, \bar{2}), (\bar{1}, \bar{3})\}$. So, the corresponding graphs are given in Figure 1 below.

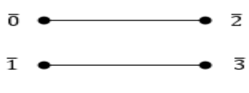

Figure 1 Graph $T_\Gamma(\mathbb{Z}_4)$

Girth of the graph G, denoted by $gr(G)$, is the length of the shortest cycle in G ($gr(G) = \infty$ if G does not contain any cycle). The definition of a cycle here is a closed walk where each edge is different, and all vertices in it are different [4]. The regular graph is denoted by $r - regular$ if the degree of each vertex is r, this graph with n vertices has $\frac{nr}{2}$ edges. A complete graph with n vertices denoted by K_n, is a graph where each vertex is joined to one another with exactly one edge. A complete bipartite graph with r vertex in A and s vertex in B is denoted by $K_{r,s}$. General references for the graph theory are [4–6].

Most of the publications concerning the form of the total graph in ring R underlined the diameter and the girth in ring R with some example in $\mathbb{Z}_n$ [2], [4]. In Chelvam and Asir[7], they have been studied about fundamental properties of total graph on $\mathbb{Z}_n$ without disscuss about the girth of the total graph of $\mathbb{Z}_n$. In this paper, we will discuss about girth of total graph from $\mathbb{Z}_n$.

2. GIRTH OF THE TOTAL GRAPH OF $\mathbb{Z}_n$

In this section, we present some properties of the total graph of $\mathbb{Z}_n$. First of all, we discuss the total graph of R. The following observation is due to Anderson and Badawi [2].

Theorem 2.1 [2]. Let R be a commutative ring such that $Z(R)$ is an ideal of R. Then $Z_\Gamma(R) = K_{|Z(R)|}$ is a complete subgraph of $T_\Gamma(R)$ with $|Z(R)|$ vertices and $Z_\Gamma(R)$ is disjoint from $Reg_\Gamma(R)$.

Theorem 2.2 [2]. Let R be commutative ring such that $Z(R)$ is an ideal of R, and let $|Z(R)| = \lambda$, and $|R/Z(R)| = \mu$. Then

$$Reg_{\Gamma}(R) = \begin{cases} \underbrace{K_{\lambda} \cup K_{\lambda} \cup \ldots \cup K_{\lambda}}_{(\mu-1)kali} & if\ 2 \in Z(R); \\ \underbrace{K_{\lambda,\lambda} \cup K_{\lambda,\lambda} \cup \ldots \cup K_{\lambda,\lambda}}_{(\frac{\mu-1}{2})kali} & if\ 2 \notin Z(R). \end{cases}$$

Example 2.3 Some examples for the total graph when $Z(R)$ is an ideal of R, $Z(R)$ is not an ideal of R, and R be an integral domain $(Z(R) = \{0\})$ are given here Figure 2, Figure 3, and Figure 4.

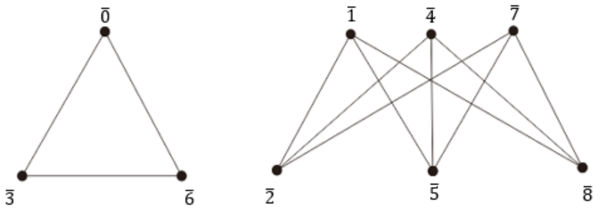

Figure 2 Graph $T_{\Gamma}(\mathbb{Z}_9)$

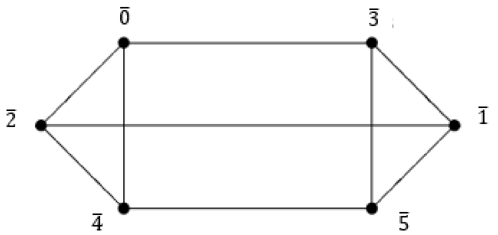

Figure 3 Graph $T_{\Gamma}(\mathbb{Z}_6)$

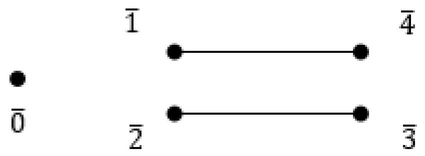

Figure 4 Graph $T_{\Gamma}(\mathbb{Z}_5)$

Next, we are interested in the total graph of $\mathbb{Z}_n$, there are some basic properties on the total graph of $\mathbb{Z}_n$ that refer to [7].

Lemma 2.4 . Let $x \in \mathbb{Z}_n$. Then $x \in Z(\mathbb{Z}_n)$ if and only if $\gcd(x, n) > 1$.

Lemma 2.5 . For $n \in \mathbb{Z}^+, n \geq 1, |Z(\mathbb{Z}_n)| = n - \phi(n)$, where ϕ is Euler's function.

Remark 2.6. If $x \in Z(\mathbb{Z}_n)$, then $\deg(x) = n - \phi(n) - 1$.

3. MAIN RESULT

In this section, we will discuss the characterization of the total graph of $\mathbb{Z}_n$

Theorem 3.1 For $n > 1, n \in \mathbb{Z}^+$, n is even, the following statements are true:

(i) If $n = 2^k, k \in \mathbb{Z}^+$, then $T_{\Gamma}(\mathbb{Z}_n) = K_{2^{k-1}} \cup K_{2^{k-1}}$.

(ii) Otherwise, $T_{\Gamma}(\mathbb{Z}_n)$ is a $(n - \phi(n) - 1) - regular$

Proof.

(i) If $n = 2^k$, $Z(\mathbb{Z}_n) = \langle 2 \rangle$ is the ideal of $\mathbb{Z}_n$, where $|Z(\mathbb{Z}_n)| = 2^k - \phi(2^k)$

$$|Z(\mathbb{Z}_n)| = 2^k - \phi(2^k) = 2^k - (2^k - 2^{k-1})$$
$$= 2^k - 2^k + 2^{k-1} = 2^{k-1}$$

$Z(\mathbb{Z}_n)$ is an ideal $\mathbb{Z}_n$, according to Theorem 2.1, then $Z_{\Gamma}(\mathbb{Z}_n) = K_{2^{k-1}}$ and $Z_{\Gamma}(\mathbb{Z}_n)$ disjoint from $Reg_{\Gamma}(\mathbb{Z}_n)$. And through Theorem 2.2, where $|Z(\mathbb{Z}_n)| = 2^{k-1}$ and $|\mathbb{Z}_n/Z(\mathbb{Z}_n)| = 2$, if $2 \in Z(\mathbb{Z}_n)$, then $Reg_{\Gamma}(\mathbb{Z}_n) = K_{2^{k-1}}$. So, $T_{\Gamma}(\mathbb{Z}_n) = K_{2^{k-1}} \cup K_{2^{k-1}}$.

(ii) If n is even, then $2 \in Z(\mathbb{Z}_n)$. For every $x \notin Z(\mathbb{Z}_n), 2x \in Z(\mathbb{Z}_n)$, x is adjacent to $y - x \in Z(\mathbb{Z}_n)$, for every $y \in Z(\mathbb{Z}_n)$, where $y \neq 2x$. According to Remark 2.6 $\deg(y) = n - \phi(n) - 1$. Therefore $\deg(x) = n - \phi(n) - 1$. So, $T_{\Gamma}(\mathbb{Z}_n)$ is $(n - \phi(n) - 1) - regular$.

Theorem 3.2 For $n, p, k \in \mathbb{Z}^+$, p be a prime number, the following statements are true:

(i) If $n = p, p > 2$, then $T_{\Gamma}(\mathbb{Z}_n) = K_1 \cup \underbrace{K_2 \cup K_2 \cup \ldots \cup K_2}_{\frac{n-1}{2}kali}$

(ii) If $n = p^k, p > 2, k > 1$, then $T_{\Gamma}(\mathbb{Z}_n) = K_{p^{k-1}} \cup \underbrace{K_{p^{k-1},p^{k-1}} \cup K_{p^{k-1},p^{k-1}} \cup \ldots \cup K_{p^{k-1},p^{k-1}}}_{(\frac{p-1}{2})kali}$

Proof.

(i) Let $n = p$, then $\mathbb{Z}_n$ be an integral domain, where $Z(\mathbb{Z}_n) = \{0\}$, $|Z(\mathbb{Z}_n)| = 1$. So, $Z_{\Gamma}(\mathbb{Z}_n) = K_1$. $Z_{\Gamma}(\mathbb{Z}_n)$ and $Reg(\mathbb{Z}_n)$ are disjoint, because for $x \in Reg(\mathbb{Z}_n), x + 0 = x \notin Z(\mathbb{Z}_n)$. For every, $x \in Reg(\mathbb{Z}_n)$, x adjacent to $-x$, because $x + (-x) = 0 \in Z(\mathbb{Z}_n)$. Therefore, $Reg_{\Gamma}(\mathbb{Z}_n) =$

$\underbrace{K_2 \cup K_2 \cup \ldots \cup K_2}_{\frac{n-1}{2} kali}$. So, $T_\Gamma(\mathbb{Z}_n) = K_1 \cup \underbrace{K_2 \cup K_2 \cup \ldots \cup K_2}_{\frac{n-1}{2} kali}$.

(ii) Let $n = p^k$, $p > 2, k > 1$, then $Z(\mathbb{Z}_n) = \langle p \rangle$ is the ideal of $\mathbb{Z}_n$ and $2 \notin Z(\mathbb{Z}_n)$, where $|Z(\mathbb{Z}_n)| = p^k - \phi(p^k) = p^k - (p^k - p^{k-1}) = p^{k-1}$. Because $Z(\mathbb{Z}_n)$ is the ideal of $\mathbb{Z}_n$, according to Theorem 2.1, then $Z_\Gamma(\mathbb{Z}_n) = K_{p^{k-1}}$ and $Z_\Gamma(\mathbb{Z}_n)$ disjoint from $Reg_\Gamma(\mathbb{Z}_n)$. Through the Theorem 2.2, where $|Z(\mathbb{Z}_n)| = p^{k-1}$ and $|\mathbb{Z}_n/Z(\mathbb{Z}_n)| = p$, if $2 \in Z(\mathbb{Z}_n)$, then $Reg_\Gamma(\mathbb{Z}_n) = \underbrace{K_{p^{k-1},p^{k-1}} \cup K_{p^{k-1},p^{k-1}} \cup \ldots \cup K_{p^{k-1},p^{k-1}}}_{\left(\frac{p-1}{2}\right) kali}$.

So, $T_\Gamma(\mathbb{Z}_n) = K_{p^{k-1}} \cup \underbrace{K_{p^{k-1},p^{k-1}} \cup K_{p^{k-1},p^{k-1}} \cup \ldots \cup K_{p^{k-1},p^{k-1}}}_{\left(\frac{p-1}{2}\right) kali}$.

From the Theorem 3.1 and Theorem 3.2 above, we can obtain that

Corollary 3.3 For $n, p, k \in \mathbb{Z}^+$, p be a prime number, the following statements are true:

(i) If $n = 2^k, k > 2,\ k \in \mathbb{Z}^+$

then $gr(T_\Gamma(\mathbb{Z}_n)) = 3$

(ii) If $n = p$, p prime, then $gr(T_\Gamma(\mathbb{Z}_n)) = \infty$

(iii) If $n = p^k$, $p > 2$, p prime, $k \in \mathbb{Z}^+$, then $gr(T_\Gamma(\mathbb{Z}_n)) = 3$

The following example of Corollary 3.3 in $T_\Gamma(\mathbb{Z}_n)$.

Example 3.4

(a) Let $n = 2^3 = 8$, then $T_\Gamma(\mathbb{Z}_8) = K_4 \cup K_4$. So, $gr(T_\Gamma(\mathbb{Z}_8)) = 3$.

(b) Let $n = 5$. From Figure 4, we can see that $T_\Gamma(\mathbb{Z}_5) = K_1 \cup K_2 \cup K_2$. An (undirected) graph $T_\Gamma(\mathbb{Z}_5)$ have no cycle, then $gr(T_\Gamma(\mathbb{Z}_5)) = \infty$.

(c) Let $n = 3^2 = 9$. From Figure 2, we can see that $T_\Gamma(\mathbb{Z}_9) = K_3 \cup K_{3,3}$, then $gr(T_\Gamma(\mathbb{Z}_8)) = 3$.

4. CONCLUSION

Let $\boldsymbol{R}$ be a commutative ring with a non-zero identity, and $\boldsymbol{Z(R)}$ is a set of zero-divisors of $\boldsymbol{R}$. The total graph of $\boldsymbol{R}$, denoted $\boldsymbol{T_\Gamma(R)}$, is an (undirected) graph with all elements $\boldsymbol{R}$ as vertices of $\boldsymbol{T_\Gamma(R)}$ and for distinct vertices $\boldsymbol{x, y \in R}$ are adjacent if and only if $\boldsymbol{x + y \in Z(R)}$. The girth of $\boldsymbol{T_\Gamma(R)}$ is the length of the shortest cycle in $\boldsymbol{T_\Gamma(R)}$, its denoted by $\boldsymbol{gr(T_\Gamma(R))}$. We obtain the characterization of $\boldsymbol{T_\Gamma(\mathbb{Z}_n)}$ and $\boldsymbol{gr(T_\Gamma(\mathbb{Z}_n))}$ for $\boldsymbol{n > 1, n \in \mathbb{Z}^+}$, $\boldsymbol{n}$ is even; for $\boldsymbol{n, p, k \in \mathbb{Z}^+}$, $\boldsymbol{p}$ be a prime number; for $\boldsymbol{n, p, k \in \mathbb{Z}^+}$, $\boldsymbol{p}$ be a prime number.

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