

Research Article

Transitive Closures of Ternary Fuzzy Relations

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ABSTRACT

Recently, we have introduced six types of composition of ternary fuzzy relations. These compositions are close in spirit to the composition of binary fuzzy relations. Based on these types of composition, we have introduced several types of transitivity of a ternary fuzzy relation and investigated their basic properties. In this paper, we prove additional properties and characterizations of these types of transitivity of a ternary fuzzy relation. Also, we provide a representation theorem for ternary fuzzy relations satisfying these types of transitivity. Finally, we focus on the problem of closing a ternary fuzzy relation with respect to the proposed types of transitivity.

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1. INTRODUCTION

Probably the most important property of binary relations is the transitivity property. This classical concept has been generalized in fuzzy set theory by the $*$ -transitivity property of fuzzy relations, where $*$ is a t -norm [1]. Using the definition of composition of relations, the transitivity property can be formulated more concisely, such that a binary (fuzzy) relation R is transitive if and only if $R \circ R \subseteq R$. In addition to its key role as a condition for various important classes of binary (fuzzy) relations (such as preorder relations, order relations and equivalence relations), it is an essential tool in many fields of application, for instance, in the fields of (fuzzy) preference modelling and multi-criteria decision making [2–4], the study of indistinguishability/equality relations on the real line [5–7] and in fuzzy control [8].

In recent years, the interest in ternary relations is on the rise, as they play an important role in many theoretical and applied areas. From a theoretical point of view, ternary relations have been studied in algebra [9,10], (fuzzy) triadic formal concept analysis [11–13] and logic [14]. In applications, ternary (fuzzy) relations can be encountered in various areas, such as social sciences (e.g., philosophy [15]), biology (e.g., modelling of phylogenies [16]) and computer science (e.g., the Resource Description Framework (RDF) [17]). Some classes of ternary relations recently came to play an important role in specific applications, e.g., betweenness relations in models for decision making [18] and aggregation [19], ternary order relations in string matching [20], cyclic orders in qualitative spatial reasoning [21] and particular ternary fuzzy relations in models of choice behavior [22].

In the ternary setting, the notion of transitivity has received far less attention and has appeared only in a few papers. For instance, Pitcher and Smiley [23] have defined several notions of four-point transitivity and five-point transitivity of a betweenness relation. Also, Novák and Novotný [24], Chajda *et al.* [9], Barkat *et al.* [25] and Zedam *et al.* [26] have defined other types of transitivity of a ternary relation.

In practice, however, the transitivity of a (binary or ternary) relation is quite often violated. The most common repair strategy is to consider its transitive closure: the smallest transitive relation (if it exists) that includes the given relation. Such transitive closures play an important role in many different areas in mathematics and computer science. Closures (and openings) of ternary relations have been studied recently by Zedam *et al.* [27] for a broad range of properties, including various transitivity properties.

Motivated by the increasing interest in ternary relations and the usefulness of the transitivity property of binary (fuzzy) relations, in a recent conference paper [28], we have introduced six types of composition of ternary fuzzy relations with the aim of studying the transitivity of ternary fuzzy relations in a similar way as in the binary case. The introduced types of composition of ternary fuzzy relations are generalizations of the types of composition introduced by Bakri *et al.* [29] in the crisp case. In this paper, we further extend the study of the six corresponding types of transitivity of ternary fuzzy relations. Also, we provide a representation theorem for ternary fuzzy relations possessing one of these types of transitivity, thus addressing one of the recurrent questions in the study of fuzzy relations. The main contribution of this paper is the study of the problem of closing a ternary fuzzy relation with respect to the proposed types of transitivity. It will turn out that such transitive closure always exists

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for the six types of transitivity. However, it can only be written as the union of powers for the types of transitivity corresponding to associative compositions.

This paper is organized as follows. In Section 2, we recall some basic concepts related to residuated lattices and t-norms, as well as binary and ternary fuzzy relations. In Section 3, we extend the compositions of ternary relations to ternary fuzzy relations and investigate their basic properties. In Section 4, we introduce several types of transitivity of a ternary fuzzy relation and show their properties. Also, we study the interaction of these properties with the binary projections and cylindrical extensions. In Section 5, we provide a representation of a given transitive (resp. reflexive and transitive) ternary fuzzy relation in terms of its binary projections and an appropriate family of functions. We study the problem of closing a ternary fuzzy relation with respect to the proposed types of transitivity in Section 6. Finally, we present some concluding remarks in Section 7.

2. PRELIMINARIES

First, we recall some basic concepts related to residuated lattices. Second, we present some basic definitions related to binary and ternary fuzzy relations.

2.1. Complete Residuated Lattices

A poset (L, \leq) (see, e.g., Davey and Priestley [30]) is called a lattice if any two elements x and y have a greatest lower bound, denoted $x \wedge y$ and called the meet (infimum) of x and y , as well as a smallest upper bound, denoted $x \vee y$ and called the join (supremum) of x and y . Note that $x \leq y$ if and only if $x \wedge y = x$ if and only if $x \vee y = y$.

A bounded lattice $(L, \wedge, \vee, 0, 1)$ is a lattice that has a bottom element 0 and a top element 1 , i.e., $0 \leq x \leq 1$ for any $x \in L$. A lattice (L, \wedge, \vee) is called complete if any nonempty subset A of L has an infimum (i.e., greatest lower bound) and a supremum (i.e., smallest upper bound), denoted $\bigwedge A$ and $\bigvee A$, respectively. Any complete lattice is bounded.

A complete residuated lattice (see, e.g., Bělohlávek [31]) is an algebra $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ such that:

- (i) $(L, \wedge, \vee, 0, 1)$ is a complete lattice;
- (ii) $(L, *, 1)$ is a commutative monoid;
- (iii) $*$ and \rightarrow , called multiplication and residuum, satisfy the adjointness property: $a * c \leq b$ if and only if $c \leq a \rightarrow b$, for any $a, b, c \in L$.

The following properties of complete residuated lattices will be used in this paper without explicit indication (see, e.g., Bělohlávek [31] and Hájek [32]). For any $a, b, c \in L$ and $B \subseteq L$, it holds that

- (i) $a \rightarrow b = \bigvee \{c \in L \mid a * c \leq b\}$;
- (ii) $1 \rightarrow a = a$;
- (iii) $a \leq b$ if and only if $a \rightarrow b = 1$;
- (iv) $a * (a \rightarrow b) \leq b$;

$$(v) \quad b \leq c \text{ implies } \begin{cases} a * b \leq a * c \\ a \rightarrow b \leq a \rightarrow c \\ b \rightarrow a \geq c \rightarrow a \end{cases}$$

$$(vi) \quad a * (\bigvee B) = \bigvee \{a * b \mid b \in B\}.$$

From here on, L always denotes a complete residuated lattice $(L, \wedge, \vee, *, \rightarrow, 0, 1)$.

The operation $*$ is also commonly called a triangular norm (t-norm, for short) and the operation \rightarrow its residual implication. T-norms were originally introduced on the real unit interval $[0, 1]$, but are readily extended to posets and lattices (see Refs. [33,34] for more details). A t-norm on a bounded lattice L is a binary operation $*$ on L that is commutative (i.e., $x * y = y * x$, for any $x, y \in L$) and associative (i.e., $x * (y * z) = (x * y) * z$, for any $x, y, z \in L$), has neutral element 1 (i.e., $x * 1 = x$, for any $x \in L$) and is order-preserving (i.e., if $x \leq y$, then $x * z \leq y * z$, for any $x, y, z \in L$). Additionally, in the case of a complete residuated lattice, a t-norm is supremum-preserving.

2.2. Binary Fuzzy Relations

The notion of an L -fuzzy relation on a set X generalizes the classical notion of a $\{0, 1\}$ -relation by expressing degrees of relationship in some complete residuated lattice $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ [35]. A binary L -fuzzy relation (binary L -relation, for short) R on a set X is a mapping $R : X \times X \rightarrow L$. Obviously, if $L = \{0, 1\}$, then binary relations are retrieved, often referred to as crisp relations.

A binary L -relation R_1 is said to be included in a binary L -relation R_2 , denoted $R_1 \subseteq R_2$, if $R_1(x, y) \leq R_2(x, y)$, for any $x, y \in X$. The intersection of two binary L -relations R_1 and R_2 on X is the binary L -relation $R_1 \cap R_2$ on X defined by $R_1 \cap R_2(x, y) = R_1(x, y) \wedge R_2(x, y)$, for any $x, y \in X$. Similarly, the union of two binary L -relations R_1 and R_2 on X is the binary L -relation $R_1 \cup R_2$ on X defined by $R_1 \cup R_2(x, y) = R_1(x, y) \vee R_2(x, y)$, for any $x, y \in X$. The transpose R^t of a binary L -relation R is the binary L -relation defined by $R^t(x, y) = R(y, x)$. The composition of two binary L -relations R_1 and R_2 on X is the binary L -relation $R_1 \circ R_2$ on X defined by:

$$R_1 \circ R_2(x, z) = \bigvee_{y \in X} R_1(x, y) * R_2(y, z).$$

2.3. Ternary Fuzzy Relations

A ternary (or triadic) L -fuzzy relation (ternary L -relation, for short) T on a set X is a mapping $T : X^3 \rightarrow L$. If $L = \{0, 1\}$, then ternary relations are retrieved. Inclusion, intersection and union are defined in a similar way as for binary L -relations. For more details on ternary (fuzzy) relations, we refer to Refs. [9,10,22,36].

A ternary L -relation T_1 is said to be included in a ternary L -relation T_2 , denoted $T_1 \subseteq T_2$, if $T_1(x, y, z) \leq T_2(x, y, z)$, for any $x, y, z \in X$. The intersection of two ternary L -relations T_1 and T_2 on X is the ternary L -relation $T_1 \cap T_2$ on X defined by $T_1 \cap T_2(x, y, z) = T_1(x, y, z) \wedge T_2(x, y, z)$, for any $x, y, z \in X$. Similarly, the union of two ternary L -relations T_1 and T_2 on X is the ternary L -relation $T_1 \cup T_2$ on X defined by $T_1 \cup T_2(x, y, z) = T_1(x, y, z) \vee T_2(x, y, z)$, for any $x, y, z \in X$.

As in the crisp case [26], we define the ternary L -relations associated with a given ternary L -relation T on X obtained by permutation as follows:

- (i) The transpose of T is the ternary L -relation T^t on X defined by $T^t(x, y, z) = T(z, y, x)$;
- (ii) The right-converse of T is the ternary L -relation T^{-1} on X defined by $T^{-1}(x, y, z) = T(x, z, y)$;
- (iii) The left-converse of T is the ternary L -relation T^+ on X defined by $T^+(x, y, z) = T(y, x, z)$;
- (iv) The right-rotation of T is the ternary L -relation T^+ on X defined by $T^+(x, y, z) = T(z, x, y)$;
- (v) The left-rotation of T is the ternary L -relation T^- on X defined by $T^-(x, y, z) = T(y, z, x)$.

3. COMPOSITIONS OF TERNARY FUZZY RELATIONS

In this section, we extend the types of composition of crisp ternary relations introduced by Bakri *et al.* [29] to the fuzzy setting, and investigate their properties.

Definition 1. [29] Let T and S be two ternary relations on a set X . The \circ_i -compositions of T and S , with $i \in \{1, \dots, 6\}$, are defined as:

- (i) $T \circ_1 S = \{(x, y, z) \in X^3 \mid (\exists t, s \in X) ((x, y, t) \in T \wedge (s, t, z) \in S)\}$;
- (ii) $T \circ_2 S = \{(x, y, z) \in X^3 \mid (\exists t, s \in X) ((x, y, t) \in T \wedge (t, s, z) \in S)\}$;
- (iii) $T \circ_3 S = \{(x, y, z) \in X^3 \mid (\exists t, s \in X) ((x, y, t) \in T \wedge (t, z, s) \in S)\}$;
- (iv) $T \circ_4 S = \{(x, y, z) \in X^3 \mid (\exists t, s \in X) ((s, x, t) \in T \wedge (t, y, z) \in S)\}$;
- (v) $T \circ_5 S = \{(x, y, z) \in X^3 \mid (\exists t, s \in X) ((x, s, t) \in T \wedge (t, y, z) \in S)\}$;
- (vi) $T \circ_6 S = \{(x, y, z) \in X^3 \mid (\exists t, s \in X) ((x, s, t) \in T \wedge (s, y, z) \in S)\}$.

The following definition extends the above types of composition to the fuzzy setting.

Definition 2. Let T and S be two ternary L -relations on a set X . The \circ_i -compositions of T and S , with $i \in \{1, \dots, 6\}$, are defined by:

- (i) $T \circ_1 S(x, y, z) = \bigvee_{s,t \in X} T(x, y, t) * S(s, t, z)$;
- (ii) $T \circ_2 S(x, y, z) = \bigvee_{s,t \in X} T(x, y, t) * S(t, s, z)$;
- (iii) $T \circ_3 S(x, y, z) = \bigvee_{s,t \in X} T(x, y, t) * S(t, z, s)$;

- (iv) $T \circ_4 S(x, y, z) = \bigvee_{s,t \in X} T(s, x, t) * S(t, y, z)$;
- (v) $T \circ_5 S(x, y, z) = \bigvee_{s,t \in X} T(x, s, t) * S(t, y, z)$;
- (vi) $T \circ_6 S(x, y, z) = \bigvee_{s,t \in X} T(x, s, t) * S(s, y, z)$.

Example 1.

Consider a bounded lattice $L = \{0, \alpha, \beta, 1\}$, with α and β either being comparable (i.e., L is a chain) or incomparable (i.e., L is a diamond) and let $*$ = \wedge . Let $X = \{a, b, c, d\}$ and consider the ternary L -relations T_1 and T_2 on X given by:

$$T_1(x, y, z) = \begin{cases} \alpha, & \text{if } (x, y, z) = (a, a, b) \\ \beta, & \text{if } (x, y, z) = (a, b, c) \\ 0, & \text{otherwise} \end{cases}$$

and

$$T_2(x, y, z) = \begin{cases} 1, & \text{if } (x, y, z) = (a, a, a) \\ \alpha, & \text{if } (x, y, z) = (b, d, a) \\ \beta, & \text{if } (x, y, z) = (c, b, b) \\ 0, & \text{otherwise} \end{cases}$$

One easily computes the \circ_i -compositions of T_1 and T_2 , with $i \in \{1, \dots, 6\}$:

$$T_1 \circ_1 T_2(x, y, z) = \begin{cases} \alpha \wedge \beta, & \text{if } (x, y, z) = (a, a, b) \\ 0, & \text{otherwise} \end{cases}$$

$$T_1 \circ_2 T_2(x, y, z) = \begin{cases} \alpha, & \text{if } (x, y, z) = (a, a, a) \\ \beta, & \text{if } (x, y, z) = (a, b, b) \\ 0, & \text{otherwise} \end{cases}$$

$$T_1 \circ_3 T_2(x, y, z) = \begin{cases} \alpha, & \text{if } (x, y, z) = (a, a, d) \\ \beta, & \text{if } (x, y, z) = (a, b, b) \\ 0, & \text{otherwise} \end{cases}$$

$$T_1 \circ_4 T_2(x, y, z) = \begin{cases} \alpha, & \text{if } (x, y, z) = (a, d, a) \\ \beta, & \text{if } (x, y, z) = (b, b, b) \\ 0, & \text{otherwise} \end{cases}$$

$$T_1 \circ_5 T_2(x, y, z) = \begin{cases} \alpha, & \text{if } (x, y, z) = (a, d, a) \\ \beta, & \text{if } (x, y, z) = (a, b, b) \\ 0, & \text{otherwise} \end{cases}$$

$$T_1 \circ_6 T_2(x, y, z) = \begin{cases} \alpha, & \text{if } (x, y, z) = (a, a, a) \\ \alpha \wedge \beta, & \text{if } (x, y, z) = (a, d, a) \\ 0, & \text{otherwise} \end{cases}$$

Next, we investigate some properties of the above compositions. First, the following proposition shows that the \circ_i -composition is associative for any $i \in \{1, 2, 5, 6\}$.

Proposition 1. *Let T_1, T_2 and T_3 be three ternary L -relations on a set X and $i \in \{1, 2, 5, 6\}$. Then it holds that*

$$(T_1 \circ_i T_2) \circ_i T_3 = T_1 \circ_i (T_2 \circ_i T_3).$$

Proof. Consider $i = 1$, the other cases being similar. Let $(x, y, z) \in X^3$, then

$$\begin{aligned} & (T_1 \circ_1 T_2) \circ_1 T_3(x, y, z) \\ &= \bigvee_{s,t \in X} T_1 \circ_1 T_2(x, y, t) * T_3(s, t, z) \\ &= \bigvee_{s,t \in X} \left(\bigvee_{m,n \in X} T_1(x, y, m) * T_2(n, m, t) \right) * T_3(s, t, z). \end{aligned}$$

Since $*$ is associative and supremum-preserving, it follows that

$$\begin{aligned} & (T_1 \circ_1 T_2) \circ_1 T_3(x, y, z) \\ &= \bigvee_{m,n,s,t \in X} (T_1(x, y, m) * T_2(n, m, t)) * T_3(s, t, z) \\ &= \bigvee_{m,n,s,t \in X} T_1(x, y, m) * (T_2(n, m, t) * T_3(s, t, z)) \\ &= \bigvee_{m,n \in X} T_1(x, y, m) * T_2 \circ_1 T_3(n, m, z) \\ &= T_1 \circ_1 (T_2 \circ_1 T_3)(x, y, z). \end{aligned}$$

In the following example, we show that the compositions \circ_3 and \circ_4 are not associative.

Example 2.

Let L (resp. T_1, T_2) be the lattice (resp. the ternary L -relations on $X = \{a, b, c, d\}$) given in Example 1, $*$ = \wedge and T_3 be the ternary L -relation on X given by:

$$T_3(x, y, z) = \begin{cases} \alpha, & \text{if } (x, y, z) = (d, d, b) \\ \beta, & \text{if } (x, y, z) = (b, c, a) \\ 0, & \text{otherwise} \end{cases}$$

One easily verifies that

$$(T_1 \circ_3 T_2) \circ_3 T_3(x, y, z) = \begin{cases} \alpha, & \text{if } (x, y, z) = (a, a, d) \\ \beta, & \text{if } (x, y, z) = (a, b, c) \\ 0, & \text{otherwise} \end{cases}$$

and

$$T_1 \circ_3 (T_2 \circ_3 T_3)(x, y, z) = \begin{cases} \beta, & \text{if } (x, y, z) = (a, b, b) \\ 0, & \text{otherwise} \end{cases}$$

It is clear that

$$(T_1 \circ_3 T_2) \circ_3 T_3 \neq T_1 \circ_3 (T_2 \circ_3 T_3).$$

Similarly, one can also verify that

$$(T_1 \circ_4 T_2) \circ_4 T_3 \neq T_1 \circ_4 (T_2 \circ_4 T_3).$$

The following proposition lists the interaction of the compositions with inclusion and set-theoretical operations. The proof is straightforward.

Proposition 2. *Let T_1, T_2, S_1, S_2 and S be ternary L -relations on a set X . For any $i \in \{1, \dots, 6\}$, the following statements hold:*

- (i) *If $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$, then $T_1 \circ_i S_1 \subseteq T_2 \circ_i S_2$;*
- (ii) *$(T_1 \cap T_2) \circ_i S = (T_1 \circ_i S) \cap (T_2 \circ_i S)$ and $S \circ_i (T_1 \cap T_2) = (S \circ_i T_1) \cap (S \circ_i T_2)$;*
- (iii) *$(T_1 \cup T_2) \circ_i S = (T_1 \circ_i S) \cup (T_2 \circ_i S)$ and $S \circ_i (T_1 \cup T_2) = (S \circ_i T_1) \cup (S \circ_i T_2)$.*

The following proposition lists the interaction of the compositions with the permutations. The proofs are again straightforward.

Proposition 3. *Let T and S be two ternary L -relations on a set X . The following equalities hold:*

- (i) *$(T \circ_i S)^- = T \circ_i S^-$, for any $i \in \{4, 5, 6\}$;*
- (ii) *$(T \circ_i S)^+ = T^+ \circ_i S$, for any $i \in \{1, 2, 3\}$;*
- (iii) *$(T \circ_i S)^+ = S^+ \circ_{i-4} T^+$, for any $i \in \{5, 6\}$;*
- (iv) *$(T \circ_i S)^- = S^- \circ_{i+4} T^-$, for any $i \in \{1, 2\}$;*
- (v) *$(T \circ_i S)^t = S^t \circ_{7-i} T^t$, for any $i \in \{1, \dots, 6\}$.*

4. TRANSITIVITY OF TERNARY FUZZY RELATIONS

In this section, based on the compositions of ternary fuzzy relations, we introduce several types of transitivity of a ternary fuzzy relation and investigate their properties.

4.1. Definitions and Basic Properties

For a given binary L -relation R on a set X , the $*$ -transitivity is defined as: $R(x, y) * R(y, z) \leq R(x, z)$, for any $x, y, z \in X$. This property is equivalent to the condition that $R \circ R \subseteq R$, where \circ denotes the composition of binary L -relations. Analogously to the binary case, we introduce the types of transitivity of a ternary fuzzy relation based on the compositions of ternary fuzzy relations introduced above.

Definition 3. Let $i \in \{1, \dots, 6\}$. A ternary L -relation T on a set X is called \circ_i -transitive if $T \circ_i T \subseteq T$. Explicitly:

- (i) *T is \circ_1 -transitive, if, for any $x, y, z, s, t \in X$, it holds that $T(x, y, t) * T(s, t, z) \leq T(x, y, z)$;*
- (ii) *T is \circ_2 -transitive, if, for any $x, y, z, s, t \in X$, it holds that $T(x, y, s) * T(s, t, z) \leq T(x, y, z)$;*
- (iii) *T is \circ_3 -transitive, if, for any $x, y, z, s, t \in X$, it holds that $T(x, y, s) * T(s, t, z) \leq T(x, y, t)$;*

- (iv) T is \circ_4 -transitive, if, for any $x, y, z, s, t \in X$, it holds that $T(x, y, s) * T(s, t, z) \leq T(y, t, z)$;
- (v) T is \circ_5 -transitive, if, for any $x, y, z, s, t \in X$, it holds that $T(x, y, s) * T(s, t, z) \leq T(x, t, z)$;
- (vi) T is \circ_6 -transitive, if, for any $x, y, z, s, t \in X$, it holds that $T(x, s, t) * T(s, y, z) \leq T(x, y, z)$.

Example 3.

Let L (resp. T_1, T_2) be the lattice (resp. the ternary L -relations on $X = \{a, b, c, d\}$) given in Example 1 and $*$ be a t -norm on L . One easily verifies that

- (i) T_1 is \circ_1 -transitive if and only if $\alpha * \beta = 0$;
- (ii) T_1 is \circ_i -transitive, for any $i \in \{2, \dots, 6\}$.

Also,

- (i) T_2 is \circ_1 -transitive;
- (ii) T_2 is \circ_i -transitive (with $i \in \{2, 3, 6\}$) if and only if $\alpha * \beta = 0$;
- (iii) T_2 is \circ_i -transitive (with $i \in \{4, 5\}$) if and only if $\alpha = 0$.

The following proposition discusses the transitivity of the transpose of a ternary relation.

Proposition 4. *Let T be a ternary L -relation on a set X and $i \in \{1, \dots, 6\}$. It holds that T is \circ_i -transitive if and only if T^t is \circ_{7-i} -transitive.*

Proof. If T is \circ_i -transitive, then $T \circ_i T \subseteq T$. Hence, $(T \circ_i T)^t \subseteq T^t$. Proposition 3 implies that $T^t \circ_{7-i} T^t \subseteq T^t$. Hence, T^t is \circ_{7-i} -transitive. The converse can be proved similarly.

The following proposition shows that transitivity is preserved under intersection.

Proposition 5. *Let $(T_j)_{j \in J}$ be a family of ternary L -relations on a set X and $i \in \{1, \dots, 6\}$. It holds that the \circ_i -transitivity of $(T_j)_{j \in J}$ implies the \circ_i -transitivity of $\bigcap_{j \in J} T_j$.*

Proof. It suffices to prove that $\bigcap_{j \in J} T_j \circ_i \bigcap_{j \in J} T_j \subseteq \bigcap_{j \in J} T_j$. Suppose that $(T_j)_{j \in J}$ is \circ_i -transitive, i.e., $T_j \circ_i T_j \subseteq T_j$, for any $j \in J$. This implies that $\bigcap_{j \in J} (T_j \circ_i T_j) \subseteq \bigcap_{j \in J} T_j$. It is clear that $\bigcap_{j \in J} T_j \circ_i \bigcap_{j \in J} T_j \subseteq \bigcap_{j \in J} (T_j \circ_i T_j)$ and hence $\bigcap_{j \in J} T_j \circ_i \bigcap_{j \in J} T_j \subseteq \bigcap_{j \in J} T_j$. Thus, $\bigcap_{j \in J} T_j$ is \circ_i -transitive.

4.2. Additional Properties

In this subsection, we discuss some additional properties of the types of transitivity of ternary fuzzy relations introduced above. More specifically, we investigate the interaction with the binary projections and the cylindrical extensions. To that end, we first introduce the binary projections of a ternary fuzzy relation and the cylindrical extensions of a binary fuzzy relation. They are immediate generalizations of the corresponding notions in the crisp case.

Definition 4. Let T be a ternary L -relation on a set X .

- (i) The left projection of T with respect to $z \in X$ is the binary L -relation ${}_z T$ on X defined by

$${}_z T(x, y) = T(z, x, y);$$

- (ii) The middle projection of T with respect to $z \in X$ is the binary L -relation T_z on X defined by

$$T_z(x, y) = T(x, z, y);$$

- (iii) The right projection of T with respect to is the binary L -relation T_z on X defined by

$$T_z(x, y) = T(x, y, z).$$

Definition 5. Let T be a ternary L -relation on a set X .

- (i) The left projection of T is the binary L -relation $P_\ell(T)$ on X defined by

$$P_\ell(T)(x, y) = \bigvee_{z \in X} T(x, y, z);$$

- (ii) The middle projection of T is the binary L -relation $P_m(T)$ on X defined by

$$P_m(T)(x, y) = \bigvee_{z \in X} T(x, z, y);$$

- (iii) The right projection of T is the binary L -relation $P_r(T)$ on X defined by

$$P_r(T)(x, y) = \bigvee_{z \in X} T_z(x, y).$$

Definition 6. Let R be a binary L -relation on a set X .

- (i) The left cylindrical extension of R is the ternary L -relation $C_\ell(R)$ on X defined by

$$C_\ell(R)(x, y, z) = R(y, z);$$

- (ii) The middle cylindrical extension of R is the ternary relation $C_m(R)$ on X defined by

$$C_m(R)(x, y, z) = R(x, z);$$

- (iii) The right cylindrical extension of R is the ternary relation $C_r(R)$ on X defined by

$$C_r(R)(x, y, z) = R(x, y).$$

Next, we express the projections of the compositions of ternary fuzzy relations in terms of the compositions of their binary projections. This proposition is a natural generalization of that in the crisp case [29].

Proposition 6. *Let T and S be two ternary L -relations on a set X . The left, middle and right projections of the compositions $T \circ_i S$, for any $i \in \{1, \dots, 6\}$, are listed in the following table:*

Table 1 | Left, middle and right projections of the compositions $T \circ_i S$, for any $i \in \{1, \dots, 6\}$.

Comp.	$P_\ell(\cdot)$	Proj. $P_m(\cdot)$	$P_r(\cdot)$
$T \circ_1 S$	$P_\ell(T) \circ P_\ell(S)$	$P_m(T) \circ P_\ell(S)$	–
$T \circ_2 S$	$P_\ell(T) \circ P_m(S)$	$P_m(T) \circ P_m(S)$	–
$T \circ_3 S$	$P_\ell(T) \circ P_r(S)$	$P_m(T) \circ P_r(S)$	–
$T \circ_4 S$	–	$P_\ell(T) \circ P_m(S)$	$P_\ell(T) \circ P_r(S)$
$T \circ_5 S$	–	$P_m(T) \circ P_m(S)$	$P_m(T) \circ P_r(S)$
$T \circ_6 S$	–	$P_r(T) \circ P_m(S)$	$P_r(T) \circ P_r(S)$

Similarly, the following proposition expresses the cylindrical extensions of the composition of binary fuzzy relations in terms of the compositions of their cylindrical extensions.

Proposition 7. *Let R_1 and R_2 be two binary L-relations on a set X . The left, middle and right cylindrical extensions of the composition $R_1 \circ R_2$ are listed in the following table:*

The following proposition expresses the relationship between the \circ_i -transitivity corresponding to the associative compositions of a given ternary fuzzy relation and the $*$ -transitivity of its binary projections.

Proposition 8. *Let T be a ternary L-relation on a set X . The following implications hold:*

- (i) *If T is \circ_1 -transitive, then $P_\ell(T)$ is $*$ -transitive;*
- (ii) *If T is \circ_2 -transitive, then $P_m(T)$ is $*$ -transitive;*
- (iii) *If T is \circ_5 -transitive, then $P_m(T)$ is $*$ -transitive;*
- (iv) *If T is \circ_6 -transitive, then $P_r(T)$ is $*$ -transitive.*

Proof. We only give the proof for the first implication, the other ones being similar. Suppose that T is \circ_1 -transitive, i.e., $uT \circ_1 T \subseteq T$. Hence, $P_\ell(T \circ_1 T) \subseteq P_\ell(T)$. It is clear from Table 1 that $P_\ell(T \circ_1 T) = P_\ell(T) \circ P_\ell(T)$. Hence, $P_\ell(T) \circ P_\ell(T) \subseteq P_\ell(T)$. Thus, $P_\ell(T)$ is $*$ -transitive.

The following proposition discusses the relationship between the $*$ -transitivity of a given binary fuzzy relation and the \circ_i -transitivity corresponding to the associative compositions of its cylindrical extensions.

Proposition 9. *Let R be a binary L-relation on a set X . The following equivalences hold:*

- (i) *R is $*$ -transitive if and only if $C_\ell(R)$ is \circ_1 -transitive;*
- (ii) *R is $*$ -transitive if and only if $C_m(R)$ is \circ_2 -transitive;*
- (iii) *R is $*$ -transitive if and only if $C_m(R)$ is \circ_5 -transitive;*
- (iv) *R is $*$ -transitive if and only if $C_r(R)$ is \circ_6 -transitive.*

Proof. We only give the proof for the first equivalence, the other ones being similar. Suppose that R is $*$ -transitive, i.e., $R \circ R \subseteq R$. Hence, $C_\ell(R \circ R) \subseteq C_\ell(R)$. From Table 2, it is clear that $C_\ell(R \circ R) = C_\ell(R) \circ_1 C_\ell(R)$. Hence, $C_\ell(R) \circ_1 C_\ell(R) \subseteq C_\ell(R)$. Thus, $C_\ell(R)$ is \circ_1 -transitive.

Conversely, suppose that $C_\ell(R)$ is \circ_1 -transitive, then it holds that $C_\ell(R) \circ_1 C_\ell(R) \subseteq C_\ell(R)$. This implies that $P_\ell(C_\ell(R) \circ_1 C_\ell(R)) \subseteq P_\ell(C_\ell(R))$. From Table 1, we know that $P_\ell(C_\ell(R) \circ_1 C_\ell(R)) = P_\ell(C_\ell(R)) \circ P_\ell(C_\ell(R))$ and this guarantees that $P_\ell(C_\ell(R)) \circ P_\ell(C_\ell(R)) \subseteq P_\ell(C_\ell(R))$. Since $P_\ell(C_\ell(R)) = R$, it follows that $R \circ R \subseteq R$. Thus, R is transitive.

□

5. REPRESENTATION OF TRANSITIVE TERNARY FUZZY RELATIONS

The representability problem in the study of fuzzy relations has a long history. It was first addressed in the eighties [37,38] for binary fuzzy relations and is regularly taken up again [39,40]. It primarily deals with the representation of a transitive fuzzy relation in terms of a family of score functions. In this section, we provide related representation theorems for transitive (resp. reflexive and transitive) ternary fuzzy relations in terms of the binary projections and a related family of (score) functions.

Theorem 10. *Let T be a ternary L-relation on a set X . The following equivalences hold:*

- (i) *T is \circ_1 -transitive if and only if there exists a family $(f_a)_{a \in X}$ of functions from X to L such that $T_a(x, y) \leq f_a(y)$, for any $x, y, a \in X$, and*

$$T(x, y, z) = \bigwedge_{a \in X} f_a(z) \rightarrow T_a(x, y),$$

for any $x, y, z \in X$;

- (ii) *T is \circ_2 -transitive if and only if there exists a family $(f_a)_{a \in X}$ of functions from X to L such that $T_a(x, y) \leq f_a(x)$, for any $x, y, a \in X$, and*

$$T(x, y, z) = \bigwedge_{a \in X} f_a(z) \rightarrow T_a(x, y),$$

for any $x, y, z \in X$;

- (iii) *T is \circ_3 -transitive if and only if there exists a family $(f_a)_{a \in X}$ of functions from X to L such that $T_a(x, y) \leq f_a(y)$, for any $x, y, a \in X$, and*

$$T(x, y, z) = \bigwedge_{a \in X} f_a(y) \rightarrow T_a(x, z),$$

for any $x, y, z \in X$;

- (iv) *T is \circ_4 -transitive if and only if there exists a family $(f_a)_{a \in X}$ of functions from X to L such that $T_a(x, y) \leq f_a(x)$, for any $x, y, a \in X$, and*

$$T(x, y, z) = \bigwedge_{a \in X} f_a(y) \rightarrow T_a(x, z),$$

for any $x, y, z \in X$;

Table 2 | Left, middle and right cylindrical extensions of the composition

$R_1 \circ R_2$.			
Comp. Cyl. ext.	$C_\ell(\cdot)$	$C_m(\cdot)$	$C_r(\cdot)$
$R_1 \circ R_2$	$C_\ell(R_1) \circ_1 C_\ell(R_2)$	$C_m(R_1) \circ_1 C_\ell(R_2)$	
	$C_\ell(R_1) \circ_2 C_m(R_2)$	$C_m(R_1) \circ_2 C_m(R_2)$	
	$C_\ell(R_1) \circ_3 C_r(R_2)$	$C_m(R_1) \circ_3 C_r(R_2)$	
		$C_\ell(R_1) \circ_4 C_m(R_2)$	$C_\ell(R_1) \circ_4 C_r(R_2)$
		$C_m(R_1) \circ_5 C_m(R_2)$	$C_m(R_1) \circ_5 C_r(R_2)$
		$C_r(R_1) \circ_6 C_m(R_2)$	$C_r(R_1) \circ_6 C_r(R_2)$

(v) T is \circ_5 -transitive if and only if there exists a family $(f_a)_{a \in X}$ of functions from X to L such that ${}_a T(x, y) \leq f_a(y)$, for any $x, y, a \in X$, and

$$T(x, y, z) = \bigwedge_{a \in X} f_a(x) \rightarrow_a T(y, z),$$

for any $x, y, z \in X$;

(vi) T is \circ_6 -transitive if and only if there exists a family $(f_a)_{a \in X}$ of functions from X to L such that ${}_a T(x, y) \leq f_a(x)$, for any $x, y, a \in X$, and

$$T(x, y, z) = \bigwedge_{a \in X} f_a(x) \rightarrow_a T(y, z),$$

for any $x, y, z \in X$.

Proof. We only give the proof for the first equivalence, the other ones being similar. First, for any $x, y, z \in X$ we show that

$$T(x, y, z) = \bigwedge_{a \in X} f_a(z) \rightarrow T_a(x, y),$$

with $f_a(z) = \bigwedge_{(s,t) \in X^2} T(s, t, z) \rightarrow T(s, t, a)$, for any $a \in X$. Note that $f_a(a) = 1$.

On the one hand, for any $x, y, z, a \in X$, it holds that

$$\begin{aligned} & T(x, y, z) * f_a(z) \\ &= T(x, y, z) * \bigwedge_{(s,t) \in X^2} T(s, t, z) \rightarrow T(s, t, a) \\ &\leq T(x, y, z) * (T(x, y, z) \rightarrow T(x, y, a)). \end{aligned}$$

The property $a * (a \rightarrow b) \leq b$ guarantees that

$$T(x, y, z) * f_a(z) \leq T(x, y, a) = T_a(x, y).$$

Hence, by the adjointness property, it holds that $T(x, y, z) \leq f_a(z) \rightarrow T_a(x, y)$. Thus,

$$T(x, y, z) \leq \bigwedge_{a \in X} f_a(z) \rightarrow T_a(x, y),$$

for any $x, y, z \in X$.

On the other hand, for any $x, y, z \in X$, it holds that

$$\begin{aligned} & \bigwedge_{a \in X} f_a(z) \rightarrow T_a(x, y) \\ & \leq f_z(z) \rightarrow T_z(x, y) \\ & = 1 \rightarrow T(x, y, z) = T(x, y, z). \end{aligned}$$

Thus,

$$T(x, y, z) = \bigwedge_{a \in X} f_a(z) \rightarrow T_a(x, y),$$

for any $x, y, z \in X$.

Next, suppose that T is \circ_1 -transitive, i.e., for any $x, y, a, s, t \in X$, it holds that $T(s, t, y) * T(x, y, a) \leq T(s, t, a)$. By the adjointness property, it holds that $T(x, y, a) \leq T(s, t, y) \rightarrow T(s, t, a)$. Hence,

$$T(x, y, a) \leq \bigwedge_{(s,t) \in X^2} T(s, t, y) \rightarrow T(s, t, a),$$

for any $x, y, a \in X$, i.e., $T_a(x, y) \leq f_a(y)$.

Conversely, suppose that $T_a(x, y) \leq f_a(y)$, for any $x, y, a \in X$. Let $x, y, z, s, t \in X$, then taking into account that $T(x, y, t) = \bigwedge_{a \in X} f_a(t) \rightarrow T_a(x, y)$, it follows that

$$\begin{aligned} & T(x, y, t) * T(s, t, z) \\ &= \left(\bigwedge_{a \in X} f_a(t) \rightarrow T_a(x, y) \right) * T(s, t, z) \\ &\leq (f_z(t) \rightarrow T_z(x, y)) * T(s, t, z). \end{aligned}$$

Since $T(s, t, z) = T_z(s, t) \leq f_z(t)$, it follows that

$$T(x, y, t) * T(s, t, z) \leq (f_z(t) \rightarrow T_z(x, y)) * f_z(t).$$

The property $a * (a \rightarrow b) \leq b$ guarantees that

$$T(x, y, t) * T(s, t, z) \leq T_z(x, y) = T(x, y, z).$$

Thus, T is \circ_1 -transitive.

□

Similarly, we provide representation theorem for reflexive and transitive ternary fuzzy relations. We recall that a ternary fuzzy relation T on a set X is called reflexive if $T(x, x, x) = 1$, for any $x \in X$.

Theorem 11. Let T be a ternary L -relation on a set X . The following equivalences hold:

- (i) T is reflexive and \circ_1 -transitive if and only if $T_a(x, y) \leq T_a(y, y)$, for any $x, y, a \in X$, and

$$T(x, y, z) = \bigwedge_{a \in X} T_a(z, z) \rightarrow T_a(x, y),$$

for any $x, y, z \in X$;

- (ii) T is reflexive and \circ_2 -transitive if and only if $T_a(x, y) \leq T_a(x, x)$, for any $x, y, a \in X$, and

$$T(x, y, z) = \bigwedge_{a \in X} T_a(z, z) \rightarrow T_a(x, y),$$

for any $x, y, z \in X$;

- (iii) T is reflexive and \circ_3 -transitive if and only if $T_a(x, y) \leq T_a(y, y)$, for any $x, y, a \in X$, and

$$T(x, y, z) = \bigwedge_{a \in X} (T_a(y, y) \rightarrow T_a(x, z)),$$

for any $x, y, z \in X$;

- (iv) T is reflexive and \circ_4 -transitive if and only if $T_a(x, y) \leq T_a(x, x)$, for any $x, y, a \in X$, and

$$T(x, y, z) = \bigwedge_{a \in X} (T_a(y, y) \rightarrow T_a(x, z)),$$

for any $x, y, z \in X$;

- (v) T is reflexive and \circ_5 -transitive if and only if $T_a(x, y) \leq_a T(y, y)$, for any $x, y, a \in X$, and

$$T(x, y, z) = \bigwedge_{a \in X} ({}_a T(x, x) \rightarrow_a T(y, z)),$$

for any $x, y, z \in X$;

- (vi) T is reflexive and \circ_6 -transitive if and only if ${}_a T(x, y) \leq_a T(x, x)$, for any $x, y, a \in X$, and

$$T(x, y, z) = \bigwedge_{a \in X} ({}_a T(x, x) \rightarrow_a T(y, z)),$$

for any $x, y, z \in X$.

Proof. We only give the proof for the first equivalence, the other ones being similar. From Theorem 10, it follows that T is \circ_1 -transitive if and only if there exists a family $(f_a)_{a \in X}$ of functions from X to L such that $T_a(x, y) \leq f_a(y)$, for any $x, y, a \in X$ and:

$$T(x, y, z) = \bigwedge_{a \in X} f_a(z) \rightarrow T_a(x, y),$$

for any $x, y, z \in X$. For the implication from left to right, it suffices to show that $f_a(z) = T_a(z, z)$, for any $a, z \in X$, while it suffices to show that T is reflexive for the converse implication.

Let $a, z \in X$. On the one hand, since $T_a(x, y) \leq f_a(y)$, for any $x, y, a \in X$, it follows that $T_a(z, z) \leq f_a(z)$. On the other hand,

$$f_a(z) = \bigwedge_{(s,t) \in X^2} (T(s, t, z) \rightarrow T(s, t, a)) \leq T(z, z, z) \rightarrow T(z, z, a).$$

The reflexivity of T guarantees that

$$f_a(z) \leq T(z, z, a) = T_a(z, z).$$

Thus, $f_a(z) = T_a(z, z)$, for any $a, z \in X$.

Conversely, the reflexivity of T is obvious from the fact that:

$$T(x, x, x) = \bigwedge_{a \in X} T_a(x, x) \rightarrow T_a(x, x) = 1,$$

for any $x \in X$.

The following example illustrates the above representation theorem.

Example 4.

Let $L = [0, 1]$ and T be the ternary L -relation on the set of real numbers \mathbb{R} given by:

$$T(x, y, z) = \begin{cases} 1, & \text{if } x \leq y \text{ and } z \leq y \\ \lambda(z), & \text{otherwise} \end{cases},$$

where $\lambda : \mathbb{R} \rightarrow L$ is a decreasing mapping. One easily verifies that T is reflexive and \circ_1 -transitive for any t -norm $*$. According to Theorem 11, the ternary L -relation T can be represented as:

$$T(x, y, z) = \bigwedge_{a \in \mathbb{R}} T_a(z, z) \rightarrow T_a(x, y),$$

where

$$T_a(x, y) = \begin{cases} 1, & \text{if } x \leq y \text{ and } a \leq y \\ \lambda(a), & \text{otherwise} \end{cases},$$

for any $x, y, z, a \in \mathbb{R}$.

6. TRANSITIVE CLOSURES

In this section, we focus on the transitive closures of a ternary fuzzy relation with respect to the types of transitivity introduced above. We recall that the transitive closure of a ternary fuzzy relation is the smallest transitive ternary fuzzy relation that includes the given ternary fuzzy relation.

6.1. Transitive Closures of a Ternary Fuzzy Relation

In this subsection, we show that the \circ_i -transitive closures of a ternary fuzzy relation can be written as the union of \circ_i -powers of this ternary fuzzy relation. However, this only holds for the associative compositions. First, we discuss the existence of the \circ_i -transitive closures of a given ternary fuzzy relation.

In general, for a given property P , the following theorem states the conditions under which the P -closure exists for all ternary fuzzy relations. This theorem is obtained in the same way as in Ref. [41] for binary fuzzy relations.

Theorem 12. *A P -closure exists for all ternary L -relations on a set X if and only if*

- (i) *The universal relation X^3 possesses property P ;*
- (ii) *The intersection of every (nonempty) family of ternary L -relations, each of which possesses property P , also possesses property P .*

According to this theorem, Proposition 5 and the fact that the universal relation is \circ_i -transitive for any $i \in \{1, \dots, 6\}$ guarantee the existence of the \circ_i -transitive closures of a given ternary fuzzy relation, for any $i \in \{1, \dots, 6\}$. In order to explicitly describe these \circ_i -transitive closures, we introduce the notion of power of a ternary L -relation.

Definition 7. Let T be a ternary L -relation on a set X and $i \in \{1, 2, 5, 6\}$. The (n, i) -th powers T^{n, \circ_i} of T are recursively defined as

$$T^{1, \circ_i} = T \quad \text{and} \quad T^{n, \circ_i} = T^{n-1, \circ_i} \circ_i T,$$

for any $n \in \mathbb{N}^*$, $n > 1$.

Note that for the nonassociative compositions \circ_3 and \circ_4 , the notion of powers cannot be defined unambiguously. For instance, one could imagine right powers being defined as $T_r^{n, \circ_i} = T_r^{n-1, \circ_i} \circ_i T$ or left powers being defined as $T_\ell^{n, \circ_i} = T \circ_i T_\ell^{n-1, \circ_i}$, but they would generally not coincide for $i \in \{3, 4\}$.

As in the case of binary fuzzy relations, the following proposition shows that the powers of a \circ_i -transitive ternary fuzzy relation are included in this ternary fuzzy relation, for any $i \in \{1, \dots, 6\}$. The proof is straightforward.

Proposition 13. *Let T be a ternary L -relation on a set X and $i \in \{1, 2, 5, 6\}$. It holds that T is \circ_i -transitive if and only if $T^{n, \circ_i} \subseteq T$, for any $n \in \mathbb{N}^*$.*

Remark 1. One easily observes that any reflexive ternary L -relation T is included in its powers, i.e.,

$$T \subseteq T^{2, \circ_i} \subseteq \dots \subseteq T^{n, \circ_i},$$

for any $n > 1$ and any $i \in \{1, 2, 5, 6\}$.

In the following theorem, we characterize the \circ_i -transitive closures of a ternary fuzzy relation, for any $i \in \{1, \dots, 6\}$.

Theorem 14. *Let T be a ternary L -relation on a set X and $i \in \{1, 2, 5, 6\}$. The \circ_i -transitive closure $\text{Tr}_{\circ_i}^{\text{cl}}(T)$ of T is given as*

$$\text{Tr}_{\circ_i}^{\text{cl}}(T) = \bigcup_{n \geq 1} T^{n, \circ_i}.$$

Proof. First, we prove that $\bigcup_{n \geq 1} T^{n, \circ_i} \subseteq \text{Tr}_{\circ_i}^{\text{cl}}(T)$. Since $T \subseteq \text{Tr}_{\circ_i}^{\text{cl}}(T)$, it is easy to verify that $T^{n, \circ_i} \subseteq (\text{Tr}_{\circ_i}^{\text{cl}}(T))^{n, \circ_i}$, for any $n \in \mathbb{N}^*$. Since $\text{Tr}_{\circ_i}^{\text{cl}}(T)$ is \circ_i -transitive, it follows from Proposition 13 that $(\text{Tr}_{\circ_i}^{\text{cl}}(T))^{n, \circ_i} \subseteq \text{Tr}_{\circ_i}^{\text{cl}}(T)$, and hence, $T^{n, \circ_i} \subseteq \text{Tr}_{\circ_i}^{\text{cl}}(T)$, for any $n \in \mathbb{N}^*$. Thus, $\bigcup_{n \geq 1} T^{n, \circ_i} \subseteq \text{Tr}_{\circ_i}^{\text{cl}}(T)$.

Conversely, to prove that $\text{Tr}_{\circ_i}^{\text{cl}}(T) \subseteq \bigcup_{n \geq 1} T^{n, \circ_i}$, it suffices to prove that $S = \bigcup_{n \geq 1} T^{n, \circ_i}$ is \circ_i -transitive. We only give the proof for $i = 1$, the other cases being similar. Let $x, y, z, s, t \in X$, then $S(x, y, t) * S(s, t, z) =$

$$\left(\bigcup_{n \geq 1} T^{n, \circ_1}(x, y, t) \right) * \left(\bigcup_{m \geq 1} T^{m, \circ_1}(s, t, z) \right).$$

Since $*$ is supremum-preserving, it follows that $S(x, y, t) * S(s, t, z) =$

$$\bigcup_{m, n \geq 1} T^{m, \circ_1}(x, y, t) * T^{n, \circ_1}(s, t, z).$$

By definition of the composition \circ_1 , it holds that

$$\begin{aligned} & S(x, y, t) * S(s, t, z) \\ & \leq \bigcup_{m, n \geq 1} T^{m, \circ_1} \circ_1 T^{n, \circ_1}(x, y, z) \\ & = \bigcup_{m, n \geq 1} T^{m+n, \circ_1}(x, y, z) \\ & \leq \bigcup_{k \geq 1} T^{k, \circ_1}(x, y, z) = S(x, y, z). \end{aligned}$$

Thus, S is \circ_1 -transitive. Since $\text{Tr}_{\circ_1}^{\text{cl}}(T)$ is the smallest \circ_1 -transitive ternary L -relation that includes T , it follows that $\text{Tr}_{\circ_1}^{\text{cl}}(T) \subseteq \bigcup_{n \geq 1} T^{n, \circ_1}$.

Therefore, $\text{Tr}_{\circ_i}^{\text{cl}}(T) = \bigcup_{n \geq 1} T^{n, \circ_i}$.

□

The following example illustrates the \circ_i -transitive closures of a ternary fuzzy relation on a finite set, for any $i \in \{1, \dots, 6\}$.

Example 5.

Let L be the lattice given in Example 1, $* = \wedge$ and T be the ternary L -relation on $X = \{a, b, c, d\}$ given by:

$$T(x, y, z) = \begin{cases} 1, & \text{if } (x, y, z) = (c, d, b) \\ \alpha, & \text{if } (x, y, z) = (d, a, d) \\ \beta, & \text{if } (x, y, z) = (a, b, c) \\ 0, & \text{otherwise} \end{cases}$$

One easily computes the \circ_1 -powers of T :

$$T^{2,\circ_1}(x, y, z) = \begin{cases} \alpha, & \text{if } (x, y, z) = (d, a, b) \\ \beta, & \text{if } (x, y, z) = (c, d, c) \\ 0, & \text{otherwise} \end{cases}$$

$$T^{3,\circ_1}(x, y, z) = \begin{cases} \alpha \wedge \beta, & \text{if } (x, y, z) = (d, a, c) \\ 0, & \text{otherwise} \end{cases}$$

$$T^{4,\circ_1} = \emptyset.$$

Thus, $\text{Tr}_{\circ_1}^{\text{cl}}(T) =$

$$\begin{cases} 1, & \text{if } (x, y, z) = (c, d, b) \\ \alpha, & \text{if } (x, y, z) \in \{(d, a, b), (d, a, d)\} \\ \beta, & \text{if } (x, y, z) \in \{(a, b, c), (c, d, c)\} \\ \alpha \wedge \beta, & \text{if } (x, y, z) = (d, a, c) \\ 0, & \text{otherwise} \end{cases}$$

In a similar way, the transitive closures of T corresponding to the other associative compositions can be written as a union of powers:

$$\text{Tr}_{\circ_2}^{\text{cl}}(T) = \begin{cases} 1, & \text{if } (x, y, z) = (c, d, b) \\ \alpha, & \text{if } (x, y, z) = (d, a, d) \\ \beta, & \text{if } (x, y, z) \in \{(a, b, c), (a, b, b)\} \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Tr}_{\circ_5}^{\text{cl}}(T) = \begin{cases} 1, & \text{if } (x, y, z) = (c, d, b) \\ \alpha, & \text{if } (x, y, z) = (d, a, d) \\ \beta, & \text{if } (x, y, z) \in \{(a, b, c), (a, d, b)\} \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Tr}_{\circ_6}^{\text{cl}}(T) = \begin{cases} 1, & \text{if } (x, y, z) = (c, d, b) \\ \alpha, & \text{if } (x, y, z) \in \{(d, a, d), (c, a, d)\} \\ \beta, & \text{if } (x, y, z) = (a, b, c) \\ \alpha \wedge \beta, & \text{if } (x, y, z) \in \{(c, b, c), (d, b, c)\} \\ 0, & \text{otherwise} \end{cases}$$

For the nonassociative compositions, i.e., for $i \in \{3, 4\}$, one computes the smallest \circ_i -transitive ternary relation containing T in an

iterative manner without computing a union of powers. This leads to:

$$\text{Tr}_{\circ_3}^{\text{cl}}(T) = \begin{cases} 1, & \text{if } (x, y, z) = (c, d, b) \\ \alpha, & \text{if } (x, y, z) \in \{(d, a, a), (d, a, d)\} \\ \beta, & \text{if } (x, y, z) \in \{(a, b, c), (a, b, d)\} \\ \alpha \wedge \beta, & \text{if } (x, y, z) \in \{(a, b, a), (a, b, b), \\ & \quad (d, a, b)\} \\ 0, & \text{otherwise} \end{cases}$$

Note that in this case the union of the right powers would have successfully yielded the transitive closure, while the union of the left powers would have missed all triplets with degree of relationship $\alpha \wedge \beta$.

Similarly, one finds

$$\text{Tr}_{\circ_4}^{\text{cl}}(T) = \begin{cases} 1, & \text{if } (x, y, z) = (c, d, b) \\ \alpha, & \text{if } (x, y, z) \in \{(a, a, d), (d, a, d)\} \\ \beta, & \text{if } (x, y, z) \in \{(a, b, c), (b, d, b), \\ & \quad (d, d, b)\} \\ \alpha \wedge \beta, & \text{if } (x, y, z) = (a, d, b) \\ 0, & \text{otherwise} \end{cases}$$

In this case the union of the left powers would have successfully yielded the transitive closure, while the union of the right powers would have missed the triplet (d, d, b) with degree of relationship β and the triplet (a, d, b) with degree of relationship $\alpha \wedge \beta$.

6.2. Properties of Transitive Closures

In this subsection, we investigate various properties of the transitive closures of a ternary fuzzy relation. The following proposition expresses the transitive closures of ternary fuzzy relations obtained by permutation.

Proposition 15. *Let T be a ternary L -relation on a set X . It holds that*

- (i) $\text{Tr}_{\circ_i}^{\text{cl}}(T^+) = (\text{Tr}_{\circ_{i+4}}^{\text{cl}}(T))^+$, for any $i \in \{1, 2\}$;
- (ii) $\text{Tr}_{\circ_i}^{\text{cl}}(T^-) = (\text{Tr}_{\circ_{i-4}}^{\text{cl}}(T))^-$, for any $i \in \{5, 6\}$;
- (iii) $\text{Tr}_{\circ_i}^{\text{cl}}(T^\dagger) = (\text{Tr}_{\circ_{7-i}}^{\text{cl}}(T))^\dagger$, for any $i \in \{1, 2, 5, 6\}$.

Proof. We only prove the first equality, the other ones being similar. Let $i \in \{1, 2\}$, then it follows from Theorem 14 that

$$\text{Tr}_{\circ_i}^{\text{cl}}(T^+) = \bigcup_{n \geq 1} (T^+)^{n,\circ_i}.$$

From Proposition 3, it follows that $(T^+)^{n,\circ_i} = (T^{n,\circ_{i+4}})^+$. This implies that

$$\begin{aligned} \text{Tr}_{\circ_i}^{\text{cl}}(T^+) &= \bigcup_{n \geq 1} (T^{n,\circ_{i+4}})^+ \\ &= \left(\bigcup_{n \geq 1} T^{n,\circ_{i+4}} \right)^+ \\ &= (\text{Tr}_{\circ_{i+4}}^{\text{cl}}(T))^+. \end{aligned}$$

□

The following proposition expresses the left, middle and right projections of the transitive closures of a ternary fuzzy relation.

Proposition 16. *Let T be a ternary L -relation on a set X . The following equalities hold:*

- (i) $P_\ell(\text{Tr}_{\circ_1}^{\text{cl}}(T)) = \text{Tr}^{\text{cl}}(P_\ell(T));$
- (ii) $P_m(\text{Tr}_{\circ_2}^{\text{cl}}(T)) = P_m(\text{Tr}_{\circ_5}^{\text{cl}}(T)) = \text{Tr}^{\text{cl}}(P_m(T));$
- (iii) $P_r(\text{Tr}_{\circ_6}^{\text{cl}}(T)) = \text{Tr}^{\text{cl}}(P_r(T)).$

Proof. We only prove the first equality, the other ones being similar. From Theorem 14, it follows that

$$P_\ell(\text{Tr}_{\circ_1}^{\text{cl}}(T)) = P_\ell\left(\bigcup_{n \geq 1} T^{n, \circ_1}\right).$$

One easily verifies that $P_\ell\left(\bigcup_{n \geq 1} T^{n, \circ_1}\right) = \bigcup_{n \geq 1} P_\ell(T^{n, \circ_1})$. From Proposition 6, it follows that

$$\bigcup_{n \geq 1} P_\ell(T^{n, \circ_1}) = \bigcup_{n \geq 1} (P_\ell(T))^n.$$

Thus,

$$P_\ell(\text{Tr}_{\circ_1}^{\text{cl}}(T)) = \bigcup_{n \geq 1} (P_\ell(T))^n = \text{Tr}^{\text{cl}}(P_\ell(T)).$$

□

Remark 2. For the nonassociative composition \circ_3 (resp. \circ_4), the binary projections of the \circ_3 -transitive (resp. \circ_4 -transitive) closure of a ternary fuzzy relation T are in general not equal to the transitive closure of any projection of T .

The following proposition expresses the transitive closures of the left, middle and right cylindrical extensions of a binary fuzzy relation.

Proposition 17. *Let R be a binary L -relation on a set X . The following equalities hold:*

- (i) $\text{Tr}_{\circ_1}^{\text{cl}}(C_\ell(R)) = C_\ell(\text{Tr}^{\text{cl}}(R));$
- (ii) $\text{Tr}_{\circ_2}^{\text{cl}}(C_m(R)) = \text{Tr}_{\circ_5}^{\text{cl}}(C_m(R)) = C_m(\text{Tr}^{\text{cl}}(R));$
- (iii) $\text{Tr}_{\circ_6}^{\text{cl}}(C_r(R)) = C_r(\text{Tr}^{\text{cl}}(R)).$

Proof. We only prove the first equality, the other ones being similar. Since $\text{Tr}^{\text{cl}}(R) = \bigcup_{n \geq 1} R^n$, it follows that

$$C_\ell(\text{Tr}^{\text{cl}}(R)) = C_\ell\left(\bigcup_{n \geq 1} R^n\right).$$

One easily verifies that

$$\bigcup_{n \geq 1} C_\ell(R^n) = \bigcup_{n \geq 1} (C_\ell(R))^{n, \circ_1}.$$

Thus,

$$C_\ell(\text{Tr}^{\text{cl}}(R)) = \bigcup_{n \geq 1} (C_\ell(R))^{n, \circ_1} = \text{Tr}_{\circ_1}^{\text{cl}}(C_\ell(R)).$$

□

Remark 3. For the nonassociative composition \circ_3 (resp. \circ_4), the cylindrical extensions of the transitive closure of a binary fuzzy relation R are in general not equal to the \circ_3 -transitive (resp. \circ_4 -transitive) closure of any cylindrical extension of R .

7. CONCLUSION

In this paper, we have extended the types of composition of ternary relations to the fuzzy setting. We have introduced several types of transitivity of a ternary fuzzy relation close in spirit to the $*$ -transitivity of a binary fuzzy relation. We have investigated their basic properties and their interaction with the binary projections of ternary fuzzy relations and cylindrical extensions of binary fuzzy relations. Also, we have provided representations of transitive (resp. reflexive and transitive) ternary fuzzy relations for the four types of transitivity corresponding to the associative compositions. Furthermore, we have studied the problem of closing a ternary fuzzy relation with respect to the proposed types of transitivity.

In future work, we intend to exploit the results obtained in this paper to study some important classes of ternary fuzzy relations, such as the fuzzy betweenness relations recently introduced by Zhang *et al.* [42] and ternary fuzzy equivalence relations.

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

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